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SELECTIVE PROPERTIES OF IDEALS ON COUNTABLE SETS

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1. Introduction

We study property (S), which is a combinatorial property of ideals on countable sets. It was first defined by Zakrzewski in [11]. However in the literature one can find similar or even the same properties under different names. For example Grigorieff in [5] studies weak T-ideals, which turn out to be ideals without property (S). The theory of such ideals is still incomplete. In this paper we summarize some facts concerning property (S).

After some necessary definitions and notations listed in Section 2, we proceed to show equivalent definitions of property (S) (3.1). This part is based on [5], where Grigorieff proved that an ideal is a T-ideal if and only if it is selective if and only if it is inductive. We fill up some gaps and correct minor errors and omissions in his proofs.

The next section is devoted to giving a proof of the fact, that every analytic P-ideal, which is not $\mathbf{\Sigma^0_2}$ has property (S) (4.3). This was shown by Zakrzewski, but we give a much simpler proof using a result from [10].

In Section 4 we show that there exists a $\mathbf{\Sigma^0_\alpha}$ ideal without property (S) for every $\alpha > 1$ (5.1). This was known only for $\alpha = 2$. In the proof we use ideals defined by Lutzer, van Mill and Pol in [6] and a result of Calbrix from [1] concerning these ideals.

In the last section we investigate weak selectiveness of ideals, which is a property closely related to (S). In particular, following Grigorieff (cf. [5]) we give some equivalent definitions of weak selectiveness. We also study a special class of weakly selective ideals, which is connected with σ-ideals on Polish spaces and was introduced by Sabok and Zapletal in [9].

2. Preliminaries

In this section we collect some definitions, notations and basic facts, which will be useful.

2.1. **Ideals.** $\mathcal{I}$ is an ideal on a countable set X, if it is a collection of subsets of X closed under subsets and finite unions. We assume additionally that $\mathcal{P}(X)$ is not an ideal. By $\mathrm{dom}\,(\mathcal{I})$ we denote the domain of $\mathcal{I}$, i.e., the set X. Fin is the ideal of all finite subsets of X.

An ideal on X is countably generated if it has a countable basis, i.e., a family $\{B_n : n \in \omega\}$ such that for every member A of the ideal there is n such that $A \subset B_n$. It is dense, if every infinite subset of X contains an infinite subset in the ideal.

For $A \subset X \times Y$ and $x \in X$ by A_x we denote the set

$$A_x = \{y \in Y : (x, y) \in A\}.$$

By $X \oplus Y$ we denote the disjoint union of X and Y, i.e., $X \oplus Y = \{0\} \times X \cup \{1\} \times Y$. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on countable sets X and Y, respectively. Then we define the ideal $\mathcal{I} \oplus \mathcal{J}$ on $X \oplus Y$ by

$$A \in \mathcal{I} \oplus \mathcal{J} \Leftrightarrow \{x \in X : (0, x) \in A\} \in \mathcal{I} \wedge \{y \in Y : (1, y) \in A\} \in \mathcal{J}.$$

Similarly $\mathcal{I} \otimes \mathcal{J}$ is an ideal on $X \times Y$ given by:

$$A \in \mathcal{I} \otimes \mathcal{J} \Leftrightarrow \{x \in X : A_x \notin \mathcal{J}\} \in \mathcal{I}.$$

2.2. **Preorderings on ideals.** We say that $\mathcal{I}$ is below $\mathcal{J}$ in the Rudin-Blass order ($\mathcal{I} \leq_{RB} \mathcal{J}$), if there is finite-to-one $f : Y \to X$ such that

$$A \in \mathcal{I} \Leftrightarrow f^{-1}[A] \in \mathcal{J}.$$

The Katětov order is defined similarly: we say that $\mathcal{I}$ is below $\mathcal{J}$ in the Katětov order ($\mathcal{I} \leq_K \mathcal{J}$), if there is $f : Y \to X$ (not necessarily finite-to-one) such that

$$A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}.$$

2.3. **Combinatorial properties of ideals.** A set A is almost included in a set B ($A \subset^* B$), if $A \setminus B$ is finite. An ideal $\mathcal{I}$ is a P-ideal, if for every sequence $(A_n)_{n \in \omega} \subset \mathcal{I}$ there is a set $A \in \mathcal{I}$ such that $A_n \subset^* A$ for each n.

