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Selected problems involving Borel filters

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Chapter 1

Preliminaries

1.1 Introduction

In this paper we study Borel filters $\mathcal{F}$ and sets of functions, which can be represented as a pointwise limit relatively to $\mathcal{F}$ of some sequence of continuous functions. The most important paper in this field is [1], in which Gabriel Debs and Jean Saint Raymond introduce the rank of a filter. We present main results published in this paper. In section 3.2, as well as in the last section we present our results, which generalize and improve some statements proved in [1]. We rectify in section 2.2 a slight misstatement from [1] related to inductive limits of filters. We focus also on some problems, which are related to recent papers published during last years by Solecki, Mazur, Miller or Dobrowolski and Marciszewski. In the last chapter we investigate filters of countable rank, which were studied extensively by David Fremlin.
The paper starts with chapter containing some basic definitions and facts, which we need in further parts. In the second chapter we introduce some operations on families of filters like Fubini sums or inductive limits. This chapter also contains the definition of Katětov filters and filters $\mathcal{N}_\alpha$ studied by Debs and Saint Raymond. Third chapter in devoted to the theory of ranks of filters. It is based on [1], but contains few new observations. Finally, in the fourth chapter we investigate filters of countable type. They were introduced by Mauldin, Preiss and Weizsäcker, but the idea comes from Katětov and Grimeisen. This part is based on Fremlin's article ([4]). We conclude this paper with some new facts, which give better estimation of rank for some filters.

1.2 Basic definitions and facts

Let I be a set. A family of sets $\mathcal{F} \subset \mathcal{P}(I)$ is a filter on I, if it is closed under taking finite intersections and supersets. Throughout this paper for a family $\mathcal{A} \subset \mathcal{P}(I)$ we denote

$$\mathcal{A}^* = \{A \subset I : I \setminus A \in \mathcal{A}\}.$$

If $\mathcal{F}$ is a filter on I, then the family $\mathcal{F}^*$ is an ideal (i.e., a family closed under taking finite unions and subsets) called the dual ideal to $\mathcal{F}$. We denote $Fin = [\omega]^{<\omega}$ and $\mathcal{F}_{Fr} = Fin^*$. Clearly Fin is an ideal and $\mathcal{F}_{Fr}$ its dual filter called the Frechét filter.

A filter $\mathcal{F}$ is free, if $\bigcap \mathcal{F} = \emptyset$. It is principal, if it is of the form

$$\mathcal{F}_E = \{A \subset I : E \subset A\}$$

for some subset $E \subset I$. Maximal filters are called ultrafilters. They can be characterized by following condition: a filter $\mathcal{F}$ is an ultrafilter if and only if $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ for each A. All principal ultrafilters are of the form

$$\mathcal{F}_{\{x\}} = \{A \subset I : x \in A\}$$

for some $x \in I$.

Two filters $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, if there is a bijection $\pi : \mathrm{dom}\mathcal{F} \to \mathrm{dom}\mathcal{G}$ such that

$$A \in \mathcal{G} \Leftrightarrow \pi^{-1}(A) \in \mathcal{F}$$

for each $A \subset \mathrm{dom}\mathcal{G}$.

If $\{X_i\}_{i \in I}$ is a family of sets, then by $\Sigma \{X_i\}_{i \in I}$ (or $\Sigma_{i \in I} X_i$) we denote its disjoint sum, i.e., the set of all pairs (i, x) where $i \in I$ and $x \in A_i$.

In the sequel we will use extensively the following definition: for a Polish space X, two subsets $A, B \subset X$ and a pointclass Γ we say that A is Γ-separated from B if there exists a subset $S \subset X$ in Γ, for which $A \subset S$ and $B \cap S = \emptyset$.

Let (X, ρ) be a metric space. A sequence $(x_i)_{i \in I}$ is $\mathcal{F}$-convergent to $x \in X$ ($x = \mathcal{F} - \lim x_i$), if for every $\epsilon > 0$ we have

$$\{i \in I : \rho(x_i, x) < \epsilon\} \in \mathcal{F}.$$

For functions f and f_i on X we write $f = \mathcal{F} - \lim f_i$ and say that f is a limit of the sequence $(f_i)_{i \in I}$ relatively to $\mathcal{F}$, if $f(x) = \mathcal{F} - \lim f_i(x)$ for every $x \in X$. By $\mathcal{C}_{\mathcal{F}}(X)$ we denote the family of all real valued functions on the space X, which can be represented as a limit relatively to $\mathcal{F}$ of a sequence of continuous funtions. In the sequel we also denote by $\mathcal{B}_\alpha(X)$ ($\mathcal{B}^{(\alpha)}(X)$) the family of all real valued functions on the space X of Borel (Baire) class $\alpha < \omega_1$.

Chapter 2

Some special filters

2.1 Fubini sums and products of filters

In this chapter we introduce some operations on families of filters, which give us a new filter. The first one is quite standard. We introduce it in the same way as Miroslav Katětov in [6], but it can also be found in [1].

Definition 1 *(cf. [6])*
For a filter $\mathcal{G}$ on I and a family of filters $(\mathcal{F}_i)_{i\in I}$, family of all sets of the form

$$\bigcup_{i\in G} F_i \cup \bigcup_{i\in I\setminus G} H_i$$

for $G \in \mathcal{G}$, $F_i \in \mathcal{F}_i$ and $H_i \subset \operatorname{dom}(\mathcal{F}_i)$ forms a filter on the set $\Sigma\{\operatorname{dom}(\mathcal{F}_i)\}_{i\in I}$. We denote this filter $\mathcal{G} - \Sigma\{\mathcal{F}_i\}_{i\in I}$ and call it the $\mathcal{G}$-Fubini sum of the family $(\mathcal{F}_i)_{i\in I}$.
If $(I,\leq)$ is a directed set and $\mathcal{G}$ is generated by sets $L_i = \{j : j \geq i\}$, then we call $\mathcal{G} - \Sigma\{\mathcal{F}_i\}_{i\in I}$ the Fubini sum of the family $(\mathcal{F}_i)_{i\in I}$.

