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Murray von Neumann order in W^*-algebras

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Abstract. The Murray-von Neumann order in W^*-algebras is widely discussed. A steering projection of a von Neumann algebra is introduced and the role it plays is emphasized. A general decomposition of arbitrary projections with respect to a steering projection is established.

1. Introduction

The Murray-von Neumann order (in the set of all [equivalence classes of] projections) is a useful tool in studies of W^*-algebras. For example, this order is involved to define types of von Neumann algebras. In most of classical textbooks on operator algebras (see e.g. [8, 7, 10, 4, 5, 1]) this order is introduced mainly for this purpose. However, even from a purely set-theoretic point of view the Murray-von Neumann order is an interesting subject of investigations. Although it was carefully studied by Tomiyama [11] and Griffin Jr. [2, 3] in general von Neumann algebras, none of the books mentioned above discuss this topic so generally. The aim of this paper is to give a new form as well as new proofs of the results by Tomiyama and Griffin Jr. As we will see, the road to the main result (Theorem 9.5), which—roughly speaking—reduces the Murray-von Neumann order to comparing central elements of a W^*-algebra, is very long and highly nontrivial. Beside this, we include here fundamental results on the Murray-von Neumann order, in particular: Cantor-Bernstein type theorem (Theorem 3.9), Comparison Theorem (Theorem 4.11), Halving Theorem (Theorem 5.2) and others. In a part of the presentation we involve the traces on von Neumann algebras of type II_1. To reduce the size of the paper, the existence and fundamental properties of them are stated without proofs. For details the reader is referred to any of the books cited above.

The presented material mainly comes from [4, 5] and [6]. The idea of steering projections was introcuded in [6]. Also the approach to the so-called dimension theory of W^*-algebras by means of steering projections comes from that paper.

2. Preliminaries

In this paper all vector spaces as well as algebras are over the field $\mathbb{C}$ of all complex numbers. We are interested in particular type of algebras, namely W^*-algebras. Recall the definition:

2.1. **Definition.** A *W^*-algebra* $\mathfrak{A}$ is a (unital) C^*-algebra which is linearly isometric to the dual of a Banach space E. The space E is called the *predual* of $\mathfrak{A}$.

The most important example of a commutative W^*-algebra is $L^\infty(X)$ with σ-finite measure space X. On the other hand, the algebra of all linear bounded operators $B(H)$ on a Hilbert space H is a canonical noncommutative (if $\dim H > 1$) example. Note that it is possible to embed $L^\infty(X)$ in $B(H)$ where $H = L^2(X)$ quite naturally: by identifying $f \in L^\infty(X)$ with the multiplication operator M_f.

2.2. **Definition.** A self-adjoint (i.e. closed under involution), unital[1] subalgebra $\mathfrak{A} \subset B(H)$ is called a *von Neumann algebra* when it is closed in the weak operator topology.

Because the weak operator topology is coarser than the norm topology, we see that a von Neumann algebra is automatically a C^*-algebra. Now we present (without proving!) theorems showing the importance of the examples mentioned above: the proofs of next two statement may be found in [7]:

2.3. **Theorem.** *Any commutative W^*-algebra is isometrically *-isomorphic to the algebra $L^\infty(X)$ for some measure space X.*

In the noncommutative case we have the following:

2.4. **Theorem.** *(Sakai) Every W^*-algebra is isometrically *-isomorphic to a von Neumann algebra acting on a certain Hilbert space.*

This theorem allows us to think about W^*-algebras as concrete subalgebras of the operator algebra $B(H)$ and shows the importance of the notion of a von Neumann algebra. This identification will be constantly used. Thus from now on, $\mathfrak{A}$, $\mathfrak{A}'$ and $\mathfrak{Z}(\mathfrak{A})$ are, respectively, a von Neumann algebra, its commutant and center, unless otherwise stated. The next theorem is the point where the theory of von Neumann algebras really starts (for the proof see e.g. [4, 10])

2.5. **Theorem.** *(Double Commutant Theorem) When $\mathfrak{A}_0 \subset B(H)$ is a self-adjoint algebra, then the closures of $\mathfrak{A}_0$ with respect to the strong and the weak operator topologies coincide with the double commutant $\mathfrak{A}_0''$ of $\mathfrak{A}_0$.*

Double Commutant Theorem can be used to prove the following result concerning the polar decomposition stated below. Recall that any operator $T \in B(H)$ may uniquely be written as

$$T = U|T| \tag{2.1}$$

where $|T| := (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with the same kernel as T. For the proof of the next statement we refer the reader to [4, 10]:

2.6. **Theorem.** *If* (2.1) *is the polar decomposition of an operator $T \in \mathfrak{A}$, then both $|T|$ and U belong to $\mathfrak{A}$ as well.*

There are various methods of constructing new von Neumann algebras from a given collection of them: direct sums, direct integrals, tensor products, crossed products, etc. Not all of the above constructions will be used by us. The following two constructions will prove to be very useful for us. The proof of the first statement is easy, while the second is surprisingly nontrivial (the part a) can be found in [7] while the part b) is a kind of folklore):

2.7. **Proposition.** *The center $\mathfrak{Z}(\mathfrak{A})$ of any von Neumann algebra $\mathfrak{A}$ is again a von Neumann algebra.*

2.8. **Theorem.** a) *Let $P \in \mathfrak{A}$ be a projection in a von Neumann algebra. Then $P\mathfrak{A}P$ is also a von Neumann algebra.*
b) *$\mathfrak{Z}(P\mathfrak{A}P)$, the center of $P\mathfrak{A}P$, coincides with $\mathfrak{Z}(\mathfrak{A})P$.*

2.9. *Remark.* It is important to remember in which Hilbert space we are working: if $\mathfrak{A} \subset B(H)$, then $P\mathfrak{A}P \subset B(P(H))$ and P becomes the unit element in $P\mathfrak{A}P$. It is also worth mentioning that the proof of the above theorem is much simpler in case P is a central projection, i.e. if P belongs to the center of $\mathfrak{A}$—then we could write just $P\mathfrak{A}$ or $\mathfrak{A}P$ instead of $P\mathfrak{A}P$.

[1]In the sense that the unit element I in $\mathfrak{A}$ coincide with identity operator in $B(H)$.

3. Partial ordering

The standard ordering on the self-adjoint part of a C^*-algebra is defined as follows: for two self-adjoint elements x, y we write $x \leq y$ iff $y - x$ is positive (that is, if $\sigma(y-x) \subset [0, \infty)$) and $x < y$ iff $x \leq y$ and $x \neq y$. In case x, y are projections the relation $x \leq$ may simply be characterized as shown by the following result:

3.1. **Proposition.** *Let p, q be two projections in a C^*-algebra $\mathfrak{A}$. The following conditions are equivalent:*

(i) $p \leq q$,
(ii) $q - p$ *is a projection,*
(iii) $pq = qp = p$.

Conditions $p \leq q$ and $q \leq p$ clearly imply $p = q$; however, it may happen that two projections are not comparable. This is because of the fact that, when comparing (in the above sense) two projections in $B(H)$, the placement of their ranges plays a key role. This is in contrast with the *Murray-von Neumann order*, soon to be defined. However, first we have to introduce the notion of a suitable equivalence.

3.2. **Definition.** We say that two projections $P, Q \in \mathfrak{A}$ are *Murray-von Neumann equivalent* if there exists $V \in \mathfrak{A}$ such that:

$$V^*V = P, \; VV^* = Q. \tag{3.2}$$

Then we will write $P \sim Q$.

Condition (3.2) automatically implies that V is a partial isometry. The relation '$\sim$' is indeed an equivalence:

3.3. **Proposition.** *'$\sim$' is an equivalence relation on the set of projections.*

Proof. For reflexivity take $V := P$; then $V^*V = VV^* = P^2 = P$.

For symmetry, observe that, if $P \sim Q$ i.e $P = V^*V$, $Q = VV^*$, it is sufficent to take $U := V^*$.

For transitivity, assume that $P = V^*V$, $Q = VV^* = U^*U$, $R = UU^*$ and take $W := UV$. Then we get

$$W^*W = (UV)^*UV = V^*U^*UV = V^*QV =$$
$$V^*(VV^*)V = (V^*V)(V^*V) = P^2 = P$$

and quite similary we get $WW^* = R$. □

3.4. *Remarks.* 1. Note that, in general, the product of two partial isometries U, V need not be a partial isometry.

2. If $\mathfrak{A}$ is an algebra of operators on a Hilbert space H, then the foregoing equivalence can be formulated in the following way: $P \sim Q \iff \dim P(H) = \dim Q(H)$. The main reason for this is that $\mathfrak{A}$ contains *all* partial isometries on H. In general, a suitable partial isometry need not belong to $\mathfrak{A}$, so the above characterization is no longer valid in arbitrary von Neumann algebras.

3.5. **Example.** From the fact that two projections P, Q are unitary equivalent, meaning that $P = U^*QU$ with some unitary $U \in \mathfrak{A}$, it follows that they are Murray-von Neumann equivalent: indeed, $V := QU$ establishes this equivalence. However, the converse is not true, as explained below: let H be a separable Hilbert space. If $M_1, M_2 \subset H$ are infinite-dimensional closed subspaces such that $\dim M_1^\perp < \infty$ and[2] $\dim M_2^\perp = \infty$, and P_1, P_2 denote the coresponding to them projections, then $P_1 \sim P_2$ but $I - P_1 \nsim I - P_2$. So, P_1, P_2 cannot be unitary equivalent.

[2]In a separable Hilbert space we use the symbol ∞; otherwise we would write $\aleph_0$ instead.

The relation '$\sim$' is stable under addition in the following sense:

3.6. **Proposition.** *If $\{P_i : i \in I\}$ and $\{Q_i : i \in I\}$ are families of mutually orthogonal projections and*

$$P_i \sim Q_i, \quad i \in I$$

then $P := \sum_{i\in I} P_i \sim Q := \sum_{i\in I} Q_i$ (the sums of projections are always understood in the strong operator topology).

Proof. Let V_i be such that $P_i = V_i^* V_i$, $Q_i = V_i V_i^*$ which means that $V_i : P_i(H) \to Q_i(H)$ are unitary isomorphisms. Define V as V_i on each subspace $P_i(H)$ and 0 on $(I-P)(H)$. Then V^* coincides with V_i^* on each subspace $Q_i(H)$ hence for $x_i \in P_i(H)$, as $V_i(x_i) \in Q_i(H)$ we have:

$$V^* V x_i = V_i^* V_i x_i = x_i = P x_i.$$

For $x \in (I-P)(H)$ we have $V^* V x = 0 = Px$ hence $V^* V$ agrees with P on each summand of $H = \bigoplus_{i\in I} \big(P_i(H)\big) \oplus (I-P)(H)$; similarly we get $VV^* = Q$. It remains to prove that $V \in \mathfrak{A}$. We fix $\varepsilon > 0$ and a finite number of points $x_k = \sum_{i\in I} x_{ki} + y_k$ with $x_{ki} \in P_i(H)$, $y_k \in (I-P)(H)$. Observe that for a suitably chosen finite subset $I_0 \subset I$ we have:

$$\|\Big(V - \sum_{i\in I_0} V_i\Big) x_k\| = \|\sum_{i\in I\setminus I_0} V_i x_{ki}\| < \varepsilon, \quad k = 1, \ldots, n.$$

We showed that V belongs to the closure of $\mathfrak{A}$ in the strong operator topology, hence $V \in \mathfrak{A}$. □

We proceed to define the Murray-von Neumann order:

3.7. **Definition.** Given two projections P, Q in $\mathfrak{A}$ we say that P is *weaker* that Q iff $P \sim Q_0$ for certain subprojection $Q_0 \leq Q$ belonging to $\mathfrak{A}$. We write then $P \preceq Q$.

We will establish that '$\preceq$' is a partial ordering (when two Murray-von Neumann equivalent projections are identified). Reflexivity is immediate. Transitivity may be proved as follows: suppose that $P \preceq Q$, $Q \preceq R$ i.e.

$$P = V^*V, \quad Q_0 = VV^* \leq Q = W^*W, \quad WW^* = R_0 \leq R.$$

Let $U := WV$ and $R_1 := WQ_0W^*$. Then it follows that:

$$U^*U = (WV)^*WV = V^*W^*WV =$$
$$= V^*QV = V^*Q_0V = V^*VV^*V = P^2 = P$$

and

$$WV(WV)^* = WVV^*W^* = WQ_0W^* = R_1.$$

Moreover R_1 is a projection, since

$$R_1^2 = WQ_0W^*WQ_0W^* = WQ_0QQ_0W^* = WQ_0W^* = R_1 (= R_1^*)$$

and obviously $R_1 \leq R_0$, hence $P \sim R_1 \leq R_0 \leq R$, so $P \preceq R$.

Before proving the antisymmetry, we make simple use of Proposition 3.6 to get the following useful property:

3.8. **Lemma.** *If $\{P_i; i \in I\}$ and $\{Q_i : i \in I\}$ are families of (mutually) orthogonal projections and*

$$P_i \preceq Q_i, \quad i \in I$$

then $P := \sum_{i\in I} P_i \preceq Q := \sum_{i\in I} Q_i$.

Proof. If $P_i \sim Q_{i0} \leq Q_i$ then according to Proposition 3.6 $P \sim Q_0 := \sum_{i\in I} Q_{i0} \leq Q$ (obviously Q_0 is a projection and $Q_0 \in \mathfrak{A}$). □

The next result together with our previous considerations will establish that '$\preceq$' is a partial ordering on the set of Murray-von Neumann equivalence classes of projections.

3.9. **Theorem.** *(Noncommutative Cantor-Bernstein type theorem) If two projections P, Q satisfy $P \preceq Q$ and $Q \preceq P$ then $P \sim Q$.*

Proof. Denote $P_0 := P$, $Q_0 := Q$ and let V_0, W_0 be such that:

$$V_0^* V_0 = P_0,\ V_0 V_0^* = Q_1 \leq Q_0,\ W_0^* W_0 = Q_0,\ W_0 W_0^* = P_1 \leq P_0.$$

Put $V_1 := V_0 P_1$; then V_1 satisfies:

$$V_1^* V_1 = P_1^* V_0^* V_0 P_1 = P_1 P_0 P_1 = P_0,\ V_1 V_1^* = V_0 P_1 P_1^* V_0 = V_0 P_1 V_0^* =: P_2.$$

Since $P_2^2 = (V_0 P_1 V_0^*)(V_0 P_1 V_0^*) = V_0 P_1 P_0 P_1 V_0^* = V_0 P_1 V_0^*$ and P_2 is easily seen to be self-adjoint, we see that P_2 is a projection. Moreover, from inclusion $P_2(H) \subset Q_1(H)$ we get $P_2 \leq Q_1$. In a similar manner we obtain a partial isometry W_1 and a projection $Q_2 \leq P_1$. We see that

$$\Big(V_0(P_0 - P_1)\Big)^* V_0(P_0 - P_1) = (P_0 - P_1)V^* V(P_0 - P_1) =$$
$$= (P_0 - P_1)P_0(P_0 - P_1) = (P_0 - P_1)^2 = P_0 - P_1,$$

$$V_0(P_0 - P_1)(V_0(P_0 - P_1))^* = V_0(P_0 - P_1)^2 V_0^* = V_0(P_0 - P_1)V_0^* = Q_1 - Q_2,$$

hence $P_0 - P_1 \sim Q_1 - Q_2$ and analogously $Q_0 - Q_1 \sim P_1 - P_2$. Continuing this procedure we construct two sequences of projections:

$$P_0 \geq P_1 \geq P_2 \geq ..., \qquad Q_0 \geq Q_1 \geq Q_2 \geq ...$$

such that $P_n - P_{n+1} \sim Q_{n+1} - Q_{n+2}$, $Q_n - Q_{n+1} \sim P_{n+1} - P_{n+2}$. Applying Proposition 3.6 we can sum up them to get:

$$\sum_{n=0}^{\infty}(P_{2n} - P_{2n+1}) \sim \sum_{n=0}^{\infty}(Q_{2n+1} - Q_{2n+2})$$

and

$$\sum_{n=0}^{\infty}(Q_{2n} - Q_{2n+1}) \sim \sum_{n=0}^{\infty}(P_{2n+1} - P_{2n+2})$$

Thus

$$P_0 = \sum_{n=0}^{\infty}(P_{2n} - P_{2n+1}) + \sum_{n=0}^{\infty}(P_{2n+1} - P_{2n+2}) \sim$$
$$\sim \sum_{n=0}^{\infty}(Q_{2n} - Q_{2n+1}) + \sum_{n=0}^{\infty}(Q_{2n} - Q_{2n+1}) = Q_0.$$

□

From now on, whenever we speak about order between projections, we mean the Murray-von Neumann order '$\preceq$'. For simplicity, we denote the set of all projections in $\mathfrak{A}$ by $\mathfrak{E}(\mathfrak{A})$.

3.10. **Theorem.** *Let $P_0, Q_0 \in \mathfrak{E}(\mathfrak{A})$ be two equivalent projections. Then the sets $\mathfrak{P} := \{P \in \mathfrak{E}(\mathfrak{A}) : P \leq P_0\}$ and $\mathfrak{Q} := \{Q \in \mathfrak{E}(\mathfrak{A}) : Q \leq Q_0\}$ are order-isomorphic. Isomorphism may be chosen to preserve orthogonality.*

Proof. Let $V \in \mathfrak{A}$ satisfy $P_0 = V^* V,\ Q_0 = V V^*$. Let

$$\iota : \mathfrak{P} \ni P \mapsto VPV^* \in \mathfrak{Q}.$$

We claim that ι is the isomorphism we searched for: as $(VPV^*)^* = VPV^*$, $VPV^* VPV^* = VPP_0 PV^* = VPV^*$ and $VPV^* \leq Q$, we see that ι is well defined.

Since $\iota(P) = VP(H)$ and V acts as a unitary isomorphism on $P(H) \subset P_0(H)$, thus ι is one-to-one. To see that ι is surjective, we fix $Q \in \mathfrak{Q}$ and take $P := V^*QV \in \mathfrak{P}$; then:

$$\iota(P) = VPV^* = VV^*QVV^* = Q_0QQ_0 = Q.$$

Now, let $P, P' \in \mathfrak{P}$ be such that $P \sim P'$. Take $U \in \mathfrak{A}$ which establishes the equivalence: $P = U^*U$, $P' = UU^*$ and put $W := VUV^*$. Then:

$$W^*W = VU^*V^*VUV^* = VU^*P_0UV^* = VU^*UV^* = VPV^* = \iota(P)$$

and analogously $WW^* = \iota(P')$, hence $\iota(P) \sim \iota(P')$.

Now, let $P, P' \in \mathfrak{P}$ satisfy $P' \leq P$. Then:

$$VP'V^*VPV^* = VP'P_0PV^* = VP'V^*(= VPV^*VP'V^*),$$

which yields $\iota(P') \leq \iota(P)$.

Now, if $P' \preceq P$ for $P', P \in \mathfrak{P}$, then $P' \sim P'' \leq P$ and, from the above argument, $\iota(P') \sim \iota(P'') \leq \iota(P)$, i.e. $\iota(P') \preceq \iota(P)$.

Finally, if $P, P' \in \mathfrak{P}$ are such that $PP' = 0$, then

$$VPV^*VP'V^* = VPP_0P'V^* = VPP'V^* = 0,$$

which means that $\iota(P) \perp \iota(P')$. □

We need also the following simple result:

3.11. **Lemma.** *If $P, Q, Z \in \mathfrak{A}$ are projections and Z is central, then:*

a) *$P \sim Q$ implies $PZ \sim QZ$,*

b) *$P \preceq Q$ implies $PZ \preceq QZ$.*

Proof. a) If $V^*V = P,\ VV^* = Q$ then

$$(VZ)^*(VZ) = Z^*V^*VZ = V^*VZ^2 = V^*VZ = PZ$$

and similarly $VZ(VZ)^* = QZ$.

Point b) follows from a):

$$P \preceq Q \Rightarrow P \sim Q_0 \leq Q \Rightarrow PZ \sim Q_0Z \leq QZ \Rightarrow PZ \preceq QZ.$$

□

Below $\prod_{i\in\mathcal{I}} \mathfrak{E}(\mathfrak{A}_i)$ is equipped with the coordinatewise order inherited from the orders on $\mathfrak{E}(\mathfrak{A}_i)$'s.