A selector of a collection of sets $(X_n)_{n \in \omega}$ is a set contained in the union of X_n's intersecting each of them in exactly one point.

We say that an ideal $\mathcal{I}$ on X has property (S), if there is a partition $(X_n)_{n \in \omega}$ of X such that:
(S_1) $\bigcup_{m \geq n} X_m \notin \mathcal{I}$, for each $n \in \omega$,
(S_2) $S \in \mathcal{I}$, for each selector S of the family $(X_n)_{n \in \omega}$.

An ideal $\mathcal{I}$ is weakly selective, if for every $A \notin \mathcal{I}$, any function $f : A \to \omega$ is either constant or one-to-one on some subset B of A such that $B \notin \mathcal{I}$. Equivalently, $\mathcal{I}$ is weakly selective, if for every partition of X, which satisfies (S_1) and has at most one element not in $\mathcal{I}$, there is a selector of that partition not in $\mathcal{I}$. Obviously, every ideal without property (S) is weakly selective.

2.4. **Descriptive complexity of ideals.** If X is a countable set, then the space 2^X of all functions $f : X \to 2$ is equipped with the product topology, where each space $2 = \{0, 1\}$ carries the discrete topology. We also treat the power set $\mathcal{P}(X)$ as the space 2^X by identifying subsets of X with their characteristic functions. Therefore we can talk about descriptive complexity of ideals on X.

2.5. **Trees.** If $s \in [\omega]^{<\omega}$, i.e., $s = (s(0), \dots, s(k))$ is a finite sequence of natural numbers, then by $\mathrm{lh}(s)$ we denote its length, i.e., $k+1$. If $s, t \in [\omega]^{<\omega}$ and $\mathrm{lh}(s) \leq \mathrm{lh}(t)$, then we write $s \preceq t$ in the case, when $s(i) = t(i)$ for all $i = 0, \dots, \mathrm{lh}(s) - 1$.

$T \subset [\omega]^{<\omega}$ is a tree, if for each element $s \in T$ and each $t \in [\omega]^{<\omega}$ such that $t \preceq s$, we have $t \in T$. A branch of a tree T is a function $b : \omega \to \omega$ such that

$$(b(0), \dots, b(k)) \in T$$

for all $k \in \omega$. We sometimes identify a branch with the set of all finite sequences of the form $(b(0), \dots, b(k))$ for $k \in \omega$ and therefore a branch can be treated as a subset of T.

Recall also that a ramification of a tree $T \subset [\omega]^{<\omega}$ at $s \in T$ is the set

$$\{n \in \omega : (s(0), \dots, s(\mathrm{lh}(s) - 1), n) \in T\}.$$

3. Equivalent definitions of property (S)

Grigorieff in [5] gave some equivalent definitions of property (S), which are listed below. The proofs can also be found in [5].

Proposition 3.1. The following are equivalent for any ideal $\mathcal{I}$:
(a) $\mathcal{I}$ does not have property (S) (i.e., in the terminology of [5], $\mathcal{I}$ is selective).
(b) For every strictly decreasing sequence $(X_n)_{n\in\omega}$ of subsets of ω, which are not in $\mathcal{I}$, there exists a strictly increasing function $f : \omega \to \omega$, with range not in $\mathcal{I}$ and such that $f(n+1) \in X_{f(n)}$ for each $n \in \omega$ ($\mathcal{I}$ is inductive, cf. [5]).
(c) Every tree $T \subset [\omega]^{<\omega}$, such that no finite intersection of its ramifications is in the ideal $\mathcal{I}$, has a branch not in $\mathcal{I}$ ($\mathcal{I}$ is a weak T-ideal, cf. [5]).

