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# Graph sharing games and Minors

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# GRAPH SHARING GAMES AND MINORS

ADAM GAGOL

## 1. Introduction

*Graph sharing games* are played on a simple connected graph with non-negative weights placed on vertices. Two players make their moves consecutively by pointing vertices one by one and removing them from the graph. The vertices available to play depend on the game variant:

- *Game T (Taken part connected)*: after each move the taken part of the graph should stay connected.
- *Game R (Remaining part connected)*: after each move the remaining part of the graph should stay connected.

The game ends when there are no more vertices. Both players' goal is to maximize the total weight they have collected at the end.

The two variants of the game coincide when the game is played on a cycle. This case was studied by Brown, who conjectured that the first player can always secure $\frac{1}{2}$ of the weight on any cycle. This is true for cycles with even number of vertices but occurred to be false in general. Later Winkler conjectured that the first player can secure at least $\frac{4}{9}$ of the weight of any cycle, and this has been proved by Micek and Walczak [4].

After solving the problem for cycles it was natural to ask what happens for general graphs. Game T and game R are two ways to generalise the game to a broader family of graphs and both have been studied. In both games there are trees on which the first player can capture only one non-zero vertex and the second player can take arbitrarily many, but this can happen only for odd trees in game R and for even trees in game T. Micek and Walczak [6, 7] showed that the first player can capture $\frac{1}{4}$ of the total weight in game T on odd trees and in game R on even trees. Later Seacrest and Seacrest [9] improved the result for game R: the first player can capture $\frac{1}{2}$ of the weight on even trees.

Parity plays an important role but is not enough to explain the behaviour of the game in the general case. There are examples, which we will explain in section 2, showing that the first player can be left with just one non-zero vertex on general graphs with both games and parities. In 2008 Micek and Walczak stated the following conjecture, which will be the main topic of this paper.

**Conjecture 1.1.** *There is a function $f(n) > 0$ such that the first player can secure at least $f(n)$ of the weight at game R of any graph with an even number of vertices and with no $K_n$-minor.*

We will prove the following, weaker statement:

**Theorem 1.2.** *There is a function $f(n) \to \infty$ such that if $G$ is a {0,1}-weighted graph with an even number of vertices and $K_n \not\prec G$ then the first player can secure at least $f(n)$ weight of the graph.*

## 2. Basics: definitions and tools

A *graph* is a pair $G = (V, E)$ where $V$ is a set of *vertices* and $E \subseteq \binom{V}{2}$ is a set of *edges*. Given a graph $G$ we denote by $V(G)$ the set of vertices of $G$ and by $E(G)$ the set of edges of $G$. All graphs considered have no loops (edges from a vertex to itself). Two vertices are *adjacent* if there is an edge between them. A *path* in a graph $G$ is a sequence $v_0, \ldots, v_k$ of vertices of $G$ such that $v_i$ is adjacent to $v_{i+1}$ for every $i$. We call $k$ the *length* of the path $v_0, \ldots, v_k$. A *cycle* is a path in which $v_0 = v_k$. The *distance* between two vertices is the length of a shortest path connecting them. The *neighbourhood* of a set $B \subseteq V(G)$ is the set of all vertices adjacent to any vertex in $B$. We denote it as $N_G(B) = \{v \in V(G) \setminus B : \exists_{b \in B}\ v$ and $b$ are adjacent$\}$. We define $N_{+(}B) = N_(B) \cup B$. We write simply $N(B)$ and $N_+(B)$ when the graph $G$ is clear from the context. Second and further neighbourhoods we denote as $N(B)^n = N(N(B)^{n-1}) - N(B)^{n-2}$ and $N(B)^n_+ = N_+(N(B)^{n-1}_+)$ where $N^0(B) = B$ and $N^{-1}(B) = \emptyset$.

A *multigraph* is a pair $(V, E)$ where $V$ is a set of vertices and $E$ is a multiset of edges (that is, two vertices can share more than one edge). If $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$ then $G'$ is a subgraph of $G$. If $V(G') \subseteq V(G)$ and $E(G')$ contains all the edges $xy \in E$ for $x, y \in V'$, then $G'$ is an *induced subgraph* of $G$. For a set $A \subseteq V(G)$ by $G[A]$ we denote the induced subgraph of $G$ with vertices taken from $A$. A graph $G$ is *connected* if there is a path connecting any two vertices of $G$. A clique $K_n$ is a graph with $n$ vertices and all possible edges, an anticlique is a graph with no edges. Ramsey theorem conects the facts of having clique and anticlique as induced subgraph.

