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FS and RB domains - a distinguishing example

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FS and RB domains - a distinguishing example

1 Basic definitions

Partially ordered set or a *poset* is a set P with reflexive, transitive and antisimmetric binary relation $\leqslant$. We say that u is an *upper bound* of a set $X \subseteq P$ if $x \leqslant u$ for all $x \in X$. If the set of all upper bounds of X has a least element, we call it *supremum* and denote by sup. We say that a subset D of P is directed if it is nonempty and for every $x, y \in D$ there exists an element $z \in D$ such that $x \leqslant z$ and $y \leqslant z$. The notions of a lower bound, infimum and filtered set are defined dually. A poset D in which every directed subset has a supremum we call a directed-complete partial order, or dcpo for short. A dcpo is *pointed* iff it has a least element. A function $f : P \to S$ between two posets is *order preserving* or *monotone* iff $x \leqslant y$ implies $f(x) \leqslant f(y)$. A function $g : P \to S$ between two dcpos is *Scott-continous* if it preserves suprema of directed sets. We say that x is *way below* y (and denote it $x \ll y$) iff for any directed set $D \subseteq P$ such that $\sup D$ exists and $y \leqslant \sup D$, there exists a $d \in D$ with $x \leqslant d$. A *basis* for P os a subset B of P such that for all $x \in P$ the intersection $B_x := B \cap \{y | y \ll x\}$ is directed and $x = \sup B_x$. A poset P is called *continous* if it has a basis. a continous dcpo is a *domain.*

If S is a dcpo then directed set of functions $D \subseteq [S \to S]$ is called an *approximate identity* if it satisfies $\sup D = 1_S$, i.e. its spremum is the identity on S. A Scott continous function on dcpo is *finitely separating* if there exists a finite set F_δ such that for all $x - inS$, there exists $y \in F_\delta$ such that $\delta(x) \leqslant y \leqslant x$. A function $f : P \to S$ is a *idempotent deflation* if it is below identity and its image is finite. Main role in this paper will be played by two related categories of dcpos:

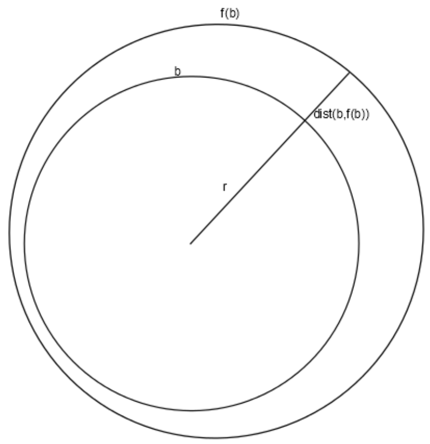
- RB: Pointed dcpos in which the identity map is a supremum of a directed set of idempotent deflations
- FS: Pointed dcpos in which the identity map is a supremum of a directed set of finitely separating maps

The main goal of this paper is to disprove following conjecture of Jung([2],[3]):

Conjecture 1.1. *The classes of RB and FS domains are equal.*

2 Main proof

Let D_C be a domain of closed discs in the plane with origins in set C and radii smaller than radius of C, ordered by reversed inclusions and with the plane added as the least element, which is known to be an FS-domain([1]). We will call a function $f : D_C \to D_C$ *stable* if it preserves origins of all balls. For function $f : D_C \to D_C$ and $b \in D_C$ lets define $\mathrm{dist}_\phi(b, f(b))$ as a length of the part of extension of radius r of b between boundary of b and boundary of $f(b)$(see picture 1).



By $\mathrm{rad}(b)$ and $\mathrm{org}(b)$ we will denote radius and origin of ball b. By $\pi_r(A)$ for $A \in D_C$ we will mean $A \cap \{b \in A; \mathrm{rad}(b) = r\}$. For a deflation with finite image f we will define following functions measuring its distance from identity:

1. $\mathrm{err}_f(x, r) = \int_0^{2\pi} \mathrm{dist}(b, f(b)) \mathrm{d}\phi$
2. $\mathrm{err}_f(C, r) = \int_C \mathrm{err}(x, r) \mathrm{d}x$
3. $\mathrm{Err}(C) = \max_{D_C}(\mathrm{dist}(\mathrm{org}(b), \mathrm{org}(f(b))))$

We will disprove conjecture of Jung by proving following theorem:

Theorem 2.1. *If $f : D_C \to D_C$ for $C = \mathrm{B}(0, 1)$ is a stable function and a supremum of directed set of idempotent deflations then for every r there is $\mathrm{err}(C, r) \geqslant \pi(\pi - 2)r$.*

It is much easier to consider limits of sequences than supremas of directed sets so first we prove the following:

Lemma 2.2. *If $f : D_C \to D_C$ is a supremum of a directed set S of idenpotent deflations and C is compact then there exists sequence of idempotent deflations f_i such that $\lim_{n\to\infty} f_n = f$ uniformly.*

Proof. Since C is compact, $C \times < 0, \mathrm{rad}(C) >$ is also compact so its enough to show thesis for pointwise convergence. Let $b \in D_C$ be any ball. Take any three diffrent angles ϕ_1, ϕ_2, ϕ_3 and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\mathrm{dis}_{\phi_i}(b, f(b)) \geqslant \alpha_i$ for every $f \in S$ and α_i is smallest possible. Let $(f_{\phi_i,k})_{k\in\mathbb{N}}$ be a sequence converging to f_i such that $\mathrm{dis}_{\phi_i}(b, f_i(b)) = \alpha_i$. We can construct sequence converging to function f' with $\mathrm{dis}_{\phi_i}(b, f(b)) = \alpha_i$ by taking $f_i = \sup(f_{\phi_1,i}\, f_{\phi_2,i}, f_{\phi_3,i})$. Now if there is $f'' \in S$ such that $f'(b) \not\subseteq f''(b)$ then we can take $\sup(f_n, f'') \in S$ for n big enough to have contradiction with $\mathrm{dis}_{\phi_i}(b, f(b)) \geqslant \alpha_i$, because there is precisely one minimal ball containing any three points. $\square$

For some technical reasons to be explained later we need also the following:

Lemma 2.3. *If $f : D_C \to D_C$ is a deflation then there exists a deflation f' such that* $\sup_{x\in C, r\in\mathbb{R}}(\mathrm{err}_{f'}(x,r)) \leqslant \sup_{x\in C, r\in\mathbb{R}} \mathrm{err}_f(x,r)$, $\partial(f^{-1}(r))$ *is nowhere dense for every r and* $| f'(D_C) | \leqslant | f(D_C) |$

Proof. Let $f_0 := f$ and $b_1, \ldots, b_n \in D_C$ be all balls in image of f. We define the following sequence of functions and proof in three steps that it's a proper construction of $f' = f_n$:

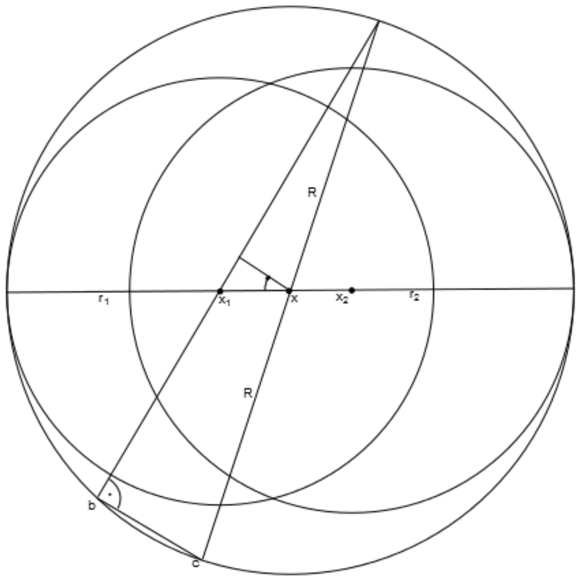
$$f_{n+1}(b) = \begin{cases} b_{i+1}, & \text{if } b \in \mathrm{int}(\mathrm{cl}(f_i^{-1}(b_{i+1}))) \\ f(b), & \text{if } b \notin \mathrm{int}(\mathrm{cl}(f_i^{-1}(b_{i+1}))) \end{cases}$$