In the proof we will use the following lemma.

Lemma 3.2. *If $\mathcal{I}$ is an inductive ideal and $\{X_s\}_{s\in[\omega]^{<\omega}}$ is a family such that no finite intersection of its elements is in $\mathcal{I}$, then there is a strictly increasing function $h : \omega \to \omega$, with range not in $\mathcal{I}$ and such that $h(n) \in X_{(h(0),\dots,h(n-1))}$ for each $n \in \omega$.*

Proof. Since no finite intersection of elements of $\{X_s\}_{s\in[\omega]^{<\omega}}$ is in $\mathcal{I}$, we can assume, that this family has the following property: if $s, t \in [\omega]^{<\omega}$ are such that $\mathrm{lh}(s) \leq \mathrm{lh}(t)$ and $\max_{k<\mathrm{lh}(s)} s(k) \leq \max_{k<\mathrm{lh}(t)} t(k)$, then $X_t \subset X_s$. Indeed, we can let X'_t be the (finite) intersection of X_t with all X_s such that $\mathrm{lh}(s) \leq \mathrm{lh}(t)$ and $\max_{k<\mathrm{lh}(s)} s(k) \leq \max_{k<\mathrm{lh}(t)} t(k)$. Then $X'_s \notin \mathcal{I}$ and

$X'_s \subset X_s$, so the desired function for the family $\{X'_s\}_{s\in[\omega]^{<\omega}}$, is also good for $\{X_s\}_{s\in[\omega]^{<\omega}}$.
Let s_n be the constant sequence of value n and length $n+1$. Then $(X_{s_n})_{n\in\omega}$ is a decreasing family of sets not in the ideal. Since $\mathcal{I}$ is inductive, there is a strictly increasing $h : \omega \to \omega$, with range not in $\mathcal{I}$ and such that $h(n+1) \in X_{s_{h(n)}}$ for each $n \in \omega$.
The sequence $(h(0), \ldots, h(n))$ is of length $n+1$ and its maximum is $h(n)$. The sequence $s_{h(n)}$ is of length $h(n)+1 \geq n+1$ (since h is strictly increasing) and its maximum is also $h(n)$. Hence $X_{s_{h(n)}} \subset X_{(h(0),\ldots,h(n))}$ and $h(n+1) \in X_{(h(0),\ldots,h(n))}$. □

Now we can proceed to proving Proposition 3.1

Proof. **(a) ⇒ (b):** Assume that $\mathcal{I}$ is selective and let $(X_n)_{n\in\omega}$ be a strictly decreasing sequence of subsets of ω, which are not in $\mathcal{I}$. Consider the partition $(Y_n)_{n\in\omega}$ of ω, where $Y_0 = \omega \setminus X_0$ and $Y_{n+1} = X_n \setminus X_{n+1}$. It satisfies condition (S_1) since X_n's are not in the ideal, so there is a selector $S \notin \mathcal{I}$ of this partition. Since $\mathcal{I}$ contains Fin, we can assume, that $0 \notin S$. Notice that $S \setminus X_n$ is finite, but not empty for each n. Moreover for $n \neq m$ sets $S \setminus X_n$ and $S \setminus X_m$ are different.
Define an increasing function $g : \omega \to \omega$ by $g(n) = \max(S \setminus X_n)$. Let us point out, that $m > g(n)$ and $m \in S$ implies $m \in X_n$. Let also $g_0 = 0$ and $g_{p+1} = g(g_p)$. If $a \in \omega$ and $b \in S$ are such that $a \leq g_p \leq g_{p+1} < b$ for some p, then $X_{g_p} \subset X_a$ (since $a \leq g_p$) and $b \in X_{g_p}$ (since $g_{p+1} < b$). Hence $b \in X_a$.
Notice that the sequence $(g_n)_{n\in\omega}$ is strictly increasing. It follows from the fact, that $g(n) > n$ for each n. Indeed, the set $S \setminus X_n$ has exactly $n+1$ elements and does not contain 0 (since $0 \notin S$), so $g(n) = \max(S \setminus X_n) > n$.
Consider the partition given by $\omega \setminus S$ and sets of the form $S \cap (g_{2p}, g_{2p+2}]$. A cofinite union of such sets contains S modulo some finite set, so this partition satisfies (S_1). Let S' be the selector of it, which is not in the ideal. Denote by a_p the element of S', which is in $S \cap (g_{2p}, g_{2p+2}]$ and define on $\{a_p : p \in \omega\}$ the equivalence relation $\sim$ by:

$$a_p \sim a_q \Leftrightarrow |p-q| \leq 1 \wedge \left[a_{\min\{p,q\}}, a_{\max\{p,q\}}\right] \subset \left(g_{p+q}, g_{p+g+2}\right].$$

Then $a_p \sim a_{p+1}$ implies $[a_p, a_q] \subset (g_{2p+1}, g_{2p+3}]$ and the equivalence classes have at most two elements. These classes together with the complement of $\{a_p : p \in \omega\}$ form another partition of ω satisfying (S_1) for the same reason as the previous one. Therefore there is a selector $S'' \notin \mathcal{I}$ of it.
Define $f : \omega \to \omega$ as follows: $f(n)$ is the n-th point of $S'' \cap \{a_p : p \in \omega\}$.

Recall that all a_p's are in S. Moreover by the definition of $\sim$ between $f(n)$ and $f(n+1)$ there is always an interval $(g_p, g_{p+1}]$ for some p. Therefore, as noticed above, $f(n+1) \in X_{f(n)}$ and since $f[\omega]$ is equal to S'' without one point, f is a desired function.

(b) $\Rightarrow$ (c): Assume that $\mathcal{I}$ is inductive. Let $T \subset [\omega]^{<\omega}$ be a tree with no finite intersection of its ramifications in the ideal. Define $(X_s)_{s\in[\omega]^{<\omega}}$ as follows: if $s \in T$ then X_s is the ramification of T at s and otherwise $X_s = \omega$. No finite intersection of this family is in the ideal, so we can apply Lemma 3.2 and we get a strictly increasing function $h : \omega \to \omega$, with range not in $\mathcal{I}$ and such that $h(n) \in X_{(h(0),\ldots,h(n-1))}$ for each $n \in \omega$. We will show inductively, that h is a branch of T. $h(0) \in X_\emptyset$ and $\emptyset$ is in every tree, so $(h(0)) \in T$. Suppose now, that $(h(0), \ldots, h(k-1)) \in T$. Then

$$h(k) \in X_{(h(0),\ldots,h(k-1))} = \{n \in \omega : (h(0), \ldots, h(k-1), n) \in T\},$$

so $(h(0), \ldots, h(k-1), h(k)) \in T$.

(c) $\Rightarrow$ (a): Assume that $\mathcal{I}$ is a weak T-ideal. Let $(X_n)_{n\in\omega}$ be a partition of X satisfying (S_1) and define a tree $T \subset [\omega]^{<\omega}$ as follows:

$$s = (s(0), \ldots, s(k)) \in T \Leftrightarrow \forall_{n\in\omega} \left|\{s(0), \ldots, s(k)\} \cap X_n\right| \leq 1.$$

Then T has no finite intersection of its ramifications in the ideal $\mathcal{I}$. Indeed, a ramification at $s = (s(0), \ldots, s(k))$ is a set

$$\bigcup \{X_n : \{s(0), \ldots, s(k)\} \cap X_n \neq \emptyset\},$$

so it is not in $\mathcal{I}$ as a cofinite union of X_n's. A finite intersection of such sets is also not in $\mathcal{I}$ for the same reason. Since $\mathcal{I}$ is a weak T-ideal, there is a branch b not in $\mathcal{I}$. By the definition of T every branch is a subset of some selector S of the family $(X_n)_{n\in\omega}$, so this selector is also not in the ideal. $\square$

4. Property (S) and analytic P-ideals

In this section we investigate ideals with property (S). In particular, we give a simple proof of the fact, that every analytic P-ideal, which is not $\mathbf{\Sigma^0_2}$ has property (S). This was first observed by Zakrzewski in [11].