For every pair of positive integers $k$ and $l$ there exists number $R(k, l)$ (known as a Ramsey number) such that any graph with at least $R(k, l)$ vertices contains a clique with $k$ vertices or independent set (induced anticlique) with $l$ vertices.

Let $H$ be a connected subgraph of $G$. By $G/H$ we denote the graph obtained from $G$ by *contracting* $H$ into a single vertex $v_H$, that is, by replacing $H$ by a vertex $v_H$ which becomes adjacent to all neighbours of any vertex from $H$. A graph $G'$ is a *minor* of $G$ (denoted by $G' \prec G$) if $G'$ can be obtained from $G$ as a result of a sequence of the following *basic contractions*:

- deleting an edge,
- deleting a vertex,
- subgraph contraction.

There is an equivalent definition of a minor using a notion of *touching* sets. Two sets $A, B \subseteq V(G)$ are *touching* if $N^+(A) \cap N^+(B) \neq \emptyset$. Note that this definition is symmetric. Now, $H \prec G$ if and only if there exists a family of disjoint sets $\{A_v : v \in H\}$ such that if $(u, v) \in E(H)$ then $A_u$ and $A_v$ are touching. An *induced* minor of $G$ is a graph $H$ obtained from $G$ by a sequence of subgraph contractions.

A *tree* is a connected graph with no cycles. A *spanning tree* of a (multi)graph $G$ is a tree $T$ which is a subgraph of $G$ such that $V(T) = V(G)$. A *rooted tree* is a tree with one selected vertex—the *root*. In a rooted tree the neighbour of a vertex $v$ in the direction towards the root is called the *parent* of $v$, while all other neighbours of $v$ are called *children* of $v$. An *ancestor* of a vertex $v$ in a rooted tree is any vertex on the tree-path connecting $v$ with the root.

A *weighted graph* is a graph $G$ equipped with a weight function $w : V(G) \to [0, \infty)$. We say that $G$ is $\{0,1\}$-weighted if $w : V(G) \to \{0,1\}$. For a $\{0,1\}$-weighted graph with vertex set $V$ we denote by $V_1$ the set of vertices of weight 1 and by $V_0$ the set of vertices of weight 0. We will also call vertices with positive weigth *heavy* and the ones with weigth 0 - *light*.

**Observation 2.1.** *Let $T$ be a weighted tree. Then there is a vertex $v \in V(T)$ such that every component $C$ of $T - \{v\}$ has weight $\mathrm{w}(C) \leqslant \frac{1}{2}\mathrm{w}(T)$. We call $v$ a weighted center of $T$.*

*Proof.* Think of every edge as directed towards the heavier component of $T - e$. Then there is a vertex such that all its incident edges are directed inwards. Such a vertex is a good candidate for a weighted center of $T$. □

## 3. State of the art

3.1. **Game T.** In the variant with taken part connected all vertices are available for the first player at the beginning, so obviously he can secure at least one heavy vertex. The following example shows that even on trees he cannot be sure to get more.

