A LINEAR BOUND ON THE DIMENSION IN GREEN-RUZSA’S THEOREM

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Abstract. In this paper a linear bound on the dimension in the Green-Ruzsa version of Freiman’s theorem is obtained. This result is best possible up to a constant.

1. Introduction

Throughout this paper we will consider finite subsets of a (not necessarily finite) abelian group $G$. For any subsets $A, B \subseteq G$ we define the sumset $A + B = \{a + b : a \in A, b \in B\}$ and call $K(A) = |A + A|/|A|$ the doubling of $A$. In the paper, $C$ denotes a constant which can vary from line to line.

The family of Freiman’s-type theorems deals with finite sets $A$ of small doubling, when compared with $|A|$. If that is the case then $A$ forms a big part of a (proper) coset progression of dimension at most $d(K)$ and size at most $f(K)|A|$.

Let us recall the definition of a coset progression. It is any subset of $G$ of the form $P + H$ where $H$ is a subgroup of $G$ and

$$P = P(x_0; x_1, \ldots, x_d; L_1, \ldots, L_d) = \left\{ x_0 + \sum_{i=1}^{d} l_i x_i : |l_i| \leq L_i \right\}$$

is a generalized arithmetic progression of dimension $d$ and size $(2L_1 + 1) \cdot \ldots \cdot (2L_d + 1)$. The dimension $d(P + H)$ of a coset progression $P + H$ is the dimension $d(P)$ of its underlying generalized arithmetic progression $P$ and size($P + H$) is size($P$)$|H|$. We say that a progression is proper if its cardinality equals its size.

As can be easily verified, the best possible bound for $d(P)$ is $|K - 1|$. Similarly, one cannot hope to obtain anything better than size($P$) = $\exp(O(K))|A|$.

Freiman’s original result, which originates to the late 60s and appearance of monograph [1], concerns torsion-free groups only and is very inefficient in bound for $f(K)$. We owe to Sander’s work [6], built upon Ruzsa’s [2] and Chang’s [3], the following formulation of Freiman’s theorem.

Theorem 1 (Sander). Let $A$ be a finite subset of a torsion-free group $G$. If $|A + A| \leq K|A|$, then Freiman’s theorem holds with $d(K) = CK^{7/4}\log^3 K$ and $f(K) = \exp(CK^{7/4}\log^3 K)$.

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Chang obtains even sharper bound on the dimension, at the cost of a slightly higher degree of a polynomial in the exponent of $f$. Moreover, some additional conditions on $|A|$ must be imposed.

**Theorem 2** (Chang). Under the assumptions of Theorem 1, if, additionally, for some $\epsilon > 0$, $|A| \geq \max(CK^2 \log^2 K_1(K + \epsilon)^2/2\epsilon)$, then $A \subseteq P$ where $P$ is a proper generalized arithmetic progression of dimension $d(P) \leq [K - 1 + \epsilon]$ and $\text{size}(P) \leq \exp(CK^2 \log^3 K)|A|$.

Observe that in the torsion-free setting every finite coset progression is in fact a generalized arithmetic progression.

In a recent paper [3], Green and Ruzsa established a generalization of Theorem 1 for arbitrary abelian groups.

**Theorem 3** (Green-Ruzsa). Let $A \subseteq G$ be finite and $|A + A| \leq K|A|$. Then $A$ is contained in a coset progression $P + H$ of dimension $d(P + H) \leq CK^4 \log(K + 2)$ and $\text{size}(P + H) \leq \exp(CK^4 \log^2(K + 2)|A|)$.

In the abelian groups setting, the necessity of using coset progressions, in place of generalized ones, follows from consideration of a family of examples with $A = G = \mathbb{Z}^d_2$. In this case, the doubling of $A$ equals 1 independently of $d$. Hence, the dimension cannot be bounded by any function of the doubling.

In what follows we show an analog of Theorem 2 in the general abelian groups setting, which is this.

**Theorem 4.** Under the assumptions of Theorem 3, either $A \subseteq P + H$ for a proper coset progression such that $d(P + H) \leq 2K$ and $\text{size}(P + H) \leq \exp(CK^4 \log^2(K + 2)|A|)$, or $A$ is fully contained in at most $CK^3 \log^2 K$ cosets, whose total cardinality is bounded by $\exp(CK^4 \log^2(K + 2)|A|)$, of some subgroup of $G$.

A weaker result $d(P + H) \leq K^2$ was communicated by Green and Ruzsa and announced in [3, Exercise 6.5.18].