The next proposition collects some simple observations concerning property (S), which will be useful in further considerations.

Proposition 4.1. The following properties hold:
(a) For any ideals $\mathcal{I}$ and $\mathcal{J}$ the ideal $\mathcal{I} \otimes \mathcal{J}$ has property (S).
(b) If $\mathcal{I}$ has property (S), then for any ideal $\mathcal{J}$ the ideal $\mathcal{I} \oplus \mathcal{J}$ has property (S).

Proof. Let X and Y be the domains of $\mathcal{I}$ and $\mathcal{J}$ respectively.
(a) Define for each $x \in X$ the set $A_x = (x) \times Y$. Then $\{A_x\}_{x \in X}$ is a partition of $X \times Y$ witnessing that $\mathcal{I} \otimes \mathcal{J}$ has property (S).
(b) Let $(A_n)_{n \in \omega}$ be a partition of X witnessing property (S) for $\mathcal{I}$. Then

$$(A_n)_{n \in \omega} \cup \{1\} \times Y$$

is a partition witnessing that $\mathcal{I} \oplus \mathcal{J}$ has property (S).

□

Proposition 4.2. If $\mathcal{I}$ has property (S), then any ideal $\mathcal{J}$ satisfying $\mathcal{I} \leq_{RB} \mathcal{J}$ also has property (S).

Proof. Let $f : \mathrm{dom}\,(\mathcal{J}) \to \mathrm{dom}\,(\mathcal{I})$ be a function witnessing $\mathcal{I} \leq_{RB} \mathcal{J}$ and let $(X_n)_{n \in \omega}$ be a partition witnessing property (S) for $\mathcal{I}$. Set $Y_n = f^{-1}[X_n]$ for $n \in \omega$. Then the family $(Y_n)_{n \in \omega}$ is a partition of $\mathrm{dom}\,(\mathcal{J})$ satisfying (S_1) for $\mathcal{J}$. Let now $S \subset \mathrm{dom}\,(\mathcal{J})$ be a selector of this family. Then $f[S]$ is a selector of $(X_n)_{n \in \omega}$ and therefore is an element of $\mathcal{I}$. This implies that $S \in \mathcal{J}$ since

$$S \subset f^{-1}\left[f[S]\right] \in \mathcal{J}.$$

□

The following corollary follows from [11, Proposition 3.1], but we give a much simpler proof.

Corollary 4.3. *Every analytic P-ideal, which is not* $\mathbf{\Sigma^0_2}$ *has property* (S).

Proof. Let $\mathcal{J}$ be an analytic P-ideal, which is not $\mathbf{\Sigma^0_2}$. By [10, Theorem 3.3] we have $\emptyset \otimes Fin \leq_{RB} \mathcal{J}$, where $\emptyset \otimes Fin$ is the ideal of all sets $A \subset \omega \times \omega$ such that A_n is finite for all $n \in \omega$. It is also easy to see that $\emptyset \otimes Fin$ has property (S) using the same arguments as in the proof of Proposition 4.1, part (b). To finish the proof, notice that Proposition 4.2 implies property (S) for $\mathcal{J}$. □

Although all analytic P-ideal, which are not $\mathbf{\Sigma^0_2}$ have property (S), there are several examples of not countably generated ideals with property (S), which are not P-ideals.

Example 4.4. Ideals

$$NWD(\mathbb{Q}) = \{A \subset \mathbb{Q} \cap [0,1] : A \text{ is nowhere dense}\}$$

and

$$NULL(\mathbb{Q}) = \left\{A \subset \mathbb{Q} \cap [0,1] : \overline{A} \text{ is of Lebesgue measure zero}\right\},$$

where the closure is taken in $\mathbb{R}$, have property (S) (cf. [11]), but are not P-ideals. Moreover both are $\mathbf{\Pi^0_3}$-complete (cf. [4]).

