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Spectra of noncommutative C^*-algebras

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SPECTRA OF NON-COMMUTATIVE C^*-ALGEBRAS

ADAM WEGERT

1. Introduction

The aim of this work is to describe the possibility of generalizing the notion of spectrum of commutative C^*-algebras to the non-commutative case and discuss some its properties. Every commutative C^*-algebra is, roughly speaking, the same as locally compact Hausdorff space (via identifying it with the space of continuous functions). This is no longer true for non-commutative algebras, however, there is still a 'dual' object corresponding to those algebras. Moreover, this object may be regarded in many different ways which are all equivalent for commutative algebras. We shall discuss why it is necessary to search for some substitutes of the classical spectrum. We indicate the similarity between this topic and the theory of representations for locally compact topological groups. We also discuss probably the broadest, reasonable class of C^*-algebras for which the elementary analysis of the spectrum can be carried out and we point out that there are some classical examples for which the effective analysis of the spectrum seems to be impossible. Finally, we gather all necessary, measure-theoretic facts and notions in the appendix.

2. Commutative case

Although this paper is mainly devoted to the investigation of non-commutative algebras, we first briefly discuss some aspects of commutative C^*-algebras. The well known Gelfand-Naimark theorem states that if $\mathfrak{A}$ is a commutative C^*-algebra, then A is *-isomorphic to $C_0(\Lambda)$ for some locally compact Hausdorff space Λ. Moreover, if $\mathfrak{A}$ has a unit, then the space Λ turns out to be compact. We can identify the space Λ with the spectrum of $\mathfrak{A}$, i.e. the set of all non-zero, linear, multiplicative functionals (called *linear characters*) equipped with the *-weak topology (the topology of pointwise convergence). We denote this space by $\hat{\mathfrak{A}}$ or $\Lambda(\mathfrak{A})$. Every such functional ω gives rise to the maximal, two-sided ideal $I_\omega = \ker \omega$ and vice versa: every such ideal produces an element in Λ. The correspondence between commutative C^*-algebras and locally compact Hausdorff spaces is *functorial* meaning that the category of locally compact Hausdorff spaces with continuous and *proper* maps is equivalent to the opposite category of the C^*-algebras with morphisms preserving approximate units. Similarly, the category of compact Hausdorff spaces with continuous maps is equivalent to the opposite category of unital C^*-algebras with unital morphisms. For details see [2]. Although it may seem that the description of spectra in the commutative case is rather simple, there are plenty of examples of commutative C^*-algebras for which the spectrum is rather exotic:

1. Let X be an arbitrary set. We define $\ell^\infty(X)$ to be the space of all bounded functions $f\colon X \to \mathbb{C}$. Regarding X as a discrete topological space we see that every $x \in \ell^\infty(X)$ extends uniquely to a continuous map defined on the Stone-Čech compactification βX of X, thus $\Lambda(\ell^\infty(X)) = \beta X$. In particular, for $\ell^\infty := \ell^\infty(\mathbb{N})$ we have $\Lambda(\ell^\infty) = \beta\mathbb{N}$. Each non-zero, linear, multiplicative functional on ℓ^∞ may be identified with the limit with respect to an ultrafilter. We point out that those limits are not invariant with respect to the shift operation (the property shared by Banach

limit): if $(x_n)_{n\in\mathbb{N}} \in \ell^\infty$ and $\omega \in \beta\mathbb{N}$, then it may happen that $\lim_\omega x_n \neq \lim_\omega x_{n+1}$. On the other hand, Banach limits are not multiplicative; to see this, consider two sequences $x := (0, 1, 0, 1, ...)$ and $y := (1, 0, 1, 0, ...)$. Then, for any Banach limit LIM we have

$$1 = \mathrm{LIM}(x + y) = \mathrm{LIM}(x) + \mathrm{LIM}(y) = 2\mathrm{LIM}(x),$$

so we get $\mathrm{LIM}(x) = \frac{1}{2}$. However, $xy = 0$ and $\mathrm{LIM}(xy) = 0 \neq \mathrm{LIM}(x) \cdot \mathrm{LIM}(y) = \frac{1}{4}$. Notice that for every simple function (that is, a sequence taking only finitely many values) any limit corresponding to an ultrafilter must be equal to one of the values of the underlying function, which is plainly not true for Banach limits.

The space βX, for discrete X, is of huge cardinality, $|\beta X| = 2^{2^{|X|}}$. It may be identified with the space of all ultrafilters on X. Every point x of X corresponds to a fixed ultrafilter which is generated by the singleton $\{x\}$, while every point of the remainder $\beta X \setminus X$ is identified with a free ultrafilter—the existence of such is guaranteed by the Tarski theorem which relies on the axiom of choice.

2. Let (X, Σ) be a measurable space and let $\mathfrak{A} = \mathcal{L}^\infty(X, \Sigma)$ stand for the space of all measurable bounded functions equipped with the supremum norm. One can prove that $\Lambda(\mathfrak{A})$ coincides with the Stone space of the Boolean algebra (X, Σ) (see for instance [18]). Note that each $x \in X$ gives rise to an element of $\Lambda(\mathfrak{A})$ via the formula $\delta_x(f) := f(x)$; however, not every $\omega \in \Lambda(\mathfrak{A})$ is of this form. It is also worth mentioning that for the space $X = [0, 1]$ considered with the σ-algebra $\mathcal{B}$ of all Borel subsets of X and, alternatively, with the σ-algebra $\mathcal{L}$ of all Lebesgue measurable subsets of X we obtain two non-isomorphic C^*-algebras, as their spectra do not coincide. Moreover, neither $([0, 1], \mathcal{B})$ nor $([0, 1], \mathcal{L})$ are complete (as Boolean algebras), hence their Stone spaces are not extremally disconnected and therefore $\mathcal{L}^\infty([0, 1], \mathcal{B})$ and $\mathcal{L}^\infty([0, 1], \mathcal{L})$ are not W^*-algebras (recall that the spectrum of any W^*-algebra is always extremally disconnected).

3. Now, for a measure space (X, Σ, μ) let $\mathfrak{A} = L^\infty(X, \Sigma, \mu)$ be the space of all equivalence classes of measurable, essentially bounded functions with the essential supremum norm. The spectrum of $\mathfrak{A}$ is homeomorphic to the Stone space of the so-called *measure algebra* of (X, Σ, μ) being the collection of all equivalence classes of measurable sets where the equivalence is defined by saying that $E \sim F$ iff $\mu(E \triangle F) = 0$. Even in the simple case, where $X = [0, 1]$ and μ is the Lebesgue measure, it is difficult to write an explicit formula for any $\omega \in \Lambda(\mathfrak{A})$. If we assume that the underlying measure is complete, then with the aid of the so-called *lifting theorem* (see [18]) we may get a partial access to the form of the spectrum of $\mathfrak{A}$. If μ is a σ-finite measure, then $\mathfrak{A}$ is a W^*-algebra and its predual is $L^1(X, \Sigma, \mu)$. Unlike in the previous example, both for $\Sigma = \mathcal{B}$ and $\Sigma = \mathcal{L}$ we obtain *the same* C^*-algebras.

In all the above examples (with the exception of some trivial choices of X) the constructed C^*-algebra is not separable, hence its spectrum is a nonmetrizable compact space. The existence of plenty of projections in each of those C^*-algebras corresponds to the fact that the spectrum is not a connected topological space. Many other algebraical properties may be translated into appropriate topological properties of the spectrum and vice versa. Therefore, the study of non-commutative C^*-algebras leads to an extension of standard topology—this is the reason for referring to the theory of C^*-algebras as to *non-commutative topology*. Our aim is to describe some properties of spectra of non-commutative C^*-algebras which can be regarded as *non-commutative spaces*.