2. GEOMETRY OF NUMBERS

In this section, we aim to prove the following two lemmas. Basically, they state that coset progressions are economically contained inside proper (convex) coset progressions.

**Lemma 5.** Let $X + H$ be a convex coset progression of dimension $d$. Then, for every integer $s \geq 1$, there exists an $s$-proper convex coset progression $X' + H'$ containing $X + H$ of dimension $d' \leq d$ and $\text{size}(X' + H') \leq s^d d^{C\ell \ell} \text{size}(X + H)$.

**Lemma 6.** Under the assumptions of Lemma 5, there exists an $s$-proper coset progression $P' + H'$ containing $X + H$ of dimension $d' \leq d$ and $\text{size}(P' + H') \leq s^d d^{C\ell \ell} \text{vol}(X)|H|$.

Here we provide some necessary definitions.

Suppose that $B \subseteq \mathbb{R}^d$ is closed, centrally symmetric and convex, $B \cap \mathbb{Z}^d$ spans $\mathbb{R}^d$ as vector space and $\phi : \mathbb{Z}^d \to G$ is a homomorphism. Then we refer to the image $X = \phi(B \cap \mathbb{Z}^d)$ as a convex progression of dimension $d$. The size of $X$ is simply $\text{size}(X) = |B \cap \mathbb{Z}^d|$, and the volume is $\text{vol}(X) = \text{vol}_d(B)$, the $d$-dimensional volume of $B$ in $\mathbb{R}^d$. 


Let $X$ be a convex progression and $H$ be a subgroup of $G$. Then we call $X + H$ a **convex coset progression**. By analogy with coset progressions, we define $\text{size}(X + H) = \text{size}(X)|H|$.

If $s \geq 1$ is an integer and if $\phi(x_1) - \phi(x_2) \in H$ implies $x_1 = x_2$ for all $x_1, x_2 \in sB \cap \mathbb{Z}^d$, then we say that $X + H$ is $s$-proper.

In order to relate progression’s size to its volume we quote the following lemma.

**Lemma 7** ([8, Lemma 3.26 and Inequality 3.14]). Suppose that $X$ is a convex progression. Then

$$\frac{1}{2^d} \leq \frac{\text{size}(X)}{\text{vol}(X)} \leq \frac{3^d d!}{2^d}.$$

**Proof of Lemma 7** We proceed by induction on $d$ by reducing progression’s dimension whenever it is not $s$-proper. Obviously, any zero-dimensional progression is so.

Fix $s$ and let $X = \phi(B \cap \mathbb{Z}^d)$ for some $d > 0$. If $X + H$ is not $s$-proper then there exists a non-zero $x_h \in 2sB \cap \mathbb{Z}^d$ such that $\phi(x_h) \in H$. Consider $x_{\text{irr}} \in 2sB \cap \mathbb{Z}^d$ such that $x_h = m x_{\text{irr}}$ for $m \in \mathbb{N}$ as big as possible. Then, as an immediate consequence of [8, Lemma 3.4], there exists a completion $(x_1, \ldots, x_{d-1}, x_{\text{irr}})$ of $x_{\text{irr}}$ to an integral basis of $\mathbb{Z}^d$.

Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be the linear transformation satisfying $\psi(x_i) = e_i, i = 1, \ldots, d-1$ and $\psi(x_{\text{irr}}) = e_d$ for $(e_i)$ the canonical basis of $\mathbb{Z}^d$. For such transformation, $\psi(\mathbb{Z}^d) = \mathbb{Z}^d$ and $\text{vol}_d(\psi(B)) = \text{vol}_d(B)$.

Let $B' = \pi_{\mathbb{R}^{d-1} \times \{0\}}(\psi(B))$ and $H' = \langle H, \phi(x_{\text{irr}}) \rangle$ be, respectively, the projection of $\psi(B)$ onto the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ and the subgroup of $G$ generated by $H$ and $\phi(x_{\text{irr}})$.

