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A note on ranks of filters

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Chapter 1

Preliminaries

1.1 Introduction

In this paper we study ranks of filters obtained from a family of filters as $\mathcal{F}$-limits. The most important paper in this field is [1], in which Gabriel Debs and Jean Saint Raymond introduce the concept of rank of a filter. We present some results published in this paper and show how they correspond to our results.
First chapter is devoted to introduce some basic definitions and facts, which will be used in the sequel, including $\mathcal{F}$-limits of families of filters and inductive limits of quasi-inductive systems. In the second chapter we present some new results concerning estimation of rank of some special filters. In particular we estimate from above the rank of any filter obtained as a limit of family of filters of given ranks and give an example that the rank of such a filter can fall to 1 even for a family of filters of arbitrarily high finite ranks. The last part is devoted to some facts concerning inductive limits of filters.

1.2 Basic definitions and facts

In the sequel we will use extensively the following definition: for a Polish space X, two subsets $A, B \subset X$ and a family $\Gamma \subset \mathcal{P}(X)$ we say that A is Γ-separated from B if there exists a subset $S \subset X$ in Γ, for which $A \subset S$ and $B \cap S = \emptyset$.

Let I be a set. A family of sets $\mathcal{F} \subset \mathcal{P}(I)$ is a filter on I, if it is closed under taking finite intersections and supersets. Throughout this paper for a family $\mathcal{A} \subset \mathcal{P}(I)$ we denote

$$\mathcal{A}^* = \{A \subset I : I \setminus A \in \mathcal{A}\}.$$

If $\mathcal{F}$ is a filter on I, then the family $\mathcal{F}^*$ is an ideal (i.e., a family closed under taking finite unions and subsets) called the dual ideal to the filter $\mathcal{F}$. We denote $Fin = [\omega]^{<\omega}$ and $\mathcal{F}_{Fr} = Fin^*$. Clearly Fin is an ideal and $\mathcal{F}_{Fr}$ its dual filter called the Frechét filter.

Let (X, ρ) be a metric space and $\mathcal{F}$ a filter on I. A sequence $(x_i)_{i\in I}$ is $\mathcal{F}$-convergent to $x \in X$ ($x = \mathcal{F} - \lim x_i$), if for every $\epsilon > 0$ we have

$$\{i \in I : \rho(x_i, x) < \epsilon\} \in \mathcal{F}.$$

We write $f = \mathcal{F} - \lim f_i$ and say that function f on X is a limit of the sequence of functions on X $(f_i)_{i\in I}$ relatively to $\mathcal{F}$, if $f(x) = \mathcal{F} - \lim f_i(x)$ for every $x \in X$. By $\mathcal{C}_{\mathcal{F}}(X)$ we denote the family of all real valued functions on the space X, which can be represented as a limit relatively to $\mathcal{F}$ of a sequence of continuous functions. In the sequel we also denote by $\mathcal{B}_\alpha(X)$ the family of all real valued functions on the space X of Borel class $\alpha < \omega_1$.

A filter $\mathcal{F}$ is free, if $\bigcap \mathcal{F} = \emptyset$. It is principal, if it is of the form

$$\mathcal{F}_E = \{A \subset I : E \subset A\}$$

for some subset $E \subset I$. Maximal filters are called ultrafilters. They can be characterized by following condition: a filter $\mathcal{F}$ is an ultrafilter if and only if $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ for each A. All principal ultrafilters are of the form

$$\mathcal{F}_{\{x\}} = \{A \subset I : x \in A\}$$

for some $x \in I$.

Two filters $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, provided that there is a bijection $\sigma : \mathrm{dom}(\mathcal{G}) \to \mathrm{dom}(\mathcal{F})$ such that for all $A \subset \mathrm{dom}(\mathcal{F})$

$$A \in \mathcal{F} \Leftrightarrow \sigma^{-1}[A] \in \mathcal{G}.$$

Definition 1 *(cf. [1] and [5])*
The rank of a filter $\mathcal{F}$ on a Polish space is the ordinal:

$$\mathrm{rk}(\mathcal{F}) = \min\left\{\alpha < \omega_1 : \mathcal{F} \textit{ is } \Sigma^0_{1+\alpha}\textit{-separated from } \mathcal{F}^*\right\}$$