If $\mathfrak{A}$ is a commutative C^*-algebra, then the existence of maximal ideals in $\mathfrak{A}$ may be proved by using the commutativity of $\mathfrak{A}$. It is by no means obvious, whether for non-commutative C^*-algebra such ideals exist, hence the existence of non-zero, linear, multiplicative functional is also not trivial.

2.1. **Examples.** 1. Let $\mathfrak{A} = M_n(\mathbb{C})$ be the C^*-algebra of $n \times n$ complex matrices. Then $\mathfrak{A}$ is *simple* i.e. it contains no non-trivial two-sided ideals which implies that there are no linear non-zero characters on $\mathfrak{A}$, that is, $\Lambda(\mathfrak{A}) = \emptyset$. Note that still there exists a non-trivial multiplicative (but non-linear) map, namely the determinant.

2. Let $\mathfrak{A} = B(\ell^2)$ be the C^*-algebra of all bounded linear operators acting on the separable Hilbert space ℓ^2. We define three operators $T_1, T_2, T_3 \in \mathfrak{A}$ by the formulas:

$$T_1(x_1, x_2, x_3 ...) = (x_2, x_3, x_4, ...),$$
$$T_2(x_1, x_2, ...) = (0, x_1, 0, x_2, 0, ...),$$
$$T_3(x_1, x_2, x_3 ...) = (x_1, x_3, x_5, ...).$$

Notice that $T_3T_1T_2 = I$ and $T_1T_3T_2 = 0$. Therefore, if ω is multiplicative, then $\omega(I) = \omega(0)$, thus ω is constant: $\omega(T) \equiv 1$ or $\omega(T) \equiv 0$. Hence, there is no non-zero linear and multiplicative functional on $\mathfrak{A}$. Obviously, one can replace ℓ^2 by an arbitrary infinite-dimensional Hilbert space. Note also that the above argument does not work with only two operators; when $T_1T_2 = 0$ and $T_2T_1 = I$ we have

$$T_2 = IT_2 = T_2T_1T_2 = T_20 = 0,$$

thus $T_2T_1 = 0$ which gives a contradiction. Moreover, in every unital normed algebra $\mathfrak{A}$ it is impossible to find $x, y \in \mathfrak{A}$ such that $xy - yx = e$ (see Chapter 13 in [27]). However, one can construct two *unbounded* operators A, B satisfying this relation (Example 13.5. in [27]). The possibility of such a construction is related to the Heisenberg uncertainty principle.

3. Consider the shift operator $S\colon \ell^2 \to \ell^2$ defined by the formula

$$S(x_1, x_2, ...) := (0, x_1, x_2, ...).$$

Then $S^*(x_1, x_2, x_3 ...) = (x_2, x_3, x_4, ...)$. Hence, $S^*S = I$ (thus, S is an isometry) but $SS^*(x_1, x_2, x_3, ...) = (0, x_2, x_3, ...)$; in particular S is not a normal operator, so the C^*-algebra $\mathfrak{A}$ generated by S is non-commutative. We claim that $\mathfrak{A}$ contains all compact operators. In fact, $P_{1,1} := S^*S - SS^* \in \mathfrak{A}$ satisfies $P_{1,1}(x_1, x_2, x_3 ...) = (x_1, 0, 0, ...)$ and $P_{1,k} := S^{k-1}P_{1,1} \in \mathfrak{A}$ acts as $P_{1,k}(x_1, x_2, ...) = (0, ..., 0, x_1, 0, ...)$ where x_1 occurs at the k-th place. Proceeding similarly, we infer that $P_{n,k} := P_{1,k}(S^*)^{n-1} \in \mathfrak{A}$ satisfies $P_{n,k}(x_1, x_2, ...) = (0, ..., 0, x_n, 0, ...)$, i.e. $P_{n,k}(x) = \langle x, e_n \rangle e_k$ where $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of ℓ^2. Therefore, every rank-one operator of the form $T(x) = \langle x, y \rangle z$ belongs to $\mathfrak{A}$, because every such operator is a norm limit of linear combinations of $P_{n,k}$'s. Consequently, every finite-rank operator also belongs to $\mathfrak{A}$, hence $\mathfrak{A}$ contains all compact operators.

Now, suppose $\omega\colon \mathfrak{A} \to \mathbb{C}$ is linear and multiplicative. Then ω is automatically continuous and preserves involution, that is, $\omega(T^*) = \overline{\omega(T)}$. Therefore,

$$\omega(P_{1,1}) = \omega(S^*S - SS^*) = |\omega(S)|^2 - |\omega(S)|^2 = 0$$

and hence

$$\omega(P_{n,k}) = (\omega(S))^{k-1}\omega(P_1, 1)\bar{(}\omega(S))^{n-1} = 0.$$

Consequently, ω vanishes on the C^*-algebra $K(H)$ of all compact operators. It is known that $\mathfrak{A}/K(H) \simeq C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle (see [22]). If f is the corresponding $*$-isomorphism, then all linear, multiplicative functionals are precisely of the form $\varphi(T) = \delta_t(f(\pi(T)))$, where $t \in \mathbb{T}$ and π is the canonical projection onto $\mathfrak{A}/K(H)$

4. Despite of the lack of linear, multiplicative funcionals on $B(H)$ (H being a separable Hilbert space), still there exist a maximal ideal in $B(H)$, namely, the

ideal of compact operators $K(H)$. In fact, one can prove that if $T \in B(H)$ is not compact, then $ATB = I$ for some $A, B \in B(H)$. So, the space of all maximal ideals is nonempty, whereas the 'classical' spectrum (understood as the set of all characters) is! It is also worth mentioning what follows:

(1) Maximal ideals need not to be prime. In fact, one can find two noncompact operators A, B such that $AB = 0$—just take $A\colon \ell^2 \to \ell^2$ to be the projection onto $\mathrm{lin}\{e_{2i}\colon i \in \mathbb{N}\}$ and let $B = I - A$.
(2) The theorem stating that $\mathfrak{I}$ is a maximal ideal iff $\mathfrak{A}/\mathfrak{I}$ is a field is no longer valid in the non-commutative case. However, every field contains no nontrivial ideals and this is still true for $\mathfrak{A}/\mathfrak{I}$ when $\mathfrak{I}$ is maximal. C^*-algebras without nontrivial ideals are called *simple*.

2.2. *Remarks.* 1. In fact, more is true: any non-zero ideal in $B(H)$ contains all finite-rank operators. Jointly with the above statement, and the fact that compact operators are precisely norm limits of finite rank operators, this implies that $K(H)$ is the only two-sided norm closed ideal. However, there are plenty of other ideals between the ideal of finite rank operators and the ideal of compact operators, the so-called *Schatten classes* (see [21], Chapter 11).

2. The quotient $B(H)/K(H)$ is a very interesting C^*-algebra, the so-called *Calkin algebra*. The theory of Calkin algebra is related to the concept of *index* of an operator. The Calkin algebra may be considered as a non-commutative analogue of the corona of the Stone-Čech compactification, $\beta X \setminus X$.

3. Translating ordinary topological statements into the language of commutative C^*-algebras we may state that being a W^*-algebra implies that the underlying space is zero dimensional, i.e. has many clopen sets. On the other hand, any closed set $Y \subset X$ determines a two-sided ideal of $C(X)$ and, conversely, each (norm closed, two-sided) ideal produces a closed subset Y of X. Thus, we may naivly think that there ought to be many closed ideals in $B(H)$, which we know is not the case. So, in the case of non-commutative C^*-algebras we observe a phenomenon which does not appear in the commutative cases.