Example 4.5. Ideal of the form $\mathcal{I}_{\omega^\beta}$ is an ideal of all subsets of ω^β of strictly smaller ordinal type (so $\mathcal{I}_{(1)} = Fin$). If $\beta > 1$, then $\mathcal{I}_{(\beta)}$ has property (S) (cf. [11]) and is not a P-ideal (cf. [3]). Moreover each $\mathcal{I}_{(\beta)}$ is $\mathbf{\Sigma^0_{2\beta}}$-complete (cf. [3]).

5. Descriptive complexity of ideals without property (S)

It is known, that every countably generated ideal does not have property (S) (cf. [5] or [11]). On the other hand every analytic P-ideal, which is not countably generated, has property (S) (see [11, Proposition 3.1]). Zakrzewski in [11] gave an example of an $\mathbf{\Sigma^0_2}$ ideal, which is not countably generated and does not have property (S) (for a proof, see [8]). In this section we give a generalization of that result: for every $\alpha > 1$ there is a $\mathbf{\Sigma^0_\alpha}$ ideal, which does not have property (S).

Proposition 5.1. For every $\alpha > 1$ there exists an ideal $\mathcal{I}_\alpha$ without property (S), which is $\mathbf{\Sigma^0_\alpha}$ but not $\mathbf{\Sigma^0_\beta}$ for any $\beta < \alpha$.

The proof is based on arguments from [5].

Proof. Set $\alpha > 1$ and let A be a $\mathbf{\Sigma^0_\alpha}$ but not $\mathbf{\Sigma^0_\beta}$ for any $\beta < \alpha$ subset of 2^ω. Consider the ideal $\mathcal{I}_\alpha$ on $2^{<\omega}$ generated by all finite subsets and branches of the tree $2^{<\omega}$ given by elements of A, i.e., sets of the form

$$B_x = \left\{ s \in 2^{<\omega} : \forall_{i \leq \mathrm{lh}(s)} s(i) = x(i) \right\}$$

for $x \in A$ (cf. [6]). Calbrix proved in [1] that A is $\mathbf{\Sigma^0_\beta}$ if and only if $\mathcal{I}_\alpha$ is $\mathbf{\Sigma^0_\beta}$. Hence $\mathcal{I}_\alpha$ is $\mathbf{\Sigma^0_\alpha}$ but not $\mathbf{\Sigma^0_\beta}$ for any $\beta < \alpha$.
Suppose that $\mathcal{I}_\alpha$ has property (S) and let $(X_n)_{n\in\omega}$ be a partition of $2^{<\omega}$ satisfying conditions (S_1) and (S_2). First we will construct a family $(H_n)_{n\in\omega}$ of distinct elements of $\mathcal{I}_\alpha$, each of which is a branch and contains an infinite subset of some selector of $(X_n)_{n\in\omega}$. Suppose that $H_0, \ldots, H_n$ are constructed. Then $\bigcup_{k\leq n} H_k \in \mathcal{I}_\alpha$, so there is an infinite subset T of some selector of $(X_n)_{n\in\omega}$, which is disjoint from $\bigcup_{k\leq n} H_k$ (since $(X_n)_{n\in\omega}$ satisfies condition (S_1)). The set T is in the ideal as a subset of a selector of $(X_n)_{n\in\omega}$, so it is contained in finitely many branches given by elements of A and some finite subset of $2^{<\omega}$. Let H_{n+1} be one of those branches including infinitely many elements of T. Then it has required properties.
Now we define a function $f : \omega \to \omega$ and an increasing, in the sense of $\preceq$, sequence of finite sequences $(s_n)_{n\in\omega} \subset 2^{<\omega}$, such that each s_n is of length k_n,