In particular, putting the same filter as members of the family from the above definition, we obtain a standard product of filters:

Definition 2 *(cf. [6])*
For filters $\mathcal{F}$ and $\mathcal{G}$ on I and X respectively, if we put $\mathcal{F}_i = \mathcal{G}$ for every $i \in I$, then we obtain a filter $\mathcal{F} \times \mathcal{G} = \mathcal{F} - \Sigma\{\mathcal{F}_i\}_{i\in I}$, which we call the product of filters $\mathcal{F}$ and $\mathcal{G}$.

2.2 Inductive limits of filters

The second operation is not so common. We present it in the same way as in [1]. First we need to define quasi-homomorphism between filters and coherent systems of quasi-homomorphisms.

Definition 3 *(cf. [1])*
If $\mathcal{F}$ and $\mathcal{G}$ are filters on A and B respectively, then a quasi-homomorphism from $\mathcal{F}$ to $\mathcal{G}$ is a mapping $\pi : F \to B$, where $F \in \mathcal{F}$, such that for all $G \in \mathcal{G}$ its preimage $\pi^{-1}(G) \in \mathcal{F}$.

Definition 4 *(cf. [1])*
For a family of filters $(\mathcal{F}_i)_{i\in I}$*, where* $(I, \leq)$ *is a directed set, the family* $(\pi_{i,j})_{i\leq j}$*, where* $\pi_{i,j}$ *is a quasi-homomorphism from* $\mathcal{F}_j$ *to* $\mathcal{F}_i$*, is a coherent system of quasi-homomorphisms for* $(\mathcal{F}_i)_{i\in I}$*, provided that for all* $i, j, k \in I$ *such that* $i \leq j \leq k$ *the following condition is satisfied:*

$$\pi_{i,k}(a) = \pi_{i,j}\left(\pi_{j,k}(a)\right)$$

for every $a \in \operatorname{dom}(\pi_{i,k}) \cap \operatorname{dom}(\pi_{j,k}) \cap \pi_{j,k}^{-1}\left(\operatorname{dom}(\pi_{i,j})\right)$.

Now we are prepared to define inductive limits:

Definition 5 *(cf. [1])*
Let $(\pi_{i,j})_{i\leq j}$ *be a coherent system of quasi-homomorphisms for the family of filters* $(\mathcal{F}_i)_{i\in I}$*. The set*

$$\lim_{\leftarrow} \mathcal{F}_i = \left\{ A \subset \Sigma\left\{\operatorname{dom}(\mathcal{F}_i)\right\}_{i\in I} : \exists_{i\in I}\exists_{M\in\mathcal{F}_i} A \supset \bigcup_{j\geq i} \pi_{i,j}^{-1}(M) \right\}$$

is called inductive limit of the system $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\leq j\\ i,j\in I}}$.

Definition of inductive limits admits some comment. In [1] it is stated, that inductive limit is always a filter (Proposition-Definition 5.4). Next example shows that this is not true.

Example 1 *Put* $I = \{1, 2, 3, \ldots\}$*, let* $\mathcal{F}_1 = \mathcal{F}_{Fr}$ *be the Frechét filter on* ω *and for* $i > 1$ *let* $\mathcal{F}_i = \mathcal{F}_{Fr} \times \mathcal{F}_{i-1}$ *be filter on* ω^i*. Next choose for each* $i > 1$ *the set* $F_i = \omega^i \setminus (\{0\} \times \omega^{i-1})$ *and define a quasi-homomorphism* $\pi_{1,i}$ *from* $\mathcal{F}_i$ *to* $\mathcal{F}_1$ *with domain* F_i *and such that* $\pi_{1,i}(x) = (x_1, \ldots, x_{i-1})$ *for* $x = (0, x_1, \ldots, x_{i-1})$*. For indexes* $1 < i \leq j$ *define quasi-homomorphisms* $\pi_{i,j}$ *from* $\mathcal{F}_j$ *to* $\mathcal{F}_i$ *with domain* ω^j *and such that* $\pi_{i,j}(x) = (x_{j-i}, \ldots, x_{j-1})$ *for* $x = (x_0, x_1, \ldots, x_{j-1}) \in \omega^j$*. Then* $(\pi_{i,j})_{i\leq j}$ *is a coherent system of quasi-homomorphisms for the family of filters* $(\mathcal{F}_i)_{i\in I}$*, but denoting for* $X \subset \omega^i$

$$\tilde{X} = \Sigma_{j\geq i}\pi_{i,j}^{-1}(X),$$

we have $\tilde{A} \cap \tilde{B} \notin \lim_{\leftarrow} \mathcal{F}_i$ *for* $A = \omega \in \mathcal{F}_1$ *and* $B = \omega^2 \in \mathcal{F}_2$*. Indeed, the union*

$$\tilde{A} \cap \tilde{B} = \bigcup_{j>1} \{i\} \times F_j$$

doesn't include any set $\tilde{P}$ *for* $P \in \mathcal{F}_i$*,* $i > 1$*, because* $\pi_{i,j}^{-1}(P)$ *is not a subset of* F_j *for every* $j > i$*. This shows that inductive limit of the system* $(\mathcal{F}_i, \pi_{i,j})_{i\leq j}$ *is not a filter.*

Notice that the above example is a slight modification of the filter $\mathcal{N}_\omega$ defined in section 2.3.