3.12. **Theorem.** *Let $\{Z_i\}_{i\in\mathcal{I}}$ be a family of mutually orthogonal central nonzero projections such that $\sum_{i\in\mathcal{I}} Z_i = I$ and put $\mathfrak{A}_i := \mathfrak{A}Z_i$. Then the sets $\mathfrak{E}(\mathfrak{A})$ and $\prod_{i\in\mathcal{I}} \mathfrak{E}(\mathfrak{A}_i)$ are order-isomorphic with respect to both the standard ordering and the Murray-von Neumann one. Isomorphism may be chosen to preserve orthogonality.*

Proof. Put $\iota : \mathfrak{E}(\mathfrak{A}) \ni P \mapsto \iota(P) := (PZ_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathfrak{E}(\mathfrak{A}_i)$. It is a well defined mapping, as each PZ_i is a projection. The function ι is one-to-one: $(PZ_i)_{i\in\mathcal{I}} = (QZ_i)_{i\in\mathcal{I}}$ implies $PZ_i = QZ_i$ for all $i \in \mathcal{I}$ and:

$$P = \sum_{i\in\mathcal{I}} PZ_i = \sum_{i\in\mathcal{I}} QZ_i = Q.$$

To see that ι is onto, we take an arbitrary collection of projections from $\prod_{i\in\mathcal{I}} \mathfrak{E}(\mathfrak{A}_i)$ which is of the form $(Q_i)_{i\in\mathcal{I}}$ with $\mathfrak{E}(\mathfrak{A}_i) \ni Q_i \leq Z_i$. As Q_i's are mutually orthogonal, we can form $P := \sum_{i\in\mathcal{I}} Q_i$ and then:

$$\iota(P) = (PZ_i)_{i\in\mathcal{I}} = (\sum_{j\in\mathcal{I}} Q_jZ_i)_{i\in\mathcal{I}} = (Q_iZ_i)_{i\in\mathcal{I}} = (Q_i)_{i\in\mathcal{I}}.$$

The fact that ι is $\leq$-order isomorphism is immediate. It is also $\preceq$-order isomorphism, i.e. $P \preceq Q \iff \iota(P) \preceq \iota(Q)$ due to the Lemmas 3.11 and 3.8. Finally, $PQ = 0$ obviously implies $(P_iQ_iZ_i)_i = (P_iZ_i)_i(Q_iZ_i)_i = 0$ so $\iota(P)\iota(Q) = 0$. □

4. Total ordering

We infer from Remarks 3.4 and the fact that cardinal numbers are well (hence also linear) ordered that for any $P, Q \in B(H)$ one of the two possibilities holds: $P \preceq Q$ or $Q \preceq P$. We will learn that it holds in any von Neumann algebra which is a so-called factor.

4.1. **Definition.** A von Neumann algebra $\mathfrak{A}$ is called a *factor* when the center $\mathfrak{Z}(\mathfrak{A})$ of $\mathfrak{A}$ consists of scalar multiples of the identity operator:

$$\mathfrak{Z}(\mathfrak{A}) = \mathbb{C}I.$$

Factors may be regarded as purely non-commutative algebras—the only factor which is simultaneously a commutative algebra is the field $\mathbb{C}$ of complex numbers.

Before showing that Murray-von Neumann order is total in factors, we must develop some of techniques involving projections. Recall that projections lying in the center of $\mathfrak{A}$ are called *central* projections. For $A \in \mathfrak{A}$ denote by $\mathcal{P}_A$ the family of all central projections P with the property $PA = 0$.

4.2. **Definition.** The *central carrier* of $A \in \mathfrak{A}$ is defined as follows: $C_A := I - P$ where $P = \bigvee \mathcal{P}_A$.

4.3. *Remark.* For each $P \in \mathcal{P}_A$, $PA = 0$, which means that the range of A is ortoghonal to the range of each $P \in \mathcal{P}_A$, so $A(H)$ is orthogonal to $(I - C_A)(H)$ and thus $(I - C_A)A = 0$. Consequently, $C_A A = A$. This justifies the terminology. Moreover, it shows that the central carrier may be alternatively defined as $C_A := \bigwedge \mathcal{Q}_A$ where $\mathcal{Q}_A$ consists of all central projections Q satisfying $QA = A$. In particular, $C_P = P$ for any central projection P. Note also that the following useful property holds: if Q and Z are projections and Z is central then:

$$QZ = 0 \iff C_Q Z = 0.$$

Indeed, ' $\Longleftarrow$ ' is obvious, because $Q \leq C_Q$. On the other hand, if $QZ = 0$, then $Z \leq P := \bigvee \mathcal{P}_A$. Hence $PZ = Z$ and $C_Q Z = (I - P)Z = Z - PZ = Z - Z = 0$. We will also use later two following properties of the central carrier (without comment!):

(i) for a family $\{P_i\}_{i \in \mathcal{I}}$ of projections $C_{\bigvee_{i \in \mathcal{I}} P_i} = \bigvee_{i \in \mathcal{I}} C_{P_i}$,
(ii) for two projections $P, Z \in \mathfrak{A}$ where $Z \in \mathfrak{Z}(\mathfrak{A})$, $C_{PZ} = ZC_P$.

Two proofs of the above statements may be found in [4].

Simplest examples of central carriers are: $C_0 = 0$, $C_I = I$. If $\mathfrak{A}$ is a factor, then $C_A = I$ for any $0 \neq A \in \mathfrak{A}$.

The range of the central carrier may be described by the following:

4.4. **Proposition.** *The range $C_T(H)$ of the central carrier of $T \in \mathfrak{A}$ coincides with $\overline{\mathfrak{A}T(H)}$, where $\mathfrak{A}T(H) := \{ATx : \ A \in \mathfrak{A}, \ x \in H\}$.*

Proof. Since $C_T T = T$, thus $T(H) \subset C_T(H)$. So, for any $A \in \mathfrak{A}$ we have $AT(H) \subset AC_T(H) = C_T A(H) \subset C_T(H)$. Hence $\mathfrak{A}T(H) \subset C_T(H)$ and consequently $\overline{\mathfrak{A}T(H)} \subset C_T(H)$.

Further, let Q be the projection onto $\overline{\mathfrak{A}T(H)}$. Since this subspace is invariant for $\mathfrak{A}$ and $\mathfrak{A}'$, we conclude that Q commutes with any member of $\mathfrak{A}$ and $\mathfrak{A}'$ as well. In other words, $Q \in \mathfrak{A}' \cap \mathfrak{A}'' = \mathfrak{A} \cap \mathfrak{A}' = \mathfrak{Z}(\mathfrak{A})$, i.e. Q is central. Moreover, $T(H) \subset \overline{\mathfrak{A}T(H)}$, so $QT = T$ and consequently $C_T \leq Q$ (see Remark 4.3). Finally, it follows from the first part of the proof that $Q \leq C_T$ which yields $Q = C_T$. □

4.5. **Proposition.** *Two equivalent projections have the same central carrier: if $P \sim Q$, then $C_P = C_Q$.*

Proof. Let V be a partial isometry establishing the equivalence of P and Q, that is, $V^*V = P,\ VV^* = Q$. In particular, we have

$$P = P^2 = V^*VV^*V = V^*QV. \tag{4.3}$$

Since $C_P(H) = \overline{\mathfrak{A}P(H)}$ and $C_Q(H) = \overline{\mathfrak{A}Q(H)}$, we get:

$$\overline{\mathfrak{A}P(H)} \subset \overline{\mathfrak{A}V^*QV(H)} \subset \overline{\mathfrak{A}Q(H)}.$$

Similarly we have $\overline{\mathfrak{A}Q(H)} \subset \overline{\mathfrak{A}P(H)}$. Thus $C_P(H) = C_Q(H)$, so $C_P = C_Q$. □

Next, we need a couple of lemmas. Denote by P_T the projection onto the subspace $\overline{T(H)}$. This operator is sometimes called the *range projection* of T. It is worth noting that $P_T \in \mathfrak{A}$ for any $T \in \mathfrak{A}$ (see the proof below).

4.6. **Lemma.** *If $T \in \mathfrak{A}$, then $P_T \sim P_{T^*}$.*

Proof. Since U from the polar decomposition of T is a unitary isomorphism between $\overline{T^*(H)}$ and $\overline{T(H)}$, we obtain $U^*U = P_{T^*}$ and $UU^* = P_T$. That $U \in \mathfrak{A}$ is confirmed by Theorem 2.6. □

4.7. **Lemma.** *Let $P, Q \in \mathfrak{A}$ be two projections. Then P and Q have equivalent, nonzero subprojections (in $\mathfrak{A}$) iff $C_P C_Q \neq 0$.*

Proof. We begin with the 'only if' part. Suppose, for the contrary, that $C_P C_Q = 0$. Denote by P_0 and Q_0 nonzero equivalent subprojections of P and Q, and by U a partial isometry establishing their equivalence; that is, $U^*U = P_0,\ UU^* = Q_0$. From Remark 4.3 we obtain

$$0 < P_0 \leq P \leq C_P, \quad 0 < Q_0 \leq Q \leq C_Q.$$

Thus:

$$U = Q_0 U P_0 = Q_0 C_Q U C_P P_0 = Q_0 U C_Q C_P P_0 = Q_0 U C_P C_Q P_0 = 0$$

and consequently $P_0 = Q_0 = 0$.

We turn to the 'if' part. Since C_P and C_Q are in the center of $\mathfrak{A}$, they commute, hence $C_P C_Q$ is a projection and due to Proposition 4.4, $C_P C_Q(H) = \overline{\mathfrak{A}P(H)} \cap \overline{\mathfrak{A}Q(H)} \neq 0$. Hence we can choose a nonzero element z which can be written in the form

$$z = \lim_{n\to\infty} A_n P x_n = \lim_{n\to\infty} B_n Q y_n$$

for some $A_n, B_n \in \mathfrak{A}$ and $x_n, y_n \in H$. Thus, we have:

$$0 \neq \|z\|^2 = \lim_{n\to\infty} \langle A_n P x_n, B_n Q y_n \rangle = \lim_{n\to\infty} \langle Q B_n^* A_n P x_n, y_n \rangle$$

and, in particular, $T := Q B_n^* A_n P \in \mathfrak{A}$ is nonzero for some n. But then $0 \neq P_T, P_{T^*} \in \mathfrak{A}$ and by Lemma 4.6, $P_T \sim P_{T^*}$. Now it suffices to note that $P_T \leq P,\ P_{T^*} \leq Q$. □

4.8. **Corollary.** *If $\mathfrak{A}$ is a factor, then any two nonzero projections in $\mathfrak{A}$ contain nonzero Murray-von Neumann equivalent subprojections.*

Proof. If $\mathfrak{A}$ is a factor and $P, Q \in \mathfrak{E}(\mathfrak{A})$ are nonzero, we have $C_P = C_Q = I$. So, Lemma 4.7 finishes the proof. □

We are now ready to show that Murray-von Neumann order is linear in factors:

4.9. **Theorem.** *Any two projections in a factor are comparable: if $\mathfrak{A}$ is a factor and $P, Q \in \mathfrak{A}$ are projections, then $P \preceq Q$ or $Q \preceq P$. Conversely, if any two projections in $\mathfrak{A}$ are comparable, then $\mathfrak{A}$ is a factor.*

4.10. **Lemma.** *If P and Z are projections and Z is central, then*

$$P \leq Z \iff P \preceq Z \tag{4.4}$$

Proof. Implication ' $\implies$ ' is obvious. To see the converse one, suppose that $P \sim Z_0 \leq Z$ and take $V \in \mathfrak{A}$ with $V^*V = P$ and $VV^* = Z_0 \leq Z$. Since Z is central, we have:

$$ZP = PZ = P^2Z = (V^*V)(V^*V)Z = V^*(VV^*Z)V =$$
$$= V^*(Z_0Z)V = V^*Z_0V = V^*(VV^*)V = P^2 = P,$$

hence $P \leq Z$. □

Proof of Theorem 4.9. The second part follows from Lemma 4.10. Indeed, assume that $\mathfrak{A}$ is not a factor, and take Z to be a nontrivial central projection. Then $I - Z$ and Z are (nonzero) incomparable projections (in the usual sense), i.e. neither $Z \leq I - Z$ nor $I - Z \leq Z$. By Lemma 4.10, Z and $I - Z$ are not comparable in the Murray-von Neumann sense.

For the proof of the main part of the theorem, fix two nonzero projections P and Q in $\mathfrak{A}$ and take a maximal family $\{(P_j, Q_j) : \ j \in J\}$ of pairs of nonzero projections such that:

1. $P_j \sim Q_j$, $P_j \leq P$ and $Q_j \leq Q$;
2. P_j and $P_{j'}$ are orthogonal for $j \neq j'$, and the same for Q_j's.

If $P' := P - \sum_{j\in J} P_j$ and $Q' := Q - \sum_{j\in J} Q_j$ are nonzero, then they have equivalent subprojections, say P_0 and Q_0 (according to Corollary 4.8). Joining them to $\{(P_j, Q_j) : \ j \in J\}$ contradicts the maximality of $\{(P_j, Q_j) : \ j \in J\}$. Hence (without loss of generality) e.g. $P = \sum_{j\in J} P_j$. On the other hand, $\sum_{j\in J} P_j \sim \sum_{j\in J} Q_j$ by Proposition 3.6 and we are done. □

Our next aim is to generalize the first part of the previous result for arbitrary von Neumann algebras. This generalization is very powerful and is known as Comparison Theorem.

4.11. **Theorem.** *(Comparison Theorem) For any pair $P, Q \in \mathfrak{A}$ of projections there exists a central projection $Z \in \mathfrak{A}$ such that*

$$QZ \preceq PZ, \quad P(I - Z) \preceq Q(I - Z).$$

Since in a factor the only central projections are 0 and I, Theorem 4.9 is in fact a special case of the above result.

Proof. Let $\{(P_j, Q_j) : \ j \in J\}$, P' and Q' be as in the proof of Theorem 4.9. Then $P - P' \sim Q - Q'$ (by Proposition 3.6) and P' and Q' cannot have equivalent nonzero subprojections. So, by Lemma 4.7, $C_{P'}C_{Q'} = 0$. Since $P' \leq C_{P'}$ and $Q' \leq C_{Q'}$, we have then

$$C_{P'}P' = P', \quad C_{P'}Q' = 0.$$

Let $Z := C_{P'}$ and note that

$$P(I - Z) = (P - P')(I - Z).$$

Thus:

$$QZ = (Q - Q')Z \sim (P - P')Z \leq PZ,$$
$$P(I - Z) = (P - P')(I - Z) \sim (Q - Q')(I - Z) \leq Q(I - Z).$$

□

4.12. **Corollary.** *Suppose that $P \leq C_Q$. There exists an orthogonal family $\{Q_i\}_{i\in\mathcal{I}}$ such that $P = \sum_{i\in\mathcal{I}} Q_i$ and $Q_i \preceq Q$ for $i \in \mathcal{I}$. In particular, when $\mathfrak{A}$ is a factor and $Q \neq 0$, such a decomposition of P is possible.*

Proof. Let $\{Q_i\}_{i\in\mathcal{I}}$ be a maximal family of nonzero, mutually orthogonal subprojections of P such that $Q_i \preceq Q$ for $i \in \mathcal{I}$. Denote $P_0 := P - \sum_{i\in\mathcal{I}} Q_i$ and apply Comparison Theorem to find a central projection Z such that:

$$P_0 Z \preceq QZ, \quad Q(I-Z) \preceq P_0(I-Z). \tag{4.5}$$

This yields $Q \geq Q(I-Z) \sim P_1 \leq P_0(I-Z) \leq P_0$ for some P_1 and thus $P_1 \leq P$ and $P_1 \preceq Q$. So, by the maximality of the family chosen above we have $P_1 = 0$ and $Q(I-Z) = 0$ as well. This means that $Q \leq Z$, thus $P_0 \leq P \leq C_Q \leq Z$. Consequently, $P_0 Z = P_0$. As $QZ = Q$, (4.5) means that $P_0 \preceq Q$ and again, by the maximality of the family chosen above, P_0 must be 0. □

4.13. *Remark.* Note that, if P and Q are not comparable (this could happen only if $\mathfrak{A}$ is not a factor!), then Z can be chosen such that

$$QZ \prec PZ, \quad P(I-Z) \prec Q(I-Z).$$

Indeed, if one of the above relations was not strict (for example, if $QZ \sim PZ$), then we would get:

$$PZ + P(I-Z) = P \preceq QZ + Q(I-Z) = Q.$$

This yields a contradition (cf. also Lemma 3.11).

5. Finite and infinite projections

Some of classical notions of theory of cardinal numbers may be adapted to the context of projections in von Neumann algebras. Namely,

5.1. **Definition.** Suppose that $P \in \mathfrak{E}(\mathfrak{A})$. Then P is called:

a) *infinite* if there is projection $P_0 \in \mathfrak{A}$ such that $P \sim P_0 < P$,
b) *finite* if it is not infinite,
c) *properly infinite* if P is infinite and for any central projection $Z \in \mathfrak{A}$ the projection PZ is either 0 or infinite.

A von Neumann algebra is *finite* (resp. *infinite*; *properly infinite*) iff its unit is such.

If X is an infinite set, then we can 'halve' X meaning that we can find $X_1, X_2 \subset X$ such that $X_1 \cap X_2 = \emptyset$, $X_1 \cup X_2 = X$ and $|X_1| = |X_2|$. In noncommutative setting we have similar situation however the notion of 'infiniteness' must be strengthened to 'proper infiniteness':

5.2. **Theorem.** *(Halving Theorem) Each properly infinite projection $P \in \mathfrak{A}$ can be halved in the following sense: there exists a subprojection $Q \in \mathfrak{A}$ of P such that $Q \sim P - Q \sim P$.*

Proof. Step 1. First we will show that PZ can be 'halved' for some central projection Z.

Since P is infinite, we can find $P > P_1 \sim P$ i.e. $V^*V = P$, $VV^* = P_1$. Put $P_2 := VP_1V^*$. Then P_2 is again a projection and moreover, $P_2 < P_1$. Indeed, $P_2 \leq P_1$ and since $V^* : P_1(H) \to P(H)$ is an isomorphism, we can take nonzero $x \in P_1(H)$ such that $V^*x \in P(H) \ominus P_1(H)$ to get $P_2 x = 0$. Also we have $Q_1 := P - P_1 \sim Q_2 := P_1 - P_2$. To convince of that, take $W := V(P - P_1)$ and observe that:

$$W^*W = (P - P_1)V^*V(P - P_1) = (P - P_1)P(P - P_1) = P - P_1$$

and quite similarly (since $P_1 = (VV^*)(VV^*) = VPV^*$) we get:

$$WW^* = V(P - P_1)(P - P_1)V^* = V(P - P_1)V^* = VPV^* - VP_1V^* = P_1 - P_2.$$

Putting $P_n := VP_{n-1}V^*$ and $Q_n := P_{n-1} - P_n$ for $n > 2$, we get an infinite family $\{Q_n\}_{n\in\mathbb{N}}$ of mutually orthogonal and equivalent (nonzero!) proper subprojections of

P. This family is contained in a maximal family $\{Q_j\}_{j\in J}$ with all these properties. We claim that $Q_1 \not\preceq P_0 := P - \sum_{j\in J} Q_j$, because otherwise $Q_1 \sim \tilde{Q} \leq P_0$ for some subprojection $\tilde{Q}$ and therefore we could add $\tilde{Q}$ to our family ($\tilde{Q}$ is orthogonal to each Q_j as a suprojection of P_0), which is a contradiction. Now using Theorem 4.11 and Remark 4.13 we can find a nonzero (!) central projection $Z \in \mathfrak{A}$ such that:

$$P_0 Z \prec Q_1 Z \sim Q_j Z, \tag{5.6}$$

where the latter relation follows from $Q_j \sim Q_1$ ($j \in J$). Divide the infinite set J into two disjoint sets J_0, J_1 such that all of the sets $J_0, J_1, J_0 \setminus \{j_0\}$ have the same cardinality (here $j_0 \in J_0$ is an arbitrary element). Using Lemma 3.11 and Proposition 3.6 we compute:

$$\begin{aligned} PZ &= \sum_{j\in J} Q_j Z + P_0 Z \preceq \sum_{j\in J_0\setminus\{j_0\}} Q_j Z + Q_{j_0} Z = \\ &= \sum_{j\in J_0} Q_j Z \sim \sum_{j\in J_1} Q_j Z \leq \sum_{j\in J_1} Q_j Z + P_0 Z \leq PZ. \end{aligned} \tag{5.7}$$

Thus, everywhere in (5.7) we may put the sign '$\sim$'. So, if we let $P' := \sum_{j\in J_1} Q_j Z + P_0 Z$, then we have (from the relation $PZ - P' = \sum_{j\in J_0} Q_j Z$) that:

$$PZ \sim P' \sim PZ - P'. \tag{5.8}$$

which is the 'halving' of PZ. Note also that $Z \leq C_{Q_{j_0}} \leq C_P$ (cf. the proof of Comparison Theorem).