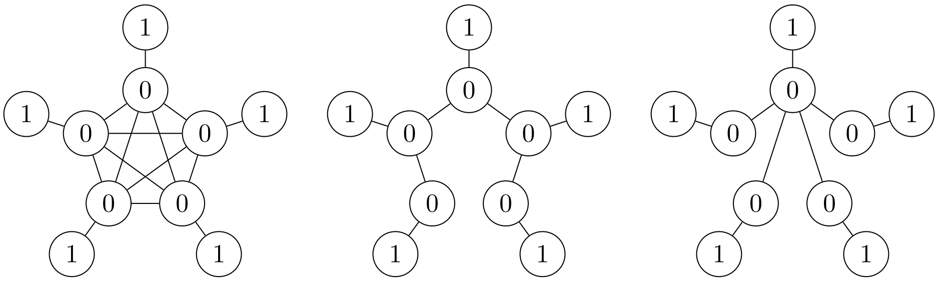



Figure 1. $G_5$ and related examples

**Example 3.1.** $G_n = (V, E, w)$ is a weighted graph with $2n$ vertices $V = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. The $a_i$'s forms a clique and the only neighbour for each $b_i$ is $a_i$. The weight is equally distributed only on $b_i$'s, so $w(b_i) = 1$ and $w(a_1) = 0$ for every $i$. There is no strategy for the first player in game T to secure more than 1 in $G_n$. When he starts with some $b_i$, the second player takes the corresponding $a_i$. In next moves the first player is forced to take only $b_i$'s and second player can always take corresponding $a_i$'s.

Note that the clique on $a_i$'s can be replaced by any other connected graph and the argument still works. This way we obtain similar examples belonging to simple classes of graphs (trees, caterpillars, subdivided stars, etc.). However, all such examples have an even number of vertices. If we request the graph to have an odd number of vertices, things get a bit more complicated, but it is still possible to construct an example for which the first player has no strategy to take more than one heavy vertex. The following example was found independently by two groups of researchers [**?**, **?**].

**Example 3.2.** $H_n = (V, E, w)$ is a weighted graph with $2n + 2^n - 1$ vertices $V = \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \cup \{c_X : X \in P(\{b_1, \ldots, b_n\}) - \{\emptyset\}\}$. The neighbourhoods of the vertices are: $N(a_i) = \{b_i\}, N(b_i) = \{a_i\} \cup \{c_X : b_i \in X\}, N(c_X) = X$. The

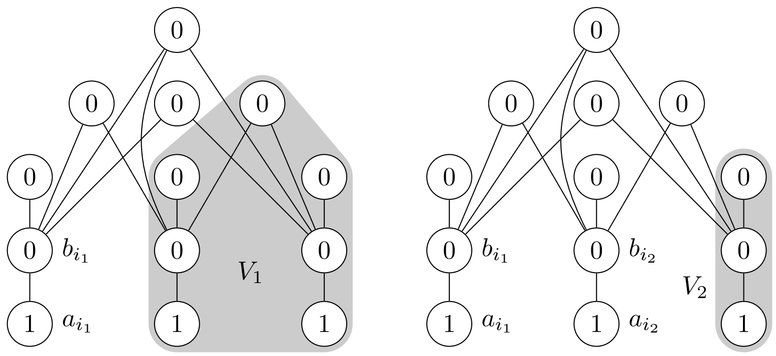



FIGURE 2. $H_3$: The first player is forced to enter sets $V_1$ and $V_2$.

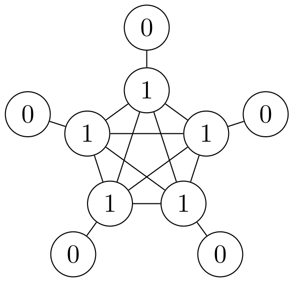
weight is equally distributed on $a_i$'s, so $\mathrm{w}(a_i = 1)$ for all $i$'s. The first player can secure at most 1 in game T on $H_n$. Suppose that he starts with $a_{i_1}$ (or $b_{i_1}$). Then the second player takes the corresponding vertex $b_{i_1}$ (or $a_{i_1}$). The set of vertices $c_X$ which are available to play depends only on the vertices $b_i$ that are already taken. For a non-empty set $X \subseteq \{b_1, \ldots, b_n\}$ denote by $V_X$ the set of these $c_Y$'s that are neighbours of some $b_i \in X$. We have $|V_X| = (2^n - 1) - (2^{n-|X|} - 1) = 2^n - 2^{n-|X|}$ even. It is still even after adding couples of $a_i$'s and $b_i$'s which are already taken. Thus the second player can ensure that he is not the one who takes a new $b_i$, and he can take $a_i$ just after the first player takes $b_i$.

Note that $K_n \prec H_n$, so this is not a counterexample for Conjecture 1.1, but a chromatic and clique numbers are both equal to 2.

3.2. **Game R.** In the variant with remaining part connected the set of vertices available for the first player in the first move is restricted, so we can hope for an example leaving the first player with no weight. To see that this is possible it is enough to consider a path with vertices weighted alternately with 0 and 1.