Step 1. f_i is monotonic. Suppose that $f_{i-1}$is monotonic (its true for $i = 1$). If $f_i(\mathrm{B}(x,r)) \neq f_{i-1}(\mathrm{B}(x,r))$ than there is a sequence $x_i \to x$ such that $f_i(\mathrm{B}(x_i)) = f_{i-1}(\mathrm{B}(x_i))$
Step 2. $\sup_{x\in C, r\in\mathbb{R}}(\mathrm{err}_{f_i}(x,r)) \leqslant \sup_{x\in C, r\in\mathbb{R}} \mathrm{err}_{f_{i-1}}(x,r)$ is obvious since for every ball such that $f_i(\mathrm{B}(x,r)) \neq f_{i-1}(\mathrm{B}(x,r))$ there is a sequence $\mathrm{B}(x_k,r)$ such that $f_{i-1}(\mathrm{B}(x_k,r)) = f_i(\mathrm{B}(x,r))$ so it holds also for $\mathrm{B}(x,r)$.
Step 3. $\partial(\mathrm{org}(\pi_r(f_n^{-1}(b_i))))$ is nowhere dense for every r and k where π_r is a projection on of balls with radii r. Straigth from construction it follows that $\partial(\mathrm{org}(\pi_r(f_i^{-1}(b_i))))$ is nowhere dense. Note that in construction of sequence f_i we changed value of one ball at most once so the thesis follows also for f_n.

□

Lemma 2.4. *If $b(x_1,r_1)$ and $b(x_2,r_2)$ are balls and $x_i \to x$ and $x'_i \to x$ sequences such that* $\forall_i f(\mathrm{B}(x_i,r)) = \mathrm{B}(x_1,r_1) \wedge f(\mathrm{B}(x'_i,r)) = \mathrm{B}(x_2,r_2)$ *then* $\lim_{\epsilon\to 0} \mathrm{err}(\mathrm{B}(x,r+\epsilon)) - \mathrm{err}(\mathrm{B}(x_1,r)) = \pi(\mathrm{dist}(x_1,x_2) + r_2 - r_1) + o(\mathrm{dist}(x_1,x_2)^2)$.

Proof. Let $\alpha = \mathrm{dist}(x_1,x_2) + r_2 - r_1$ and $\beta = \mathrm{dist}(x_1,x)$, where $\mathrm{B}(x,R)$ is the smallest ball containing $\mathrm{B}(x_1,r_1)$ and $\mathrm{B}(x_2,r_2)$. Note that $2r_1 + \alpha = 2R$ and $\mathrm{dist}(b,c) = 2\beta\cos\phi$(see picture) so $\mathrm{dist}_\phi(\mathrm{B}(x_1,r_1), f(\mathrm{B}(x_1,r_1))) = \sqrt{(2r_1+\alpha)^2 - (2\beta\cos(\phi))^2} - 2r_1$



We estimate using Taylor series for square root as follows: $\mathrm{err}(\mathrm{B}(x, r+\epsilon)) - \mathrm{err}(\mathrm{B}(x_i, r)) = \int_0^\pi (\sqrt{(2r_1+\alpha)^2 - (2\beta\cos(\phi))^2} - 2r_1)\mathrm{d}\phi = \int_0^\pi (\sqrt{(2r_1+\alpha)^2} - \frac{\sqrt{(2r_1+\alpha)^2}(2\beta\cos(\phi))^2}{2(2r_1+\alpha)^2} - o(\beta^2) - 2R)\mathrm{d}\phi = \int_0^\pi (2r_1+\alpha - \frac{(2\beta\cos(\phi))^2}{2(2r_1+\alpha)} - o(\beta^2) - 2r_1)\mathrm{d}\phi = \int_0^\pi (2r_1 + \alpha - o(\beta^2) - 2r_1)\mathrm{d}\phi = \pi\alpha - o(\alpha^2)$ □

If $x \in \partial(f_r^{-1})$, $x_i \to x$ and $x_i' \to x$ such that $\forall_i f(\mathrm{B}(x_i, r)) = \mathrm{B}(x_1, r_1) \wedge f(\mathrm{B}(x_i', r)) = \mathrm{B}(x_2, r_2)$ then let $\mathrm{dis}_{\phi,r}(x)$ be the distance between the origins of $f(\mathrm{B}(x_i, r))$ and $f(\mathrm{B}(x_i', r)$ projected on the line of angle ϕ and $\mathrm{dis}_r(x)$ be the total distance between their origins.

Lemma 2.5. *There is* $\lim_{\epsilon\to 0_+} \frac{\int_C \max_{x_0\in\mathrm{B}(x,\epsilon)}(\mathrm{dis}_r(x_0))\mathrm{d}x}{\epsilon} \geqslant \pi\mathrm{Ar}(C) - o(\mathrm{Err}(r))$.