Step 2: Now define $\{Z_j\}_{j\in J} \subset \mathfrak{Z}(\mathfrak{A})$ to be a maximal orthogonal family of nonzero subprojections of C_P with the property: each of projections PZ_j can be halved—such a family exists, by what we have proved. Denote by Q'_j the projections establishing the halving, i.e.

$$Q'_j \sim PZ_j - Q'_j \sim PZ_j.$$

We claim that

$$C_P = \sum_{j\in J} Z_j. \tag{5.9}$$

If it is not the case, then, since P is properly infinite and $\big(C_P - \sum_{j\in J} Z_j\big)$ is central, we conclude that $\big(C_P - \sum_{j\in J} Z_j\big)P$ is properly infinite as well (cf. Remark 4.3). From the first part of the proof we can find a central nonzero subprojection Z_0 of[3] $C_P - \sum_{j\in J} Z_j$ such that

$$\Big(C_P - \sum_{j\in J} Z_j\Big) P Z_0 = \underbrace{\Big(C_P - \sum_{j\in J} Z_j\Big) Z_0}_{Z_0} P = P Z_0$$

can be halved—this contradicts the maximality of $\{Z_j\}_{j\in J}$. So, the proof of (5.9) is complete. Now put $Q := \sum_{j\in J} Q'_j$ and observe that:

$$\begin{aligned} Q &= \sum_{j\in J} Q'_j \sim \sum_{j\in J} (PZ_j - Q'_j) = (\sum_{j\in J} Z_j)P - Q = \\ &= C_P P - Q = P - Q \sim \sum_{j\in J} PZ_j = PC_P = P. \end{aligned}$$

□

[3]From the first part of the proof it follows that $Z_0 \leq C_{(C_P - \sum_{j\in J} Z_j)P}$ but $C_{(C_P - \sum_{j\in J} Z_j)P} \leq C_{C_P - \sum_{j\in J} Z_j} = C_P - \sum_{j\in J} Z_j$, cf. Remark 4.3.

5.3. **Lemma.** *If $P, Q \in \mathfrak{A}$ are two equivalent projections and P is infinite (finite, properly infinite), then Q is also infinite (resp. finite, properly infinite).*

Proof. First, let P be finite and let $Q_0 \leq Q$, $Q_0 \sim Q \sim P$, i.e. $V^*V = P$, $VV^* = W^*W = Q$, $WW^* = Q_0$. We have to prove that $Q_0 = Q$. Put $U := V^*WV$ and compute:

$$U^*U = V^*W^*VV^*WV = V^*W^*QWV = V^*W^*WV = V^*QV = V^*V = P,$$

$$UU^* = V^*WVV^*W^*V = V^*WQW^*V = V^*WW^*V = V^*Q_0V.$$

It follows that U is a partial isometry: indeed, UU^* is clearly self-adjoint and since $UU^*UU^* = UPU^* = UU^*$ (as $U(H) = P(H)$), UU^* is a projection. Therefore we have $P \sim V^*Q_0V \leq P$; from the finiteness of P we get $P = V^*Q_0V$. Using this we obtain $Q_0 = Q$ since:

$$Q_0 = QQ_0Q = VV^*Q_0VV^* = VPV^* = VV^*VV^* = Q^2 = Q.$$

Now the case of infinite projections follows from the above.

Finally, let P be properly infinite and Z be a central projection—then $PZ \sim QZ$ (Lemma 3.11). Since PZ is either 0 or infinite, the same is true for QZ and we are done. □

In particular, when we halve properly infinite projection P as $P_1 + P_2$, we get again properly infinite projections and therefore we can get $P = P_1 + \ldots + P_n$ for any $n \in \mathbb{N}$ where $P_k \sim P$, $k \in \mathbb{N}$ and P_k's are mutually orthogonal. However, more can be said—we can get $P = \sum_{n=1}^{\infty} P_n$ where $P_n \sim P$, $n \in \mathbb{N}$:

5.4. **Theorem.** *Let $P \in \mathfrak{A}$ be a (nonzero) projection. Then P is properly infinite $\iff$ $P = \sum_{n\in\mathbb{N}} P_n$ where each $P_n \sim P$.*

Proof. Succesively applying Halving Theorem we obtain

$$P = P_1 + Q_1 = P_1 + P_2 + Q_2 = \ldots$$

and so on. Therefore we get a sequence of mutually orthogonal projections $(P_n)_{n\in\mathbb{N}}$ such that $P_n < P$ and $P_n \sim P$. We can ensure ourselves that $\sum_{n=1}^{\infty} P_n = P$ by possibly redefining P_1 to be $P_1' := P_1 + P'$ where $P' := P - \sum_{n=1}^{\infty} P_n$. Still we have $P_1' \sim P$ since:

$$P \sim P_1 \leq P_1 + P' \leq P.$$

Now, let us prove the converse implication: suppose that $P = \sum_{n=1}^{\infty} P_n$ with mutually orthogonal $P_n \sim P$ and let Z be a central projection. If $PZ = 0$ then we are done, and if $PZ \neq 0$ then PZ must be infinite: indeed, it follows that $PZ \sim P_nZ$ for $n \in \mathbb{N}$, in particular each $P_nZ \neq 0$ hence

$$PZ \sim P_nZ < PZ.$$

□

In a set-theoretic situation, i.e. when speaking of sets in place of projections, the notion of 'proper infiniteness' could be paraphrased as follows: X is properly infinite if there is partition $X = \bigcup_{n=1}^{\infty} X_n$ where $|X_n| = |X|$ for $n \in \mathbb{N}$ and obviously it holds iff X is infinite. Therefore this notion has no typical counterpart in the classical set theory.

Before we proceed with next results, we recall the definition of some special kind of projections:

5.5. **Definition.** For a cardinal number α, a projection $P \in \mathfrak{A}$ is called *α-decomposable* if the cardinality of any family of mutually orthogonal nonzero subprojections $Q_i \in \mathfrak{A}$ of P is at most α. For $\alpha = \aleph_0$ we call P *countably decomposable*.

Note that in the above situation the cardinal number α may depend on a von Neumann algebra $\mathfrak{A}$ from which we take Q_j. However, we will sometimes not stress this fact, when it will cause no confusion.

5.6. **Definition.** A projection $P \in \mathfrak{A}$ is called *cyclic* when $P(H) = \overline{\mathfrak{A}'x}$ for some vector $x \in H$.

The following is one of key results for proving the main result of the paper. It will allow us to define the so-called steering projection in a type III von Neumann algebra (details will be explained later).

5.7. **Theorem.** *Suppose that $P, Q \in \mathfrak{A}$ are two projections, P is properly infinite and Q is countably decomposable. If $C_Q \leq C_P$, then $Q \preceq P$.*

Proof. The case $Q = 0$ is trivial. If $0 = C_P C_Q = C_Q$, then, since $Q \leq C_Q$, we get $Q = 0 \preceq P$. We can therefore assume that $C_P C_Q \neq 0 \neq Q$.

Step 1: We claim that there is a central projection Z such that $0 \neq QZ \preceq PZ$.

From our assumption, by Lemma 4.7, P and Q have equivalent (nonzero!) subprojections P_0 and Q_0. Using Halving Lemma repeatedly we construct an orthogonal family $\{P_n\}_{n\in\mathbb{N}}$ such that $P = \sum_{n\in\mathbb{N}} P_n$ and $P_n \sim P$ for each $n \in \mathbb{N}$ (here it is assumed that $0 \notin \mathbb{N}$ since Q_0 was already defined). Denote by $\{Q_n\}_{n\in\mathbb{N}}$ a maximal family of orthogonal subprojections of Q with the property $Q_n \sim Q_0$. The countability of this family follows from the fact that Q is countably decomposable. As in the proof of Halving Theorem, we infer that $Q_0 \not\preceq Q - \sum_{n\in\mathbb{N}} Q_n$ and (again, as in Theorem 5.2) we can find a central projection $Z \in \mathfrak{Z}(\mathfrak{A})$ such that:

$$(Q - \sum_{n\in\mathbb{N}} Q_n)Z \prec Q_0 Z. \tag{5.10}$$

In particular, $QZ \geq Q_0 Z \neq 0$. Applying Lemma 3.11 to the relations $Q_n \sim Q_0 \sim P_0 \preceq P_m (\sim P)$ we get:

$$(Q - \sum_{n\in\mathbb{N}} Q_n)Z \preceq Q_0 Z \preceq P_1 Z,$$
$$Q_n Z \preceq P_{n+1} Z.$$

Thus:

$$QZ = (Q - \sum_{n\in\mathbb{N}} Q_n)Z + \sum_{n\in\mathbb{N}} Q_n Z \preceq P_1 Z + \sum_{n=2}^{\infty} P_n Z = PZ.$$

Note also that, as in proof of the Halving Theorem, we have $Z \leq C_{Q_0} \leq C_Q$.

Step 2: Let $\{Z_j\}_{j\in J}$ be a maximal orthogonal family of nonzero central subprojections of C_Q with the property $QZ_j \preceq PZ_j$. The existence and nonemptiness of such a family follows from Zorn's lemma and the first part of the proof. We have $\sum_{j\in J} Z_j = C_Q$. Otherwise (as $\big(C_Q - \sum_{j\in J} Z_j\big)Q$ is still countably decomposable) we can find a nonzero central projection $Z_0 \leq C_Q - \sum_{j\in J} Z_j$ such that:

$$QZ_0 = Q\Big(C_Q - \sum_{j\in J} Z_j\Big)Z_0 \preceq PZ_0$$

Adding Z_0 to our family contradicts its maximality. Thus $C_Q = \sum_{j\in J} Z_j$ and hence:

$$Q = QC_Q = Q\Big(\sum_{j\in J} Z_j\Big) \preceq P\Big(\sum_{j\in J} Z_j\Big) = PC_Q \leq PC_P = P.$$

□

It is instructive to review the proofs of Theorems 5.2 and 5.7 in case when $\mathfrak{A}$ is a factor—then the suitable central projection Z coincides with I and thus we can omit 'maximal arguments' used in the second part of the proof. It is also worth noting that in a factor each infinite projection is automatically properly infinite.

As a consequence we get the following:

5.8. **Theorem.** *Suppose that $P, Q \in \mathfrak{A}$ are two properly infinite, countably decomposable projections. Then:*

$$P \sim Q \iff C_P = C_Q.$$

Proof. If $C_P = C_Q$, we can apply previous theorem to get $P \preceq Q$ and $Q \preceq P$. Hence $P \sim Q$. The direct implication was established earlier. □

As we have already mentioned, in a factor every infinite projection is properly infinite. Obviously, when H is separable, then every (infinite) projection is countably decomposable. Therefore we get:

5.9. **Corollary.** *When $\mathfrak{A} \subset B(H)$ is a factor and H is a separable Hilbert space, then any two infinite projections are equivalent.*

5.10. **Theorem.** *Suppose that $\{P_i\}_{i\in J_1}$ and $\{Q_j\}_{j\in J_2}$ are two infinite families consisting of mutually orthogonal nonzero projections, each P_i is cyclic and $Q \preceq P$ where $Q := \sum_{j\in J_2} Q_j$, $P := \sum_{i\in J_1} P_i$. Then $|J_1| \leq |J_2|$. If moreover each Q_j is cyclic and $P \sim Q$ then $|J_1| = |J_2|$.*

Proof. Let V be a partial isometry satisfying $V^*V = Q$, $VV^* = P_0 \leq P$. Note that $\{VQ_jV^*\}_{j\in J_2}$ is again orthogonal family of nonzero projections: it follows from the fact that $V : Q(H) \to P_0(H)$ is unitary. Obviously the cardinality of this family is the same as of $\{Q_j\}_{j\in J_2}$; also we have

$$\sum_{j\in J_2} VQ_jV^* = V\Big(\sum_{j\in J_2} Q_J\Big)V^* = VQV^* = (VV^*)(VV^*) = P_0.$$

Therefore, we can assume that $Q \leq P$ instead of $Q \preceq P$. Denote by x_i a vector such that $P_i(H) = \overline{\mathfrak{A}'x_i}$. We claim that for every $j \in J_2$ there is $i \in J_1$ such that $Q_jx_i \neq 0$. Otherwise we will have that

$$\{0\} = \overline{\mathfrak{A}'Q_jx_i} = \overline{Q_j\mathfrak{A}'x_i} = Q_jP_i(H), \qquad i \in J_1$$

hence $Q_jP = 0$. Since $Q_j \leq P$, we obtain $Q_j = 0$ which is a contradiction. Therefore we have $J_2 = \bigcup_{i\in J_1} J_2^i$ where $J_2^i := \{j \in J_2 : Q_jx_i \neq 0\}$. What is more, each J_2^i is at most countable, because

$$\sum_{j\in J_2} \|Q_jx_i\|^2 \leq \|x_i\|^2.$$

Thus $|J_2| \leq |J_1| \cdot \aleph_0 = |J_1|$. The remaining part follows from the above argument applied twice. □

Next we proceed to prove an intuitively clear result that the union of two finite projection is again finite. However, it is not straightforward and requires several lemmas—some of them are interesting in itself. First result shows that a finite von Neumann algebra need not be finite dimensional (as a vector space).

5.11. **Lemma.** *If $\{Z_j\}_{j\in J}$ is a family of central projections and P is such that each of PZ_j is finite then PZ is also finite, where $Z = \bigvee_{j\in J} Z_j$.*

Proof. Assume, for the contrary, that $PZ \sim Q < PZ$. Then $0 \neq PZ - Q \leq Z$. Note that $(PZ - Q)Z_j \neq 0$ for some $j \in J$: indeed, if not, then for all $j \in J$ we would have $(PZ - Q)Z_j = 0$, i.e. $(PZ - Q)(H) \perp Z_j(H)$, hence $(PZ - Q)(H) \perp Z(H)$, yielding $0 = (PZ - Q)Z = PZ - Q$, contrary to the hypothesis. For those j's we have

$$QZ_j < PZZ_j = PZ_j. \tag{5.11}$$

Moreover, $Q \sim PZ$ gives

$$QZ_j \sim PZZ_j = PZ_j. \tag{5.12}$$

Relations (5.11) and (5.12) yield that PZ_j is infinite. □

As a consequence we obtain the following:

5.12. **Proposition.** *For every infinite projection P there exists a central subprojection Z of C_P with the property: PZ is properly infinite and $P(I - Z)$ is finite.*

Proof. Let $\{Z'_j\}_{j\in J}$ be a maximal orthogonal family of central projections such that each of PZ'_j is finite (this family is non-empty since 0 is finite). From the previous lemma PZ' is also finite where $Z' := \bigvee Z'_j$. Put $Z := I - Z'$. Then we have that $PZ' = P(I - Z)$ is finite and PZ is properly infinite due to the maximality of $\{Z_j\}_{j\in J}$ (otherwise, if PZZ_0 is finite and nonzero for some central projection Z_0 then $\tilde{Z} := ZZ_0$ is orthogonal to each Z'_j and thus the family $\{Z'_j\}_{j\in J}$ may be enlarged). □

5.13. **Lemma.** *(Kaplansky Formula) If $Q, R \in \mathfrak{E}(\mathfrak{A})$, then $(Q\vee R-R) \sim (Q-Q\wedge R)$.*

Proof. We claim that $Q\vee R-R = P_{(I-R)Q}$ and $Q-Q\wedge R = P_{Q(I-R)} = P_{((I-R)Q)^*}$. Since $P_{Q_1Q_2} = Q_1 - Q_1 \wedge (I - Q_2)$ for any two projections Q_1 and Q_2, we compute:

$$P_{Q(I-R)} = Q - Q \wedge (I - (I - R)) = Q - Q \wedge R, \tag{5.13}$$

$$P_{(I-R)Q} = I - R - (I - R) \wedge (I - Q) = I - (I - R) \wedge (I - Q) - R = Q \vee R - R \tag{5.14}$$

(here we used the fact that $(M_1^\perp \wedge M_2^\perp)^\perp = M_1 \vee M_2$). Relations (5.13) and (5.14) prove our claim. The statement of the lemma now follows from Lemma 4.6. □

5.14. **Theorem.** *If P, Q are finite projections, then $P \vee Q$ is also finite.*

Proof. Without loss of generality we may assume that $P \perp Q$. Indeed, from Kaplansky Formula, $P \vee Q - Q \sim P - P \wedge Q$. Since $P - P \wedge Q$ is finite as a subprojection of a finite projection P, the same is true for $P \vee Q - Q$ by Lemma 5.3. Since $P \vee Q = Q + (P \vee Q - Q)$, we may assume that P and Q are orthogonal. Now suppose (for the contrary) that $P + Q$ is infinite. We may also assume that $P + Q$ is properly infinite: in fact, by Proposition 5.12 we can find a central projection Z such that $(P+Q)Z$ is properly infinite and PZ, QZ are still finite as subprojections of finite projections. We can now use Halving Theorem to construct a subprojection R of $P + Q$ such that

$$R \sim R' := P + Q - R \sim P + Q. \tag{5.15}$$

Comparison Theorem gives us a central projection Z such that:

$$(P \wedge R)Z \preceq (Q \wedge R')Z, \quad (Q \wedge R')(I - Z) \preceq (P \wedge R)(I - Z) \tag{5.16}$$

At least one of the projections $(P+Q)Z$ and $(P+Q)(I-Z)$ is nonzero—for example $(P + Q)Z \neq 0$. As $P + Q$ is properly infinite, $(P + Q)Z$ is also properly infinite, PZ and QZ are still orthogonal and finite, and moreover:

$$PZ \wedge RZ = (P \wedge R)Z \preceq (Q \wedge R')Z = QZ \wedge R'Z. \tag{5.17}$$

This allows us to assume that $P \wedge R \preceq Q \wedge R'$. This and Kaplansky Formula imply:

$$R = R - (P \wedge R) + (P \wedge R) \preceq (P \vee R) - P + (Q \wedge R') \leq Q. \tag{5.18}$$

(Here we use Lemma 3.6 which is possible since $(P \vee R) - P$ and $Q \wedge R'$ are orthogonal. To see this, note that $R(H) \perp R'(H)$ and $P(H) \perp Q(H)$, so $(R' \wedge Q)(H) \perp (P \vee R)(H)$. Also we have that $P \vee R \leq P \vee Q = P + Q$ and thus $(P \vee R) - P \leq Q$ and obviously $Q \wedge R' \leq Q$.) As $R \preceq Q$, we have that R is finite as well—but $R \sim P + Q$, so $P + Q$ should be finite as well, by Lemma 5.3. This gives a contradition. □

5.15. **Lemma.** *Suppose that $R \neq 0$ is a finite projection such that C_R is countably decomposable with respect to the center $\mathfrak{Z}(\mathfrak{A})$. Then $R = \sum_{n=1}^{\infty} C_n$ where each of C_n's is cyclic in $\mathfrak{A}$.*

Proof. Step 1. Denote by R_0 a cyclic nonzero subprojection of R. Let $\{R_j\}_{j \in J}$ be a maximal orthogonal family of subprojections of R each of which is Murray-von Neumann equivalent to R. This family is finite—otherwise $|J| = |J_0|$ for some proper subset $J_0 \subset J$ and therefore, using Lemma 3.6 we get:

$$\sum_{j \in J} R_j \sim \sum_{j \in J_0} R_j < \sum_{j \in J} R_j$$

contradicting the fact that R is finite. From the maximality of $\{R_j\}_{j \in J}$ we have $R_0 \not\preceq R - \sum_{j \in J} R_j$ (cf. the proof of Halving Theorem). Using Comparison Theorem we find a nonzero central projection Z such that:

$$\Big(R - \sum_{j \in J} R_j\Big) Z \prec R_0 Z. \tag{5.19}$$

By (5.19), $\big(R - \sum_{j \in J} R_j\big) Z$ is cyclic; since $R_j Z \leq R_j \sim R_0$, each $R_j Z$ is also cyclic. Hence RZ is a sum of finite number of cyclic projections. Note also that Z may be chosen so that $Z \leq C_R$.

Step 2. Let $\{Z_j\}_{j \in \mathbb{N}}$ be an orthogonal family of nonzero central subprojections of C_R which is maximal with respect to the property that each RZ_j is a sum of finite number of cyclic projections, say $RZ_j = \sum_{k=1}^{n_j} C_{k,j}$. The countability of this family follows from the fact that C_R is countably decomposable. We claim that:

$$C_R = \sum_{j \in \mathbb{N}} Z_j. \tag{5.20}$$

If it is not the case, then $R\big(C_R - \sum_{j \in \mathbb{N}} Z_j\big)$ is nonzero. Arguing as above, we obtain a subprojection Z' of $C_{R(C_R - \sum_{j \in \mathbb{N}} Z_j)} \leq C_R - \sum_{j \in \mathbb{N}} Z_j$ such that

$$R\Big(C_R - \sum_{j \in \mathbb{N}} Z_j\Big) Z' = RZ'$$

is a sum of finite number of cyclic projections, contrary to the construction. Hence, relation (5.20) holds. Thus:

$$R = C_R R = \sum_{j \in \mathbb{N}} RZ_j = \sum_{j \in \mathbb{N}} \sum_{k=1}^{n_j} C_{k,j}. \tag{5.21}$$

As the sum in (5.21) is countable, the assertion follows. □

The next result is another key tool in proving the main result of the paper.