We have seen so far that the classical spectrum is inadequate when dealing with non-commutative C^*-algebras. So, we shall look for some substitutes of linear characters. One possible candidate is the notion of *state*. Recall that by a state (on a given C^*-algebra) we mean any linear, positive funcional with norm one; the space of all states on $\mathfrak{A}$ will be denoted by $S(\mathfrak{A})$. The state is called *pure* if it is an extremal point of $S(\mathfrak{A})$. The space of pure states will be denoted by $P(\mathfrak{A})$. Some equivalent definitions follow from the following simple facts:

2.3. **Theorem.** *Let $\mathfrak{A}$ be a C^*-algebra with a unit e and let $\varphi \in \mathfrak{A}^*$. Then:*

(1) If φ is positive, then φ is continous and $\|\varphi\| = \varphi(e)$. In particular, a positive functional φ is a state iff $\varphi(e) = 1$.
(2) If $\varphi(e) = 1$ and $\|\varphi\| = 1$, then φ is positive.

In the case where $\mathfrak{A}$ is non-unital we shall replace the condition $\varphi(e) = 1$ by $\lim_\alpha \varphi(e_\alpha) = \|\varphi\|$, where $(e_\alpha)_\alpha$ is an approximate unit of $\mathfrak{A}$ (every C^*-algebra admits such an approximate unit and it is countable if $\mathfrak{A}$ is separable; see [8]).

Note that each C^*-algebra admits at least one pure state (see, e.g., [2]). Hence, $P(\mathfrak{A})$ is always non-empty, unlike the classical spectrum defined as in the commutative case. The space $S(\mathfrak{A})$ of all states turns out to be a compact Hausdorff space (provided that $\mathfrak{A}$ is unital), but this is no longer true for the space of *pure* states which may even fail to be locally compact! Moreover, the space of all pure states is not a sufficent tool to distinguish between C^*-algebras. For a simple example, just consider any non-commutative C^*-algebra $\mathfrak{A}$ and the commutative algebra $\mathfrak{B} = C(P(\mathfrak{A}))$. Then $P(\mathfrak{B}) \simeq P(\mathfrak{A})$ but obviously $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic.

2.4. *Remarks.* 1. From the above discussion one observes that the set of extremal points of a compact set may fail to be compact. However, one do not need the very abstract space as $P(\mathfrak{A})$—this phenomenon can happen even in the finite-dimensional case; see [23].

2. For non-unital C^*-algebras the lack of compactness of $P(\mathfrak{A})$ is easy to obtain: just set $\mathfrak{A} = C_0(\mathbb{R})$; then $P(\mathfrak{A}) \simeq \mathbb{R}$ is locally compact, yet not compact.

The next example shows why the space of pure states may be regarded as a generalization of the spectrum:

2.5. **Examples.** 1. Let $\mathfrak{A} = C(X)$ be the space of complex valued functions on a compact Hausdorff space X. Then, by the Riesz representation theorem, the dual space $\mathfrak{A}^*$ can be identified with the space of complex valued Radon measures on X while $S(\mathfrak{A})$—with the space of all probabilistic measures. Finally, if the support of such a measure has at least two points, then this measure can be expressed as a convex combination of two probabilistic measures and thus, it does not correspond to any pure state. Therefore, the space of pure states consists exclusively of Dirac measures, $P(\mathfrak{A}) = \{\delta_x : x \in X\}$. All Dirac's measures can be viewed as linear characters via the formula $\delta_x(f) =: f(x)$, and so in the commutative case we have $P(\mathfrak{A}) = \Lambda(\mathfrak{A})$.

2. Now, let $\mathfrak{A} = B(H)$ for a separable Hilbert space H. Fix a unit vector $\xi \in H$ and consider the map $\varphi_\xi : \mathfrak{A} \ni A \mapsto \langle A\xi, \xi\rangle$. One easily verify that this formula defines a state; any state of this form is called a *vector state.* Moreover, vector states are always pure states, which is a special case of Theorem I.9.8 in [12], but not every pure state is a vector state, as we shall explain below. Note also that if $\xi_n \to \xi$ where ξ_n, ξ are unit vectors, then $\varphi_{\xi_n} \to \varphi_\xi$, so we see that the space $P(\mathfrak{A})$ contains a huge connected component. In particular, it is not zero dimensional, as it is the case for commutative von Neumann algebras.

3. Let us identify the commutative C^*-algebra $\mathfrak{A}_0 = \ell^\infty$ with the algebra of diagonal operators on a separable Hilbert space H. The space of all pure states of $\mathfrak{A}_0$ is homeomorphic to $\beta\mathbb{N}$, hence $|P(\mathfrak{A}_0)| = 2^{\mathfrak{c}}$. Every pure state of $\mathfrak{A}_0$ extends to the pure state on the whole of $B(H)$ (this is true in much general context; see [8], Lemma 4.1.7) and thus $|P\big(B(H)\big)| = 2^{\mathfrak{c}}$. It should be also noted that the problem of uniqueness of such extensions is related to the so called Kadison–Singer hypothesis: if we replace $\mathfrak{A}_0$ by any closed C^*-algebra, then the answer is negative (there are more then one possible extension). For $\mathfrak{A}_0$, as defined above, the answer is still unknown. This problem is strongly related to some parts of discrete mathematics. For further information, the reader is referred to [10, 9, 11].

States are closely related to *representations* of C^*-algebras. Namely, with each state one can associate the so-called GNS representation (see [5, 8]). Let us note some basic facts about representations which connect algebraical and topological structures of C^*-algebras.

2.6. **Theorem.** *Let $\pi\colon \mathfrak{A} \to B(H)$ be a representation. Then:*

(1) π is continuous and $\|\pi\| \le 1$,
(2) if π is faithful then π is isometric,
(3) the image $\pi(\mathfrak{A})$ is norm closed in $B(H)$—in particular, it is a C^*-subalgebra.

The large supply of pure states on a given C^*-algebra $\mathfrak{A}$ allows us to construct a faithful (i.e. injective) representation of $\mathfrak{A}$, that is, to embed $\mathfrak{A}$ as a closed *-subalgebra into $B(H)$. This representation in constructed as the direct sum[1] $\pi_u := \bigoplus_{\varphi \in P(\mathfrak{A})} \pi_\varphi$ on the Hilbert space $H_u := \bigoplus_{\varphi \in P(\mathfrak{A})} H_\varphi$. This representation is called a *universal* representation. Now, we sketch some instructive examples:

[1]Some authors consider the drect sum over all states, not necessarily pure

2.7. **Examples.** 1. Let $\mathfrak{A} = C(X)$ where X is a compact Hausdorff space. Given a state φ on $\mathfrak{A}$ we consider the Hilbert space H_φ that arises from the GNS construction. Then $H_\varphi \simeq L^2(X, \mu)$ where μ is the probabilistic measure corresponding to φ. In particular, for $\varphi = \delta_x$ we have $H_\varphi \simeq \mathbb{C}$, which is connected with the fact that pure states induce *irreducible* representations (defined below) and the fact that irreducible representations of commutative C^*-algebras are one-dimensional.

2. If $\mathfrak{A}$ is separable, then it is possible to consider only (at most) countable collection of states to obtain an isometric embedding into $B(H)$. Consequently, $\mathfrak{A}$ can be embedded into $B(H)$ with H being a separable Hilbert space. Probably the most important examples of separable C^*-algebras are $C(X)$-algebras with X being metrizable and compact, and $K(H)$—the algebra of compact operators on a separable Hilbert space H.