Since one can treat $\phi \circ \psi^{-1}_{|\mathbb{R}^{d-1} \times \{0\}}$ as some $\phi' : \mathbb{R}^{d-1} \to G$, we have $X + H \subseteq X' + H'$ for $X' = \phi'(B' \cap \mathbb{Z}^{d-1})$. Indeed, for an arbitrary element of $X + H$ we have the following representation, with $x \in \mathbb{R}^{d-1} \equiv \mathbb{R}^{d-1} \times \{0\}$, $l \in \mathbb{Z}$ and $h \in H$:

$$\phi(\psi^{-1}(x + lx_{\text{irr}}) + h = \phi'(x) + (l \phi(x_{\text{irr}}) + h) \in X' + H'.$$

Next, we estimate the size of $X' + H'$ but, for technical reasons, we prefer to consider $\text{vol}(X')|H'|$ instead. These two quantities are related by Lemma 7.

Since $$m \phi(x_{\text{irr}}) = \phi(x_h) \in H,$$

it follows that

$$|H'| = |\langle H, \phi(x_{\text{irr}}) \rangle| = |H + \{0, \phi(x_{\text{irr}}), \ldots, (m-1) \phi(x_{\text{irr}})\}| \leq m|H|.$$
Notice that the inequality $\text{vol}_d(O) \leq \text{vol}_d(\psi(B))$ is a non-trivial one because, in general,
\[ B' = \pi_{\mathbb{R}^{d-1} \times \{0\}}(\psi(B)) \nsubseteq \psi(B) \cap (\mathbb{R}^{d-1} \times \{0\}) \]
and therefore $O \nsubseteq \psi(B)$. Instead, let us consider the convex set $\tau(\psi(B))$, where
\[ \tau(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1}, x_d - CM_{\psi(B)}(x_1, \ldots, x_{d-1})) \]
$CM_{\psi(B)}(\cdot)$ denoting the center of mass of the corresponding fibre of $\psi(B)$. Obviously, in the spirit of Fubini’s theorem, $\text{vol}_d(\tau(\psi(B))) = \text{vol}_d(\psi(B))$. Moreover,
\[ B' \subset \tau(\psi(B)) \quad \text{and} \quad \pm \psi(x_h/2s) = \pm m/2s \cdot e_d \in \psi(B) \cap \tau(\psi(B)) \]
so $O \subseteq \tau(\psi(B))$ and hence $\text{vol}_d(O) \leq \text{vol}_d(\psi(B))$.

By inductive argument and Lemma 4, we can obtain an $s$-proper convex coset progression $X'' + H'' \supset X + H$ of dimension $d'' \leq d$, such that
\[
\text{size}(X'')|H''| \leq \frac{3^d d!}{2^d} \text{vol}(X'')|H''| \\
\leq \left(\frac{3s}{2}\right)^d (d!)^2 \text{vol}(X)|H| \\
\leq (3s)^d (d!)^2 \text{size}(X)|H| \\
= s^d \exp(C d \log d) \text{size}(X)|H|.
\]

We prove Lemma 6 in much the same way as [4, Theorem 2.5] with an application of [4, Lemma 2.3] replaced by that of Lemma 3. One can check that both proofs result in the same asymptotic bounds on $\text{size}(P' + H')$ as both [4, Lemma 2.3] and Lemma 5 establish them asymptotically the same.

3. The main argument

Let us first introduce a notion of projection. For any $s$-proper convex coset progression $X + H$ we define the canonical projection $\pi_{sX}(\cdot)$ of $sX + H$ onto $sX$ in the following way: $\pi_{sX}(x + h) = x$ for $x \in sX$ and $h \in H$. Since $X + H$ is $s$-proper, this definition is unambiguous. Of course any $s$-proper progression is so for all $s' \leq s$ and we can consider relevant projections $\pi_{s'X}(\cdot)$ for $s' \leq s$.

We will now show an auxiliary lemma which roughly relates the doubling of a set to additive properties of its projection.

Lemma 8. Let $A \subseteq X + H$, where $X + H$ is a 2-proper convex coset progression and $K_{\min} = \min_{Y \subseteq \pi_X(A)} |Y + \pi_X(A)|/|Y|$. Then $K(A) \geq K_{\min}$.