Clearly for every filter $\mathcal{F}$ its rank $\mathrm{rk}(\mathcal{F})$ is unique. By Souslin separation Theorem, every analytic filter has a countable rank. Note also that

$$\mathrm{rk}(\mathcal{F}) = \min\left\{\alpha < \omega_1 : \mathcal{F} \text{ is } \Pi^0_{1+\alpha}\text{-separated from } \mathcal{F}^*\right\}$$

since the canonical involution $A \mapsto A^c$ is a homeomorphism. Let us make one more easy observation: if $\mathcal{F}$ and $\mathcal{G}$ are filters satisfying $\mathcal{F} \subset \mathcal{G}$, then $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{G})$ ([1], section 3). Note also that a filter is of rank 0 if and only if it is not free ([1], section 3).

Solecki in [5] and Debs with Saint Raymond in [1] observed that the rank of a filter plays a fundamental role in studying the class of pointwise limits relatively to a filter of sequences of continuous functions. This is because of the following theorem.

Theorem 1 *(cf. [1], section 2 and 3)*
Let $\mathcal{F}$ be an analytic filter and $\alpha < \omega_1$ a countable ordinal. Then
(a) $\mathcal{C}_{\mathcal{F}}(X) \subset \mathcal{B}_\alpha(X)$ for any Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \leq \alpha$.
(b) $\mathcal{C}_{\mathcal{F}}(X) \supset \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) \geq \alpha$.
(c) $\mathcal{C}_{\mathcal{F}}(X) = \mathcal{B}_\alpha(X)$ for any zero-dimensional Polish space X if and only if $\mathrm{rk}(\mathcal{F}) = \alpha$.

If $(X_i)_{i\in I}$ is a family of sets, then by $\Sigma_{i\in I} X_i$ we denote its disjoint sum, i.e., the set of all pairs (i, x) where $i \in I$ and $x \in A_i$.

Definition 2 *(cf. [4])*
For a filter $\mathcal{G}$ on I and a family of filters $(\mathcal{F}_i)_{i\in I}$, family of all sets of the form

$$\sum_{i\in G} F_i$$

for $G \in \mathcal{G}$ and $F_i \in \mathcal{F}_i$ forms a filter on the set $\sum_{i\in I} \operatorname{dom}(\mathcal{F}_i)$. We denote this filter by $\mathcal{G} - \sum_{i\in I} \mathcal{F}_i$ and call it the $\mathcal{G}$-Fubini sum of the family $(\mathcal{F}_i)_{i\in I}$.
If $(I, \leq)$ is a directed set and $\mathcal{G}$ is generated by sets $L_i = \{j : j \geq i\}$, then we call $\mathcal{G} - \sum_{i\in I} \mathcal{F}_i$ the Fubini sum of the family $(\mathcal{F}_i)_{i\in I}$.

In particular, putting the same filter as members of the family from the above definition, we obtain a standard product of filters:

Definition 3 *(cf. [4])*
For filters $\mathcal{F}$ and $\mathcal{G}$ on I and X respectively, if we put $\mathcal{F}_i = \mathcal{G}$ for every $i \in I$, then we obtain a filter $\mathcal{F} \times \mathcal{G} = \mathcal{F} - \sum_{i\in I} \mathcal{F}_i$, which we call the product of filters $\mathcal{F}$ and $\mathcal{G}$.

1.3 Two other operations on families of filters

We will also use two operations on families of filters defined below, which are not so common: $\mathcal{F}$-limit of the family of filters and inductive limit of family of filters with a coherent system of quasi-homomorphisms. The first operation was studied in [2] by Fremlin and the second operation in [1] by Debs and Saint Raymond.

Definition 4 *(cf. [2])*
Let X be a set, $\mathcal{F}$ a filter on a set I and $(\mathcal{F}_i)_{i\in I}$ a family of filters on X. Then

$$\lim_{i\to\mathcal{F}} \mathcal{F}_i = \{A \subset X : \{i \in I : A \in \mathcal{F}_i\} \in \mathcal{F}\}$$

is a filter on X. We sometimes denote it also by $\lim_{\mathcal{F}} \mathcal{F}_i$ and call it the $\mathcal{F}$-limit of the family of filters $(\mathcal{F}_i)_{i\in I}$.

To define inductive limits we need to introduce the concept of coherent system of quasi-homomorphisms.