Remark 1 *Although inductive limit is not always a filter, it is easy to show, that it is a filter, if for all indexes* $i \leq j$ *from* I *the domain* $\operatorname{dom}(\pi_{i,j}) = \operatorname{dom}(\mathcal{F}_j)$*. It is so, since this condition implies that the family of filters of the form*

$$\mathcal{G}_n = \left\{ A \subset \Sigma \left\{ \operatorname{dom}\left(\mathcal{F}_i\right) \right\}_{i \in I} : \exists_{M \in \mathcal{F}_n} A \supset \bigcup_{j \geq n} \pi_{n,j}^{-1}\left(M\right) \right\}$$

for $n \in I$*, is increasing. We can also write more responsive condition: for all indexes* $i \leq j$ *from* I

$$\forall_{k>j} \pi_{j,k}^{-1}\left(\operatorname{dom}(\pi_{i,j})\right) \subset \operatorname{dom}(\pi_{i,k}).$$

This also implies that the family $\left(\mathcal{G}_i\right)_{i \in I}$ *is increasing. Notice also that* I *being finite is another case, which guarantees inductive limit to be a filter.*

2.3 The Katětov filters

First family of filters, which we define in this section, comes from Katětov ([6]) and Grimeisen ([5]), who worked on this problem independently. It is an important family since it generates the Baire classes $\mathcal{B}^{(\alpha)}$. This fact is stated in the consecutive theorem.

Definition 6 *(cf. [6])*
For every $\alpha < \omega_1$ *we define inductively filters* $\mathcal{N}^{(\alpha)}$*:*

- $\mathcal{N}^{(0)} = \{\{0\}\}$ *is a filter on the set* $\{0\}$*,*
- $\mathcal{N}^{(\alpha+1)} = \mathcal{F}_{Fr} \times \mathcal{N}^{(\alpha)}$*,*
- *for* λ *being limit ordinal*

$$\mathcal{N}^{(\lambda)} = \mathcal{F} - \Sigma \left\{ \mathcal{N}^{(\alpha)} \right\}_{\alpha < \lambda},$$

where $\mathcal{F}$ *is a filter on* λ *generated by sets* $L_\alpha = \{\xi : \xi \geq \alpha\}$*.*

Flters defined above are called Katětov filters.

Theorem 1 *(cf. [6])*
Let $\alpha < \omega_1$*. The Katětov filter* $\mathcal{N}^{(\alpha)}$ *generates the Baire class* $\mathcal{B}^{(\alpha)}$ *i.e.,* $\mathcal{C}_{\mathcal{N}^{(\alpha)}} = \mathcal{B}^{(\alpha)}$*.*

The second family of filters is similar, but for limit ordinals requires introducing quasi-homomorphisms between Katětov filters: since every $\mathcal{N}^{(\alpha)}$ is obtained as a Fubini sum of previous filters (all of them or only one used many times), by putting the identity mapping as a quasi-homomorphism between the same filters and inductively as a quasi-homomorphism from $\mathcal{N}^{(\alpha)}$ to $\mathcal{N}^{(\beta)}$ (for $\alpha > \beta$) the mapping $\pi(a) = \pi_{\alpha,\beta}(a)$, where $a \in \operatorname{dom}\left(\mathcal{N}^{(\beta)}\right) \subset \Sigma\left\{\operatorname{dom}\left(\mathcal{N}^{(\xi)}\right)\right\}_{\xi<\alpha}$, we obtain a coherent system of quasi-homomorphisms for the family $\left(\mathcal{N}^{(\alpha)}\right)_{\alpha<\omega_1}$.
The following family was defined in [1] and it generates the Borel classes $\mathcal{B}_\alpha$.

Definition 7 *(cf. [1])*
For every $\alpha < \omega_1$ *we define inductively filters* $\mathcal{N}_\alpha$*:*

- $\mathcal{N}_\alpha = \mathcal{N}^{(\alpha)}$*, if* α *is successor ordinal,*
- $\mathcal{N}_\lambda$ *is an inductive limit of the family* $\left(\mathcal{N}^{(\alpha)}\right)_{\alpha<\lambda}$*, if* λ *is limit ordinal.*

Theorem 2 *(cf. [1])*
Let $\alpha < \omega_1$*. The filter* $\mathcal{N}_\alpha$ *is Borel and generates the Borel class* $\mathcal{B}_\alpha$ *i.e.,* $\mathcal{C}_{\mathcal{N}_\alpha} = \mathcal{B}_\alpha$.

Chapter 3

Rank of a filter

3.1 Definition and basic properties

In this section we characterize filters $\mathcal{F}$ satisfying following inclusions: $\mathcal{C}_{\mathcal{F}} \subset \mathcal{B}_\alpha$ and $\mathcal{C}_{\mathcal{F}} \supset \mathcal{B}_\alpha$. At first we define the rank of a filter, which was studied by Gabriel Debs and Jean Saint Raymond in [1].

Definition 8 *(cf. [1])*
The rank of a filter $\mathcal{F}$ is the ordinal:

$$\mathrm{rk}(\mathcal{F}) = \min\left\{\alpha < \omega_1 : \mathcal{F} \textit{ is } \Sigma^0_{1+\alpha}\textit{-separated from } \mathcal{F}^*\right\}$$

Clearly for every filter $\mathcal{F}$ its rank $\mathrm{rk}(\mathcal{F})$ is unique. By Souslin separation Theorem, every analytic filter has a countable rank. Note also that

$$\mathrm{rk}(\mathcal{F}) = \min\left\{\alpha < \omega_1 : \mathcal{F} \text{ is } \Pi^0_{1+\alpha}\text{-separated from } \mathcal{F}^*\right\}$$

since the canonical involution $A \mapsto A^c$ is a homeomorphism. Let us make few more easy observations:

Proposition 1 *(cf. [1])*
If $\mathcal{F}$ and $\mathcal{G}$ are filters satisfying $\mathcal{F} \subset \mathcal{G}$, then $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{G})$.

Proof. Let S be a set separating $\mathcal{G}$ from its dual ideal. We have: $\mathcal{F} \subset \mathcal{G} \subset S$ and $\mathcal{F} \cap S \subset \mathcal{G} \cap S = \emptyset$, so S separates also $\mathcal{F}$ from its dual ideal and $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{G})$. □

Proposition 2 *(cf. [1])*
If $\mathcal{F}$ and $\mathcal{G}$ are filters and there is a quasi-homomorphism from $\mathcal{F}$ to $\mathcal{G}$, then $\mathrm{rk}(\mathcal{G}) \leq \mathrm{rk}(\mathcal{F})$.