Proof. The inequality $\lim\limits_{\epsilon\to 0_+} \frac{1}{\epsilon}\int_C \max_{x_0\in\mathrm{B}(x,\epsilon)}(\mathrm{dis}_{r,\phi}(x_0))\mathrm{d}x \geqslant 2\mathrm{Ar}(C) - o(\mathrm{Err}(r))$ follows simply from the fact, that on every line of angle ϕ differences sums at least to length of this line minus $2\mathrm{Err}(r)$. To compute integral for dis_r note that if distance $\mathrm{dis}_r(x)$ is realised in angle α_x then from basic trygonometry we get $\mathrm{dis}_{\phi,r}(x) = |\cos(\alpha - \phi)|\mathrm{dis}_r(x)$. It follows that:
$\pi\mathrm{Ar}(C) - o(\mathrm{Err}(r)) \leqslant \lim\limits_{\epsilon\to 0} \frac{1}{4\epsilon}\int_0^{2\pi}\int_C \max_{x_0\in\mathrm{B}_\phi(x,\epsilon)}(\mathrm{dis}_{\phi,r}(x))\mathrm{d}x\mathrm{d}\phi =$
$\lim\limits_{\epsilon\to 0} \frac{1}{4\epsilon}\int_C\int_0^{2\pi} \max_{x_0\in\mathrm{B}_\phi(x,\epsilon)}(\mathrm{dis}_{\phi,r}(x))\mathrm{d}\phi\mathrm{d}x = \lim\limits_{\epsilon\to 0} \frac{1}{4\epsilon}\int_C\int_0^{2\pi} \max_{x_0\in\mathrm{B}_\phi(x,\epsilon)}$
$(|\cos(\alpha_{x_0} - \phi)|\mathrm{dis}_r(x))\mathrm{d}\phi\mathrm{d}x = \lim\limits_{\epsilon\to 0} \frac{1}{\epsilon}\int_C \max_{x_0\in\mathrm{B}(x,\epsilon)}(\mathrm{dis}_r(x_0))\mathrm{d}x$

□

Now we can proceed to proof of the main theorem 2.1:

Proof. From lemma 2.3 we get that $A = [(x, -\sqrt{1-x^2}), (x, \sqrt{1-x^2})] \cap \partial(f^{-1}(r))$ is a nowwhere dense so for every $a \in A$ there exists ϵ such that $\mathrm{B}(a,\epsilon) \cap A = a$. Let $\mathrm{B}(x_1, r_1), \mathrm{B}(x_2, r_2) \in f(D_C)$ be an image of balls with radius r and origins close to a and $\mathrm{B}_a(a,\epsilon)$ be a projection of ball $\mathrm{B}(a,\epsilon)$ onto interval containing a of angle 0. Then from lemma 2.4 applied to both sides of a we also have $\int_{\mathrm{B}_x(a,\epsilon)} \mathrm{err}(y, r+\epsilon)\mathrm{d}y \geqslant \int_{\mathrm{B}_x(a,\epsilon)} \mathrm{err}(y,r)\mathrm{d}y - 4\pi\epsilon + \pi(\mathrm{dist}(x_1,x_2) + r_2 - r_1) + \pi(\mathrm{dist}(x_1,x_2) + r_1 - r_2) = \int_{\mathrm{B}_x(a,\epsilon)} \mathrm{err}(y,r)\mathrm{d}y - 4\pi\epsilon + 2\pi(\mathrm{dist}(x_1,x_2))$. Applied to every point in $\partial(f^{-1}(r))$ and in connection with 2.5 it gives us

$$\lim_{\epsilon \to 0} \frac{\epsilon \to 0 \int_C \mathrm{err}(x,r+\epsilon)\mathrm{d}x - \int_C \mathrm{err}(x,r)\mathrm{d}x}{\epsilon} \geqslant \pi^2 \mathrm{Ar}(C) - 2\pi \mathrm{Ar}(C) + o(\mathrm{err}(r))$$

□

Bibliography:

[1] Abramsky, S., Jung A.(1994). "Domain theory". In S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, (PDF). Handbook of Logic in Computer Science. III. Oxford University Press. ISBN 0-19-853762-X.

[2] Jung, A., "Cartesian Closed Categories of Domains (Dissertation) (1989), Vol. 66 of CWI Tracts, Centrum voor Wiskunde en Informatica, Amsterdam, 107 pp.

[3] Keilem, K.(2010) "Bicontinuous Domains and Some Old Problems in Domain Theory", Journal Electronic Notes in Theoretical Computer Science, Volume 257, p. 35-54