5.16. **Theorem.** *(Generalized Invariance of Dimension) Suppose that $P, R \in \mathfrak{A}$ are two projections and R is finite and nonzero. Let $\mathfrak{P} = \{P_i\}_{i \in J_1}$ and $\mathfrak{Q} = \{Q_j\}_{j \in J_2}$ be two orthogonal families of subprojections of P maximal with respect to the property: $P_i \sim Q_j \sim R,\ i \in J_1,\ j \in J_2$. Then $|J_1| = |J_2|$.*

Proof. We divide the proof into several steps.

Step 1: Relation $R \not\preceq P$ means that P has no subprojection equivalent to R and hence $J_1 = J_2 = \emptyset$ and the theorem is proved. Thus we may assume that $R \preceq P$. In this case both J_1 and J_2 have at least one element (by maximality of $\mathfrak{P}$ and $\mathfrak{Q}$). Replacing R by any of P_i we can therefore assume that $R \leq P$. Two projections are equivalent in $\mathfrak{A}$ iff they are equivalent in $P\mathfrak{A}P$ (see the remark after the proof). Thus, we can assume that $P = I$. Since $P_i \sim Q_j \sim R$ then, by Theorem 5.8, we get $C_{P_i} = C_{Q_j} = C_R$. Thus, the families $\{P_i\}_{i\in J_1}, \{Q_j\}_{j\in J_2}$ lie in $\mathfrak{A}C_R$. So, replacing $\mathfrak{A}$ by $\mathfrak{A}C_R$, we may and do assume that $C_{P_i} = C_{Q_j} = C_R = I$. Using Proposition 5.12, we obtain a central projection Z such that Z is finite and $I - Z$ is either 0 or properly infinite (in case when I is finite we can not use this proposition and just put $Z = I$). We consider two cases: when $Z = 0$ or not. Denote $\tilde{P} := \sum_{i\in J_1} P_i$ and $\tilde{Q} := \sum_{j\in J_2} Q_j$.

In Steps 2–4 below we assume that Z is nonzero.

Step 2: We claim that both J_1 and J_2 are finite. In fact, if, for example J_1 is infinite then we can find a proper subset $J_1' \subsetneq J_1$ such that $|J_1'| = |J_1|$. Then:

$$\Big(\sum_{i\in J_1} P_i\Big)Z \sim \Big(\sum_{i\in J_1'} P_i\Big)Z < \Big(\sum_{i\in J_1} P_i\Big)Z.$$

This is a contradiction, since $(\sum_{i\in J_1} P_i)Z$ is finite. By a similar argument J_2 is finite.

Step 3: We will show that:

$$R(I-Z) \preceq (I-\tilde{P})(I-Z). \tag{5.22}$$

Suppose that (5.22) does not hold. In particular, $I - Z \neq 0$, thus $I - Z$ is properly infinite due to the choice of Z. On the other hand, $\tilde{P}(I - Z)$ is finite—this follows from the fact that $\tilde{P}$ is finite (as $P_i \sim R$ where R is finite, and $|J_1| < \infty$). Let Z' be a nonzero central subprojection of $I - Z$; then:

$$Z'(I-Z) = Z' = \tilde{P}Z' + (I-\tilde{P})Z'.$$

As $I - Z$ is properly infinite, Z' is infinite and since $\tilde{P}$ is finite, $\tilde{P}Z'$ is also finite. Therefore $(I-\tilde{P})Z' = \Big((I-\tilde{P})(I-Z)\Big)Z'$ is infinite (cf. Theorem 5.14). This proves that $(I - \tilde{P})(I - Z)$ is properly infinite. From the asumption $R(I - Z) \not\preceq (I - \tilde{P})(I - Z)$, by Comparison Theorem we can find a central projection $\tilde{Z} \leq C_{R(I-Z)} \leq C_{I-Z} = I - Z$ such that:

$$(I-\tilde{P})(I-Z)\tilde{Z} \prec R(I-Z)\tilde{Z} = R\tilde{Z} \tag{5.23}$$

Since the left-hand side of (5.23) is infinite and the right-hand side is finite we get a contradiction, hence (5.22) holds, meaning that, for some P' we have:

$$R(I-Z) \sim P' \leq (I-\tilde{P})(I-Z).$$

Step 4: Now, we will show that:

$$RZ \not\preceq (I-\tilde{P})Z. \tag{5.24}$$

Indeed, if it is not the case, we can find $P'' \leq (I-\tilde{P})Z$ such that $P'' \sim RZ$. Hence we get:

$$R = RZ + R(I-Z) \sim P'' + P' \leq (I-\tilde{P})Z + (I-\tilde{P})(I-Z) = I - \tilde{P}.$$

This means that $P' + P''$ is orthogonal to each P_i, $i \in J_1$ and equivalent to R, in contrast with maximality of $\mathfrak{P}$. Thus (5.24) holds and there is a central projection $Z' \leq Z$ such that:

$$(I-\tilde{P})Z' \prec RZ'$$

(in particular, $Z' \neq 0$).

Now the assertion of the theorem easily follows: suppose that $|J_1| < |J_2|$ and take a set $J_2' \subset J_2$ such that

$$|J_1| + 1 = |J_2'|.$$

Summing up the relations $(I - \tilde{P})Z' \prec RZ' \sim Q_{j_0}Z'$ (where $j_0 \in J_2'$) and $P_iZ' \sim Q_{j(i)}Z'$ for $i \in J_1$, where $\{j(i) : i \in J_1\} = J_2' \setminus \{j_0\}$, we get that Z' is equivalent to a proper subprojection of itself—contradicting the fact that Z' was finite (as $Z' \leq Z$).

Step 5: Assume now that $Z = 0$. We claim that J_1 and J_2 are infinite. From the fact that $Z = 0$ it follows that I is properly infinite. If we assume that J_1 is finite then, arguing as in Step 3, from (5.22) (with I instead of $I - Z$) we get:

$$R \preceq I - \tilde{P}.$$

In this case, we can enlarge the family $\{P_i\}_{i \in J_1}$ contradicting its maximality. Repeat this argument for J_2 to get that both J_1 and J_2 must be infinite. Moreover, this shows that $R \not\preceq I - \tilde{P}$, and hence there is a central projection Z such that $(I - \tilde{P})Z \prec RZ$ (in particular, $Z \neq 0$). Together with $P_iZ \sim RZ$, $i \in J_1$, this yields $Z \sim \tilde{P}Z$ and thus $\tilde{Q}Z \preceq \tilde{P}Z$.

Step 6: Let Z_0 be a nonzero central subprojection of Z, cyclic in $\mathfrak{Z}(\mathfrak{A})$. We apply Lemma 5.15 to RZ_0 (it is possible, since $RZ_0 \leq R$ is finite, $C_{RZ_0} \leq C_{Z_0} = Z_0$ and Z_0 is countably decomposable as a cyclic projection, see Proposition 5.5.15 in [4]). Thus RZ_0 is a countable sum of cyclic projections. The same is true for P_iZ_0, $i \in J_1$. The cardinality of the family of *all* those cyclic projections (when i runs over J_1) is $\aleph_0 \cdot |J_1| = |J_1|$ and their sum is $\tilde{P}Z_0$. As $\tilde{Q}Z_0 \preceq \tilde{P}Z_0$ we can apply Theorem 5.10 to get $|J_2| \leq |J_1|$. By symmetry, we have $|J_1| \leq |J_2|$ as well. □

5.17. *Remark.* We have used the fact that if $P, Q \in \mathfrak{E}(R\mathfrak{A}R)$ where $R \in \mathfrak{A}$ is a projection, then P and Q are equivalent with respect to $\mathfrak{A}$ iff they are equivalent with respect to $R\mathfrak{A}R$. Let us briefly show this. If $P \sim_{R\mathfrak{A}R} Q$, then $V^*V = P$, $VV^* = Q$ for some $V \in R\mathfrak{A}R$, hence $V \in \mathfrak{A}$ and therefore $P \sim_{\mathfrak{A}} Q$. Conversely, if $V^*V = P$, $VV^* = Q$ for some $V \in \mathfrak{A}$, then $V \in R\mathfrak{A}R$. Indeed, if $P = RP_0R$ and $Q = RQ_0R$, then $V : RP_0R(H) \to RQ_0R(H)$ is unitary, V vanishes on the orthogonal complement of $R(H)$ and R acts as the identity on $V(H)$. This means that

$$RVRx = VRx = VRx + V(I - R)x = Vx$$

thus $V = RVR \in R\mathfrak{A}R$ and $P \sim_{R\mathfrak{A}R} Q$.

However, it is not true in general that if $P \sim Q$ with respect to some von Neumann algebra $\mathfrak{A}$, then $P \sim Q$ with respect to each von Neumann subalgebra of $\mathfrak{A}$ containing P and Q; if it was the case, then since any von Neumann algebra is contained in $B(H)$ for a certain Hilbert space, the equivalence would always be characterized by the dimensions of the ranges of projections as described in Remark 3.4.

6. Abelian projections

Another important type of projections are so-called *abelian projections*. They play the same role in general von Neumann algebras as minimal projections in case of factors.

6.1. **Definition.** A projection $P \in \mathfrak{A}$ is called *abelian* if $P\mathfrak{A}P$ is a commutative algebra.

6.2. **Lemma.** *For any $T \in \mathfrak{A}$ and $T' \in \mathfrak{A}'$ we have: $TT' = 0 \iff C_TC_{T'} = 0$.*

Proof. We refer reader to [4], Theorem 5.5.4. □

6.3. **Proposition.** *Suppose that $P \in \mathfrak{A}$ is an abelian projection. Then*
a) *if $Q \leq P$ is a projection, then $Q = C_Q P$,*
b) *P is finite.*

Proof. a) By Theorem 2.8, $\mathfrak{Z}(P\mathfrak{A}P) = \mathfrak{Z}(\mathfrak{A})P$ and since P is an abelian projection, thus $\mathfrak{Z}(P\mathfrak{A}P) = P\mathfrak{A}P$, consequently $P\mathfrak{A}P = \mathfrak{Z}(\mathfrak{A})P$. If $Q \leq P$ then $Q = PQP \in P\mathfrak{A}P = \mathfrak{Z}(\mathfrak{A})P$. Thus, Q must be of the form $Q = Z_0 P$ for some element $Z_0 \in \mathfrak{Z}(\mathfrak{A})$ (not necessarily a projection!). Write Q as

$$Q = Z_0 P = Z_0 C_P P =: ZP,$$

where $Z := Z_0 C_P$. We claim that Z is a central projection. Obviously $Z \in \mathfrak{Z}(\mathfrak{A})$ and Z is normal as a member of the center of $\mathfrak{A}$. To see that $Z^2 = Z$, note first that $ZC_P = Z_0 C_P C_P = Z_0 C_P = Z$. From the equality $Q = ZP = Q^2 = Z^2 P$ we have $(Z - Z^2)P = 0$. As $Z - Z^2 \in \mathfrak{Z}(\mathfrak{A}) = \mathfrak{A} \cap \mathfrak{A}' \subset \mathfrak{A}'$ and $P \in \mathfrak{A}$ thus Lemma 6.2 applies. Therefore we have $C_{Z-Z^2} C_P = 0$, meaning that $C_{Z-Z^2}(H) \perp C_P(H)$. But $(Z - Z^2)(H) \subset C_{Z-Z^2}(H)$, hence $0 = (Z - Z^2)C_P = Z - Z^2$. Finally, since C_Q is the smallest central projection majorizing Q, we obtain

$$ZP = Q \leq C_Q P \leq ZP,$$

which gives $Q = C_Q P$.

b) If $P \sim Q \leq P$, then $C_P = C_Q$ and by the above argument $P = C_P P = C_Q P = Q$. □

6.4. **Proposition.** *If $\{P_i\}_{i \in \mathcal{I}}$ is a family of abelian projections and moreover the family $\{C_{P_i}\}_{i \in \mathcal{I}}$ is orthogonal then $\sum_{i \in \mathcal{I}} P_i$ is abelian.*

Proof. Put $P := \sum_{i \in \mathcal{I}} P_i$ (this sum makes sense since $P_i \leq C_{P_i}$) and take two elements $PAP, PBP \in P\mathfrak{A}P$. For $i \neq j \in \mathcal{I}$ we have

$$P_i A P_j = C_{P_i} P_i A C_{P_j} P_j = P_i A P_j C_{P_i} C_{P_j} = 0,$$

thus we can write PAP in the form:

$$PAP = \Big(\sum_{i \in \mathcal{I}} P_i\Big) A \Big(\sum_{j \in \mathcal{I}} P_j\Big) = \sum_{i,j \in \mathcal{I}} P_i A P_j = \sum_{i \in \mathcal{I}} P_i A P_i.$$

Moreover, it turns out that $P_i A P_i P_j B P_j = P_j B P_j P_i A P_i$: if $i = j$ then it follows from the assumption that $P_i = P_j$ is abelian; while if $i \neq j$, then both sides are equal to zero by the similar argument as above. So we have

$$\sum_{i \in \mathcal{I}} (P_i A P_i) \sum_{j \in \mathcal{I}} (P_j B P_j) = \sum_{i,j \in \mathcal{I}} P_i A P_i P_j B P_j =$$
$$\sum_{i,j \in \mathcal{I}} P_j B P_j P_i A P_i = \sum_{j \in \mathcal{I}} P_j B P_j \sum_{i \in \mathcal{I}} P_i A P_i$$

meaning that $(PAP)(PBP) = (PBP)(PAP)$, i.e. P is abelian. □

6.5. **Proposition.** *Suppose that $P, Q \in \mathfrak{A}$ are two projections:*
a) *if P is abelian and $P \sim Q$ then Q is also abelian,*
b) *if both P, Q are abelian and $C_P = C_Q$, then $P \sim Q$.*

Proof. a) Take $QAQ, QBQ \in Q\mathfrak{A}Q$ and V such that $V^*V = P$, $VV^* = Q$. Then compute:

$$(QAQ)(QBQ) = VV^*AVV^*BVV^* =$$
$$VPV^*AVPV^*BVPV^* = VPV^*BVPV^*AVPV^* = (QBQ)(QAQ).$$

b) We will show that $C_P \leq C_Q$ implies $P \preceq Q$. Indeed, if not, then by Comparison Theorem we find a central projection $Z \leq C_P \leq C_Q$ such that $QZ \prec PZ$, which means that $0 \neq QZ \sim P_1 < PZ$ for some subprojection P_1 of PZ. This contradicts point (a) of Proposition 6.3 as P_1 and PZ have the same central carrier:

$$C_{P_1} = C_{QZ} = ZC_Q = Z = ZC_P = C_{PZ}.$$

Repeat this argument to get $Q \preceq P$ and thus $P \sim Q$. □

7. Type decomposition

7.1. **Definition.** A von Neumann algebra $\mathfrak{A}$ is called *type* I if there is an abelian projection P such that $C_P = I$. If $\mathfrak{A}$ has no nonzero abelian projection but it has a finite projection Q with $C_Q = I$ then $\mathfrak{A}$ is called a *type* II von Neumann algebra—of *type* II_1 if I is finite and of *type* II_∞ if I is properly infinite. Finally, if $\mathfrak{A}$ has no finite projection then it is called a *type* III von Neumann algebra.

7.2. **Example.** Suppose that $\mathfrak{A}_{\mathrm{I}}$ is of type I, $\mathfrak{A}_{\mathrm{II}}$ is of type II and $\mathfrak{A}_{\mathrm{III}}$ is of type III. Then:

(a) $\mathfrak{A} := \mathfrak{A}_{\mathrm{I}} \oplus \mathfrak{A}_{\mathrm{II}}$ is neither of type I nor of type II: it cannot be type II, since it has a nonzero abelian projection: just take $P \oplus 0$ where $P \in \mathfrak{A}_{\mathrm{I}}$ is abelian—but none of abelian projections in $\mathfrak{A}$ has $I_{\mathfrak{A}}$ as the central carrier. This follows from the fact that any abelian projection in $\mathfrak{A}$ is of the form $P \oplus 0$ where $P \in \mathfrak{A}_{\mathrm{I}}$ is abelian. For such an abelian projection we have $C_{P\oplus 0} = C_P \oplus C_0 = C_P \oplus 0$ which is never equal to $I_{\mathrm{I}} \oplus I_{\mathrm{II}}$. Moreover, $\mathfrak{A}$ is not of type III, as any abelian projection is finite. So, we see that $\mathfrak{A}$ is none of the types defined above.

(b) As in previous example, $\mathfrak{A} := \mathfrak{A}_1 \oplus \mathfrak{A}_{\mathrm{III}}$ is none of types I,II,III: it has nonero abelian (hence finite) projections, so is neither of type II nor type III—but the central carrier of any such projection is of the form $Z \oplus 0$, thus $\mathfrak{A}$ is not of type I. Finally, $\mathfrak{A}_{\mathrm{II}} \oplus \mathfrak{A}_{\mathrm{III}}$ is also none of the types defined above, by a similar argumentation.

Type II and III von Neumann algebras are highly exotic and far from being elementary. What is more, no type III von Neumann algebra was known to von Neumann himself for about ten years. In the example above we have explained that there are von Neumann algebras which are none of type I, II and III. However, the next result tells us that any von Neumann algebra is the direct sum of von Neumann algebras of these types:

7.3. **Theorem.** *(Type Decomposition) Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space H. There are mutually orthogonal central projections*

$$I_{\mathrm{I}},\ I_{\mathrm{II}_1},\ I_{\mathrm{II}_\infty},\ I_{\mathrm{III}}$$

with sum I and such that either $I_\varepsilon = 0$ or $\mathfrak{A}I_\varepsilon$ is a type ε von Neumann algebra where $\varepsilon \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$.

Proof. Let $\{P_i\}_{i\in\mathcal{I}}$ be a maximal family with properties:

(i) each P_i is a nonzero abelian projection,
(ii) central carriers C_{P_i} are mutually orthogonal (in particular P_i are mutually orthogonal).

Put $P := \sum_{i\in\mathcal{I}} P_i$: then P is abelian by (ii) (Proposition 6.4). Moreover,

$$C_P = C_{\sum_{i\in\mathcal{I}} P_i} = C_{\bigvee_{i\in\mathcal{I}} P_i} = \bigvee_{i\in\mathcal{I}} C_{P_i} = \sum_{i\in\mathcal{I}} C_{P_i}. \tag{7.25}$$

We define a central projection $I_{\mathrm{I}} := C_P = \sum_{i\in\mathcal{I}} C_{P_i}$. It follows that $I - I_{\mathrm{I}}$ contains no nonzero abelian subprojections. In fact, if $R \le I - I_{\mathrm{I}}$ is abelian, then $R \perp I_{\mathrm{I}}$, hence $R \perp C_{P_i}$ for every $i \in \mathcal{I}$ thus $C_R \perp C_{P_i}$ for every $i \in \mathcal{I}$. Now it follows from the maximality of the family $\{P_i\}_{i\in\mathcal{I}}$ that $R = 0$.

Since the unit element of $\mathfrak{A}I_{\mathrm{I}}$ is $I_{\mathrm{I}} = C_P$, it follows that a von Neumann algebra $\mathfrak{A}I_{\mathrm{I}}$ is of type I. We consider two cases:

a) $I - I_{\mathrm{I}}$ is finite: then define $I_{\mathrm{II}_1} := I - I_{\mathrm{I}}$ and note that $I - I_{\mathrm{I}} - I_{\mathrm{II}_1} = 0$,
b) $I - I_{\mathrm{I}}$ is infinite: then by Proposition 5.12 we find a central projection $Z \leq C_{I-I_{\mathrm{I}}} = I - I_{\mathrm{I}}$ such that $Z(I - I_{\mathrm{I}}) = Z$ is properly infinite and $(I - Z)(I - I_{\mathrm{I}}) = I - I_{\mathrm{I}} - Z$ is finite.

By putting $I_{\mathrm{II}_1} := I - I_{\mathrm{I}} - Z$ in this second case we obtain a finite central projection such that $Z = I - I_{\mathrm{I}} - I_{\mathrm{II}_1}$ is properly infinite. Observe that $\mathfrak{A}I_{\mathrm{II}_1}$ is a type II_1 von Neumann algebra: as its identity is I_{II_1}, it is finite and since $I - I_{\mathrm{I}}$ contains no nonzero abelian subprojections, the same is true for I_{II_1}, so there is no nonzero abelian projection in $\mathfrak{A}I_{\mathrm{II}_1}$.

Now, let $\{Q_k\}_{k\in\mathcal{K}}$ be a maximal family with properties:

(i) each $Q_k \leq I - I_{\mathrm{I}} - I_{\mathrm{II}_1}$ is a finite nonzero projection,
(ii) the central carriers C_{Q_k} are mutually orthogonal (in particular Q_k are mutually orthogonal).