Proof. Let $y_1, y_2, \ldots \in \pi_X(A)$ be all elements of $Y = \pi_X(A)$ in decreasing order with respect to the cardinality $|A_H(y_i)|$ of $A_H(y_i) = A \cap (y_i + H)$. Write $Y_i = \{y_1, \ldots, y_i\}$.
Then, by assumption, \(|Y_i + Y| \geq iK_{\text{min}}\) and there are at least \(|A_H(y_i)|\) elements of \(A + A\) in every \(H\)-coset of \(Y_i + Y + H\). Hence

\[
|A + A| \geq \sum_i (|Y_i + Y| - |Y_{i-1} + Y|) \cdot |A_H(y_i)|
= \sum_i |Y_i + Y| \cdot (|A_H(y_i)| - |A_H(y_{i+1})|)
\geq \sum_i iK_{\text{min}} \cdot (|A_H(y_i)| - |A_H(y_{i+1})|)
\geq K_{\text{min}} \sum_i (i - (i - 1)) |A_H(y_i)|
= K_{\text{min}} |A|
\]

\[\square\]

Notice that, as direct consequence, this lemma allows us to prove some version of the Green-Ruzsa theorem provided we can bound \(K_{\text{min}} = \min_{Y \subseteq X} |Y + X|/|Y|\) in terms of the doubling \(K(X)\). While Plünnecke’s inequality \([8, \text{Corollary 6.28}]\) leads to a quadratic bound on dimension, we need some more elaborate reasoning to obtain a linear one.

Here we prove a slightly more general version of Theorem 4.

**Theorem 9.** Let \(A \subseteq G\) satisfy \(|A + A| \leq K |A|\). Then for any integer \(s \geq 1\) either there exists an \(s\)-proper coset progression \(P + H\) of dimension \(d(P + H) \leq 2 |K|\) and \(\text{size}(P + H) \leq s^{2K} \exp(C K^4 \log^2(K + 2)) |A|\) such that \(A \subseteq P + H\), or \(A\) is fully contained in at most \(CK^3 \log^2 K\) cosets, whose total cardinality is bounded by \(\exp(C K^4 \log^2(K + 2)) |A|\), of some subgroup of \(G\).

**Proof.** By Theorem 3 and Lemma 5, \(A\) is contained in a 2-proper convex coset progression \(X + H\) of dimension \(d \leq C K^4 \log(K + 2)\) and \(\text{size}(X + H) \leq \exp(C K^4 \log^2(K + 2)) |A|\). Write \(X = \phi(B \cap \mathbb{Z}^d)\).

Consider \(Z = \phi^{-1}(\pi_X(A)) \subset \mathbb{Z}^d\). Let

\[
K_{\text{min}} = \min_{T \subseteq Z} |T + Z|/|T| = |S + Z|/|S|
\]

for some \(S \subseteq Z\). Obviously, \(|S + Z|/|S| \leq K_{\text{min}} \leq K\), the last inequality stemming from Lemma 8. We consider two cases: either \(|S| \geq CK^2_{\text{min}} \log^2 K_{\text{min}}\) and therefore \(S\) satisfies the assumptions of Theorem 2, or \(S\) is too small.

In the first case, by Chang’s theorem, there exists a generalized arithmetic progression \(Q\) containing \(S\), of dimension \(d(Q) \leq K_{\text{min}}\) and \(\text{size}(Q) \leq \exp(C K_2^{\text{min}} \log^3 K_{\text{min}}) |S|\).

By a well known Ruzsa’s covering lemma \([8, \text{Lemma 2.14}]\) there exists a subset \(Z' \subseteq Z\) such that \(|Z'| \leq |S + Z|/|S| = K_{\text{min}}\) and \(Z \subseteq Z' + S - S \subseteq Z' + Q - Q\). Therefore, \(Z\) is contained in a generalized arithmetic progression \(Q'\) of dimension \(d(Q') \leq |Z'| + d(Q) \leq 2 |K_{\text{min}}|\) and \(\text{size}(Q') \leq 3^{|Z'|} 2^D Q^{D/2} \text{size}(Q) = \exp(C K_2^{\text{min}} \log^3 K_{\text{min}}) |X|\).

The case concludes by moving back by \(\phi\) to \(G\): for \(P = \phi(Q')\) we find \(A \subseteq P + H\), the coset progression \(P + H\) is of dimension \(d(P + H) \leq 2 |K_{\text{min}}|\) and

\[
\text{size}(P + H) \leq \exp(C K_2^{\text{min}} \log^3 K_{\text{min}}) |X|/|H| \leq \exp(C K^4 \log^2(K + 2)) |A|.
\]

Application of Lemma 8 gives the desired result.
On the other hand, if $|S| < CK^2 \log^2 K_{\min}$ then $|Z| \leq |Z + S| = K_{\min}|S| < CK^3 \log^2 K$.

This concludes the proof. \hfill \Box

4. Remarks

A new version of Bogolyubov-Ruzsa’s lemma, proved in [7], results in the following bounds in Freiman’s and Green-Ruzsa’s theorems.