If $\mathcal{F}$ and $\mathcal{G}$ are filters, then a quasi-homomorphism from $\mathcal{F}$ to $\mathcal{G}$ is a mapping $\pi : F \to \operatorname{dom}(\mathcal{G})$, where $F \in \mathcal{F}$, such that for each $G \in \mathcal{G}$ its preimage $\pi^{-1}[G]$ belongs to $\mathcal{F}$. If there exists a quasi-homomorphism from $\mathcal{F}$ to $\mathcal{G}$, then $\operatorname{rk}(\mathcal{G}) \leq \operatorname{rk}(\mathcal{F})$ ([1], section 5). As a simple consequence we get that two isomorphic filters have the same rank.

Definition 5 *([1], section 5)*
For a family of filters $(\mathcal{F}_i)_{i\in I}$, where $(I, \leq)$ is a directed set, the family $(\pi_{i,j})_{i\leq j}$, where each $\pi_{i,j}$ is a quasi-homomorphism from $\mathcal{F}_j$ to $\mathcal{F}_i$, is a coherent system of quasi-homomorphisms for $(\mathcal{F}_i)_{i\in I}$, provided that for all $i, j, k \in I$ such that $i \leq j \leq k$ the following condition is satisfied:

$$\pi_{i,k}(a) = \pi_{i,j}(\pi_{j,k}(a))$$

for every $a \in \operatorname{dom}(\pi_{i,k}) \cap \operatorname{dom}(\pi_{j,k}) \cap \pi_{j,k}^{-1}(\operatorname{dom}(\pi_{i,j}))$.
In this case we call $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\leq j\\ i,j\in I}}$ a quasi-inductive system.

Definition 6 *([1], section 5)*
Let $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\leq j\\ i,j\in I}}$ *be a quasi-inductive system. The set*

$$\varinjlim \mathcal{F}_i = \left\{ M \subset \Sigma_{i\in I}\mathrm{dom}\,(\mathcal{F}_i) : \exists_{i\in I}\exists_{P\in\mathcal{F}_i} M \supset \bigcup_{j\geq i} \{j\} \times \pi_{i,j}^{-1}[P] \right\}$$

is called inductive limit of the system $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\leq j\\ i,j\in I}}$.

Definition 7 *(cf. [4])*
For $\alpha < \omega_1$ *Katětov filters* $\mathcal{N}^{(\alpha)}$ *are defined inductively by:*

- $\mathcal{N}^{(0)} = \{\{0\}\}$ *is an unique filter on the set* $\{0\}$,
- $\mathcal{N}^{(\alpha+1)} = \mathcal{F}_{Fr} \times \mathcal{N}^{(\alpha)}$,
- $\mathcal{N}^{(\lambda)}$ *is the Fubini sum of the family* $\left(\mathcal{N}^{(\alpha)}\right)_{\alpha<\lambda}$, *if* λ *is a nonzero limit ordinal.*

The idea of such filters comes from Katětov ([4]) and Grimeisen ([3]), who worked on this problem independently. The family $(\mathcal{N}_\alpha)_{\alpha<\omega_1}$ is similar, but for limit ordinals requires introducing quasi-homomorphisms between Katětov filters: by proposition 5.8 in [1] for any quasi-inductive system $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\leq j\\ i,j\in I}}$ its extension $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\leq j\\ i,j\in I\cup\{\infty\}}}$, where $\mathcal{F}_\infty$ is the Fubini sum of $(\mathcal{F}_i)_{i\in I}$ and $\pi_{i,\infty} = \sum_{j>i} \pi_{i,j}$, is also a quasi-inductive system. So setting $\nu_{\alpha,\alpha}$ the identity mapping for $\alpha < \omega_1$ and applying this proposition inductively we get a quasi-inductive system $\left(\mathcal{N}^{(\alpha)}, \nu_{\alpha,\beta}\right)_{\alpha\leq\beta<\lambda}$.