Proof. Let $\pi : F \to \mathrm{dom}(\mathcal{G})$ be a quasi-homomorphism from $\mathcal{F}$ to $\mathcal{G}$ and S be a $\Sigma^0_{1+\alpha}$ set separating $\mathcal{F}$ of rank α from its dual ideal. Then the set

$$S' = \left\{A \subset \mathrm{dom}(\mathcal{G}) : \pi^{-1}(A) \in S\right\}$$

also is a $\Sigma^0_{1+\alpha}$ set. If $A \in \mathcal{G}$, then $\pi^{-1}(A) \in \mathcal{F} \subset S$ and if $A \in \mathcal{G}^*$, then $\pi^{-1}(\mathrm{dom}(\mathcal{G}) \setminus A) \in \mathcal{F}$, so $\pi^{-1}(A) = F \setminus \pi^{-1}(\mathrm{dom}(\mathcal{G}) \setminus A) \in \mathcal{F}^*$ is not an element of S. Hence S' separates the filter $\mathcal{G}$ from its dual ideal.

□

Proposition 3 *(cf. [1])*
A filter is of rank 0 *if and only if it is not free.*

We omit the proof of the last fact. It can be found in section 3 of [1].

The rank of a filter plays a fundamental role in studying the class of pointwise limits relatively to a filter of sequences of continuous functions. This is because of the following theorem.

Theorem 3 *(cf. [1])*
Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ a countable ordinal. Then
(a) $\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ for any Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \le \alpha$.
(b) $\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \ge \alpha$.
(c) $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) = \alpha$.

Of course the last characterization is an immediate consequence of part (a) and (b). The zero-dimensional restriction on a Polish space in the above theorem can be omitted for $\alpha \le 2$. This is a consequence of results presented in section 3.4. For $\alpha > 2$ the problem of omitting the zero-dimensional restriction is still open and related to question stated at the end of section 3.4.

Remark 2 *Subject of this section was extensively studied recent years. In 1995 Dobrowolski and Marciszewski in [2] addressed a problem whether for a filter $\mathcal{F}$*

$$\{(x_n)_{n\in\omega} \in \mathbb{R}^\omega : \exists_{x\in\mathbb{R}} (x_n)_{n\in\omega} \text{ is converging to } x \text{ relatively to } \mathcal{F}\} \in \Pi^0_\alpha$$

as a subset of $\mathbb{R}^\omega$, provided that $\mathcal{F} \in \Pi^0_\alpha$. In [12] Solecki reduced this problem to separating a filter $\mathcal{F} \in \Pi^0_\alpha$ from its dual ideal by a Σ^0_β set for some $\beta < \alpha$ and answered this question possitively in case $\alpha = 3$. Debs and Saint Raymond using the theory of ranks generalize this result to the case of all α and, as a consequence, solve possitively the problems studied by Solecki as well as Dobrowolski and Marciszewski.

Some questions involving separation properties of filters were asked during last few years (for example in [11]). The following is still open:

Question 1 *(cf. [1])*
For all ordinals $\alpha = \lambda + n > 0$, where λ is limit and $n < \omega$, denote $\alpha^* = \lambda$, if $\alpha = \lambda$ is limit and $\alpha^* = 1 + \lambda + 2n - 1$, if α is succesor. Is there a $\Pi^0_{\alpha^*}$ filter of rank $\ge \alpha$?

Debs and Saint Raymond conjecture in [1], that there is no such filter.

3.2 Rank of products of filters

In this section we deal with the rank of filters of the form $\mathcal{G} - \Sigma\{\mathcal{F}_i\}_{i\in I}$.

Theorem 4 *(cf. [1])*
Let $\mathcal{F}$ be a filter on a set I and $\mathcal{F}_i$ for $i \in I$ be filters on D_i respectively. Let also $\mathcal{G} = \mathcal{F} - \Sigma\{\mathcal{F}_i\}_{i\in I}$ and $J \subset I$ be an element of the filter $\mathcal{F}$.
(a) If $\mathrm{rk}(\mathcal{F}) \geq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \geq \xi$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \geq \xi + \alpha$.
(b) If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \xi$ for all $i \in J$, then $\mathrm{rk}(\mathcal{G}) \leq \xi + 1 + \alpha$.

Proof. Denote by D the set $\Sigma_{i\in I} D_i$.
(a) Let τ_0 be the standard product topology on the set 2^ω and $g \in \mathcal{B}_{\xi+\alpha}(2^\omega)$. Then there exists a zero-dimensional Polish topology τ on 2^ω such that $\Sigma^0_\xi(\tau_0) \subset \Sigma^0_1(\tau) \subset \Sigma^0_{1+\xi}(\tau_0)$ and $g \in \mathcal{B}_\alpha(2^\omega)$ relatively to τ. Denote by X the set 2^ω equipped with the topology τ and by h the identity mapping from 2^ω onto X. Since h is of Baire class ξ there is a function $f : X \to \mathbb{R}$ of class α such that $g = f \circ h$. According to theorem 3 there is a sequence $(f_i)_{i\in I}$ of continuous functions on X such that $f = \mathcal{F} - \lim f_i$. Since for $i \in J$ the function $f_i \circ h$ is of Baire class ξ relatively to τ there is a family $(f_{i,n})_{n\in\omega}$ of continuous functions relatively to τ such that $f_i \circ h = \mathcal{F}_i - \lim f_{i,n}$. Setting $f_{i,n} = 0$ for $i \in I \setminus J$ and $n \in \omega$ we obtain that $g = \mathcal{G} - \lim f_{i,n}$.
(b) For $i \in J$ let $S_i \in \Sigma^0_{1+\xi}$ be a subset of $\mathcal{P}(D_i)$ separating $\mathcal{F}_i$ from its dual ideal and let $S \in \Sigma^0_{1+\alpha}$ be a subset of $\mathcal{P}(I)$ separating $\mathcal{F}$ from its dual ideal. Define for every $i \in I$ a function $\varphi_i : \mathcal{P}(D) \to 2$ of class $\xi + 1$:

$$\varphi_i(M) = \begin{cases} 1 & \text{, if } i \in J \wedge (M \cap D_i) \in S_i \\ 0 & \text{, else} \end{cases}$$

for $M \subset D$. Define also a function $\varphi : \mathcal{P}(D) \to 2^I$ of class $\xi + 1$:

$$\varphi(M) = (\varphi_i(M))_{i\in I}$$

for $M \subset D$. Then the set $S' = \varphi^{-1}(S)$ is of multiplicative class $1 + \xi + 1 + \alpha$. Moreover, if $A \in \mathcal{G}$, then $J' = \{i \in I : A \cap D_i \in \mathcal{F}_i\} \in \mathcal{F}$, so $\varphi(A)$ is the characteristic function of the set $J' \cap J \in \mathcal{F}$ and therefore belongs to S. Assume now that $A \in \mathcal{G}^*$. Then $J' = \{i \in I : A \cap D_i \in \mathcal{F}_i^*\} \in \mathcal{F}$. In this case $\varphi(A)$ is the characteristic function of some set included in $J \setminus J' \in \mathcal{F}^*$ and therefore doesn't belong to S.

□

We improve this estimation for Σ^0_2 filters. For this we need a lemma proved by Mazur in [10].

Definition 9 *A map $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is a submeasure on ω, if $\phi(\emptyset) = 0$, and $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$, for all $A, B \subset \omega$. It is lower semicontinuous if additionally for all $A \subset \omega$ we have $\phi(A) = \lim_{n\to\infty} \phi(A \cap n)$.*

Lemma 1 *(cf. [10])*
Let $\mathcal{F}$ be a filter on ω. Then $\mathcal{F} \in \Sigma^0_2$ if and only if $\mathcal{F} = \{A \subset \omega : \phi(A^c) < \infty\}$ for some lower semicontinuous submeasure ϕ on ω.

Lemma 2 *Let $\mathcal{F} = \{A \subset \omega : \phi(A^c) < \infty\}$ for some lower semicontinuous submeasure ϕ. Then*

$$A \in \mathcal{F}^* \Leftrightarrow \exists_{n\in\omega}\forall_{B\in Fin}\phi(B) > n \Rightarrow \exists_{i\in B} i \notin A.$$

Proof. If $A \in \mathcal{F}^*$, then set $n = \phi(A)$. If $B \in Fin$ is such that $\phi(B) > n$, then B cannot be contained in A. Conversely, for $A \notin \mathcal{F}^*$ we have $\phi(A) = \infty$ so lower semicontinuity of ϕ implies that for each $n \in \omega$ there is a finite $B \subset A$ such that $\phi(B) > n$.

□

Proposition 4 *If $\mathcal{F}$ is an Σ^0_2 filter, for $i \in I$ $\mathcal{F}_i$ are filters on D_i of rank less than or equal to ξ and $\mathcal{G} = \mathcal{F} - \Sigma\{\mathcal{F}_i\}_{i\in I}$, then $\mathrm{rk}(\mathcal{G}) \leq \xi + 1$.*

Proof. Denote by D the set $\Sigma_{i\in I} D_i$ and for $i \in I$ let $S_i \in \Pi^0_{1+\xi}$ be a subset of $\mathcal{P}(D_i)$ separating $\mathcal{F}_i$ from its dual ideal. Since $\mathcal{F}$ belongs to Σ^0_2, so does its dual ideal and therefore according to the lemma 1 the filter $\mathcal{F}$ assigns a lower semicontinuous submeasure ϕ. Set

$$S = \{A \subset D : \exists_{n\in\omega}\forall_{B\in Fin}\phi(B) > n \Rightarrow \exists_{i\in B} A \cap M_i \in S_i\}.$$

This set is of Borel additive class $1 + \xi + 1$ since the mapping $A \mapsto A \cap M_i$ is continuous. Moreover by lemma 2

$$A \in S \Leftrightarrow \{i : A \cap M_i \notin S_i\} \in \mathcal{F}^*.$$

This implies that S separates $\mathcal{G}$ from its dual ideal.

□

Debs and Saint Raymond in [1] proved the previous fact only for the Frechét filter. The generalized version of the previous fact is presented in section 4.2.

3.3 Rank of inductive limits

In this section we estimate the rank of inductive limits of filters.

Proposition 5 *(cf. [1])*
Let $\mathcal{F}$ be a filter, which is the inductive limit of the family of filters $(\mathcal{F}_i)_{i\in I}$ with the coherent system of quasi-homomorphisms $(\pi_{i,j})_{i\leq j}$. Then:
(a) If $\mathrm{rk}(\mathcal{F}_i) \geq \alpha$ for some $i \in I$, then $\mathrm{rk}(\mathcal{F}) \geq \alpha$.
(b) If $\mathrm{rk}(\mathcal{F}_i) \leq \alpha$ for infinitely many $i \in I$, then $\mathrm{rk}(\mathcal{F}) \leq \alpha + 1$.