By Lemma 5.11, $Q := \sum_{k\in\mathcal{K}} Q_k$ is finite and, as in (7.25),

$$C_Q = \sum_{k\in\mathcal{K}} C_{Q_k}. \tag{7.26}$$

We define a central projection $I_{\mathrm{II}_\infty} := C_Q$. Observe that $I_{\mathrm{II}_\infty} \leq I - I_{\mathrm{I}} - I_{\mathrm{II}_1}$: indeed, $Q_k \leq I - I_{\mathrm{I}} - I_{\mathrm{II}_1}$ implies $C_{Q_k} \leq I - I_{\mathrm{I}} - I_{\mathrm{II}_1}$ and therefore $C_Q \leq I - I_{\mathrm{I}} - I_{\mathrm{II}_1}$. In case (a) as above $I_{\mathrm{II}_\infty} = 0$; otherwise, as $I - I_{\mathrm{I}} - I_{\mathrm{II}_1}$ is properly infinite, the same holds for I_{II_∞}. Thus, the identity of $\mathfrak{A}I_{\mathrm{II}_\infty}$ is properly infinite. Together with the facts that $I_{\mathrm{II}_\infty} = C_Q$ where Q is finite and that I_{II_∞} contains no nonzero abelian subprojections, this implies that $\mathfrak{A}I_{\mathrm{II}_\infty}$ is a type II_∞ von Neumann algebra.

Finally, define a central projection $I_{\mathrm{III}} := I - I_{\mathrm{I}} - I_{\mathrm{II}_1} - I_{\mathrm{II}_\infty}$. It follows that I_{III} has no nonzero finite subprojection. In fact, if $R \leq I_{\mathrm{III}}$ is finite then in particular $R \perp I_{\mathrm{II}_\infty}$, hence $R \perp C_{Q_k}$ for every $k \in \mathcal{K}$ and consequently $C_R \perp C_{Q_k}$ for every $k \in \mathcal{K}$. As before, the maximality of the family $\{Q_k\}_{k\in\mathcal{K}}$ implies that $R = 0$. This argument shows that $\mathfrak{A}I_{\mathrm{III}}$ is a type III von Neumann algebra.

□

8. Steering projection

In the previous part we have given the definition of abelian projections and established a few facts about them. Here we proceed to introduce the concept of a so-called *quasi-abelian* projection, which is relevant to define a steering projection, the main tool for considerations in this section. However, the definition of the latter projection will be different for various types of von Neumann algebras. The simplest cases include von Neumann algebras of type I and II_1.

8.1. **Definition.** a) If $\mathfrak{A}$ is a type II_1 von Neumann algebra, we will simply call the identity of $\mathfrak{A}$ the *steering projection.*

b) If $\mathfrak{A}$ is a type I von Neumann algebra, a projection $P \in \mathfrak{A}$ is called a *steering projection* iff P is abelian and $C_P = I$.

8.2. **Definition.** A von Neumann algebra $\mathfrak{A}$ is called *quasi-commutative* iff for every projection $P \in \mathfrak{A}$ the relation $P \sim C_P$ holds. A projection P is called *quasi-abelian* iff $P = 0$ or $P\mathfrak{A}P$ is a quasi-commutative von Neumann algebra.

8.3. **Examples.** a) In a commutative von Neumann algebra $\mathfrak{A}$ each projection is central, thus $P = C_P$ for any projection, in particular $P \sim C_P$, hence any commutaive von Neumann algebra is quasi-commutative. It also follows that any abelian projection $P \in \mathfrak{A}$ is quasi-abelian.

b) In a type III factor $\mathfrak{A} \subset B(H)$ where H is a separable Hilbert space all projections are equivalent (cf. Theorem 5.8). In particular, for a projection $P \in \mathfrak{A}$ we have $P \sim C_P (= I)$ and this gives that $\mathfrak{A}$ is quasi-abelian.

c) Finally, let $\mathfrak{A}$ be finite factor, i.e. I is finite. Then for a nonzero projection $P \neq I$ we have $P \nsim C_P = I$. Thus $\mathfrak{A}$ is not quasi-abelian provided that such a projection exists, i.e. if $\mathfrak{A} \neq \mathbb{C}$.

8.4. **Definition.** Suppose that $\mathfrak{A}$ is a type III von Neumann algebra. A projection $P \in \mathfrak{A}$ is called a *steering projection* iff P is quasi-abelian and $C_P = I$.

The existence as well as uniqueness (up to equivalence) of steering projections in type I and II_1 von Neumann algebras can easily be established. In fact, in a type II_1 von Neumann algebra we have nothing to do. On the other hand, if $\mathfrak{A}$ is of type I, then two steering projections P and Q must be equivalent—as both they are abelian and have the same central carrier (also a projection equivalent to a steering projection is steering as well—cf. Proposition 6.5). From the argument in Example 8.3 b) it follows that for $\mathfrak{A}$ being a type III factor acting in a separable Hilbert space, I can be taken for a steering projection and this is unique choice up to equivalence. However, it is not (yet) clear how to deal with the general case of type III von Neumann algebras. To investigate this case we need the following auxiliary result.

8.5. **Lemma.** *If $P \in \mathfrak{A}$ is a projection, then the following conditions are equivalent:*

(i) *P is quasi-abelian,*
(ii) *for every subprojection $Q \leq P$ it holds $Q \sim PC_Q$,*
(iii) *for a projection Q we have that $P \preceq Q \iff P \leq C_Q$.*

Proof. Suppose that P is quasi-abelian and $Q \leq P$. Since $P\mathfrak{A}P$ is quasi-abelian, $Q \sim_{P\mathfrak{A}P} C'_Q$ where the latter projection is the central carrier of Q computed with respect to $P\mathfrak{A}P$. But $C'_Q = PC_Q$, so $Q \sim PC_Q$ (see also Remark 5.17). The converse implication is proved similarly.

To show that (iii) follows from (ii), suppose that $P \leq C_Q$. If $P = 0$, we are done; so, let $P \neq 0$. Let $\{P_i\}_{i\in\mathcal{I}}$ be a maximal family of projections with the following properties:

a) each P_i is a nonzero subprojection of P,
b) $P_i \preceq Q$ for every $i \in \mathcal{I}$,
c) for $i \neq j$ we have $PC_{P_i}C_{P_j} = 0$ (in particular, P_i's are mutually orthogonal).

From the maximality and Lemma 4.7 it follows that $P = \sum_{i\in\mathcal{I}} PC_{P_i}$. Note also that $C_{C_{P_i}C_{P_j}P} = C_{P_i}C_{P_j}C_P = 0$ (because $C_{P_i}C_{P_j}P = 0$) which means that $\{C_{P_i}C_P\}_{i\in\mathcal{I}}$ is orthogonal. From our assumption, condition $P_i \leq P$ implies $P_i \sim C_{P_i}P$; thus $C_{P_i}P \preceq Q$ and $(C_{P_i}C_P)C_{P_i}P = C_{P_i}P \preceq C_{P_i}C_PQ$. This yields

$$P = \sum_{i\in\mathcal{I}} C_{P_i}P \preceq \sum_{i\in\mathcal{I}} (C_{P_i}C_P)Q \leq Q.$$

The converse implication in (iii) need no any additional assumptions: $P \preceq Q$ implies $P \preceq C_Q$ and thus $P \leq C_Q$ (see Lemma 4.10).

It remains to prove (ii) under the assumption of (iii): let $Q \leq P$ and consider the projection $R := Q + (I - C_Q)P$. Then:

$$C_R = C_{Q+(I-C_Q)P} = C_Q \vee C_{(I-C_Q)P} = C_QC_P \vee (I - C_Q)C_P = \Big(C_Q \vee (I - C_Q)\Big)C_P = C_P \geq P.$$

This yields, from (iii) that $P \preceq R$, thus $C_QP \preceq C_QR = C_QQ + C_Q(I - C_Q)P = Q \leq C_QP$ i.e. $Q \sim C_QP$. □

8.6. **Theorem.** *Suppose $\mathfrak{A}$ is a type III von Neumann algebra. Then $\mathfrak{A}$ has a steering projection. Any two steering projections are equivalent.*

Proof. Let $\{P_i\}_{i\in\mathcal{I}}$ be a maximal family of nonzero projections with properties:

a) $C_{P_i}C_{P_j} = 0$ for $i \neq j$,

b) each P_i is countably decomposable.

Put $P := \sum_{i\in\mathcal{I}} P_i$ and observe that $C_P = I$. This follows from the maximality of the above family and the fact that every nonzero projection contains a nonzero subprojection which is countably decomposable (recall that any cyclic projection is such). Now observe that each P_i is quasi-abelian: taking into account the previous lemma, it is enough to show that for a projection $Q_i \leq P_i$ we have $Q_i \sim C_{Q_i}P_i$. Since we are working in a type III von Neumann algebra, all projections are properly infinite, so, by Theorem 5.8, it is enough to show that $C_{Q_i} = C_{C_{Q_i}P_i}$. But the latter is equal to $C_{Q_i}C_{P_i} = C_{Q_i}$. It remains to note that P is also quasi-abelian. In fact, for $Q \in P\mathfrak{A}P$ of the form $Q = \sum_{i\in\mathcal{I}} P_iQ_0P_i$ (see also the proof of Proposition 6.5) we have that: $C_Q = C_{\sum_{i\in\mathcal{I}} P_iQ_0P_i} = \sum_{i\in\mathcal{I}} C_{P_iQ_0P_i} \sim \sum_{i\in\mathcal{I}} P_iQ_0P_i = Q$.

For uniqueness, note that if P, Q are steering, then both are quasi-abelian and $P \leq C_Q = I = C_P \geq Q$, thus, by the previous lemma, $P \preceq Q$ and $Q \preceq P$ and consequently $P \sim Q$. □

We turn to deal with the last case—when $\mathfrak{A}$ is a type II_∞ von Neumann algebra. We introduce the following notation: for $n \in \{1, 2, \ldots\} \cup \{\omega\}$ by $n \odot Q$ we denote any projection of the form $\sum_{k=1}^{n} Q_k$ where each $Q_k \sim Q$ (when $n = \omega$, we think of a sum $\sum_{k=1}^{\infty} Q_k$). If $n = 0$ we put simply $n \odot P := 0$. For example, any properly infinite projection P could be written as $P = \omega \odot P$, as established earlier (see Halving Theorem). Before proving the next lemma, we have to introduce the notion of the *trace* on a von Neumann algebra, being a far-going generalization of the usual trace defined for matrices.

8.7. **Definition.** For a von Neumann algebra $\mathfrak{A}$ with center $\mathfrak{Z}(\mathfrak{A})$ by a *trace* we mean any linear mapping $\mathrm{tr} : \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ with the following properties:

(i) (positivity) $\mathrm{tr}(A) > 0$ for $A > 0$,

(ii) $\mathrm{tr}(Z) = Z$ for each $Z \in \mathfrak{Z}(\mathfrak{A})$,

(iii) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $A, B \in \mathfrak{A}$.

8.8. *Remarks.* We emphasize the fact that a trace acts between $\mathfrak{A}$ and $\mathfrak{Z}(\mathfrak{A})$, i.e. it is *center-valued*. A (scalar-valued) functional satysfing similar conditions[4] will also be denoted by 'tr' and called a *faithful tracial state*—this is indeed a state, and the term 'faithful' refers to the condition $\mathrm{tr}(A) > 0$ for $A > 0$. When $\mathfrak{A}$ is a factor, then $\mathfrak{Z}(\mathfrak{A}) = \mathbb{C}I$ which can be identified with $\mathbb{C}$ and then the notions of a trace and a faithful tracial state coincide.

The following is a very deep result. For factors it was proved by von Neumann. It was Dixmier who first shown this theorem in its full generality.

8.9. **Theorem.** *In a finite von Neumann algebra $\mathfrak{A}$ there always exists a center-valued trace and is unique. Moreover, except of axiomatic properties, the mapping* tr *also satisfies the following:*

a) $\|\mathrm{tr}(A)\| \leq \|A\|$,

b) $\mathrm{tr}(AZ) = Z\mathrm{tr}(A)$ *for any* $Z \in \mathfrak{Z}(\mathfrak{A})$,

c) $\mathrm{tr}(\sum_{i\in\mathcal{I}} P_i) = \sum_{i\in\mathcal{I}} \mathrm{tr}(P_i)$ *for any family* $\{P_i\}_{i\in\mathcal{I}}$ *of mutually orthogonal projections (in particular,* tr *preserves the orthogonality of projections).*

[4] Instead of (ii) we assume then that $\mathrm{tr}(I) = 1 \in \mathbb{C}$.

The proof of Theorem 8.9 may be found in any of classical textbooks, see e.g. [5, 7, 10].

8.10. *Remarks.* The above theorem (condition b) shows that tr is a *module map* when $\mathfrak{A}$ is considered as a two-sided module over the ring $\mathfrak{Z}(\mathfrak{A})$. From the positivity it is immediate that $\mathrm{tr}(A) \leq \mathrm{tr}(B)$ if $A \leq B$. It can be shown that the existence of a trace implies that $\mathfrak{A}$ is finite—nevertheless, the so-called *semi-finite trace* may be defined. It is similar to the trace introduced above but, restricted to projections, takes values ∞ (precisely) on the set of infinite projections.

Now we shall establish some properties of the trace concerning the Murray-von Neumann order. They will be applied later.

8.11. **Theorem.** *Let $\mathfrak{A}$ be a finite von Neumann algebra and $\mathrm{tr} : \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ denote its (unique) trace. If $P, Q \in \mathfrak{E}(\mathfrak{A})$, then:*
(a) $P \preceq Q \iff \mathrm{tr}(P) \leq \mathrm{tr}(Q)$,
(b) $P \sim Q \iff \mathrm{tr}(P) = \mathrm{tr}(Q)$.

Proof. (a): The ‘ $\Longrightarrow$ ’ parts follow quickly: if $P \sim Q' \leq Q$, i.e. $P = U^*U$, $Q' = UU^*$ then:

$$\mathrm{tr}(P) = \mathrm{tr}(U^*U) = \mathrm{tr}(UU^*) = \mathrm{tr}(Q') \leq \mathrm{tr}(Q).$$

In particular, this ensures us that $P \sim Q$ provided $\mathrm{tr}(P) = \mathrm{tr}(Q)$.

The ‘ $\Longleftarrow$ ’ part (in (a)) uses Comparison Theorem: this theorem allows us to find a central projection Z such that $ZP \preceq ZQ$ and $(I - Z)Q \preceq (I - Z)P$. Consequently, using what we have proved, and properties of the trace, we get:

$$\mathrm{tr}((I - Z)Q) = (I - Z)\mathrm{tr}(Q) \leq \mathrm{tr}((I - Z)P) = (I - Z)\mathrm{tr}(P). \tag{8.27}$$

But from the fact that $\mathrm{tr}(P) \leq \mathrm{tr}(Q)$ we get:

$$(I - Z)\mathrm{tr}(P) \leq (I - Z)\mathrm{tr}(Q).$$

Together with (8.27) we get

$$(I - Z)\mathrm{tr}(P) = (I - Z)\mathrm{tr}(Q). \tag{8.28}$$

Let $E \in \mathfrak{E}(\mathfrak{A})$ be such that $E \leq (I - Z)P$ and $E \sim (I - Z)Q$. Using (8.28) we conclude:

$$\mathrm{tr}(E) = \mathrm{tr}((I - Z)Q) = (I - Z)\mathrm{tr}(Q) = (I - Z)\mathrm{tr}(P) = \mathrm{tr}((I - Z)P).$$

As $E \leq (I-Z)P$ and the trace is (strictly) positive, we conclude that $E = (I-Z)P$, which means that

$$(I - Z)P \sim (I - Z)Q.$$

Together with $ZP \preceq ZQ$ it gives that $P \preceq Q$.

Part (b) is an immediate consequence of (a). □

8.12. **Lemma.** *If $\mathfrak{A}$ is a type II_1 von Neumann algebra and $P \in \mathfrak{A}$ is a nonzero projection, then there exists a projection $Q < P$ such that $C_Q = C_P$.*

Proof. Since P is non-abelian, the algebra $P\mathfrak{A}P$ is non-commutative. Since the linear span of all projections in a von Neumann algebra coincides with this algebra, we conclude that there is a projection $Q_0 \leq P$ which is not central in $P\mathfrak{A}P$. This means that $Q_0 \neq C_{Q_0}P$. So, it suffices to put $Q = (I - C_{Q_0})P + Q_0$. □

8.13. **Lemma.** *If $\mathfrak{A}$ is a type II_1 von Neumann algebra, $\mathrm{tr} : \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ is its trace, $P \in \mathfrak{A}$ is a projection and $Z \in \mathfrak{A}$ is a central element (not necessarily a projection) with $0 \leq Z \leq \mathrm{tr}(P)$, then there is a projection $Q \leq P$ such that $\mathrm{tr}(Q) = Z$.*

Proof. *Step 1.* For any nonzero projection $P \in \mathfrak{A}$ and each integer $n > 0$ there is a nonzero projection $Q \le P$ such that $\mathrm{tr}(Q) \le \frac{1}{2^n}\mathrm{tr}(P)$.

By a simple induction argument, it is enough to prove Step 1 for $n = 1$. It follows from Lemma 8.12 that there is a projection $Q_0 < P$ with $C_{Q_0} = C_P$. Then $Q_1 = P - Q_0$ is nonzero and hence $0 \ne C_{Q_1}(\le C_P = C_{Q_0})$. Now put $Q_2 = C_{Q_1}Q_0$ and notice that $Q_2 \perp Q_1$, $Q_1, Q_2 \le P$ and $C_{Q_1} = C_{Q_2} \ne 0$. Since these properties are symmetric with respect to Q_1 and Q_2, applying Comparison Theorem we may assume that there is a nonzero central projection $Z \le C_{Q_1}$ such that $ZQ_1 \preceq ZQ_2$. Then $Q = ZQ_1$ is a nonzero subprojection of P and $\mathrm{tr}(Q) \le \mathrm{tr}(ZQ_2)$, which yields $2\mathrm{tr}(Q) \le \mathrm{tr}(ZQ_1) + \mathrm{tr}(ZQ_2) \le \mathrm{tr}(P)$ and we are done.

Step 2. If $R \in \mathfrak{A}$ is a nonzero central element such that $R \le \mathrm{tr}(P)$, then there is a nonzero projection $Q \le P$ with $\mathrm{tr}(Q) \le R$.

Indeed, using Theorem 2.3 for $\mathfrak{Z}(\mathfrak{A})$, we see that there is a nonzero central projection $Z \in \mathfrak{A}$ and a positive integer n such that $Z \le 2^n R$. Since $\mathrm{tr}(Z) = Z$, we conclude from Step 1 that there is a nonzero projection $Q_0 \le Z$ with $\mathrm{tr}(Q_0) \le \frac{1}{2^n}\mathrm{tr}(Z)(\le R \le \mathrm{tr}(P))$. Now Theorem 8.11 implies that $Q_0 \preceq P$ and hence there is a (nonzero) projection $Q \le P$ which is equivalent to Q_0. Then $\mathrm{tr}(Q) = \mathrm{tr}(Q_0) \le R$ and we are done.

We are now ready to prove the whole assertion of the theorem. For $Z = 0$ just put $Q = 0$. Now assume that $Z \ne 0$. Let $\{Q_s\}_{s\in S} \subset \mathfrak{A}$ be a maximal family of mutually orthogonal nonzero projections such that $\sum_{s\in S} Q_s \le P$ and $\sum_{s\in S}\mathrm{tr}(Q_s) \le Z$. (It follows from Step 2 that $S \ne \emptyset$.) We claim that $\mathrm{tr}(Q) = Z$ where $Q = \sum_{s\in S} Q_s \le P$. Indeed, $\mathrm{tr}(Q) = \sum_{s\in S}\mathrm{tr}(Q_s) \le Z$. So, if $\mathrm{tr}(Q) < Z$, then $Q < P$ as well and then $0 \ne Z - \mathrm{tr}(Q) \le \mathrm{tr}(P - Q)$. Then an application of Step 2 would give us a nonzero projection $Q' \le P - Q$ for which $\mathrm{tr}(Q') \le Z - \mathrm{tr}(Q)$. But this contradicts the maximality of the family $\mathfrak{Q} := \{Q_s\}_{s\in S}$, since the collection $\mathfrak{Q} \cup \{Q'\}$ also satisfies all the desired conditions. □

8.14. **Proposition.** *Suppose that $\mathfrak{A}$ is a type II_∞ von Neumann algebra and E is a finite projection in $\mathfrak{A}$. Let $\mathrm{tr} : E\mathfrak{A}E \to \mathfrak{Z}(E\mathfrak{A}E)$ be the trace on $E\mathfrak{A}E$. If P and Q are two projections in $E\mathfrak{A}E$ and $\mathrm{tr}(P) \le n \cdot \mathrm{tr}(Q)$ (where n is a positive integer), then $P \preceq n \odot Q$.*

Proof. We will show the existence of a sequence of mutually orthogonal projections $P_1, P_2, ..., P_n$ such that $\sum_{i=1}^n P_i = P$ and $\mathrm{tr}(P_i) = \frac{1}{n}\mathrm{tr}(P)$ for $i = 1, 2, \dots, n$. Using Lemma 8.13 we find a projection $P_1 \le P$ with $\mathrm{tr}(P_1) = \frac{1}{n}\mathrm{tr}(P)$. Suppose that we have already constructed mutually orthogonal projections $P_1, ..., P_k$, $(k < n)$ with the properties: $\sum_{i=1}^k P_i \le P$ and $\mathrm{tr}(P_i) = \frac{1}{n}\mathrm{tr}(P)$. Then $P - \sum_{i=1}^k P_i$ is a projection with:

$$\mathrm{tr}(P - \sum_{i=1}^k P_i) = \frac{n-k}{n}\mathrm{tr}(P) \ge \frac{1}{n}\mathrm{tr}(P).$$

Again, by Lemma 8.13 we can find a projection $P_{k+1} \le P - \sum_{i=1}^k P_i$ with $\mathrm{tr}(P_{k+1}) = \frac{1}{n}\mathrm{tr}(P)$. After the construction we know that $P_1 + P_2 + \ldots + P_n \le P$ and

$$\mathrm{tr}(P_1 + P_2 + \ldots + P_n) = \mathrm{tr}(P_1) + \mathrm{tr}(P_2) + \ldots + \mathrm{tr}(P_n) = n \cdot \frac{1}{n}\mathrm{tr}(P) = \mathrm{tr}(P).$$

From the positivity of the trace we conclude that

$$P_1 + P_2 + \ldots + P_n = P.$$

By virtue of Theorem 8.11 we have

(i) $P_j \sim P_k$ for $j, k \in \{1, ..., 2^n\}$ and consequently $P \sim n \odot P_1$,
(ii) $\mathrm{tr}(P_1) \le \mathrm{tr}(Q)$, thus $P_1 \preceq Q$.