Definition 8 *(cf. [1])*
For $\alpha < \omega_1$ *the filters* $\mathcal{N}_\alpha$ *are defined inductively by:*

- $\mathcal{N}_0 = \mathcal{N}^{(0)}$
- $\mathcal{N}_{\alpha+1} = \mathcal{N}^{(1+\alpha)}$
- $\mathcal{N}_\lambda$ *is an inductive limit of the quasi-inductive system*

  $$\left(\mathcal{N}^{(\alpha)}, \nu_{\alpha,\beta}\right)_{\alpha\leq\beta<\lambda},$$

  if λ *is a nonzero limit ordinal.*

The filters $\mathcal{N}_\alpha$ were introduced by Debs and Saint Raymond in [1]. Note that actually $\mathcal{N}_1$ is the Frechét filter $\mathcal{F}_{Fr}$ and $\mathcal{N}_2$ is the dual filter to well known and extensively studied ideal $Fin \times Fin$. These filters play fundamental roles in the theory of ranks of filters since the following result:

Theorem 2 *(cf. [1], section 6)*
For every $\alpha < \omega$ *the filter* $\mathcal{N}_\alpha$ *generates the Borel class* $\mathcal{B}_\alpha(X)$ *for any zero-dimensional Polish space* X *(i.e.,* $\mathcal{C}_{\mathcal{N}_\alpha}(X) = \mathcal{B}_\alpha(X)$*) and therefore has rank* α.

Definition 9 *Let $\mathcal{F}$ and $\mathcal{G}$ be filters. We shall write $\mathcal{F} \sqsubseteq \mathcal{G}$ and say that $\mathcal{G}$ contains an isomorphic copy of $\mathcal{F}$, if there exists a bijection $\sigma : \mathrm{dom}(\mathcal{F}) \to \mathrm{dom}(\mathcal{G})$ such that $\sigma(A) \in \mathcal{G}$ for every element $A \in \mathcal{F}$.*

Given two filters $\mathcal{F}$ and $\mathcal{G}$, if $\mathcal{F} \sqsubseteq \mathcal{G}$, then $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{G})$ ([1], section 7). Moreover for $\alpha \in \{1, 2\}$ a filter $\mathcal{F}$ is of rank greater than or equal to α if and only if it contains an isomorphic copy of $\mathcal{N}_\alpha$ ([1], section 7). Debs and Saint Raymond conjecture in [1], that this is true for every $\alpha < \omega_1$.

Chapter 2

Results

2.1 Rank of limits of families of filters

In this section we investigate the rank of limits of families of filters. Next proposition shows that we can estimate from above the Borel class and rank of limits of families of filters.

Proposition 1 *Let $\mathcal{F}$ be a filter on I and $(\mathcal{F}_i)_{i\in I}$ a family of filters on a Polish space X. Let also $J \in \mathcal{F}$ and for $\xi < \omega_1$ let Γ^0_ξ be a pointclass Σ^0_ξ or Π^0_ξ.*

a) If $\mathcal{F} \in \Gamma^0_{1+\alpha}(\mathcal{P}(I))$ and for $i \in J$ $\mathcal{F}_i \in \Gamma^0_{\beta_i}(\mathcal{P}(X))$ for some $\beta_i < \beta$, then $\lim_{i\to\mathcal{F}} \mathcal{F}_i \in \Gamma^0_{1+\beta+\alpha}(\mathcal{P}(X))$.

b) If $\mathrm{rk}(\mathcal{F}) \leq \alpha$ and $\mathrm{rk}(\mathcal{F}_i) \leq \beta$ for all $i \in J$, then $\mathrm{rk}\left(\lim_{i\to\mathcal{F}} \mathcal{F}_i\right) \leq \beta + 1 + \alpha$.

Proof. To prove part a) we define functions $\varphi_i : \mathcal{P}(X) \to 2$ by

$$\varphi_i(A) = \begin{cases} 1 & \text{, if } i \in J \wedge A \in \mathcal{F}_i \\ 0 & \text{, if not} \end{cases}$$

Every such function is of class β. The function $\varphi : \mathcal{P}(X) \to \mathcal{P}(I)$ defined by $\varphi(A) = \{i \in I : \varphi_i(A) = 1\}$ is also of class β. Notice also that

$$A \in \lim_{i\to\mathcal{F}} \mathcal{F}_i \Leftrightarrow \varphi[A] \in \mathcal{F} \Leftrightarrow A \in \varphi^{-1}[\mathcal{F}].$$