Proof. (a) According to proposition 2 it is sufficient to construct a quasi-homomorphisms from $\mathcal{F}$ to $\mathcal{F}_i$ for every $i \in I$. Let $i \in I$ and define $F = \bigcup_{j\geq i} \mathrm{dom}(\pi_{i,j}) \in \mathcal{F}$ and mapping $\pi : F \to \mathrm{dom}(\mathcal{F}_i)$, $\pi(a) = \pi_{i,j}(a)$ if $a \in \mathrm{dom}(\pi_{i,j}) \subset \mathrm{dom}(\mathcal{F}_j)$. This mapping is a quasi-homomorphism from $\mathcal{F}$ to $\mathcal{F}_i$, since $\pi^{-1}(A) = \bigcup_{j\geq i} \pi^{-1}_{i,j}(A) \in \mathcal{F}$ for every $A \in \mathcal{F}_i$.
(b) Denote by D_i the sets $\mathrm{dom}(\mathcal{F}_i)$ and by D the domain of $\mathcal{F}$. Since there are quasi-homomorphisms from $\mathcal{F}_j$ to $\mathcal{F}_i$ for every pair $i \leq j$, proposition 2 implies that the rank of $\mathcal{F}_i$ for every $i \in I$ is less than or equal to α. So for $i \in I$ let S_i be the $\Pi^0_{1+\alpha}$ set separating $\mathcal{F}_i$ from its dual ideal. The set

$$S = \{A \subset D : \exists_i \forall_{j\geq i} M \cap D_j \in S_j\}$$

is $\Sigma^0_{1+\alpha+1}$ since the mapping $A \mapsto M \cap D_j$ is continuous for every $j \in I$. We will show that S separates $\mathcal{F}$ from its dual ideal.
Let $A \in \mathcal{F}$. There are $i \in I$ and $M \in \mathcal{F}_i$ such that $\tilde{M} = \bigcup_{j \geq i} \pi^{-1}_{i,j}(M) \subset A$. If $j \geq i$, then $A \cap D_j \supset \tilde{M} \cap D_j = \pi^{-1}_{i,j}(M) \in \mathcal{F}_j \subset S_j$, so we have $A \in S$.
Let now $A \in \mathcal{F}^*$. There are $i \in I$ and $M \in \mathcal{F}_i$ such that $\tilde{M} = \bigcup_{j \geq i} \pi^{-1}_{i,j}(M)$ and A are disjoint. Then for every $j \in I$ the sets $A \cap D_j$ and $\tilde{M} \cap D_j = \pi^{-1}_{i,j}(M)$ are disjoint, so $(A \cap D_j)^c \supset \tilde{M} \cap D_j \in \mathcal{F}_j$. This shows that $A \cap D_j$ is an element of $\mathcal{F}^*_j$ and since $\mathcal{F}^*_j \cap S_j = \emptyset$ we obtain that $A \notin S$.

□

3.4 Relation between $\mathcal{N}_\alpha$ and the rank of a filter

Our aim in this section is to investigate characterization of all filters of given rank by the family $(\mathcal{N}_\alpha)_{\alpha<\omega_1}$. First we make a simple observation.

Definition 10 *Let $\mathcal{F}$ and $\mathcal{G}$ be filters. We shall write $\mathcal{F} \sqsubseteq \mathcal{G}$ and say that $\mathcal{G}$ contains a copy of $\mathcal{F}$, if there exists a bijection $\sigma : \mathrm{dom}(\mathcal{F}) \to \mathrm{dom}(\mathcal{G})$ such that $\sigma(A) \in \mathcal{G}$ for every element $A \in \mathcal{F}$.*

Proposition 6 *(cf. [1])*
Given two filters $\mathcal{F}$ and $\mathcal{G}$, if $\mathcal{F} \sqsubseteq \mathcal{G}$, then $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{G})$.

Proof. Let σ be the bijection from the definition of $\mathcal{F} \sqsubseteq \mathcal{G}$ and S be a $\Sigma^0_{1+\alpha}$ set separating the filter $\mathcal{G}$ of rank α from its dual ideal. The set $S' = \{A : \sigma(A) \in S\}$ is also $\Sigma^0_{1+\alpha}$ since the mapping $A \mapsto \sigma(A)$ is continuous. Moreover, if $A \in \mathcal{F}$, then $\sigma(A) \in \mathcal{G} \subset S$ and if $A \in \mathcal{F}^*$, then also $\sigma(A) \in \mathcal{G}^*$ which implies that $\sigma(A) \notin S$. Hence S' separates $\mathcal{F}$ from its dual ideal and $\mathrm{rk}(\mathcal{F}) \leq \alpha$.

□

Now we can state two propositions validating that there is a chance of characterising all filters of given rank by the family $(\mathcal{N}_\alpha)_{\alpha<\omega_1}$.

Proposition 7 *(cf. [1])*
A filters $\mathcal{F}$ is of rank greater than or equal to 1 if and only if it contains a copy of $\mathcal{N}_1$.

Proposition 8 *(cf. [1])*
A filters $\mathcal{F}$ is of rank greater than or equal to 2 if and only if it contains a copy of $\mathcal{N}_2$.

Remark 3 *The above proposition was also proved by I. Recław and M. Laczkovich in [7], but using essentially different methods including Borel determinacy for a game introduced by C. Laflamme in [8]. Actually in [7] was shown that for any Polish space X the following are equivalent:*

- $\mathcal{C}_\mathcal{F}(X) = \mathcal{B}_1(X)$,
- *$\mathcal{F}$ does not contain a copy of $\mathcal{N}_2$,*

- *$\mathcal{F}$ is of rank* 2.

It is quite obvious to ask the next question, which is still open.

Question 2 *(cf. [1])*
Is it true that for any ordinal $\alpha < \omega_1$ a filters $\mathcal{F}$ is of rank greater than or equal to α if and only if it contains a copy of $\mathcal{N}_\alpha$.

Remark 4 *Proposition 8 is significant (since we can characterize all filters of rank $\geq$ 2) to eliminate in some cases the zero-dimensional restriction in part (b) and (c) of theorem 3. So we can state that if $\mathrm{rk}(\mathcal{F}) \geq 2$, then $\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_2(X)$ for any Polish space X. Positive answer to the question asked above would eliminate the zero-dimensional restriction in theorem 3 in all cases.*

Chapter 4

Another operation on families of filters

4.1 Filters of countable type

This section is based on [4], where David Fremlin studies the theory of filters of countable type. The idea of such filters comes probably from Katětov's filters and was introduced by Mauldin, Preiss and Weizsäcker in [9].