From (i) and (ii) we infer that:

$$P \sim n \odot P_1 \preceq n \odot Q.$$

□

8.15. **Lemma.** *Let $\mathfrak{A}$ be a type II_∞ von Neumann algebra. For a projection $P \in \mathfrak{A}$ the following conditions are equivalent:*

(i) *P is finite,*
(ii) *for any projection $Q \in \mathfrak{A}$, $P \leq C_Q$ iff there is a sequence of central projections $\{Z_n\}_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} Z_n = I$ and $PZ_n \leq n \odot Q$ for each n.*

Proof. Assume (ii) holds for a projection $P \in \mathfrak{A}$. Choose $Q_0 \in \mathfrak{A}$ to be a finite projection with $C_{Q_0} = I$. Since then $P \leq C_{Q_0}$, (ii) allows us to find central projections $Z_1, Z_2, ...$ such that $PZ_n \leq n \odot Q_0$. Since $n \odot Q_0$ is finite, thus PZ_n is finite as well and hence $P = \sum_{n\in\mathbb{N}} PZ_n$ is also finite (see Lemma 5.11). Conversely, if P is finite and $P \leq C_Q$, we can find a family $\{Q_i\}_{i\in\mathcal{I}}$ of mutually orthogonal projections such that $P = \sum_{i\in\mathcal{I}} Q_i$ and for all $i \in \mathcal{I}$ relation $Q_i \preceq Q$ holds. Since P is finite, there is a (unique) trace $\mathrm{tr} : P\mathfrak{A}P \to \mathfrak{Z}(P\mathfrak{A}P)(= \mathfrak{Z}(\mathfrak{A})P)$. For each Q_i we find central (in $\mathfrak{A}$) projections $Z_{i,n,k}$, $n \in \mathbb{N}$, $k = 1, ..., 2^n$ such that:

$$\mathrm{tr}(Q_i) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \frac{k}{2^n} Z_{i,n,k} P.$$

(This can simply be deduce from the commutative Gelfand-Naimark theorem and the total disconnectedness of the Gelfand spectrum of $\mathfrak{Z}(\mathfrak{A})$. See also the remarks after the proof.) From the above equality we obtain $\mathrm{tr}(Z_{i,n,k}) \leq 2^n \mathrm{tr}(Q_i)$. We infer from Proposition 8.14 that $Z_{i,n,k} \preceq 2^n \odot Q_i \preceq 2^n \odot Q$. Moreover, by:

$$P = \mathrm{tr}(P) = \sum_{i\in\mathcal{I}} \mathrm{tr}(Q_i) = \sum_{i\in\mathcal{I}} \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \frac{k}{2^n} Z_{i,n,k}$$

we have $PZ = P$ and thus $Z \geq P$ where $Z := \bigvee_{i,n,k} Z_{i,n,k}$ (these projections need not be mutually orthogonal!). Reindexing this family we obtain a collection $\{C_j\}_{j\in\mathcal{J}}$ of central projections with $PC_j \preceq n_j \odot Q$ and $C := \bigvee_{j\in\mathcal{J}} C_j \geq P$. We may assume that the family $\{C_j\}_{j\in\mathcal{J}}$ is orthogonal (see the remarks after the proof). We define $C_\infty := I - C$. Then the family $\{C_k\}_{k\in\mathcal{K}}$ where $\mathcal{K} := \mathcal{J} \cup \{\infty\}$ is still orthogonal, $\sum_{k\in\mathcal{K}} C_k = I$ and $PC_k \preceq n_k \odot Q$ for each $k \in \mathcal{K}$ (since $PC_\infty = P - PC = 0$, we may take any number for n_∞, e.g. $n_\infty := 1$). Putting $Z_n := \sum\{C_k : n_k = n\}$ we get the desired sequence. □

8.16. *Remarks.* The existence of $Z_{i,n,k}$ follows from the measure-theoretic argument concerning aproximation of any $f \in L^\infty(X)$ by a linear combinations of characteristic functions. One more comment about the proof is needed: the existence of *orthogonal* family $\{C_j\}_{j\in\mathcal{J}}$ can be established by transfinite induction: we equip $\mathcal{J}$ with a well ordering and denote by j_0 the smallest element. We put $C'_{j_0} := C_{j_0}$ and if C'_j's are already defined for all $j < l$ then we put $C'_l := C_l(I - \sum_{j<l} C_j)$.

8.17. **Proposition.** *Let $\mathfrak{A}$ be type II_∞ and*

$$\mathfrak{E}_\omega(\mathfrak{A}) := \{Q \in \mathfrak{E}(\mathfrak{A}) : Q \sim \omega \odot P \text{ for some finite projection } P\}.$$

Then:

(i) *if $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and Z is a central projection, then $PZ \in \mathfrak{E}_\omega(\mathfrak{A})$,*
(ii) *for $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and a properly infinite projection Q we have $P \preceq Q \iff P \leq C_Q$,*
(iii) *if $P \in \mathfrak{E}_\omega(\mathfrak{A})$ satisfies $C_P = I$, then $Q \sim C_Q P$ for every $Q \in \mathfrak{E}_\omega(\mathfrak{A})$.*

Proof. (i) Let $P \sim \omega \odot P_0$ where P_0 is finite. This allows us to write $P = \sum_{n=1}^{\infty} P_n$ where $P_n \sim P_0$ for $n \in \mathbb{N}$. Then $PZ \sim \sum_{n=1}^{\infty} P_n Z$ and $P_n Z \sim P_0 Z$ for $n \in \mathbb{N}$ (obviously $P_0 Z$ is still finite).

(ii) Suppose that $P \leq C_Q$. Take a finite projection P_0 such that $P \sim \omega \odot P_0$ and find, using Lemma 8.15, a sequence $\{Z_n\}_{n=1}^{\infty}$ of central projections such that $P_0 Z_n \preceq n \odot Q$. Since Q is properly infinite, then $Q \sim \omega \odot Q$ and thus we get:

$$P_0 Z_n \preceq n \odot Q \preceq \omega \odot Q \sim Q.$$

The above holds for any $n \in \mathbb{N}$, thus $\sum_{n=1}^{\infty} P_0 Z_n = P_0 \preceq Q$. Consequently $P \sim \omega \odot P_0 \preceq \omega \odot Q \sim Q$.

(iii) If $P, Q \in \mathfrak{E}_\omega(\mathfrak{A})$, then (in particular) P, Q are properly infinite and $Q \leq C_P (= I)$. So, we can use (ii) to get $Q \preceq P$ and consequently $Q \preceq C_Q P$. As $C_Q P \in \mathfrak{E}_\omega(\mathfrak{A})$ by (i), in order to prove $C_Q P \preceq Q$ it is enough to show that $C_Q \geq C_{C_Q P}$. But $C_{C_Q P}$ is equal to $C_Q C_P = C_Q$ and the assertion follows. □

8.18. **Definition.** Suppose that $\mathfrak{A}$ is a type II_∞ von Neumann algebra. Then $P \in \mathfrak{A}$ is called a *steering projection* if $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and $C_P = I$.

8.19. **Theorem.** *Let $\mathfrak{A}$ be a type II_∞ von Neumann algebra. Then $\mathfrak{A}$ has a steering projection and any two steering projections are equivalent.*

Proof. Since I is properly infinite, we may write it as $I = \sum_{n=1}^{\infty} P_n$ where $P_n \sim I$ for $n \in \mathbb{N}$. It then follows that for any $Q \in \mathfrak{A}$ there is a infinite sequence of mutually orthogonal projections $(Q_n)_{n=1}^{\infty}$ with $Q_n \sim Q$ for $n \in \mathbb{N}$ (see Theorem 3.2). In other words, for any $Q \in \mathfrak{E}(\mathfrak{A})$ we can define $\omega \odot Q$. Now take a finite projection $P_0 \in \mathfrak{A}$ such that $C_{P_0} = I$ and form $P := \omega \odot P_0 (= \sum_{n=1}^{\infty} P_n)$. Then $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and still $C_P = I$:

$$C_P = C_{\bigvee_{n=1}^{\infty} P_n} = \bigvee_{n=1}^{\infty} C_{P_n} = C_{P_0} = I.$$

This establish the existence. To deal with the uniqueness assume that P, Q are both steering. Then they both belong to $\mathfrak{E}_\omega(\mathfrak{A})$ and $P \leq C_Q (= I)$ and $Q \leq C_P (= I)$, so we can use Proposition 8.17 to get $P \preceq Q$ and $Q \preceq P$, i.e. $P \sim Q$. □

We have defined a steering projection in a von Neumann algebra of any of types: $\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$. Now, if $\mathfrak{A}$ is an arbitrary von Neumann algebra, then we define a steering projection to be the sum of steering projections of $\mathfrak{A}I_{\mathrm{I}}$, $\mathfrak{A}I_{\mathrm{II}_1}$, $\mathfrak{A}I_{\mathrm{II}_\infty}$, $\mathfrak{A}I_{\mathrm{III}}$ where I_ε is defined as in Theorem 7.3 for $\varepsilon \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$.

8.20. **Theorem.** *An arbitrary von Neumann $\mathfrak{A}$ algebra has a steering projection. Any two steering projections in $\mathfrak{A}$ are equivalent. If P is a steering projection in $\mathfrak{A}$, then $C_P = I$ and for any nonzero central $Z \in \mathfrak{E}(\mathfrak{A})$ the projection PZ is steering in $\mathfrak{A}Z$.*

Proof. The existence follows directly from the definition and the previous discussion, the same with uniqueness: indeed, if $P, Q \in \mathfrak{E}(\mathfrak{A})$ are steering and

$$\begin{cases} P = P_{\mathrm{I}} + P_{\mathrm{II}_1} + P_{\mathrm{II}_\infty} + P_{\mathrm{III}} \\ Q = Q_{\mathrm{I}} + Q_{\mathrm{II}_1} + Q_{\mathrm{II}_\infty} + Q_{\mathrm{III}}. \end{cases}$$

then, by the previous discusion, $P_\varepsilon \sim Q_\varepsilon$ in $\mathfrak{A}_\varepsilon := \mathfrak{A}I_\varepsilon$ for $\varepsilon \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$ and thus $P \sim Q$ (cf. Remark 5.17). Moreover, $C_{P_\varepsilon} = I_\varepsilon$ and hence

$$C_P = C_{(P_{\mathrm{I}} + P_{\mathrm{II}_1} + P_{\mathrm{II}_\infty} + P_{\mathrm{III}})} = C_{P_{\mathrm{I}}} + C_{P_{\mathrm{II}_1}} + C_{P_{\mathrm{II}_\infty}} + C_{P_{\mathrm{III}}} = I_{\mathrm{I}} + I_{\mathrm{II}_1} + I_{\mathrm{II}_\infty} + I_{\mathrm{III}} = I.$$

Now, if $Z \in \mathfrak{E}(\mathfrak{A})$ is central and nonzero, and P is as above, then

$$PZ = P_{\mathrm{I}} Z_{\mathrm{I}} + P_{\mathrm{II}_1} Z_{\mathrm{II}_1} + P_{\mathrm{II}_\infty} Z_{\mathrm{II}_\infty} + P_{\mathrm{III}} Z_{\mathrm{III}}$$

where $Z_\varepsilon := ZI_\varepsilon$ is the part of Z 'within' $\mathfrak{A}_\varepsilon$. We claim that $P_\varepsilon Z_\varepsilon$ is steering in $\mathfrak{A}Z_\varepsilon = (\mathfrak{A}Z)_\varepsilon$. For $\varepsilon = \mathrm{II}_1$ this is immediate. Now observe that

$$C_{P_\varepsilon Z_\varepsilon} = Z_\varepsilon C_{P_\varepsilon} = Z_\varepsilon I_\varepsilon = Z_\varepsilon$$

is the identity of $(\mathfrak{A}Z)_\varepsilon$ and $P_{\mathrm{I}} Z_{\mathrm{I}}$ is abelian, since P_{I} is such. To see that $P_{\mathrm{III}} Z_{\mathrm{III}}$ is quasi-abelian, take $Q \leq P_{\mathrm{III}} Z_{\mathrm{III}}$. In particular, $Q \leq P_{\mathrm{III}}$ and thus $Q \sim C_Q P_{\mathrm{III}}$, and

$$QZ_{\mathrm{III}} = Q \sim C_Q P_{\mathrm{III}} Z_{\mathrm{III}}.$$

Consequently, $P_{\mathrm{III}} Z_{\mathrm{III}}$ is quasi-abelian. Finally, as $P_{\mathrm{II}_\infty} \in \mathfrak{E}_\omega(\mathfrak{A}_{\mathrm{II}_\infty})$ and Z_{II_∞} is central, then $P_{\mathrm{II}_\infty} Z_{\mathrm{II}_\infty}$ belongs to $\mathfrak{E}_\omega(\mathfrak{A}_{\mathrm{II}_\infty})$ by Proposition 8.17. □

9. General decomposition

Before formulating the main theorem, we need some auxiliary results, which are however interesting in itself. By Card we mean the *class* of all cardinal numbers, while Card_∞ is its subclass of infinite cardinals. For $\alpha \in \mathrm{Card}$ by α^+ we denote the immediate succesor of α. The notation of the form '$Q \sim \alpha \odot P$' (where P and Q are projections and $\alpha \in \mathrm{Card}$) means that Q may be written in the form $Q = \sum_{s \in S} Q_s$ where $Q_s \sim P$ for all $s \in S$ and $|S| = \alpha$.

9.1. **Lemma.** *Let $\mathfrak{A}$ be a von Neumann algebra of type ε where $\varepsilon \in \{\mathrm{I}, \mathrm{II}_\infty, \mathrm{III}\}$ and P be a steering projection in $\mathfrak{A}$. Suppose that the projections Q, Q' satisfy: $Q \sim \alpha \odot P$ and $Q' \sim \beta \odot P$ where $\alpha, \beta \in \mathrm{Card}_\infty \cup \{0\}$. Then $Q \sim Q' \iff \alpha = \beta$. If $\mathfrak{A}$ is of type* I*, then the statement is valid for any $\alpha, \beta \in \mathrm{Card}$.*

Proof. The 'if' part is obvious. Further, if, for example, $\alpha = 0$, then $0 = Q \sim Q'$, so $Q' = 0$ and $\beta = 0$. We therefore assume that $\alpha, \beta > 0$ and consider three cases, namely when $\mathfrak{A}$ is of type ε with:

- $\varepsilon = \mathrm{I}$; then a steering projection is defined to be abelian, hence finite. Since

$$\alpha \odot P \sim Q \sim Q' \sim \beta \odot P,$$

so, by Theorem 5.16 we get $\alpha = \beta$. (Note that we have not used the fact that α, β are infinite.)

- $\varepsilon = \mathrm{II}_\infty$; in this case $P \sim \omega \odot P_0$ for some finite projection P_0. We have then, using the fact that α, β are infinite:

$$Q \sim \alpha \odot P \sim \alpha \odot (\omega \odot P_0) = (\alpha \cdot \aleph_0) \odot P_0 = \alpha \odot P_0,$$

and similarly $Q' \sim \beta \odot P_0$. As P_0 is finite, we are in the situation from the previous step and we obtain $\alpha = \beta$ (by Theorem 5.16).

- $\varepsilon = \mathrm{III}$; from the proof of Theorem 8.6 we know that $P = \sum_{i \in \mathcal{I}} P_i$ where each P_i is countably decomposable and $\{P_i\}$'s are centrally orthogonal, i.e. $C_{P_i} C_{P_j} = 0$ for $i \neq j$. Choose $i_0 \in \mathcal{I}$ and denote for simplicity $Z := C_{P_{i_0}}$. Then

$$PZ = \sum_{i \in \mathcal{I}} P_i Z = P_{i_0} Z + \sum_{i \neq i_0} \underbrace{P_i Z}_{0} = P_{i_0}$$

and thus we obtain:

$$QZ \sim (\alpha \odot P)Z \sim \alpha \odot PZ = \alpha \odot P_{i_0} \sim$$

$$\sim Q'Z \sim (\beta \odot P)Z \sim \beta \odot PZ = \beta \odot P_{i_0}.$$

Now, using Theorem 5.10 (see the remark below) we obtain $\alpha = \beta$.

□

9.2. *Remark.* Theorem 5.10 is valid for infinite orthogonal families of *cyclic* projections and in the above argument we dealt with countably decomposable projections. However, the assertion is also valid in this case: this follows from the fact that each nonzero projection is the sum of a family of nonzero cyclic projections (see for instance [4], Theorem 5.5.9.) —so, in case of a countably decomposable projection this family is countable. So, an infinite family of nonzero countably decomposable projections may be replaced by a family of nonzero cyclic projections without changing the sum and the cardinality.