**Theorem 10** ([7] Theorems 1 and 2). Let $A$ be finite and satisfy $|A + A| \leq K|A|$. Then Freiman’s theorem holds with $d(K) = K^{1+C(\log K)^{-1/2}}$ and $f(K) = \exp(d(K))$ if $G$ is torsion-free and with $d(K) = (K + 2)^{3+C(\log(K+2))^{-1/2}}$ and $f(K) = \exp(d(K))$ otherwise.

These may potentially serve to obtain still better bounds in Theorem 4. To this end, we will formulate a slightly improved version of Chang’s Theorem 2.

**Theorem 11** (Chang). Let $A \subseteq G$ be a finite subset of a torsion-free group $G$ and $|A + A| \leq K|A|$. If $|A| \geq K^{1+C(\log K)^{-1/2}}$ then $A$ is contained in a proper generalized arithmetic progression $P$ of dimension $d(P) \leq (1 + o(1)) K$ and size($P$) \leq $\exp(CK^2 \log K)|A|$. If, additionally, $|A| \geq (K + \epsilon)^2 / 2 \epsilon$, for $\epsilon > 0$, then $d(P) \leq K - 1 + \epsilon$.

**Proof (sketch).** This sketch will follow Green’s exposition [4] proof of Theorem 3.2.

By Theorem 10 and Lemma 5, $A \subseteq X$ where $X = \phi(B \cap Z^d)$ is a 2-proper convex progression of dimension $d \leq d(K) = K^{1+C(\log K)^{-1/2}}$ and size($X$) bounded by $\exp(K^{1+C(\log K)^{-1/2}})|A|$. Let us denote by $d'$ the dimension of the linear space spanned by $\phi^{-1}(A)$. If $d' \leq K - 1$, we can skip the next few steps, where we establish bounds on $d'$.

Otherwise, by Freiman’s lemma [8 Lemma 5.13],

$$K|A| \geq |A + A| \geq (d' + 1)|A| - d'(d' + 1)/2$$

and

$$|A| \leq r(d') = \frac{d'(d' + 1)}{2(d' + 1 - K)}.$$

Let us define $d''$ as the second solution to the equation $r(x) = r(d(K))$, equivalent to

$$x^2 - x \left(\frac{d(K)(d(K) + 1)}{d(K) + 1 - K} - 1\right) + (K - 1)\frac{d(K)(d(K) + 1)}{d(K) + 1 - K} = 0.$$

By Viète’s formula

$$d'' = \frac{d(K)(d(K) + 1)}{d(K) + 1 - K} - 1 - d(K) = \frac{d(K) + 1}{d(K) + 1 - K}(K - 1) = (1 + o(1))K.$$

Since $r$ is convex, $r(d') \geq |A| > r(d(K)) = K^{1+C(\log K)^{-1/2}}$ and $d' \leq d(K)$, we have

$$d' \leq d'' = (1 + o(1))K.$$

If $|A| \geq (K + \epsilon)^2 / 2 \epsilon > r(|K - 1 + \epsilon|)$, for $\epsilon > 0$, we can conclude that $d' \leq |K - 1 + \epsilon|$.

It remains to show that the slice of $B$ containing $\phi^{-1}(A)$, contained itself in the hyperplane of dimension $d'$, is reasonably small. This is done in exactly the same manner as in Green’s exposition [4]. \hfill \Box
A literal repetition of the proof of Theorem 4 gives the following result.

**Theorem 12.** Let $A \subseteq G$ satisfy $|A + A| \leq K|A|$. Then for any integer $s \geq 1$ either $A \subseteq P + H$ for an $s$-proper coset progression $P + H$ of dimension $d(P + H) \leq (2 + o(1))K$ and size $(P + H) \leq s^{2K} \exp(C(K + 2)^3 \log(K + 2))|A|$, or $A$ is fully contained in at most $K^{2+\mathcal{O}(\log(K+2))^{-1/2}}$ cosets of some subgroup of $G$ whose total cardinality is bounded by $\exp(C(K + 2)^3 \log(K + 2))|A|$.

Observe that this formulation exhibits some imperfection of characterization of unstructured sets $A$. Obviously, we would prefer to bound the number of cosets containing $A$ by $K^{1+\epsilon}$ instead of $K^{2+\mathcal{O}(\log(K+2))^{-1/2}}$. This would be near-optimal since $2K - 1$ is an obvious lower bound for this problem.

**References**