Definition 11 *(cf. [4])*
Let X be a set, $(\mathcal{F}_i)_{i\in I}$ a nonempty family of filters on X and $\mathcal{F}$ a filter on I. Then we define a filter on X:

$$\lim_{i\to\mathcal{F}} \mathcal{F}_i = \{A \subset X : \{i \in I : A \in \mathcal{F}_i\} \in \mathcal{F}\}.$$

Definition 12 *(cf. [4])*
Let X be a set. We define a family $(\mathfrak{F}_\alpha)_{\alpha<\omega_1}$:

- *$\mathfrak{F}_0$ is the family of all principal ultrafilters on X.*
- *$\mathfrak{F}_\alpha$ is the family of all filters on X, which are equal to $\lim_{i\to\mathcal{F}_{Fr}} \mathcal{F}_i$, where $\mathcal{F}_i \in \bigcup_{\xi<\alpha} \mathfrak{F}_\xi$ for every $i \in \omega$.*

The set $\mathfrak{F}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathfrak{F}_\alpha$ is the set of all filters of countable type.

Filters of countable type can be equivalently defined in the way presented below:

Definition 13 *(cf. [4])*
A filter $\mathcal{F}$ on a set X is of countable type if it belongs to the smallest class of filters on X containing the principal ultrafilters and closed under countable intersections and increasing countable unions.

Remark 5 *Mauldin, Preiss and Weizsäcker in [9] used only the second definition. They also proved that for any Polish space X*

$$\bigcup_{\alpha<\omega_1} \mathcal{B}^{(\alpha)}(X) = \bigcup_{\mathcal{F}\in\mathfrak{F}_{\omega_1}} \mathcal{C}_{\mathcal{F}}(X)$$

Note that the family $(\mathfrak{F}_\alpha)_{\alpha<\omega_1}$ is non-decreasing. This fact allows us to define a function on the set of filters on countable type:

Definition 14 *(cf. [4])*
The countable-type level of a filter $\mathcal{F}$ of countable type is the ordinal:

$$\mathrm{ct}(\mathcal{F}) = \min\{\alpha < \omega_1 : \mathcal{F} \in \mathfrak{F}_\alpha\}.$$

This defines us a function $\mathrm{ct} : \mathfrak{F}_{\omega_1} \to \omega_1$.

Proposition 9 *(cf. [4])*
Let X be a set, $(\mathcal{F}_i)_{i\in I}$ a nonempty family of filters of countable type on X and $\mathcal{F}$ a filter of countable type on I. Then $\lim_{i\to\mathcal{F}} \mathcal{F}_i$ is also a filter of countable type.

Proof. We can induce on $\mathrm{ct}(\mathcal{F})$:
If $\mathrm{ct}(\mathcal{F}) = 0$, then $\mathcal{F}$ is a principal ultrafilter generated by some n. In this case $\lim_{i\to\mathcal{F}} \mathcal{F}_i = \mathcal{F}_n$ is a filter of countable type.
Suppose that $\lim_{i\to\mathcal{G}} \mathcal{F}_i$ is of countable type for filters $\mathcal{G}$ such that $\mathrm{ct}(\mathcal{G}) < \alpha$ and let $\mathcal{F}$ be a filter of countable-type level α. Then $\mathcal{F} = \lim_{i\to\mathcal{F}_{Fr}} \mathcal{G}_i$ for some filters $\mathcal{G}_i$ of countable-type level less than α. We have:

$$\lim_{i\to\mathcal{F}} \mathcal{F}_i = \{A : \{n : \{i : A \in \mathcal{F}_i\} \in \mathcal{G}_n\} \in \mathcal{F}_{Fr}\} = \lim_{n\to\mathcal{F}_{Fr}} \lim_{i\to\mathcal{G}_n} \mathcal{F}_i.$$

Since $\mathrm{ct}(\mathcal{G}_n) < \alpha$ for every n, the filters $\lim_{i\to\mathcal{G}_n} \mathcal{F}_i$ are of countable type and so is $\lim_{i\to\mathcal{F}} \mathcal{F}_i$.

□

Example 2 *Not every filter is of countable type. Fremlin shown this fact using the Bolzano-Weierstrass property for filters: for a filter $\mathcal{F}$ on a set X and a sequence $g : \omega \to \mathbb{R}$ we say that its subsequence $g \restriction A$ for some $A \subset X$ is $\mathcal{F}$-convergent, if there exists $r \in \mathbb{R}$ such that $\{n \in A : |g(n) - r| < \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$. Following Filipów, Mrożek, Recław and Szuca ([3]) we say that $\mathcal{F}$ has the Bolzano-Weierstrass property if for every bounded sequence $g : \omega \to \mathbb{R}$ there is $A \notin \mathcal{F}^*$ such that $g \restriction A$ is $\mathcal{F}$-convergent.*
It is known that the filter of asymptotic density 1:

$$\mathcal{F}_d = \left\{A \subset \omega : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 1\right\}$$

does not satisfy the Bolzano-Weierstrass property.
In [4] it is shown that every filter included in a filter of countable type has the Bolzano-Weierstrass property. So $\mathcal{F}_d$ is a filter, which is not of countable type.

Not every filter is of countable type, although Fremlin proved in [4] that there are continuum many isomorphism classes of filters of countable type.