is included in $H_{f(n+1)}$ and infinitely many of the H_p's, but not in $H_{f(i)}$ for all $i \leq n$. Suppose that this is done for $k \leq n$. There exists $k > k_n$ such that not all of the H_p's containing s_n have the same elements of length k. Let $k_{n+1} = k$ and s_{n+1} be an element of length k_{n+1}, which is included in infinitely many H_p's containing s_n. Let also $f(n+1)$ be such that $s_{n+1} \notin H_{f(n+1)}$. Observe that if $i \leq n+1$, then $s_i \preceq s_{n+1}$. Since $s_i \notin H_{f(i)}$, it follows that $s_{n+1} \notin H_{f(i)}$ for all $i \leq n+1$.
To finish the proof choose a selector S of $(X_n)_{n\in\omega}$ containing for each $n \in \omega$ an element of $H_{f(n)}$ of length $k > k_n$. Then S is not in the ideal, since it is not contained in finitely many branches given by elements of A (actually S is even an infinite antichain in $(2^{<\omega}, \preceq)$). A contradiction. □

Remark 5.2. A special case of the ideal $\mathcal{I}_2$ is the branching ideal $\mathcal{I}_b$ generated by all branches of $2^{<\omega}$, which was investigated by Mazur in [7]. It is $\mathbf{\Sigma^0_2}$ (cf. [7]), not countably generated and does not have property (S) (cf. [8] and [11]). The above proposition is a generalization of that result: since every countably generated ideal must be $\mathbf{\Sigma^0_2}$, ideals $\mathcal{I}_\alpha$ for $\alpha > 2$ are not countably generated as well.

Remark 5.3. In Proposition 4.2 the Rudin-Blass order cannot be replaced by weaker Katětov order.

Proof. We will show that there are ideals $\mathcal{I}$ with property (S) and $\mathcal{J}$ without property (S) satisfying $\mathcal{I} \leq_K \mathcal{J}$.
Let $\mathcal{I}$ be any ideal on ω with property (S). By part (b) of proposition 4.1 the ideal $\mathcal{I} \oplus Fin$ also has property (S). Let $\mathcal{J}$ be the ideal $\mathcal{I}_2$ from proposition 5.1 and denote by T its domain. Let $\phi : \omega \oplus \omega \to T$ be any bijection, which sends $0 \times \omega$ onto one branch of T. Then

$$\mathcal{I} = \{\phi[A] : A \in \mathcal{I} \oplus Fin\}$$

is an ideal on T with property (S). Moreover $\mathcal{I} \leq_K \mathcal{J}$, since $\mathcal{I} \subset \mathcal{J}$. □

6. Weakly selective ideals

In the last section we investigate weakly selective ideals. The weak selectivity is a property closely related to (S). In particular, ideals without property (S) are very similar to weakly selective ideals, however the two classes differ. Before showing it, we give the following equivalent definitions of weak selectivity formulated by Grigorieff in [5].

Proposition 6.1. The following are equivalent for any ideal $\mathcal{I}$:
(a) $\mathcal{I}$ is weakly selective.

(b) For every decreasing sequence $(X_n)_{n\in\omega}$ of subsets of ω, which are not in $\mathcal{I}$ and such that $X_n \setminus X_{n+1} \in \mathcal{I}$ for each n, there exists a strictly increasing function $f : \omega \to \omega$, with range not in $\mathcal{I}$ and such that $f(n+1) \in X_{f(n)}$ for each $n \in \omega$ (i.e., in the terminology of [5], $\mathcal{I}$ is weak inductive).
(c) For every $X \notin \mathcal{I}$ and every tree $T \subset [\omega]^{<\omega}$ whose ramifications differ from X on a set in $\mathcal{I}$, there is a branch not in $\mathcal{I}$ ($\mathcal{I}$ is a very weak T-ideal, cf. [5]).

The proof of the previous proposition is similar to the proof of Proposition 3.1, therefore we omit it.

Recall that a σ-ideal is an ideal closed under countable unions. Let X be a Polish space, (i.e., separable completely metrizable topological space), D a countable dense subset of X and suppose that I is a σ-ideal on X. Following Sabok and Zapletal (see [9]) we define an ideal on D

$$\mathcal{J}_I = \left\{A \subset D : \overline{A} \in I\right\},$$

where the closure is taken in X. Moreover $\mathcal{J}_I$ is dense and $\mathbf{\Pi^0_3}$ provided that it is analytic and X is compact (cf. [9]). The next proposition can also be found in [9].