Since $\mathcal{F} \in \Gamma^0_{1+\alpha}(\mathcal{P}(I))$ we have $\lim_{i\to\mathcal{F}} \mathcal{F}_i = \varphi^{-1}[\mathcal{F}] \in \Gamma^0_{1+\beta+\alpha}(\mathcal{P}(X))$.
The proof of part b) is similar: let $S \in \Sigma^0_{1+\alpha}$ be a set separating the filter $\mathcal{F}$ from its dual ideal and for each $i \in J$ let $S_i \in \Sigma^0_{1+\beta}$ be a set separating the filter $\mathcal{F}_i$ from its dual ideal. Define functions $\psi_i : \mathcal{P}(X) \to 2$ by

$$\psi_i(A) = \begin{cases} 1 & \text{, if } i \in J \wedge A \in S_i \\ 0 & \text{, if not} \end{cases}$$

Then these functions, as well as the function ψ defined in the same way as the function φ above, are of class $\beta + 1$ and the set $\psi^{-1}[S] \in \Sigma^0_{1+\beta+1+\alpha}(\mathcal{P}(X))$. It is now sufficient to show that $\psi^{-1}[S]$ separates $\lim_{i\to\mathcal{F}} \mathcal{F}_i$ from its dual ideal.

Since $\mathcal{F} \subset S$ we have $\lim_{i\to\mathcal{F}} \mathcal{F}_i = \varphi^{-1}[\mathcal{F}] \subset \psi^{-1}[\mathcal{F}] \subset \psi^{-1}[S]$.
On the other hand, if $A \in (\lim_{i\to\mathcal{F}} \mathcal{F}_i)^*$, then $\psi(A) \subset \{i \in I : A \notin \mathcal{F}_i^*\} \in \mathcal{F}^*$, so $\psi(A) \notin S$.
$\square$

Rank of the filters $\mathcal{F}_i$ and the filter $\mathcal{F}$ does not imply any restriction on the rank of $\lim_{i\to\mathcal{F}} \mathcal{F}_i$. Actually for every filter $\mathcal{F}$, which is not an ultrafilter, we can obtain a $\mathcal{F}$-limit of family of filters of some finite positive rank, which is of rank 1. To prove this we need some preparations.

For a subset C of ω we will use the following notation:

$$\mathcal{N}_1^C = \{A \subset \omega : |C \setminus A| < \omega\}.$$

Obviously $\mathcal{N}_1^C$ is a filter on ω isomorphic to $\mathcal{N}_1$.

Lemma 1 *For every natural number $n > 1$ there exist two filters $\mathcal{F}$ and $\mathcal{G}$ of rank n and such that $\mathcal{F} \cap \mathcal{G}$ has rank 1.*

Proof. Let $\left(A_{i_1,\dots,i_{n-1}}\right)_{(i_1,\dots,i_{n-1})\in\omega^{n-1}}$ be a partition of ω on infinite sets. There is a bijection $\sigma_{\mathcal{F}} : \omega^n \to \omega$ such that we have $\sigma_{\mathcal{F}}\left[\{(i_1,\dots,i_{n-1})\} \times \omega\right] = A_{i_1,\dots,i_{n-1}}$ for every $(i_1,\dots,i_{n-1}) \in \omega^{n-1}$. Set

$$\mathcal{F} = \{\sigma_{\mathcal{F}}[Y] : Y \in \mathcal{N}_n\} = \left\{X \subset \omega : \exists_{t_0}\forall_{i_0>t_0} \dots \exists_{t_{n-1}}\forall_{i_{n-1}>t_{n-1}} X \in \mathcal{N}_1^{A_{i_1,\dots,i_{n-1}}}\right\}.$$

Let $\left(B_{i_1,\dots,i_{n-1}}\right)_{(i_1,\dots,i_{n-1})\in\omega^{n-1}}$ be a partition of ω on infinite sets such that for every $(i_1,\dots,i_{n-1}), \left(i'_1,\dots,i'_{n-1}\right) \in \omega^{n-1}$ the set $A_{i_1,\dots,i_{n-1}} \cap B_{i'_1,\dots,i'_{n-1}}$ is infinite. There is a bijection $\sigma_{\mathcal{G}} : \omega^n \to \omega$ such that we have $\sigma_{\mathcal{G}}\left[\{(i_1,\dots,i_{n-1})\} \times \omega\right] = B_{i_1,\dots,i_{n-1}}$ for every $(i_1,\dots,i_{n-1}) \in \omega^{n-1}$. Set

$$\mathcal{G} = \{\sigma_{\mathcal{G}}[A] : A \in \mathcal{N}_n\} = \left\{X \subset \omega : \exists_{t_0}\forall_{i_0>t_0} \dots \exists_{t_{n-1}}\forall_{i_{n-1}>t_{n-1}} X \in \mathcal{N}_1^{B_{i_1,\dots,i_{n-1}}}\right\}.$$