9.3. **Proposition.** *Suppose that $\mathfrak{A}$ is a von Neumann algebra of type ε where $\varepsilon \in \{\mathrm{I}, \mathrm{II}_\infty, \mathrm{III}\}$ with a steering projection P, Q is a properly infinite projection and $\alpha \in \mathrm{Card}_\infty \cup \{0\}$. Assume that the following condition is satisfied: $\alpha \odot P \preceq Q$ and for any central $0 \neq Z \in \mathfrak{C}(\mathfrak{A})$, QZ does not contain α^+ copies of PZ, i.e. $\alpha^+ \odot PZ \not\preceq QZ$. Then $Q \sim \alpha \odot P$. If $\mathfrak{A}$ is of type* I*, the assumption for Q being properly infinite can be dropped and α may be taken arbitrary.*

Proof. Step 1: First we will show that there is a central projection $Z \neq 0$ such that:

$$QZ \sim \alpha \odot PZ. \tag{9.29}$$

Case $\alpha = 0$. With our assumptions we have that

$$P \preceq Q \iff C_Q = I \tag{9.30}$$

Indeed, if $\varepsilon \in \{\mathrm{I}, \mathrm{III}\}$, then P is quasi-abelian (even abelian for $\varepsilon = \mathrm{I}$), thus $P \preceq Q \iff P \leq C_Q$ (by Lemma 8.5), but the latter inequality is equivalent to $C_P \leq C_Q$. Since P is steering, so $C_P = I$ which yields (9.30). In case $\varepsilon = \mathrm{II}_\infty$ we are assuming that Q is properly infinite and P, being a steering projection, belongs to $\mathfrak{C}_\omega(\mathfrak{A})$, so again we have $P \preceq Q \iff P \leq C_Q$, this time by Proposition 8.17. The argument as above gives (9.30). As we assumed that $\alpha = 0$, i.e. Q contains no copy of P, we have $P \not\preceq Q$; so, from (9.30) we obtain $C_Q \neq I$. Taking $Z := I - C_Q \neq 0$ we have

$$QZ = Q - QC_Q = 0 \sim 0 \odot (PZ).$$

Case $\alpha > 0$. From our assumption, there exists an orthogonal family $\{P_i\}_{i \in \mathcal{I}_0}$ with $|\mathcal{I}_0| = \alpha$ such that each $P_i \sim P$, $P_i \leq Q$. We extend this family to a maximal family $\{P_i\}_{i \in \mathcal{I}}$ with all the above properties—the fact that we still have $|\mathcal{I}| = \alpha$ follows from the assumption about Q (in case when $\varepsilon = \mathrm{I}$ and α is finite the first family *is already* maximal). If it happens that $Q = \sum_{i \in \mathcal{I}} P_i (= \alpha \odot P)$ then putting $Z := I$ we get (9.29). Suppose now that $Q' := Q - \sum_{i \in \mathcal{I}} P_i \neq 0$. By the maximality of the family $\{P_i\}_{i \in \mathcal{I}}$ we have $P \not\preceq Q'$ and by Comparison Theorem, we find a nonzero central projection Z such that $Q'Z \prec PZ$. It turns out that this Z can be taken to fulfill (9.29) for $\varepsilon \in \{\mathrm{II}_\infty, \mathrm{III}\}$. Indeed, for those ε the steering projection P is properly infinite, so PZ is also properly infinite and the same is true for $P_{i_0}Z$ (being equivalent to PZ; here $i_0 \in \mathcal{I}$ is fixed). Moreover, $Q'Z \preceq P_{i_0}Z$ and since $Q' \perp P_{i_0}$, also $Q'Z \perp P_{i_0}Z$. Putting these facts together we obtain:

$$P_{i_0}Z \preceq Q'Z + P_{i_0}Z \preceq 2 \odot P_{i_0}Z \sim P_{i_0}Z \sim PZ,$$

hence all of the above components are equivalent. In particular:

$$QZ = Q'Z + \sum_{i \in \mathcal{I}} P_i Z = Q'Z + P_{i_0}Z + \sum_{i \neq i_0} P_i Z \sim \alpha \odot P.$$

It remains to investigate the case when $\varepsilon = \mathrm{I}$. Now P is no longer properly infinite. As before, we have $Q'Z \preceq PZ$ and

$$QZ = Q'Z + \sum_{i \in \mathcal{I}} P_i Z.$$

If it happens that $Q'Z = 0$ then we are already done, because

$$QZ = \sum_{i\in\mathcal{I}} P_i Z \sim \alpha \odot P.$$

So, assume that $Q'Z \neq 0$ and form $0 \neq Z' := C_{Q'Z} = ZC_{Q'} \leq Z$. This implies that

$$Q'Z' = Q'ZZ' \preceq PZZ' = PZ' \sim P_{i_0}Z'$$

for each $i_0 \in \mathcal{I}$ meaning that there is P' with

$$Q'Z' \sim P' \leq P_{i_0}Z' \leq P_{i_0}.$$

Recall that now P is abelian (in particular quasi-abelian) so by Lemma 8.5 we have

$$P' = C_{P'}P_{i_0}. \tag{9.31}$$

But observe that since $P' \sim Q'Z'$, thus we have

$$C_{P'} = C_{Q'Z'} = Z'C_{Q'} = ZC_{Q'}C_{Q'} = ZC_{Q'} = Z'$$

so (9.31) turns out to be $P' = P_{i_0}Z'$. This means that

$$Q'Z' \sim P' = P_{i_0}Z' \sim PZ'$$

so we have that

$$QZ' = \underbrace{Q'Z'}_{\sim PZ'} + \sum_{i\in\mathcal{I}} \underbrace{P_iZ'}_{\sim PZ'} \sim (\alpha+1) \odot PZ'.$$

In case when α is infinite this yields $QZ' \sim \alpha \odot PZ'$, while in case of finite α we obtain a contradiction with the fact that $Q'Z \neq 0$. In that case we conclude that $Q'Z = 0$, which was discussed before.

Step 2: Let $\{Z_j\}_{j\in\mathcal{J}}$ be a maximal family of nonzero central projections such that for each $j \in \mathcal{J}$ we have $QZ_j \sim \alpha \odot PZ_j$. Then we obtain:

$$Q(\sum_{j\in\mathcal{J}} Z_j) = \sum_{j\in\mathcal{J}} QZ_j \sim \sum_{j\in\mathcal{J}} \alpha \odot PZ_j \sim \alpha \odot P(\sum_{j\in\mathcal{J}} PZ_j).$$

We will show that $\sum_{j\in\mathcal{J}} Z_j = I$. Assume the contrary, i.e. $Z' := I - \sum_{j\in\mathcal{J}} Z_j \neq 0$. Form $\mathfrak{A}_0 := \mathfrak{A}Z'$. Note that $\mathfrak{A}_0$ is again a von Neumann algebra of type ε and denote by $P' := PZ'$ a steering projection for $\mathfrak{A}_0$ (see Theorem 8.20) and $Q' := QZ'$. Take $0 \neq Z \in \mathfrak{Z}(\mathfrak{A}_0)$. In particular, $Z \leq Z'$ and

$$Q'Z = QZ'Z = QZ \quad \text{and} \quad P'Z = PZ'Z = PZ. \tag{9.32}$$

It follows from our assumption that QZ does not contain α^+ copies of PZ so, by (9.32), $Q'Z$ does not contain α^+ copies of PZ. Moreover, Q contains α copies of P and consequently $Q' = QZ'$ contains α copies of $PZ' = P'$. So, all assumptions of the theorem are satisfied for $\mathfrak{A}_0$ and Q', thus we can apply Step 1 for $\mathfrak{A}_0$ and find $0 \neq Z_0 \in \mathfrak{Z}(\mathfrak{A}_0)$ such that

$$QZ_0 = QZZ_0 = Q'Z_0 \sim \alpha \odot P'Z = PZ'Z_0 = PZ_0.$$

Since $Z_0 \leq Z' = I - \sum_{j\in\mathcal{J}} Z_j$, thus Z_0 is orthogonal to all of Z_j's and this contradicts the maximality of the taken family. In consequence, $\sum_{j\in\mathcal{J}} Z_j = I$. $\square$

9.4. **Lemma.** *Let $\mathfrak{A}$ be a von Neumann algebra, $(X, \leq_X)$ be a well ordered set with maximal and minimal elements x_{max}, x_{min} and $\{Z_x\}_{x\in X} \subset \mathfrak{Z}(\mathfrak{A})$ be a family of central projections. Suppose that this family satisfies the following conditions:*

a) *if $x \leq_X y$, then $Z_x \geq Z_y$,*
b) *for any limit element $x \in X \setminus \{x_{min}\}$, $Z_x = \bigwedge_{y<x} Z_y$.*

Denote $W_x := Z_x - Z_{x^+}$ for $x \neq x_{max}$. Then $\sum_{x\neq x_{max}} W_x = Z_{x_{min}} - Z_{x_{max}}$.

Proof. The proof is by transfinite induction: we claim that for $x \in X$ we have

$$\sum_{y <_X x} W_y = Z_{x_{min}} - Z_x. \tag{9.33}$$

For $x = x_{min}$ both sides of (9.33) are equal to 0 (we use the convention that the sumation over the empty set is 0).

For a succesor element $x = y^+$ we are assuming that $\sum_{t <_X y} W_t = Z_{x_{min}} - Z_y$, so:

$$\sum_{t <_X x} W_t = \sum_{t <_X y} W_t + W_y = Z_{x_{min}} - Z_y + Z_y - Z_{y^+} = Z_{x_{min}} - Z_x.$$

Finally, let $x \neq x_{min}$ be a limit element. From the transfinite induction hypothesis we have $\sum_{t <_X y} W_t = Z_{x_{min}} - Z_y$ for every $y < x$. We claim that:

$$\sum_{t <_X x} W_t = \bigvee_{y <_X x} \sum_{t <_X y} W_t \tag{9.34}$$

To prove (9.34), observe that obviously $\sum_{t <_X y} W_t \leq \sum_{t <_X x} W_t$ and if for each $y <_X x$ we have $\sum_{t <_X y} W_t \leq W$, then in particular $W_y \leq W$ (just put y^+ in place of y), so $W_y W = W_y$ and thus $(\sum_{y <_X x} W_y)W = \sum_{y <_X x} W_y W = \sum_{y <_X x} W_y$ which means that $\sum_{y <_X x} W_y \leq W$ and proves (9.34). Now we compute:

$$\sum_{t <_X x} W_t = \bigvee_{y <_X x} (Z_{x_{min}} - Z_y) = Z_{x_{min}} - \bigwedge_{y <_X x} Z_y = Z_{x_{min}} - Z_x.$$

Thus, we have estabilished that (9.33) is valid for any $x \in X$. Put $x := x_{max}$ to get the assertion. □

For the purpose of the next result, let $\Lambda_{\mathrm{I}} = \mathrm{Card}$, $\Lambda_{\mathrm{II}} = \{0, 1\} \cup \mathrm{Card}_\infty$ and $\Lambda_{\mathrm{III}} = \{0\} \cup \mathrm{Card}_\infty$.

9.5. **Theorem.** *Let P be a steering projection in a von Neumann algebra $\mathfrak{A}$ and denote by $Z^{\mathrm{I}}, Z^{\mathrm{II}}, Z^{\mathrm{III}}$ the central projections such that $Z^{\mathrm{I}} + Z^{\mathrm{II}} + Z^{\mathrm{III}} = I$ and $\mathfrak{A}Z^i$ is of type i for $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Additionally, let $Z^{\mathrm{II}_1} \in \mathfrak{Z}(\mathfrak{A})$ be such that $Z^{\mathrm{II}_1} \leq Z^{\mathrm{II}}$, $\mathfrak{A}Z^{\mathrm{II}_1}$ is of type II_1 and $Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ is properly infinite. Then for each $Q \in \mathfrak{E}(\mathfrak{A})$ there is a unique system of central projections*

$$\{Z^{\mathrm{I}}_\alpha(Q)\}_{\alpha \in \Lambda_{\mathrm{I}}} \cup \{Z^{\mathrm{II}}_\alpha(Q)\}_{\alpha \in \Lambda_{\mathrm{II}}} \cup \{Z^{\mathrm{III}}_\alpha(Q)\}_{\alpha \in \Lambda_{\mathrm{III}}}$$

with the following properties:

a) $Z^i = \sum_{\alpha \in \Lambda_i} Z^i_\alpha(Q)$ *for* $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$,
b) *for* $\alpha \in \Lambda_i$ *with* $(i, \alpha) \neq (\mathrm{II}, 1)$*:* $Z^i_\alpha(Q)Q \sim \alpha \odot Z^i_\alpha(Q)P$.
c) (i) $Z^{\mathrm{II}}_1(Q)Q$ *is finite,*
 (ii) *if* W *is a nonzero central projection and* $W \leq Z^{\mathrm{II}}_1(Q)$*, then* $WQ \neq 0$,
 (iii) $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q)P \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q)Q$.

Proof. **Existence** *Step 1:* First we deal with the II_1 part. Define

$$Z^{\mathrm{II}}_1(Q) := \bigvee\{W \leq Z^{\mathrm{II}} C_Q : WQ \text{ is finite}\}.$$

Then $Z^{\mathrm{II}}_1(Q)Q$ is finite, by Lemma 5.11. Obviously, $Z^{\mathrm{II}}_1(Q) \leq C_Q$ and hence if $W \leq Z^{\mathrm{II}}_1(Q) \leq C_Q$ is central and nonzero, then $C_{WQ} = WC_Q = W \neq 0$, which yields $WQ \neq 0$. For the last property of point (c), observe that $Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ is properly infinite (or 0), so the same is true for $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q)$, thus $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q)P$ is a steering projection (if nonzero!) in some type II_∞ von Neumann algebra (call it $\tilde{\mathfrak{A}}$). From the fact that $Z^{\mathrm{II}}_1(Q)Q$ is finite, we conclude that $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q)Q$ is finite and its central carrier is equal to

$$(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q)C_Q = (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q)$$

being the identity of $\tilde{\mathfrak{A}}$. Hence from the properties of steering projections in type II_∞ von Neumann algebras we have that

$$(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)P \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)Q.$$

We also conclude that

$$(Z^{\mathrm{II}} - Z_1^{\mathrm{II}}(Q))C_Q \leq (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})C_Q. \tag{9.35}$$

Indeed, (9.35) is equivalent to

$$Z^{\mathrm{II}_1}C_Q \leq Z_1^{\mathrm{II}}(Q)C_Q = Z_1^{\mathrm{II}}(Q) \tag{9.36}$$

and this follows from the definition of $Z_1^{\mathrm{II}}(Q)$ since $Z^{\mathrm{II}_1}Q = Z^{\mathrm{II}_1}C_QQ$ is finite ($\leq Z^{\mathrm{II}_1}$) and $Z^{\mathrm{II}_1}C_Q \leq Z^{\mathrm{II}}C_Q$.

Step 2: Now we construct $Z_0^i(Q) := Z^i(I - C_Q)$. It is then immediate that $Z_0^i(Q)Q = 0$ i.e.

$$Z_0^i(Q)Q \sim 0 \odot Z_0^i(Q)P.$$

Define

$$E^i := Z^i - Z_0^i(Q), \quad i = \mathrm{I}, \mathrm{III},$$
$$E^{\mathrm{II}} := Z^{\mathrm{II}} - Z_0^{\mathrm{II}}(Q) - Z_1^{\mathrm{II}}(Q),$$

(note that the last expression makes sense, since $Z_0^{\mathrm{II}}(Q) \leq I - C_Q$ and $Z_1^{\mathrm{II}}(Q) \leq C_Q$). Direct calculations show that $E^i = Z^iC_Q$ for $i = \mathrm{I}, \mathrm{III}$. For $i = \mathrm{II}$ we have that

$$E^{\mathrm{II}} \leq (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})C_Q.$$

In fact, since $Z^{\mathrm{II}_1}C_Q \leq Z_1^{\mathrm{II}}(Q)C_Q$, we have:

$$E^{\mathrm{II}} = Z^{\mathrm{II}}C_Q - Z_1^{\mathrm{II}}(Q) = \Big(Z^{\mathrm{II}} - Z_1^{\mathrm{II}}(Q)\Big)C_Q \leq (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})C_Q.$$

Finally, $E^{\mathrm{II}}Q$ is properly infinite (or 0): to see this, take a nonzero central projection $W \leq E^{\mathrm{II}}$. Obviously $W \leq Z^{\mathrm{II}}$ and also (by the above argument) $W \leq C_Q$, thus $W \leq Z^{\mathrm{II}}C_Q$. S, if WQ is finite, then $W \leq Z_1^{\mathrm{II}}(Q)$ and simultaneously $W \leq E^{\mathrm{II}}$ yielding a contradiction. Thus WQ is infinite and this means that $E^{\mathrm{II}}Q$ is properly infinite.

For $0 \neq \alpha \in \Lambda_i$ such that $(i, \alpha) \neq (\mathrm{II}, 1)$ let

$$E_\alpha^i := \bigvee\{W \leq E^i : W \in \mathfrak{Z}(\mathfrak{A}),\ \alpha \odot WP \preceq WQ\}.$$

The projections E_α^i have the following properties:

(i) if $\alpha \geq \beta$, then $E_\alpha^i \leq E_\beta^i$ (indeed, if $\alpha \odot WP \preceq WQ$, then of course $\beta \odot WP \preceq WQ$);

(ii) $\alpha \odot E_\alpha^i P \preceq E_\alpha^i$. To see this, write E_α^i as $E_\alpha^i = \bigvee_{s\in S} W_s$ where for each $s \in S$ it holds

$$\alpha \odot W_sP \preceq W_sQ \tag{9.37}$$

We can find an orthogonal family $\{V_s\}_{s\in S} \subset \mathfrak{Z}(\mathfrak{A}) \cap \mathfrak{E}(\mathfrak{A})$ such that $V_s \leq W_s$, $(s \in S)$ and $E_\alpha^i = \sum_{s\in S} V_s$ (cf. Remarks 8.16). From (9.37) we conclude that

$$V_s(\alpha \odot W_sP) \sim \alpha \odot V_sP \preceq V_sW_sQ = V_sQ$$

and therefore summing up gives:

$$\sum_{s\in S}(\alpha \odot V_sP) \sim \alpha \odot \sum_{s\in S} V_sP = \alpha \odot E_\alpha^iP \preceq \sum_{s\in S} V_sQ = E_\alpha^iQ.$$

(iii) If α is a limit (and not the first) element[5] in $\Lambda_i \setminus \{0\}$, $i = \mathrm{I}, \mathrm{III}$ (in $\Lambda_i \setminus \{0,1\}$ for $i = \mathrm{II}$), then $E^i_\alpha = \bigwedge\{E^i_\beta :\ \beta < \alpha,\ \beta \neq 0,\ (i,\beta) \neq (\mathrm{II},1)\}$. For the proof denote by Z the right-hand side of the above relation and note that $E^i_\alpha \leq Z$: indeed, from (i) we have $E^i_\alpha \leq E^i_\beta$ for $\beta < \alpha$, so $E^i_\alpha \leq Z$. For the converse, we will show that Z contains $\alpha \odot PZ$. If $Z = 0$, then it is trivial. For $Z \neq 0$ we have $PZ \neq 0$ since $C_P = I$ and

$$C_{PZ} = ZC_P = ZI = Z \neq 0.$$

Clearly (when $\beta < \alpha$) $Z \leq E^i_\beta$ and, as $\beta \odot E^i_\beta P \preceq E^i_\beta$ therefore we get:

$$Z(\beta \odot E^i_\beta P) = \beta \odot PZ \preceq E^i_\beta Z = Z, \tag{9.38}$$

in other words, Z contains $\beta \odot PZ$. But $\alpha = \sum_{0 \neq \beta < \alpha} \beta$ (in case $i = \mathrm{II}$ we exclude $\beta = 1$, all considered β will be called *admissible*). Denote by $\{P_t\}_{t \in T}$ a maximal orthogonal family of subprojections of Z with the property that $P_t \sim PZ$ for $t \in T$ (in particular $P_t \neq 0$). We claim that $|T| \geq \alpha$. To show this, it is enough to show that $|T| \geq \beta$ for each admissible $\beta < \alpha$ (because α is limit). From (9.38), there exists a maximal orthogonal family $\{Q_s\}_{s \in S}$ of subprojections of Z such that $Q_s \sim PZ$ for each $s \in S$ and $|S| \geq \beta$. Arguing as in the proof of Lemma 9.1 we conlude $|T| = |S| \geq \beta$ and we are done.

We have showed that the assumption of Lemma 9.4 are satisfied. Therefore we can form $Z^i_\alpha(Q) := E^i_\alpha - E^i_{\alpha^+}$ and with the help of this lemma conclude that:

$$\sum_{\alpha \in \mathrm{Card}_\infty} Z^i_\alpha(Q) = E^i_{\aleph_0},\ i = \mathrm{II}, \mathrm{III} \text{ and } \sum_{\alpha > 0} Z^{\mathrm{I}}_\alpha(Q) = E^{\mathrm{I}}_1 \tag{9.39}$$

(note that the second term is missing, because $E^i_\alpha = 0$ for sufficently large α, for example $\alpha := |\mathfrak{E}(\mathfrak{A})|^+$).