3. Let $\mathfrak{A} = B(H)$ where H is separable. Then, as we have seen before, $|P(\mathfrak{A})| = 2^\mathfrak{c}$ thus the Hilbert space dimension of H_u equals $2^\mathfrak{c}$ and $B(H_u)$ is much larger then $\mathfrak{A}$. What is interesting, $\mathfrak{A}$ is a von Neumann algebra, but the *-algebraic image $\pi_u(\mathfrak{A})$ in $B(H_u)$ is not closed in the weak operator topology! Moreover, it occurs that the weak operator topology closure $\overline{\pi(\mathfrak{A})}^w$ (or, equivalently, the double commutant $\pi(\mathfrak{A})''$; recall the double commutant theorem of von Neumann) in $B(H_u)$ is isomorphic to the second dual $\mathfrak{A}^{**}$ of $\mathfrak{A}$.

4. Sometimes it is possible to obtain a faithful representation from just one state; recall that the state φ is called *faithful* if $\varphi(x^*x) = 0$ implies $x = 0$. We claim that the GNS representation π_φ obtained from a faithful state is faithful. Indeed, we have

$$0 = \|\pi_\varphi(x)\|^2 = \|\pi_\varphi(x)^*\pi_\varphi(x)\| = \|\pi_\varphi(x^*x)\|$$

meaning that $\pi_\varphi(x^*x)$ is the zero operator. Let $N_\varphi = \{y \in \mathfrak{A} : \varphi(y^*y) = 0\}$ be the left ideal which occurs in the GNS construction[2]. Then, as $\pi_\varphi(x)(y+N_\varphi) = xy+N_\varphi$, we have

$$0 = \pi_\varphi(x^*x)(e + N_\varphi) = x^*x + N_\varphi.$$

Therefore, $x^*x \in N_\varphi$, that is, $\varphi((x^*x)^*x^*x) = 0$ and, since φ is faithful, we get $x^*x = 0$, so $\|x\|^2 = \|x^*x\| = 0$. We point out that the converse is *not* true: the GNS representation π_φ obtained from some state may be faithful, yet the state φ need not to be. One of the most important examples of faithful states is the *trace* (more precisely,*the finite tracial state*) which can be constructed in every finite von Neumann algebra R, for example in the so called II_1 factor (cf. the Chapter 6 in [4]). The Hilbert space obtained from the GNS construction is then usually denoted by $L^2(R)$.

5. When dealing with a W^*-algebra $\mathfrak{A}$, then in our construction of a faithful representation we can restrict ourselves to *normal* states; these are states coming from the predual of $\mathfrak{A}$. Then $\mathfrak{A}$ is *-isomorphic to a von Neumann algebra (see [5]). Hence, the W^*-algebras is an abstract version of a weakly closed *-subalgebra of $B(H)$, while C^*-algebras correspond to norm closed $*$-algebras of $B(H)$. For $\mathfrak{A} = L^\infty(X, \Sigma, \mu)$, where μ is σ-finite, the predual is $L^1(X, \Sigma, \mu)$; for $\mathfrak{A} = B(H)$ the predual is $T(H)$, the collection of all trace class operators (see [5]).

In the first of the above examples, we mentioned the so-called irreducible representation. To be precise, there are two concepts of irreducibility: algebraical and topological. Here we gather all necessary definitions:

2.8. **Definition.** Let $\pi \colon \mathfrak{A} \to B(H)$ be a representation. Then π is called:

(1) *topologically irreducible* if π does not have any non-trivial closed invariant subspaces; if π does not have any non-trivial invariant subspace (not necessarily closed), then it is called *algebraically irreducible*;

[2] This notation will be used hereinafter.

(2) *cyclic* if there is a $\xi \in H$ such that $\overline{\pi(\mathfrak{A})\xi} = H$, where $\pi(\mathfrak{A})\xi := \{\pi(x)\xi : x \in \mathfrak{A}\}$; we then call ξ a *cyclic* vector;
(3) *non-degenerate* if the only $\xi \in H$ such that $\pi(x)\xi = 0$ for every $x \in \mathfrak{A}$ is the zero vector.

2.9. **Examples.** (1) Every cyclic representation is non-degenerate. It follows from the fact that π is non-degenerate iff $\overline{\pi(\mathfrak{A})H} = H$. For, take any $\xi \perp \overline{\pi(\mathfrak{A})H}$ and notice that for all $x \in \mathfrak{A}, \eta \in H$ we have

$$0 = \langle \xi, \pi(x)\eta \rangle = \langle \pi(x)^*\xi, \eta \rangle = \langle \pi(x^*)\xi, \eta \rangle.$$

As η was arbitrary, we infer that $\pi(x^*)\xi = 0$ for all $x \in \mathfrak{A}$ and since $\mathfrak{A}$ is self adjoint, $\pi(x)\xi = 0$ for all $x \in \mathfrak{A}$. As π is assumed to be non-degenerate, we conclude that $\xi = 0$. The converse may be proved in a similar way.
(2) If $\mathfrak{A}$ is unital, with a unit e, and $\pi(e) = I$, then $H = \pi(e)H \subset \pi(A)H$ and hence π is non-degenerate. In particular, the universal representation π_u is non-degenerate. However, π_u need not to be cyclic. To see this, take $\mathfrak{A} = B(H)$ with H separable. Then, as explained before, $|P(\mathfrak{A})| = 2^{\mathfrak{c}}$ and $|H_u| = 2^{\mathfrak{c}}$ but $|\pi_u(\mathfrak{A})\xi| = |B(H)| = \mathfrak{c}$ and $|\overline{\pi(\mathfrak{A})\xi}| = \mathfrak{c}$. The same argument goes through for $\dim H > \aleph_0$. The above discussion shows that a faithful representation need not to be cyclic and that the direct sum of cyclic (and even irreducible) representations need not to be cyclic.
(3) Faithful representations need not to be non-degenerate; just take the embedding $\pi : \mathbb{C} \to M_2(\mathbb{C})$, $\pi(\lambda) = \mathrm{diag}(\lambda, 0)$ and notice that $0 \neq \xi = (0,1)^T$ satisfies $\pi(x)\xi = 0$ for all $x \in \mathbb{C}$.

2.10. *Remark.* It is fully legitimate to consider only non-degenerate representations. In fact, we have shown that $\{\xi \in H : \forall_{x \in \mathfrak{A}} \pi(x)\xi = 0\} = \pi(\mathfrak{A})H^{\perp}$ and thus we can restrict ourselves to the space $H_0 := \pi(\mathfrak{A})H^{\perp}$, because this space is invariant (hence reducing) for $\pi(\mathfrak{A})$.

As explained in the next theorem, the concept of algebraical irreducibility and topological irreducibility are the same; for simplicity, in both of these two cases we shall say that the considered representation is simply *irreducible*. The criterion for a representation to be irreducible is the following ([8], Theorem 3.13.2):

2.11. **Theorem.** *Let $\pi\colon \mathfrak{A} \to B(H)$ be a non-zero representation. Then, the following conditions are equivalent:*

(1) π is topologically irreducible.
(2) The commutant of π is trivial: $(\pi(\mathfrak{A}))' = \mathbb{C}I$.
(3) $\pi(\mathfrak{A})$ is dense in $B(H)$ with respect to the strong operator topology.
(4) For any two unit vectors $\xi, \eta \in H$ there is an $a \in \mathfrak{A}$ such that $\pi(a)\xi = \eta$.
(5) Every $0 \neq \xi \in H$ is cyclic for π.
(6) π is equivalent to some representation obtained from a pure state via the GNS construction.

2.12. *Remark.* 1. From the above criterion it is easy to deduce that all irreducible representations of any commutative C^*-algebra are one-dimensional. Indeed, if A is commutative, then so is $\pi(A)$ and hence $\pi(A) \subset (\pi(A))'$. Note that such representations are hardly ever faithful. For an example of an infinite-dimensional, irreducible, faithful representation, just consider the identity representation of $B(H)$.