Filters $\mathcal{F}$ and $\mathcal{G}$ are isomorphic with $\mathcal{N}_n$ and therefore have rank n. Notice that

$$\mathcal{F} \cap \mathcal{G} = \left\{X \subset \omega : \exists_{t_0}\forall_{i_0>t_0} \dots \exists_{t_{n-1}}\forall_{i_{n-1}>t_{n-1}} X \in \mathcal{N}^1_{C_{i_1,\dots,i_{n-1}}}\right\},$$

where $C_{i_1,\dots,i_{n-1}} = A_{i_1,\dots,i_{n-1}} \cup B_{i_1,\dots,i_{n-1}}$. Observe also that $\mathcal{F}\cap\mathcal{G}$ is not free, so $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) > 0$.
To finish the proof we will show, that $\mathcal{N}_2 \not\subseteq \mathcal{F} \cap \mathcal{G}$, hence $\mathrm{rk}\,(\mathcal{F} \cap \mathcal{G}) < 2$. Let $\pi : \omega \to \omega^2$ be any bijection and denote $E_i = \pi^{-1}\left[\{i\} \times \omega\right]$.
1. Suppose that there is $(i_1^0,\dots,i_{n-1}^0) \in \omega^{n-1}$ such that $C_{i_1^0,\dots,i_{n-1}^0}$ is covered by finitely many E_i, i.e., $C_{i_1^0,\dots,i_{n-1}^0} \subset \bigcup_{i\in T} E_i$ for some finite set T. Then the set $\bigcup_{i\in T}\left(\{i\} \times \omega\right) \in \mathcal{N}_2$, but its preimage under π is disjoint with $C_{i_1^0,\dots,i_{n-1}^0}$ and therefore is not an element of $\mathcal{F}$, since for any $(i_1,\dots,i_{n-1})$ the intersection $C_{i_1^0,\dots,i_{n-1}^0} \cap A_{i_1,\dots,i_{n-1}}$ is infinite.
2. Suppose now that none of the sets $C_{i_1,\dots,i_{n-1}}$ is covered by finitely many E_i. Let $\sigma : \omega^{n-1} \to \omega$ be any 1-1 mapping. For every $(i_1,\dots,i_{n-1}) \in \omega^{n-1}$ let $S_{i_1,\dots,i_{n-1}}$ be a selector of the family

$$\left\{C_{i_1,\dots,i_{n-1}} \cap E_k : k > \sigma(i_1,\dots,i_{n-1}) \wedge C_{i_1,\dots,i_{n-1}} \cap E_k \neq \emptyset\right\}.$$

Then

$$\pi\left[\omega \setminus \bigcup_{(i_1,\ldots,i_{n-1})\in\omega^{n-1}} S_{i_1,\ldots,i_{n-1}}\right] \in \mathcal{N}_2,$$

since for every $k \in \omega$

$$\left|E_k \cap \bigcup_{(i_1,\ldots,i_{n-1})\in\omega^{n-1}} S_{i_1,\ldots,i_{n-1}}\right| \le k < \omega.$$

However its preimage under π is not an elemnet of $\mathcal{F}\cap\mathcal{G}$, since for every $(i_1^0,\ldots,i_{n-1}^0) \in \omega^{n-1}$

$$\left|\bigcup_{(i_1,\ldots,i_{n-1})\in\omega^{n-1}} S_{i_1,\ldots,i_{n-1}} \cap C_{i_1^0,\ldots,i_{n-1}^0}\right| = \left|S_{i_1^0,\ldots,i_{n-1}^0} \cap C_{i_1^0,\ldots,i_{n-1}^0}\right| = \omega.$$

So $\mathcal{N}_2 \not\subseteq \mathcal{F}\cap\mathcal{G}$.