4.2 Limits of filters and its rank

In this section we study the operation on families of filters presented at the beginning of previous section. Our aim is to estimate the rank of $\lim_{i\to\mathcal{F}}\mathcal{F}_i$ in the way presented in sections 3.2 and 3.3 for Fubini sums and inductive limits. The first theorem estimates it from above:

Theorem 5 *If $\mathcal{F}$ is a Borel filter of rank 1, $J\in\mathcal{F}$ and $\mathcal{F}_i$ are filters such that $\mathrm{rk}(\mathcal{F}_i)\le\alpha$ for $i\in J$, then $\lim_{i\to\mathcal{F}}\mathcal{F}_i$ is of rank $\le\alpha+1$.*

Example 3 *Estimation from below of the rank of $\lim_{i\to\mathcal{F}}\mathcal{F}_i$ is more complicated than for other presented operations. To see this first consider the filters $\mathcal{F}_i=\mathcal{F}_{Fr}$ for $i\in\omega$. Then $\lim_{i\to\mathcal{F}_{Fr}}\mathcal{F}_i=\mathcal{F}_{Fr}$ is of rank 1, likewise all filters $\mathcal{F}_i$. Althought, considering filters*

$$\mathcal{F}_i=\{A\subset\omega\times\omega:|\{j:(i,j)\notin A\}|<\omega\}=\{A\subset\omega\times\omega:\exists_j\forall k>j(i,j)\in A\}$$

for $i\in\omega$, we obtain $\mathrm{rk}(\mathcal{F}_i)=1$ for all i (since $\mathcal{F}_i$ are Σ^0_2), but $\mathrm{rk}\left(\lim_{i\to\mathcal{F}_{Fr}}\mathcal{F}_i\right)=\mathrm{rk}\left(\mathcal{N}_2\right)=2$.

Remark 6 *Theorem 5 gives a better estimation (actually an exact outcome) for $\mathcal{F}-\Sigma\left\{\mathcal{F}_i\right\}_{i\in I}$, when $\mathcal{F}$ is a Borel filter of rank 1. Indeed, if we define filters*

$$\mathcal{G}_n=\left\{A\subset\Sigma_{i\in I}\mathrm{dom}(\mathcal{F}_\mathrm{i}):A\cap\mathrm{dom}(\mathcal{F}_\mathrm{n})\in\mathcal{F}_n\right\},$$

then it is easy to see, that $\mathrm{rk}(\mathcal{G}_i)=\mathrm{rk}(\mathcal{F}_i)$ for $i\in I$. Moreover

$$\mathcal{F}-\Sigma\left\{\mathcal{F}_i\right\}_{i\in I}=\left\{A:A\subset\bigcup_{i\in F}F_i \text{ for some } F\in\mathcal{F} \text{ and } F_i\in\mathcal{F}_i\right\}=$$

$$=\left\{A:\left\{i:A\cap\mathrm{dom}(\mathcal{F}_\mathrm{i})\in\mathcal{F}_i\right\}\in\mathcal{F}\right\}=\left\{A:\left\{i:A\in\mathcal{G}_i\right\}\in\mathcal{F}\right\}=\lim_{i\to\mathcal{F}}\mathcal{G}_i.$$

Combining part (a) of theorem 4 with theorem 5 we obtain, that for $\mathcal{F}$ being a Borel filter of rank 1, $J\in\mathcal{F}$ and $\mathcal{F}_i$ being filters of rank α for $i\in J$, the rank of $\mathcal{F}-\Sigma\left\{\mathcal{F}_i\right\}_{i\in I}$ is equal to $\alpha+1$.

Let us also point out, that theorem 5 is a generalization of proposition 4 (since every Σ^0_2 filter is of rank 1) and implies part (b) of theorem 5 in some cases: given a coherent system of quasi-homomorphisms $(\pi_{i,j})_{i\le j}$ for the family of filters $(\mathcal{F}_i)_{i\in I}$ set

$$\mathcal{G}_n=\left\{A\subset\Sigma\left\{\mathrm{dom}\left(\mathcal{F}_i\right)\right\}_{i\in I}:\exists_{M\in\mathcal{F}_n}A\supset\bigcup_{j\ge n}\pi^{-1}_{n,j}\left(M\right)\right\}.$$

If the sets $L_i=\{j:\mathcal{G}_i\subset\mathcal{G}_j\}$ form a basis of some filter

$$\mathcal{F}=\left\{A\subset I:\exists_{F\in Fin}\bigcap_{i\in F}L_i\subset A\right\},$$

then this filter is Borel of rank 1 *(since it is a* Σ^0_2 *set). Moreover, since for all* n *we have a quasi-homomorphism* π_n *from* $\mathcal{F}_n$ *to* $\mathcal{G}_n$ *with domain* $\operatorname{dom}(\mathcal{F}_n)$ *defined by* $\pi_n(a) = (n, a)$ *for* $a \in \operatorname{dom}(\mathcal{F}_n)$*, by proposition 2* $\operatorname{rk}(\mathcal{G}_i) \leq \operatorname{rk}(\mathcal{F}_i)$ *for all* i*. Actually even* $\operatorname{rk}(\mathcal{G}_i) = \operatorname{rk}(\mathcal{F}_i)$ *since every set* S *separating* $\mathcal{G}_i$ *from its dual ideal defines a subset*

$$S' = \{A \cap \operatorname{dom}(\mathcal{F}_i) : A \in S\}$$

of $\mathcal{P}(\operatorname{dom}(\mathcal{F}_i))$*, which is in the same pointclass as* S *(since the mapping* $A \mapsto A \cap \operatorname{dom}(\mathcal{F}_i)$ *is continuous) and separates* $\mathcal{F}_i$ *from its dual ideal. Finally observe that:*

$$\lim_{\leftarrow} \mathcal{F}_i = \left\{ A : \exists_{i \in I} \exists_{M \in \mathcal{F}_i} A \subset \bigcup_{j \geq i} \pi^{-1}_{i,j}(M) \right\} = \{A : \exists_{i \in I} A \in \mathcal{G}_i\} =$$

$$= \{A : \exists_{i \in I} \{j : A \in \mathcal{G}_j\} \supset L_i \in \mathcal{F}\} = \{A : \{i : A \in \mathcal{G}_i\} \in \mathcal{F}\} = \lim_{i \to \mathcal{F}} \mathcal{G}_i.$$

So inductive limit of a family of filters can be represented as a limit of filters of the same rank provided that the sets $L_i = \{j : \mathcal{G}_i \subset \mathcal{G}_j\}$ *form a basis of some filter.*

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