2. It is also immediate that every irreducible representation is cyclic (in particular non-degenerate). The converse is not true, since every representation associated with a state φ is cyclic but irreducible are precisely those representations which are obtained from a pure state.

3. If $\mathfrak{A} \subset B(H)$ is a concrete C^*-algebra, then all representations equivalent to the identity representation are of the form $\pi(T) = U^*TU$. Every such representation is a *-isomorphism; this isomorphism is sometimes called *spatial*.

4. Let φ be a linear, multiplicative functional. Then $\varphi(x^*x) = |\varphi(x)|^2$ and therefore the ideal N_φ appearing in the GNS construcion is $\ker\varphi$. Thus, H_φ is one-dimensional and any one-dimensional representation is obviously irreducible. Therefore, φ is a pure state, so in fact pure states generalize linear characters.

5. On account of the above theorem we can construct a surjective map from $P(\mathfrak{A})$ to the space $\hat{\mathfrak{A}}$ of all equivalence classes of irreducible representations of $\mathfrak{A}$. After introducing a suitable topology on $\hat{\mathfrak{A}}$ this map turns out to be continuous and open.

3. Primitive ideals

We have seen that in the commutative case there is a one-to-one correspondence between linear characters and maximal ideals. Moreover, we have seen that maximal ideals may exist despite of the lack of linear characters, as in the case where $\mathfrak{A} = B(H)$. So, the natural question arises whether it is possible to generalize the concept of the spectrum in terms of ideals. However, the concept of the maximal ideal has to be replaced by a more suitable property. The definition is as follows:

3.1. **Definition.** An ideal $\mathfrak{I} \subset \mathfrak{A}$ is called *primitive* if it is the kernel of an irreducible representation. The set of all such ideals is called the *primitive spectrum* of $\mathfrak{A}$ or *the structure space* of $\mathfrak{A}$ and is denoted by $\mathrm{Prim}(\mathfrak{A})$.

Note that each maximal ideal is primitive but the converse is not true. The set $Prim(\mathfrak{A})$ may be turned into a topological space as follows: for a set $X \subset \mathrm{Prim}(\mathfrak{A})$ we define its closure by the formula

$$\overline{X} := \{\mathfrak{a} \in \mathrm{Prim}(\mathfrak{A}) : \bigcap_{\mathfrak{b}\in X} \mathfrak{b} \subset \mathfrak{a}\}.$$

In other words, we declare that the set X in $\mathrm{Prim}(\mathfrak{A})$ is closed iff every $\mathfrak{a} \in \mathrm{Prim}(\mathfrak{A})$ containing $\bigcap_{\mathfrak{b}\in X} \mathfrak{b}$ belongs to X. This is sometimes called the *hull-kernel topology* or the *Jacobson topology*. It is instructive to think about the Jacobson topology as of a non-commutative generalization of the so-called *Zariski topology* known from abstract algebra and the theory of commutative rings. However, it turns out that the structure space may fail to be even a T_1 space, unlike in the case of commutative rings and the Zariski topology.

3.2. **Example.** As we have already mentioned, for $\mathfrak{A} = B(H)$, where H is separable Hilbert space, there are only three norm closed, two-sided ideals in $\mathfrak{A}$, namely: $\mathfrak{a}_0 := \{0\}, \mathfrak{a}_1 := K(H)$ and $B(H)$. However, $B(H)$ is not primitive by our agreement that we are only interested in non-degenerate representations. The singleton $\{\mathfrak{a}_1\}$ is closed, while $\{\mathfrak{a}_0\}$ is not (therefore it is dense), since always $\mathfrak{a}_0 \subset \mathfrak{a}$ for any ideal $\mathfrak{a}$. Hence, $\mathrm{Prim}(\mathfrak{A})$ is a finite, non-Hausdorff topological space and contains no interesting information about $B(H)$. Note that the same argument applies to every C^*-algebra that admits a faithful, irreducible representation; for such a C^*-algebra the singleton consisting of the zero ideal forms a dense set. Thus, the structure space turns out to be not a sufficient tool to determine much information about a given C^*-algebra in general, and so it cannot serve as an appropriate generalization of the spectrum in the non-commutative case.

4. Group representations

It is instructive to compare results on representations of C^*-algebras to the theory of representations of topological groups. We briefly discuss some aspects of this theory. Let G be a locally compact topological group, H be a separable Hilbert space. We are interested only in *unitary* representations of groups, namely continuous homomorphisms $\pi\colon G \to U(H)$ where $U(H)$ stands for the set of all unitary operators on H, equipped with the strong operator topology. If H is n-dimensional, we denote $U(H)$ by $U(n)$ and in the case $n = 1$ we write $\mathbb{T}$ for $U(1)$ identifying the set of all scalar unitary operators with the unit circle—we say then that π is a *continuous character*. The set of all continuous characters on G, with pointwise multiplication as the group operation, equipped with the topology of uniform convergence on compact subsets, is denoted by $\hat{G}$ and it is again a locally compact topological group called the *dual group* of G. We list here some basic examples of dual groups:

4.1. **Examples.** 1. Let G be a finite cyclic group, $G = \mathbb{Z}_n$. Then $\hat{G} \simeq G$, i.e. G is *self-dual*. Moreover, for any two groups G_1, G_2 we have $(G_1 \times G_2)\hat{} = \hat{G}_1 \times \hat{G}_2$. Combining these facts yields that for a finite abelian group G we have $\hat{G} \simeq G$. Note also that if we denote by $\hat{\mathfrak{A}}$ the spectrum of a commutative C^*-algebra $\mathfrak{A}$, then $\mathfrak{A} \simeq C(\hat{\mathfrak{A}})$ but $(\mathfrak{A}_1 \times \mathfrak{A}_2)\hat{} = \hat{\mathfrak{A}}_1 \sqcup \hat{\mathfrak{A}}_2$.

2. Another example of a self-dual group is the additive group of real numbers $G = \mathbb{R}$; every continuous character on $\mathbb{R}$ is of the form $t \mapsto e^{ist}$ for some real number s. However, for the additive group of real numbers with the discrete topology the dual group is the group of *all* homomorphisms and its cardinality equals $2^{\mathfrak{c}}$. In fact, we can construct $2^{\mathfrak{c}}$ additive maps $a\colon \mathbb{R} \to \mathbb{R}$ (as linear automorphisms of the $\mathfrak{c}$-dimensional linear space $\mathbb{R}$ over the field $\mathbb{Q}$ of rationals). Each additive function gives rise to the character $t \mapsto e^{ia(t)}$ and we can ensure that distinct additive maps lead to distinct characters. An interesting fact is that each discontinuous character is automatically non-measurable. The space of all characters on $\mathbb{R}$ may be viewed as the so called *Bohr compactification* of $\mathbb{R}$ which is a construction much in the same spirit as the Stone-Čech compactification, but in the category of topological groups. It can be also viewed as the spectrum of the commutative C^*-algebra of continuous *almost periodic* functions. As this is not the main topic of our presentation, we refer to [19] for further information.

3. For the infinite cyclic group $G = \mathbb{Z}$ we have $\hat{G} \simeq \mathbb{T}$. It is the most basic example of the following rule: the dual of any discrete abelian group is compact and the dual of any compact abelian group is discrete.

4. For non-abelian groups it may happen that the dual group consists only of the trivial character (identically equal to 1); the dual of every semi-simple connected Lie group is trivial. Probably the most prominent example is the three-dimensional unit sphere S^3, viewed as the subgroup of the quaternions. This group can be also viewed as $SU(2)$, i.e. the group of all unitary matrices U with $\det U = 1$.

The situation is quite similar to the C^*-algebraic setting: the (classical) spectrum of any commutative C^* is always non-empty, whereas it may be empty in the non-commutative setting. A subtle difference is that in the case of groups, we require the trivial character to belong to the dual, in order to obtain an object still belonging to the category of groups. Allowing the zero character to belong to the spectrum in the C^*-algebraic setting would spoil the fact that spectrum is the subset of the unit sphere.