□

Theorem 3 *For every $n > 0$ and every filter $\mathcal{F}$, which is not an ultrafilter, there is a family of filters $(\mathcal{F}_i)_{i\in\mathrm{dom}(\mathcal{F})}$ such that each $\mathcal{F}_i$ is of rank n and $\lim_{\mathcal{F}}\mathcal{F}_i$ is of rank 1.*

Proof. If $n = 1$, then set $\mathcal{F}_i = \mathcal{N}_1$ for $i \in \mathrm{dom}(\mathcal{F})$. If $n > 1$, then let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the filters from the previous lemma. Since $\mathcal{F}$ is not an ultrafilter, we can find a set $H \subset \mathrm{dom}(\mathcal{F})$ such that $H \notin \mathcal{F}$ and $\mathrm{dom}(\mathcal{F}) \setminus H \notin \mathcal{F}$. Set $\mathcal{F}_i = \mathcal{G}_1$ for $i \in H$ and $\mathcal{F}_i = \mathcal{G}_2$ for $i \notin H$. Then $\lim_{\mathcal{F}}\mathcal{F}_i = \mathcal{G}_1 \cap \mathcal{G}_2$, hence is of rank 1.

□

2.2 Some facts concerning inductive limits

For a quasi-inductive system $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\le j\\ i,j\in I}}$ and a set $X \subset \mathrm{dom}(\mathcal{F}_i)$ denote

$$\hat{X} = \sum_{j\ge i} \pi_{i,j}^{-1}[X].$$

Obviously $\hat{X}$ is a subset of $\sum_{i\in I}\mathrm{dom}(\mathcal{F}_i)$.

Next example shows that inductive limit is not always a filter. Notice also that the following example is a slight modification of the filter $\mathcal{N}_\omega$.

Example 1 *Put $I = \omega$ and consider the filters $\mathcal{N}_i$ for $i \in I$. Choose for each $i > 1$ the set $F_i = \omega^i \setminus (\{0\}\times\omega^{i-1}) \in \mathcal{N}_i$ and define a quasi-homomorphism $\pi_{1,i}$ from $\mathcal{N}_i$ to $\mathcal{N}_1$ with domain F_i and such that $\pi_{1,i}$ is a projection onto the last coordinate. For indexes $1 < i \le j$ let $\pi_{i,j} : \omega^j \to \omega^i$ be a projection onto the last i coordinates. Then $(\pi_{i,j})_{i\le j}$ is a coherent system of quasi-homomorphisms for the family of filters $(\mathcal{N}_i)_{i\in I}$. But for $A = \omega \in \mathcal{N}_1$ and $B = \omega^2 \in \mathcal{N}_2$ we have $\hat{A}\cap\hat{B} \notin \underleftarrow{\lim}\mathcal{N}_i$. Indeed, the union*

$$\hat{A}\cap\hat{B} = \Sigma_{j>1}F_j$$

does not include any set $\hat{P}$ for $P \in \mathcal{N}_i$, $i > 1$, because $\pi_{i,j}^{-1}(P)$ is not a subset of F_j for $j > i$. This shows that inductive limit of the quasi-inductive system $(\mathcal{N}_i, \pi_{i,j})_{i \leq j}$ is not a filter.

For a quasi-inductive system $(\mathcal{F}_i, \pi_{i,j})_{\substack{i \leq j \\ i,j \in I}}$ we say that it satisfies the $(\star)$ condition if

$$\forall_{k \geq j} \pi_{j,k}^{-1} \left(\mathrm{dom}(\pi_{i,j})\right) \subset \mathrm{dom}(\pi_{i,k})$$

for all indexes $i \leq j$.

If a quasi-inductive system is such that $\mathrm{dom}(\pi_{j,k}) \subset \mathrm{dom}(\pi_{i,k})$ for all $i \leq j \leq k$ (which is more natural condition), then it satisfies also the $(\star)$ condition.

Note that the $(\star)$ condition is satisfied for the filters $\mathcal{N}_\alpha$. It follows from more general fact:

Proposition 2 *If a quasi-inductive system $(\mathcal{F}_i, \pi_{i,j})_{\substack{i \leq j \\ i,j \in I}}$ satisfies the $(\star)$ condition, then its extension*

$$(\mathcal{F}_i, \pi_{i,j})_{i \leq j \in I \cup \{\infty\}},$$

where $\mathcal{F}_\infty$ is the Fubini sum of $(\mathcal{F}_i)_{i \in I}$ and $\pi_{i,\infty} = \sum_{j > i} \pi_{i,j}$, is also a quasi-inductive system satisfying the $(\star)$ condition.