Proposition 6.2. If I is a σ-ideal on a Polish space X with a countable dense subset D, then $\mathcal{J}_I$ is weakly selective.

Proof. Let $A \notin \mathcal{J}_I$ (so $\overline{A} \notin I$) and $f : A \to \omega$ be a function, which is not constant on any $B \notin \mathcal{J}_I$. Denote by C the set $\overline{A}$ minus all basic open sets intersecting $\overline{A}$ on a member of I. Obviously $C \notin I$. Enumerate all basic open sets with nonempty intersection with C into a sequence $(U_n)_{n\in\omega}$ and construct a sequence of points $(x_n)_{n\in\omega}$ such that f is one-to-one on it and each x_n is in $U_n \cap A$. Suppose that x_k are constructed for all $k < n$. The set $C \cap U_n$ is not in I, so $A \cap U_n$ is not in $\mathcal{J}_I$ since $C \cap U_n \subset \overline{A \cap U_n}$. Indeed, $C \cap U_n \subset \overline{A} \cap U_n$. Take x in $\overline{A} \cap U_n$. For every open neighbourhood U of x the intersection $U \cap U_n$ is an open set containing x. Since x is in the closure of A, there is an element $y \in A$ such that $y \in U \cap U_n$. Hence y is in $A \cap U_n$ and $x \in \overline{A \cap U_n}$.
Since f is not constant on any $B \notin \mathcal{J}_I$, the image of $A \cap U_n$ under f cannot be finite. Indeed, otherwise

$$f^{-1}[f[A \cap U_n]]$$

would be a union of finitely many sets from the ideal $\mathcal{J}_I$, hence $A \cap C \cap U_n \in \mathcal{J}_I$. A contradiction. Take $x_n \in A \cap U_n$ such that $f(x_n)$ is different from all $f(x_k)$, $k < n$.

The set $B = \{x_n : n \in \omega\}$ is obviously a subset of A, on which f is one-to-one. Moreover $C \subset \overline{B}$. Indeed, for every $x \in C$ and its open neighbourhood U, there is a basic open set $V \subset U$ containing x. Since $x \in C$, the set $C \cap V$ is nonempty, hence not in the σ-ideal I. So there is n such that $V = U_n$ and the point x_n is in the intersection of U and B. So x is in the closure of B. Since $C \notin I$, we have $\overline{B} \notin I$ and therefore $B \notin \mathcal{J}_I$. □

Corollary 6.3. *There are weakly selective ideals with property* (S).

Proof. Take $D = \mathbb{Q}$ as a countable dense subset of $\mathbb{R}$. Let $\mathcal{K}$ be the σ-ideal of all meager subsets of $\mathbb{R}$ and $\mathcal{N}$ be the σ-ideal of all Lebesgue measure zero subsets of $\mathbb{R}$. Observe that $\mathcal{J}_{\mathcal{K}} = NWD(\mathbb{Q})$ and $\mathcal{J}_{\mathcal{N}} = NULL(\mathbb{Q})$. Then weak selectiveness of $NWD(\mathbb{Q})$ and $NULL(\mathbb{Q})$ follows from the previous proposition (another proof of the fact that $NWD(\mathbb{Q})$ is weakly selective can be found in [2]). To finish the proof recall that in Example 4.4 it was stated that this two ideals have property (S). □

We close this section with the following conjecture formulated by Sabok and Zapletal in [9].

Conjecture 6.4. If $\mathcal{I}$ on ω is a dense $\mathbf{\Pi}^0_3$ weakly selective ideal, then there is a Polish space X with a countable dense subset D and a σ-ideal I such that $\mathcal{I}$ is isomorphic to $\mathcal{J}_I$, i.e., there is a bijection $\pi : D \to \omega$ such that

$$A \in \mathcal{J} \Leftrightarrow \pi^{-1}[A] \in \mathcal{J}_I.$$

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