Suppose now that G is an abelian locally compact group. With each $g \in G$ we can associate $\Phi_g\colon \hat{G} \to \mathbb{T}$ defined by $\Phi_g(\chi) := \chi(g)$—this construction reminds the definition of the Gelfand transform or the canonical embedding of a Banach space X into its second dual X^{**}. Unlike in the Banach space theory, it occurs that for any abelian group G the second dual $(\hat{G})\hat{}$ is always isomorphic to G. This is the statement of the celebrated van Kampen–Pontryagin duality theorem (see Theorem

24.2 in [15] for details). The case of non-abelian groups is much more difficult and requires investigating all irreducible unitary representations of G; this is done in the first chapter of [16] and is due to Tannaka and Krein.

We now proceed to a description of a very important construction giving a link between group representations and representations of C^*-algebras. Let G be a discrete countable group and let $H = \ell^2(G)$. We denote by $\mathbb{C}G$ the *group algebra* of G, i.e. the set of all functions $\xi\colon G \to \mathbb{C}$ with finite supports. We define a multiplication on $\mathbb{C}G$ as the convolution:

$$(\xi * \eta)(g) := \sum_{g_1 g_2 = g} \xi(g_1)\eta(g_2)$$

and the $*$-structure is given by defining $\xi^*(g) := \overline{\xi(g^{-1})}$. Note that under the identification of elements from the group with corresponding functions in $\mathbb{C}G$, namely $\delta_g \simeq g$, the convolution product is nothing else but the natural extension of the group operation onto the whole of $\mathbb{C}G$. Define $\pi\colon G \to U(H)$ to be the *left regular representation* which is given by the shift operator:

$$(\pi(g)\xi)(h) = \xi(g^{-1}h).$$

We can naturally extend π to $\mathbb{C}G$ obtaining an injective *-homomorphism.

4.2. **Definition.** (1) The norm closure of $\pi(\mathbb{C}G)$ in $B(H)$ is called *the reduced group C^*-algebra* and is denoted by $C_r^*(G)$.

(2) The *full group C^*-algebra* is defined as the completion of $\pi(\mathbb{C}G)$ with respect to the norm $\|\xi\|_f := \sup\{\|\pi(\xi)\|\colon \pi \text{ is a *-representation of } \mathbb{C}G\}$ and is denoted by $C^*(G)$.

(3) The bicommutant $(\pi(A))''$ is called the *group von Neumann algebra* of G and is denoted by $W^*(G)$.

4.3. *Remark.* At the first sight it may seem that $C^*(G)$ is 'smaller' than $C_r^*(G)$, as it is a completion with respect to a stronger norm. However, it is not the case: the identity mapping $id\colon (\pi(\mathbb{C}G), \|\cdot\|_f) \to (\pi(\mathbb{C}G), \|\cdot\|)$ is continuous and therefore extends to a continuous $*$-homomorhism $J\colon (C^*(G), \|\cdot\|_f) \to (C_r^*(G), \|\cdot\|)$. The range of this mapping contains $\pi(\mathbb{C}G)$ and is closed, by a theorem similar to Theorem 2.7. Therefore, J is onto, however, it may happen that J is not injective. This phenomenon is described in [20] in Chapter 1, Section 3 (however, it is done not in the context of C^*-algebras, but rather in the context of Hilbert spaces).

4.4. **Theorem.** *$C_r^*(G) \simeq C^*(G)$ if and only if the group G is amenable*

Recall that *amenable* group is the group which admits positive left invariant (eq. right invariant) mean $\lambda : \ell^\infty(G) \to \mathbb{C}$. Examples of amenable groups include compact and abelian groups (more general: solvable and even virtually solvable groups). The most prominent example of non amenable group is the free group with $n > 1$ generators.

4.5. **Example.** Let G be a (non-trivial) *i.c.c.* group (this abreviation stands for *infinite conjugacy classes*), that is, we assume that for each $g \neq e$ the set $\{hgh^{-1} : h \in G\}$ is infinite. Then the group von Neumann algebra of G turns out to be a factor of type II_1 (the converse is also true). Examples of i.c.c. groups are free groups $\mathbb{F}_n$ with $n > 1$ generators and Π, the group of all permutations of $\mathbb{Z}$ which fix all but finite number of elements of $\mathbb{Z}$. The group von Neumann algebra of Π is the (unique) *hyperfinite* II_1 factor and is non-isomorphic to $W^*(\mathbb{F}_n)$ for any $n > 1$. Note also that due to the Cayley theorem, the group Π contains all finite groups.

5. C^*-ALGEBRAS OF COMPACT OPERATORS

Most classes of operators acting on a Hilbert space (e.g. self adjoint, normal, unitary, isometric operators, projections etc.) can be characterized in a purely algebraic way. However, such a characterization is not possible for compact operators:

5.1. **Example.** Let $\mathfrak{A} = B(H)$ and $P \in \mathfrak{A}$ be a rank-one projection. Then P is obviously compact. Consider the mapping $\pi\colon \mathfrak{A} \to B(K)$ (where $K = \bigoplus_{i\in\mathcal{I}} H_i$ with each $H_i = H$ and $\mathcal{I}$ being infinite) defined by the formula: $\pi(T) := \bigoplus_{i\in\mathcal{I}} T$. Then π is a faithful representation but $\pi(P)$ is a projection with infinite rank, and therefore it is not compact.

This example suggests that investigating representations of C^*-algebras of compact operators may lead to serious difficulties. Fortunately, one can give a satisfactory description of representations of such algebras. The main technique in this context involves *minimal projections.*

5.2. **Theorem.** *Let $\mathfrak{A} \subset K(H)$ be a C^*-algebra and π be a non-degenerate representation of $\mathfrak{A}$. Then there exists an orthogonal family $\{\pi_i\}_{i\in\mathcal{I}}$ of subrepresentations of π with the following properties:*

(1) each π_i is irreducible;
(2) each π_i is equivalent to some subrepresentation of the identity representation;
(3) $\pi = \bigoplus_{i\in\mathcal{I}} \pi_i$.

If we take $\mathfrak{A} = K(H)$, then the identity representation is irreducible (since every non-zero vector is cyclic) and thus each π_i appearing in the above theorem satisfies $\pi_i \sim id$. Moreover, if π is already irreducible, then $|\mathcal{I}| = 1$ and $\pi \sim id$. Therefore, we obtain the following:

5.3. **Corollary.** *(1) Every representation of $K(H)$ is equivalent to the multiple of the identity representation.*
(2) There is only one, up to unitary equivalence, irreducible representation of $K(H)$, namely, the identity representation.

From the above results one can derive the form of possible *-isomorphisms of $\mathfrak{A}$, where $\mathfrak{A}$ is the C^*-algebra $K(H)$ of compact operators, or $\mathfrak{A} = B(H)$. However, the latter case requires some additional facts concerning extensions of irreducible representation. It turns out that any such *-isomorphism is *spatial*. More precisely, we have the following:

5.4. **Corollary.** *If $\alpha\colon K(H_1) \to K(H_2)$ is a *-isomorphism, then there exists a unitary operator $U\colon H_1 \to H_2$ such that $\alpha(T) = UTU^*$. The same is true if we replace $K(H_i)$ by $B(H_i)$.*

Proof. If $\alpha\colon K(H_1) \to K(H_2)$ is a *-isomorphism, then α is surjective and

$$\Big(\alpha(K(H_1))\Big)' = \big(K(H_2)\big)' = \mathbb{C}I$$

and thus α may be viewed as an irreducible representation. Therefore, it is equivalent to id and the conclusion follows.