Proof. Set $i < j$, $i, j \in I$. Observe that

$$\pi_{j,\infty}^{-1} \left(\mathrm{dom}(\pi_{i,j})\right) = \sum_{k > j} \pi_{j,k}^{-1} \left(\mathrm{dom}(\pi_{i,j})\right).$$

Applying the $(\star)$ condition we obtain that

$$\sum_{k > j} \pi_{j,k}^{-1} \left(\mathrm{dom}(\pi_{i,j})\right) \subset \sum_{k > j} \mathrm{dom}(\pi_{i,k}) \subset \sum_{k > i} \mathrm{dom}(\pi_{i,k}) = \mathrm{dom}(\pi_{i,\infty}),$$

which finishes the proof.

□

Proposition 3 *Let $(\mathcal{F}_i, \pi_{i,j})_{\substack{i \leq j \\ i,j \in I}}$ be a quasi-inductive system such that it satisfies the $(\star)$ condition. Then $\varinjlim \mathcal{F}_i$ is a filter.*

Proof. Define filters

$$\hat{\mathcal{F}}_i = \left\{ M \subset \Sigma_{k \in I} \mathrm{dom}\left(\mathcal{F}_k\right) : \exists_{P \in \mathcal{F}_i} M \supset \hat{P} \right\}.$$

We will show that $\hat{\mathcal{F}}_i \subset \hat{\mathcal{F}}_j$ providing $i \leq j$. Indeed, if $M \in \hat{\mathcal{F}}_i$, then there is $P \in \mathcal{F}_i$ such that $M \supset \hat{P}$. Set $P' = \pi_{i,j}^{-1}[P] \in \mathcal{F}_j$. We will show that $\hat{P}' \subset \hat{P}$. If $k \geq j$ and $x \in \pi_{j,k}^{-1}[P']$ (so (k, x) is an element of $\hat{P}'$) then we have $\pi_{i,k}(x) = \pi_{i,j}(\pi_{j,k}(x)) \in P$, since $(\pi_{i,j})_{i \leq j}$ is a coherent system of quasi-homomorphisms satisfying the $(\star)$ condition. So x

is an element of $\pi_{i,k}^{-1}[P]$, hence (k,x) is an element of $\hat{P}$.
Observe also that

$$\varinjlim \mathcal{F}_i = \left\{ M : \exists_{i\in I} M \in \hat{\mathcal{F}}_i \right\} = \bigcup_{i\in I} \hat{\mathcal{F}}_i.$$

Since the family of filters $\left(\hat{\mathcal{F}}_i\right)_{i\in I}$ is increasing, $\varinjlim \mathcal{F}_i$ is a filter, which finishes the proof. □

Remark 1 *However the above proof shows that the $(\star)$ condition implies inductive limit being a filter, negation of it does not determine if an inductive limit is a filter or not. It can be a filter for instance for finite families of filters, but example 1 shows that it is not always a filter.*

Remark 2 *Notice that inductive limit of the quasi-inductive system $(\mathcal{F}_i, \pi_{i,j})_{\substack{i\le j\\ i,j\in I}}$ satisfying the $(\star)$ condition is a special case of limit of filters $\left(\hat{\mathcal{F}}_i\right)_{i\in I}$. Indeed, for a filter $\mathcal{F}$ on I generated by sets of the form*

$$\{j \in I : j \geq i\}$$

for some $i \in I$, the following equalities hold:

$$\varinjlim \mathcal{F}_i = \left\{ M : \exists_{i\in I} M \in \hat{\mathcal{F}}_i \right\} = \left\{ M : \exists_{i\in I} \forall_{j\geq i} M \in \hat{\mathcal{F}}_j \right\} =$$

$$= \left\{ M : \exists_{i\in I} \left\{ j : M \in \hat{\mathcal{F}}_j \right\} = \{j \in I : j \geq i\} \right\} =$$

$$= \left\{ M : \exists_{i\in I} \left\{ j : M \in \hat{\mathcal{F}}_j \right\} \in \mathcal{F} \right\} = \lim_{i\to\mathcal{F}} \hat{\mathcal{F}}_i.$$

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