**Example 3.3.** $P_n = (V, E, w)$ is a weighted path with $2n - 1$ vertices. The weight is equally distributed on every second vertex of the path. It is clear that the second player can guarantee himself all the weight.

There is also an example with an even number of vertices and with a slightly more complicated structure involving $K_n$ as a subgraph. It is a slight modification of Example 3.1.




**Example 3.4.** $G'_n = (V, E, w)$ is a weighted graph with $2n$ vertices $V = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. The $a_i$'s form a clique and the only neighbour for each $b_i$ is $a_i$. The weight is equally distributed only on $a_i$'s, so $\mathrm{w}(a_i) = 1$ and $\mathrm{w}(b_1) = 0$ for every $i$. It should be clear that the second player take all heavy vertices except one, taking always $a_i$ corresponding to $b_i$ taken by the first player.

## 4. Proof of the main result

**Theorem 4.1.** *There is a function $f(n) > 0$ that if $G$ is a $\{0{,}1\}$-weighted graph with no $K_n$ minor then first player can secure at least $f(n)$-th fraction of the total weight on a graph.*

*Proof.* We construct special graph $H$. Vertices of $H$ are all heavy vertices of $G$ and there is an edge between $x, y \in H$ iff there is a path between them in $G$ not crossing any other heavy vertex. We can get this graph by retracting all edges with at least one end in light vertex, so $H \prec G$ and hence $K_n \not\prec H$. Let $T$ be a maximal induced subtree of $H$ and $G'$ be a graph occurring from $G$ by erasing all weights out of $T$. Due to theorem of Fox, Loh and Sudakov[[**?**]FLS)] size of $H$ is at least $\frac{log \mathrm{w}(G)}{log r}$.

The first player will play as if the game would be played on graph $G'$. At the beginning of the game his strategy is to simple take any available heavy vertex. If no heavy vertex is available than any light vertex not making any heavy available for the opponent. Proceeding like this we will reduce graph to the case where every heavy vertex is a cut-vertex and taking out light vertex either disconnects the graph or makes one of the heavy vertices available. We also ensure that we take at least half of the total weight up to this moment.

Let $G''$ be a graph after the reduction and $H''$ be its graph of heavy vertices. We will prove that $H''$ is a tree. Suppose that there is a cycle $C = x_1, \ldots, x_n$ in $H''$. It occurs from weighted vertices $x_1, \ldots, x_n$ in $G''$ and possibly some with weight 0. Some of them could have positive weight in $G$ and by adding them to $C$ we naturally extend it to a cycle $C'$ in $H$. Since we got rid of all cycles in $H$ which had erased weight without disconnecting $H$ so there is at least one vertex $v$ in $C' \backslash C$ such that among connected components of $G'' \backslash v$ there is at most one with non-zero weight. Due to the fact that the graph was reduced before erasing weights we get that $v$ actually disconnects $G''$ so we can erase components with zero weight without making any heavy vertex available and hence contradict the fact of $G''$ being reduced.

After the reductions we are left with the tree so we can use the game on tree [7] to get $\frac{2}{5}$ of total weight of a tree. Hence it is possible to secure $o(\frac{log \mathrm{w}(G)}{log r})$ in any graph $G$ without $K_r$ minor.

□

## 5. Summary and further directions

Although the proof is made for the {0,1}-weighted case, it may be possible to use a similar idea for graphs with general weights. The only problem is in showing that for arbitrarily weighted graph with bounded clique number there is a induced tree of linear weight.

There are more problems in the area, the most intriguing ones arising when we consider graphs with an odd number of vertices in game $T$ and an even number of vertices in game $R$. It looks to be an extremely hard problem to find the exact constant that the first player can secure under certain conditions (minor-free etc.). Easier but still very interesting are problems concerning the necessary conditions for a class of weighted graphs to ensure that the first player has a strategy to secure a linear fraction of the total weight of any graph in this class.

An instance of such question for game $R$ is Conjecture 1.1, and it seems that forbidden minors are the weakest natural graph property that can guarantee a linear gain for the first player. For example, this could be some property of $S(G)$ or some quantitative requirement on number of odd surroundings.

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