We claim that

$$Z^i_\alpha(Q)Q \sim \alpha \odot Z^i_\alpha(Q)P. \tag{9.40}$$

If $Z^i_\alpha(Q) = 0$ then (9.40) is valid, so we assume that $Z^i_\alpha(Q) \neq 0$. Define

$$\tilde{Q} := Z^i_\alpha(Q)Q \text{ and } \tilde{P} := Z^i_\alpha(Q)P \neq 0.$$

Then $\tilde{P}$ becomes a steering projection in $\tilde{\mathfrak{A}} := \mathfrak{A}Z^i_\alpha(Q)$. As $Z^i_\alpha(Q) \leq E^i_\alpha$ and $\alpha \odot E^i_\alpha P \preceq E^i_\alpha Q$, we conclude that:

$$Z^i_\alpha(Q)(\alpha \odot E^i_\alpha P) = \alpha \odot Z^i_\alpha(Q)P = \alpha \odot \tilde{P} \preceq Z^i_\alpha(Q)Q = \tilde{Q}.$$

Now take $W \in \mathfrak{Z}(\tilde{\mathfrak{A}})$. Then in particular $W \in \mathfrak{Z}(\mathfrak{A})$ and $W \leq Z^i_\alpha(Q)$. The latter gives $WE^i_{\alpha^+} = 0$. So, if $W \neq 0$, we cannot have $W \leq E^i_{\alpha^+}$ and thus $WQ = W\tilde{Q}$ does not contain $\alpha^+ \odot WP = \alpha^+ \odot W\tilde{P}$. Thus we have shown that all assumptions of Proposition 9.3 are satisfied and therefore

$$\tilde{Q} \sim \alpha \odot \tilde{P}.$$

To complete the proof of the existence, it remains to show that

$$E^{\mathrm{I}}_1 = E^{\mathrm{I}} \text{ and } E^i_{\aleph_0} = E^i,\ i = \mathrm{II}, \mathrm{III}. \tag{9.41}$$

(Once we have (9.41) we conclude that:

$$\sum_{\alpha \in \Lambda_1} Z^{\mathrm{I}}_\alpha(Q) = Z^{\mathrm{I}}_0(Q) + \sum_{\alpha > 0} Z^{\mathrm{I}}_\alpha(Q) = Z^{\mathrm{I}}_0(Q) + E^{\mathrm{I}}_1 = Z^{\mathrm{I}}_0 + E^{\mathrm{I}} = Z^{\mathrm{I}},$$

[5]For example, $\aleph_0$ is a limit cardinal, but it is not a limit element in Λ_{III}.

$$\sum_{\alpha\in\Lambda_2} Z^{\mathrm{I}}_\alpha(Q) = Z^{\mathrm{II}}_0(Q) + Z^{\mathrm{II}}_1(Q) + \sum_{\alpha\geq\aleph_0} Z^{\mathrm{II}}_\alpha(Q) =$$
$$= Z^{\mathrm{II}}_0(Q) + Z^{\mathrm{II}}_1(Q) + E^{\mathrm{II}}_{\aleph_0} = Z^{\mathrm{II}}_0(Q) + Z^{\mathrm{II}}_1(Q) + E^{\mathrm{II}} = Z^{\mathrm{II}},$$

$$\sum_{\alpha\in\Lambda_3} Z^{\mathrm{III}}_\alpha(Q) = Z^{\mathrm{III}}_0(Q) + \sum_{\alpha\geq\aleph_0} Z^{\mathrm{III}}_\alpha(Q) = Z^{\mathrm{III}}_0(Q) + E^{\mathrm{III}}_{\aleph_0} = Z^{\mathrm{III}}_0(Q) + E^{\mathrm{III}} = Z^{\mathrm{III}}.)$$

For the proof of (9.41) note that $E^{\mathrm{I}}_1 \leq E^I$ and $E^i_{\aleph_0} \leq E^i$ as a straightforward conseqence of the definition of E^i_α. For the converse inclusion it suffices to show that E^i appears in the family defining E^i_α with suitable i and α, i.e.[6]

$$E^{\mathrm{I}}P \preceq E^{\mathrm{I}}Q, \quad \omega\odot E^i_{\aleph_0}P \preceq E^i_{\aleph_0}Q. \tag{9.42}$$

Note that, as E^i are central, $E^i \leq C_Q$ and $C_P = I$, then $C_{E^iQ} = E^iC_Q = E^i = E^iC_P = C_{E^iP}$ and thus $E^iP \leq C_{E^iP} = C_{E^iQ}$. For $i = 1$ we have that PE^{I} is a steering projection in some von Neumann algebra of type I, in particular it is quasi-abelian, so Lemma 8.5 ensures us that $E^{\mathrm{I}}P \preceq E^{\mathrm{I}}Q$, giving the first part of (9.42).

For $i = \mathrm{III}$ we have that E^{III} is a steering projection in some von Neumann algebra of type III and again, from Lemma 8.5 we get $E^{\mathrm{III}}P \preceq E^{\mathrm{III}}Q$. But $E^{\mathrm{III}}P$ is properly infinite, hence $E^{\mathrm{III}}P \sim \omega\odot E^{\mathrm{III}}P$ and thus $\omega\odot E^{\mathrm{III}}P \preceq E^{\mathrm{III}}Q$.

Finally, for $i = \mathrm{II}$ recall that $E^{\mathrm{II}} \leq Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ thus $E^{\mathrm{II}}P$ is a steering projection in some von Neumann algebra of type II_∞. Recall also that $E^{\mathrm{II}}Q$ is properly infinite. This again allows us to conclude that $E^{\mathrm{II}}P \preceq E^{\mathrm{II}}Q$, this time from Proposition 8.17 (and the definition of a steering projection in a type II_∞ von Neumann algebra). Moreover, $E^{\mathrm{II}}P \sim \omega\odot E^{\mathrm{II}}P$ which gives the remaining part of (9.42).

Uniqueness: Suppose that we have two such systems $\{Z^i_\alpha\}$ and $\{W^i_\alpha\}$ (with α and i varying as before). Our task is to show

$$Z^i_\alpha = W^i_\alpha \tag{9.43}$$

For the proof of (9.43) with $\alpha = 0$ we will show that

$$Z^i_0 = (I - C_Q)Z^i. \tag{9.44}$$

Once (9.44) is proved, it gives (9.43) with $\alpha = 0$ since (9.44) gives an explicit formula for Z^i_0 and the roles of Z^i_α and W^i_α are symmetric.

We have that

$$Z^i_0Q \sim 0\odot Z^i_0P = 0$$

and therefore $Z^i_0Q = 0$, hence[7] $Z^i_0 \leq I - C_Q$. Since obviously $Z^i_0 \leq Z^i$, we have

$$Z^i_0 \leq (I - C_Q)Z^i.$$

We will now show the converse inequality.

For $(i,\alpha) \neq (\mathrm{II}, 1)$ we have $Z^i_\alpha Q \sim \alpha\odot Z^i_\alpha P$ and if $W \leq Z^i_\alpha$ is central then

$$WQ = W(Z^i_\alpha Q) \sim W(\alpha\odot Z^i_\alpha P) = \alpha\odot WP$$

and this means that $WQ = 0$ implies $WP = 0$, and as $C_P = I$, it also implies $W = 0$. In particular, for $W := (I - C_Q)Z^i_\alpha$ (being central), $WQ = 0$ holds, thus $W = 0$ so $Z^i_\alpha = Z^i_\alpha C_Q$ and obviously $Z^i_\alpha \leq Z^i$ which together yields $Z^i_\alpha \leq C_QZ^i$.

For $(i,\alpha) = (\mathrm{II}, 1)$ the projection Z^{II}_1 has the property that for every central projection $0 \neq W \leq Z^{\mathrm{II}}_1$, $WQ \neq 0$. Thus for $W := (I - C_Q)Z^{\mathrm{II}}_1$ we obtain as

[6] Here $\alpha = 1$ for $i = \mathrm{I}$ and $\alpha = \aleph_0$ for $i = \mathrm{II}, \mathrm{III}$.

[7] From the definition of the central carrier, or from the fact that $C_{Z^i_0Q} = Z^i_0C_Q = 0$.

before $W = 0$ and finally $Z_1^{\mathrm{II}} \le C_Q Z^{\mathrm{II}}$. Thus we have proved that $Z_\alpha^i \le C_Q Z^i$ for $i = \mathrm{I}, \mathrm{III}, \mathrm{III}$ and $\alpha \ne 0$. Summing up over all $\alpha \ne 0$ yields:

$$\sum_{\alpha > 0} Z_\alpha^i = Z^i - Z_0^i \le C_Q Z^i$$

which means that $Z^i(I - C_Q) \le Z_0^i$ and proves (9.44).

Now we will prove (9.43) for $(i, \alpha) = (\mathrm{II}, 1)$. For $\alpha \ge \aleph_0$ we have $Z_\alpha^{\mathrm{II}} \sim \alpha \odot Z_\alpha^{\mathrm{II}} P$ and the latter projection is properly infinite (or 0),[8] so $Z_\alpha^{\mathrm{II}} Q$ is properly infinite or 0 and the same is true for $\sum_{\alpha \ge \aleph_0} Z_\alpha^{\mathrm{II}} Q$. But the sum appearing in this exspression is equal to $Z^{\mathrm{II}} - Z_0^{\mathrm{II}} - Z_1^{\mathrm{II}}$—this means that $(Z^{\mathrm{II}} - Z_0^{\mathrm{II}} - Z_1^{\mathrm{II}})Q$ is properly infinite or 0. Since $W_1^{\mathrm{II}} Q$ is finite, then $W_1^{\mathrm{II}}(Z^{\mathrm{II}} - Z_0^{\mathrm{II}} - Z_1^{\mathrm{II}})Q$ is also finite. But at the same time, since W_1^{II} is central, we have that $W_1^{\mathrm{II}}(Z^{\mathrm{II}} - Z_0^{\mathrm{II}} - Z_1^{\mathrm{II}})Q$ is properly infinite or 0. Therefore this second possibility takes place:

$$W_1^{\mathrm{II}}(Z^{\mathrm{II}} - Z_0^{\mathrm{II}} - Z_1^{\mathrm{II}})Q = 0.$$

By taking the central carrier we get

$$W_1^{\mathrm{II}}(Z^{\mathrm{II}} - Z_0^{\mathrm{II}} - Z_1^{\mathrm{II}})C_Q = 0$$

but from the above discussion $W_1^{\mathrm{II}} \le C_Q Z^{\mathrm{II}} \le C^Q$, hence we get:

$$W_1^{\mathrm{II}}(Z^{\mathrm{II}} - Z_0^{\mathrm{II}} - Z_1^{\mathrm{II}}) = 0$$

therefore

$$\underbrace{W_1^{\mathrm{II}} Z^{\mathrm{II}}}_{=W_1^{\mathrm{II}}} - \underbrace{W_1^{\mathrm{II}}(I - C_Q)}_{=0} Z^{\mathrm{II}} - W_1^{\mathrm{II}} Z_1^{\mathrm{II}} = 0.$$

This means that $W_1^{\mathrm{II}} = W_1^{\mathrm{II}} Z_1^{\mathrm{II}} \le Z_1^{\mathrm{II}}$, and by repeating this argument (the roles of Z_1^{II} and W_1^{II} are symmetric) we obtain $W_1^{\mathrm{II}} = Z_1^{\mathrm{II}}$.
Now we deal with the case $\alpha \ne 0$ and $(i, \alpha) \ne (\mathrm{II}, 1)$. We claim that

$$Z_\alpha^i W_\beta^i = 0 \text{ whenever } \beta \in \Lambda_i, \beta \ne \alpha. \tag{9.45}$$

For $\beta = 0$ or $(i, \beta) = (\mathrm{II}, 1)$ it has been already proved, as in these cases $Z_\beta^i = W_\beta^i$ and Z_α^i's are all mutually orthogonal. For $\beta > 0$ (in case $i = \mathrm{I}$) and $\beta \in \mathrm{Card}_\infty$ (in case $i \ne \mathrm{I}$) as $Z_\alpha^i Q \sim \alpha \odot Z_\alpha^i P$ we have

$$W_\beta^i Z_\alpha^i Q \sim \alpha \odot W_\beta^i Z_\alpha^i P.$$

But at the same time, from $W_\beta^i Q \sim \beta \odot W_\beta^i P$ we get

$$W_\beta^i Z_\alpha^i Q \sim \beta \odot W_\beta^i Z_\alpha^i P.$$

This together yields

$$\alpha \odot W_\beta^i Z_\alpha^i P \sim \beta \odot W_\beta^i Z_\alpha^i P.$$

Now $W_\beta^i Z_\alpha^i \ne 0$ implies $W_\beta^i Z_\alpha^i P \ne 0$ (recall that $C_P = I$) and then $W_\beta^i Z_\alpha^i P$ is a steering projection in some von Neumann algebra of a specific type. By virtue of Proposition 9.1 we obtain $\alpha = \beta$, thus proving (9.45). From (9.45) we have

$$Z_\alpha^i \sum_{\beta \in \Lambda_i \setminus \{\alpha\}} W_\alpha^i = Z_\alpha^i (Z^i - W_\alpha^i) = Z_\alpha^i - Z_\alpha^i W_\alpha^i = 0$$

and thus $Z_\alpha^i \le W_\alpha^i$. Changing the role of Z_α^i and W_α^i we get (9.43). The whole proof is complete. □

9.6. **Corollary.** *Let $Q, Q' \in \mathfrak{A}$ be two projections. Then the following conditions are equivalent:*

a) $Q \preceq Q'$,
b) $Z_\alpha^i(Q) Z_\beta^i(Q') = 0$ *whenever* $\alpha > \beta$; *and* $Z_1^{\mathrm{II}}(Q) Z_1^{\mathrm{II}}(Q') Q \preceq Z_1^{\mathrm{II}}(Q) Z_1^{\mathrm{II}}(Q') Q'$.

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[8] Later we will not always underline this fact, althought it should be kept in mind.

Proof. Step 1: First we show the following criterion for the Murray-von Neumann equivalence: for two projections $Q, Q' \in \mathfrak{A}$ we have

$$Q \sim Q' \iff Z^i_\alpha(Q) = Z^i_\alpha(Q'),\ i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\},\ \alpha \in \Lambda_i \text{ and } Z^{\mathrm{II}}_1(Q)Q \sim Z^{\mathrm{II}}_1(Q')Q'.$$

For the proof assume first that $Q \sim Q'$. Then $Z^i_\alpha(Q')Q \sim Z^i_\alpha(Q')Q' \sim \alpha \odot Z^i_\alpha(Q')P$ for $(i, \alpha) \neq (\mathrm{II}, 1)$. For $(i, \alpha) = (\mathrm{II}, 1)$ we have $Z^{\mathrm{II}}_1(Q')Q \sim Z^{\mathrm{II}}_1(Q')Q'$; since the latter projection is finite, so is $Z^{\mathrm{II}}_1(Q')Q$. Further, $Q \sim Q'$ implies $C_Q = C_{Q'}$ and thus if $0 \neq W \leq Z^{\mathrm{II}}_1(Q')C_Q$ is a central projection, then $0 \neq W \leq Z^{\mathrm{II}}_1(Q')C_{Q'}$ and thus $WQ \neq 0$. Finally, for the last property observe that

$$(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q')P \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q')Q' \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z^{\mathrm{II}}_1(Q')Q.$$

So, we have proved that the system $\{Z^i_\alpha\}_{i,\alpha}(Q')$ have all the desired properties (for Q)— from its uniqueness we conclude that $Z^i_\alpha(Q) = Z^i_\alpha(Q')$ for all $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and $\alpha \in \Lambda_i$—then also $Z^{\mathrm{II}}_1(Q)Q \sim Z^{\mathrm{II}}_1(Q')Q'$.

For the proof of the converse implication, denote for simplicity $Z^i_\alpha := Z^i_\alpha(Q) = Z^i_\alpha(Q')$. We have then in particular $Z^{\mathrm{II}}_1 Q \sim Z^{\mathrm{II}}_1 Q'$ and for $(i, \alpha) \neq (\mathrm{II}, 1)$:

$$Z^i_\alpha Q \sim \alpha \odot Z^i_\alpha P \sim Z^i_\alpha Q'.$$

Since $\sum_{i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}} \sum_{\alpha \in \Lambda_i} Z^i_\alpha = I$ then

$$Q = \sum_{i,\alpha} Z^i_\alpha Q \sim \sum_{i,\alpha} Z^i_\alpha Q' = Q'.$$

Step 2: Now we shall check that (b) is implied by (a). The second part of our statement is immediate (see Lemma 3.11). First let $(i, \alpha), (i, \beta) \neq (\mathrm{II}, 1)$. Then, as $Z^i_\alpha(Q)Q \sim \alpha \odot Z^i_\alpha(Q)P$ and $Z^i_\beta(Q')Q' \sim \beta \odot Z^i_\beta(Q')P$ and $Q \preceq Q'$, we have:

$$\alpha \odot Z^i_\alpha(Q)Z^i_\beta(Q')P \sim Z^i_\alpha(Q)Z^i_\beta(Q')Q \preceq Z^i_\alpha(Q)Z^i_\beta(Q')Q',$$

but, as $Z^i_\beta(Q')Q' \sim \beta \odot Z^i_\beta(Q')P$, we have

$$Z^i_\alpha(Q)Z^i_\beta(Q')Q' \sim \beta \odot Z^i_\alpha(Q)Z^i_\beta(Q')P.$$

Therefore we have (if we denote, to simplify, $Z := Z^i_\alpha(Q)Z^i_\beta(Q')$) that $\alpha \odot ZP \preceq \beta \odot ZP$. Now if we assume $\beta < \alpha$, then the converse inequality is also true, namely $\beta \odot ZP \preceq \alpha \odot ZP$, hence $\alpha \odot ZP \sim \beta \odot ZP$. Now suppose, for the contrary, that $Z \neq 0$. Then $ZP \neq 0$ too and ZP is then a steering projection in some von Neumann algebra, and then from Lemma 9.1 we conclude $\alpha = \beta$ which is a contradiction.

Further, we see that $Z^{\mathrm{II}}_1(Q)Z^{\mathrm{II}}_0(Q')Q \preceq Z^{\mathrm{II}}_1(Q)Z^{\mathrm{II}}_0(Q')Q' = 0$ and hence, by point (c) of Theorem 9.5 for Q, $Z^{\mathrm{II}}_1(Q)Z^{\mathrm{II}}_0(Q') = 0$. Finally, if $\alpha \in \mathrm{Card}_\infty$, then $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q')Q \preceq Z^{\mathrm{II}}_1(Q')Q'$, which implies that $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q')Q$ is both properly infinite (since $Z^{\mathrm{II}}_\alpha(Q)Q$ is such) and finite (since $Z^{\mathrm{II}}_1(Q')Q'$ is finite). We infer that $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q')Q = 0$ and consequently $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q') = 0$, because $Z^{\mathrm{II}}_\alpha(Q) \leq C_Q$. This finishes the proof of (b).

Step 3: For the proof that (a) follows from (b), note that, since

$$\sum_i \sum_{\alpha,\beta \in \Lambda_i} Z^i_\alpha(Q)Z^i_\beta(Q') = I,$$

so it is enough to show that

$$Z^i_\alpha(Q)Z^i_\beta(Q')Q \preceq Z^i_\alpha(Q)Z^i_\beta(Q')Q',\ \alpha, \beta \in \Lambda_i \tag{9.46}$$

Fix $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and $\alpha, \beta \in \Lambda_i$ and consider the cases:

(1°) $\alpha > \beta$; then from our assumptions $Z^i_\alpha(Q)Z^i_\beta(Q') = 0$ and (9.46) is satisfied.

(2°) $(i, \alpha) = (i, \beta) = (\mathrm{II}, 1)$; then (9.46) is exactly the second part of our assumption.

(3°) $\alpha \leq \beta$ and $(i, \beta) = (\mathrm{II}, 1), (i, \alpha) \neq (\mathrm{II}, 1)$. Then, in particular, $i = \mathrm{II}$ and the only possibility is that $\alpha = 0$. In this case $Z^i_\alpha(Q)Q \sim 0 \odot Z^i_\alpha(Q)P = 0$, so $Z^i_\alpha(Q)Z^i_\beta(Q')Q = 0$ and (9.46) is satisfied.

(4°) Both (i, α) and (i, β) are different from $(\mathrm{II}, 1)$ and $\alpha \leq \beta$. Then, as before, we conclude from the relations $Z^i_\alpha(Q)Q \sim \alpha \odot Z^i_\alpha(Q)P$ and $Z^i_\beta(Q')Q' \sim \beta \odot Z^i_\beta(Q')P$ that

$$Z^i_\alpha(Q)Z^i_\beta(Q')Q \sim \alpha \odot Z^i_\alpha(Q)Z^i_\beta(Q')P,$$
$$Z^i_\alpha(Q)Z^i_\beta(Q')Q' \sim \beta \odot Z^i_\alpha(Q)Z^i_\beta(Q')P.$$

Since $\alpha \leq \beta$ then:

$$\alpha \odot Z^i_\alpha(Q)Z^i_\beta(Q')P \preceq \beta \odot Z^i_\alpha(Q)Z^i_\beta(Q')P$$

and (9.46) is satisfied.

(5°) The remaining case is $\alpha \leq \beta, (i, \alpha) = (\mathrm{II}, 1), (i, \beta) \neq (\mathrm{II}, 1)$. In this case $i = \mathrm{II}$ and hence $\beta > \alpha$ means that $\beta \geq \aleph_0$. As we have $Z^{\mathrm{II}}_\beta(Q')Q' \sim \beta \odot Z^{\mathrm{II}}_\beta(Q')P$ with infinite β, we conclude that $Z^{\mathrm{II}}_\beta(Q')Q'$ is properly infinite (or 0). But Z^{II_1} is central and finite, thus $Z^{\mathrm{II}_1}Z^{\mathrm{II}}_\beta(Q')Q' = 0$. Taking the central carrier gives

$$Z^{\mathrm{II}_1}Z^{\mathrm{II}}_\beta(Q')C_{Q'} = 0. \tag{9.47}$$

But recall[9] that $Z^{\mathrm{II}}_\beta(Q') \leq C_{Q'}$. Thus (9.47) transforms into $Z^{\mathrm{II}_1}Z^{\mathrm{II}}_\beta(Q') = 0$. We conclude that $Z^{\mathrm{II}}_\beta(Q') \leq Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ and then (by point (c) of Theorem 9.5 for Q) $Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)P \sim \omega \odot Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)Q$. So:

$$Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)Q \preceq Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)P \preceq \beta \odot Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)P \sim Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)Q'$$

and we are done.

□

9.7. *Remark.* Recently Sherman [9] proved (using Lemma 8.13 and a variation of Theorem 9.5) that the Murray-von Neumann order in an arbitrary W^*-algebra is complete. That is, every set of projections has the g.l.b. as well as the l.u.b. with respect to '$\preceq$'.

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[9]It follows from the proof of the previous theorem: $Z^{\mathrm{II}}_\beta(Q') \leq E^{\mathrm{II}} \leq (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})C_{Q'} \leq C_{Q'}$.