Now, if $\alpha\colon B(H_1) \to B(H_2)$ is a *-isomorphism, then again it is an irreducible representation. One can show that the restriction of an irreducible representation to any ideal is again irreducible, thus $\alpha_0 := \alpha|_{K(H_1)}$ is irreducible and $\alpha_0(T) = UTU^*$ for some unitary U. Define $\beta\colon B(H_1) \to B(H_2)$ by the same formula: $\beta(T) = UTU^*$. Then β is irreducible and extends α_0, so in view of the fact that such an irreducible extension is unique we infer that $\alpha = \beta$. □

In Example 5.1 we have seen that the image of a compact operator under some representation may fail to be compact. However, the representation appearing in that example is plainly not irreducible. Thus, we may still hope to say something about elements of $\mathfrak{A}$ that are mapped to compact operators under every *irreducible* representation. This leads to a certain important class of C^*-algebras:

5.5. **Definition.** A C^*-algebra $\mathfrak{A}$ is called CCR if for every irreducible representation π its image $\pi(\mathfrak{A})$ consists of compact operators.

5.6. **Examples.** 1. Any commutative C^*-algebra $\mathfrak{A}$ is CCR, since every its irreducible representation is one-dimensional. Note that $\mathfrak{A}$ need not to consist of compact operators; just take $\mathfrak{A} = C[0, 1]$ acting as multiplication operators. The point is that the identity representation is not irreducible in this case.

2. The C^*-algebra $\mathfrak{A}$ of compact operators is CCR, since the identity representation is the only one irreducible representation (up to equivalence).

3. Consider the C^*-algebra $\mathfrak{A}$ generated by the shift operator (see the third example from 2.1). Since S has no non-trivial reducing subspaces, the identity representation is irreducible. But S is non-compact, therefore $\mathfrak{A}$ is *not* CCR. However, roughly speaking, $\mathfrak{A}$ is not too bad and still there is much to say about its spectrum.

Thus CCR algebras generalize both commutative C^*-algebras and C^*-algebra of compact operators. It turns out that they share some nice properties with commutative C^*-algebras.

5.7. **Proposition.** *Let $\mathfrak{A}$ be CCR algebra. Then:*

(1) the kernel of any irreducible representation is a maximal ideal of $\mathfrak{A}$;
(2) any irreducible representation of $\mathfrak{A}$ is determined by its kernel.

This proposition may be paraphrased: every primitive ideal is automatically maximal and therefore the space $\mathrm{Prim}(\mathfrak{A})$ is T_1. Unfortunately, still it may happen that the space $\mathrm{Prim}(\mathfrak{A})$ is not Hausdorff. Note also that none of the above properties remains true for general C^*-algebras; the C^*-algebra $B(H)$ is the most natural counterexample (see the last section). However, a similar result may be proved in a more general situation.

For a C^*-algebra $\mathfrak{A}$ and its irreducible representation π on H, let us define $\mathfrak{K}_\pi := \{x \in \mathfrak{A} : \pi(x) \in K(H)\}$. Then $\mathfrak{K}_\pi$ is a two-sided, norm closed ideal of $\mathfrak{A}$ containing $\ker \pi$. Let $\mathrm{CCR}(\mathfrak{A}) = \bigcap_\pi \mathfrak{K}_\pi$, where the intersection is taken over all irreducible representation. Then $\mathrm{CCR}(\mathfrak{A})$ is an ideal of $\mathfrak{A}$ (possibly 0) and it is a CCR C^*-algebra in its own right; $\mathrm{CCR}(\mathfrak{A})$ is the largest CCR ideal in $\mathfrak{A}$.

5.8. **Definition.** A C^*-algebra $\mathfrak{A}$ is called GCR whenever for any ideal $\mathfrak{J} \subset \mathfrak{A}, \mathfrak{J} \neq \mathfrak{A}$ we have $\mathrm{CCR}(\mathfrak{A}/\mathfrak{J}) \neq 0$.

Suppose that π is a non-zero irreducible representation of $\mathfrak{A}/\mathfrak{J}$. Then π composed with the canonical projection $\mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ is again irreducible, so if the underlying algebra $\mathfrak{A}$ is CCR, so is $\mathfrak{A}/\mathfrak{J}$. Therefore, any CCR algebra is GCR, but the converse is false. The correspondence between (the classes of) irreducible representations and primitive ideals can be extended also for GCR algebras, due to the following result:

5.9. **Theorem.** *Let $\mathfrak{A}$ be a GCR algebra. Then:*

(1) for any irreducible representation π on a Hilbert space H we have $K(H) \subset \pi(\mathfrak{A})$;
(2) if π_1 and π_2 are irreducible representations with the same kernel, then $\pi_1 \sim \pi_2$.

5.10. *Remark.* Each of the above conditions is in fact equivalent for a C^*-algebra to be GCR, however, the proofs are much more complicated: for the first condition it was done first in the separable case by Glimm and, independetly, by Dixmier and the general case was treated by Sakai.

6. The Borel structure of spectrum

We have seen so far that, for certain reasons, both $P(\mathfrak{A})$ and $\mathrm{Prim}(\mathfrak{A})$ are not sufficent in order to describe non-commutative C^*-algebras. The last way in which the spectrum can be defined is the way involving representations: we define the spectrum $\hat{\mathfrak{A}}$ of $\mathfrak{A}$ to be the set of all classes of irreducible representations of $\mathfrak{A}$. Various topologies on $\hat{\mathfrak{A}}$ may be defined (see [13]). Although $\hat{\mathfrak{A}}$ turns out to be a compact space, there is still a problem with separation axioms, namely, it may happen that $\hat{\mathfrak{A}}$ is not a Hausdorff space. Due to this fact, we will focus only on the measure-theoretic aspect of the spectrum, instead of regarding it purely from the topological point of view. From now on, $\mathfrak{A}$ will stand for a *separable* C^*-algebra. In this case, we can restrict ourselves to irreducible representations on *separable* Hilbert spaces. In fact, if $\mathfrak{A}$ is separable, so is the Hilbert space H_φ obtained from the GNS construction starting from the state φ (since the GNS-norm is weaker than the norm in $\mathfrak{A}$). In particular, this argument can be applied to GNS constructions obtained from pure states and therefore each irreducible representation acts on a separable Hilbert space. We shall now describe how the Borel structure on $\hat{\mathfrak{A}}$ is defined. Denote by H_n the n-dimensional Hilbert space, where $n \in \mathbb{N} \cup \{\infty\}$ (here $\infty = \aleph_0$) and by $\mathrm{rep}(\mathfrak{A}, H_n)$ the space of all representations $\pi\colon \mathfrak{A} \to B(H_n)$. Equip this space with the following metric: choose a dense set $\{x_k\}_{k\in\mathbb{N}}$ (resp. $\{\xi_k\}_{k\in\mathbb{N}}$) in the unit ball of $\mathfrak{A}$ (resp. H_n) and define

$$d(\pi_1, \pi_2) := \sum_{k,m=1}^{\infty} 2^{-(k+m)} \|\pi_1(x_k)\xi_m - \pi_2(x_k)\xi_m\|.$$

This formula is well defined (the above series is convergent) and it can be easily checked that it defines a metric on $\mathrm{rep}(\mathfrak{A}, H_n)$. Moreover, $\mathrm{rep}(\mathfrak{A}, H_n)$ becomes a complete metric space which has a countable base (i.e. is separable). Therefore, $\mathrm{rep}(\mathfrak{A}, H_n)$ turns out to be a Polish space. By $\mathrm{irr}(\mathfrak{A}, H_n)$ we denote the set of all irreducible representations $\pi\colon \mathfrak{A} \to H_n$. The answer to the question how $\mathrm{irr}(\mathfrak{A}, H_n)$ is embedded into $\mathrm{rep}(\mathfrak{A}, H_n)$ is contained in the following result:

6.1. **Theorem.** *(1) The set of all non-degenerate representations* $\subset \mathrm{rep}(\mathfrak{A}, H_n)$ *is* G_δ *in* $\mathrm{rep}(\mathfrak{A}, H_n)$*.*
(2) The set $\mathrm{irr}(\mathfrak{A}, H_n)$ *is* G_δ *in* $\mathrm{rep}(\mathfrak{A}, H_n)$*. Therefore, it is completely metrizable and thus becomes a Polish space in its own right.*

6.2. *Remark.* We have used the fact that any subset of a complete metric space is completely metrizable iff it is G_δ. The class of G_δ sets in a metric space contains open (always) and closed (this is no longer true for non-metrizable spaces) sets, but obviously not all G_δ sets are of this form. Probably the simplest example is the set $\mathbb{R}\setminus\mathbb{Q}$ of irrationals. Note also that this set is not complete in the standard euclidean metric, however, one can modify this metric to get an equivalent one in which $\mathbb{R}\setminus\mathbb{Q}$ is complete. This shows that completness is topological (but not metric) invariant.

Having defined the topological structure on $\mathrm{irr}(\mathfrak{A}, H_n)$, we can speak about a Borel structure on this space. We have a natural equivalence relation on $\mathrm{irr}(\mathfrak{A}, H_n)$ defined by $\pi_1 \sim \pi_2$ iff π and π_2 are unitarily equivalent. Define $\hat{\mathfrak{A}_n} := \mathrm{irr}(\mathfrak{A}, H_n)/\sim$ and finally set $\hat{\mathfrak{A}} := \sum_{n\in\mathbb{N}\cup\{\infty\}} \hat{\mathfrak{A}_n}$ to be the direct sum of $\hat{\mathfrak{A}_n}$'s. Therefore, $\hat{\mathfrak{A}}$ is the disjoint sum of *quotients* of standard Borel spaces—obviously, this does not imply that $\hat{\mathfrak{A}}$ is a standard Borel space itself. However, it turns out, that for GCR algebras it is the case.

6.3. **Theorem.** *(1) Let $\mathfrak{A}$ be a separable GCR algebra and $n \in \mathbb{N} \cup \{\infty\}$. Then the canonical projection $\pi_n : \mathrm{irr}(\mathfrak{A}, H_n) \to \hat{A}_n$ admits a Borel cross section.*
(2) $\hat{\mathfrak{A}}$ is a standard Borel space

6.4. *Remark.* It is known that for C^*-algebras which are *not* GCR spectra behave patologically in the sense that they do not admit absolutely mesasurable cross sections. In particular, such a spectrum is not countably generated and cannot be a subspace of any Polish space. This suggest that the class of GCR algebras is probably the broadest class of C^*-algebras for which the effective analysis of its spectrum can be done.

7. Two classical examples

We now discuss two examples of classical C^*-algebras for which the structure of the spectrum seems to be beyond our current knowledge.

7.1. **Examples.** 1. Consider the Calkin algebra $\mathfrak{A} = \mathrm{Calk}(H)$. Since there is the only one norm closed, two-sided ideal of $B(H)$, the Calkin algebra is *simple* (it contains no proper, two-sided, closed ideals). The kernel of any representation is such an ideal, hence any non-zero representation of $\mathfrak{A}$ is automatically faithful. The natural question arises whether it is possible to construct such a representation on a separable Hilbert space. The answer is negative! It is a conseqence of the following combinatorial result:

7.2. **Lemma.** *There exist an uncountable family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ with the property that for all distinct $A, B \in \mathcal{F}$ the intersection $A \cap B$ is a finite set.*

Proof. It is irrelevant which countable set we choose instead of $\mathbb{N}$. For the desired family we may take the collection of all injective and convergent sequences consisting of rational numbers (regarded as sets rather than sequences). □

For each subset $A \subset \mathbb{N}$ denote by $P_A : \ell^2 \to \ell^2$ the orthogonal projection onto $\mathrm{lin}\{e_i : i \in A\}$. Constructing a family $\mathcal{F}$ as above we obtain an uncountable family of projections which differ only by finite-rank (therefore, compact) perturbations. This gives us an uncountable family of mutually orthogonal projections in $\mathfrak{A}$. Hence, in particular, each Hilbert space constructed from the GNS construction for $\mathfrak{A}$ is non-separable.

The spectrum of $\mathfrak{A}$ has cardinality $2^{\mathfrak{c}}$. Indeed, notice that under the natural identification we have $\ell^\infty / c_0 \subset \mathfrak{A}$. But $\ell^\infty / c_0 \simeq C(\beta\mathbb{N} \setminus \mathbb{N})$, so it contains $2^{\mathfrak{c}}$ pure states. Each such state can be extended to whole of $\mathfrak{A}$. To finish the argument we need the following result:

7.3. **Theorem.** *Let π_1, π_2 be two GNS representations of $\mathfrak{A}$ obtained from some states φ_1 and φ_2, respectively. Then, π_1 and π_2 are equivalent iff there exists a unitary element $U \in \mathfrak{A}$ such that $\varphi_1(A) = \varphi_2(U^* A U)$ for $A \in \mathfrak{A}$.*

Consequently, every pure states produces only $\mathfrak{c}$ equivalent irreducible representations, hence there are $2^{\mathfrak{c}}$ non-equivalent irreducible representations.

2. Now, we will investigate the algebra $B(H)$ where H is separable. Suppose that $\pi : B(H) \to B(H_0)$ is a (non-zero) representation on another separable Hilbert space H_0. There are only two possibilities: $ker\pi = \{0\}$ or $ker\pi = K(H)$. If $ker\pi = K(H)$, then π induces a (non-zero) representation $\tilde{\pi} : \mathrm{Calk}(H) \to B(H_0)$ on a separable Hilbert space, which gives a contradiction. To show that there is only one (up to unitary equivalence) representation on a separable Hilbert space, we will need some facts which can be easily derived from results of [3, 4] (Chapters 4 and 10):

7.4. **Theorem.** *Each pure state on $B(H)$ is either of the form $\varphi_\xi(A) = \langle A\xi, \xi\rangle$, with a unit vector $\xi \in H$, or it vanishes on the set of all compact operators $K(H)$.*

It is immediate that each two vector states are equivalent and if a pure state φ vanishes on $K(H)$, then it is not equivalent to any vector state. So, vector states produce only one class of irreducible representations. The obvious guess is that the GNS construction for any vector state leads to a representation which is equivalent to the identity representation; it can be shown by direct calculations or by using the following fact (slightly simplified Proposition 4.5.3. from [3]):

7.5. **Theorem.** *If $\pi\colon \mathfrak{A} \to B(H)$ is a cyclic representation and π_φ is the GNS representation associated with the state $\varphi = \varphi_\xi \circ \pi$, where $\|\xi\| = 1$ is some unit vector of H, then π and π_φ are equivalent.*

This result applied to $\pi = id$ implies that the GNS representation of any vector state is equivalent to the identity representation. Note that the above argumentation can be used to show that vector states are pure (since the identity representation is irreducible). To see that it is the only irreducible representation (up to equivalence) on a separable Hilbert space, note that if φ is a pure state that vanishes on $K(H)$, then $K(H) \subset N_\varphi$. Hence, for the GNS representation π_φ and $K \in K(H)$ we have $KA \in N_\varphi$ for any $A \in B(H)$, and so

$$\pi_\varphi(K)(A + N_\varphi) = KA + N_\varphi = 0.$$

Thus, $K(H) \subset ker\pi_\varphi$ and π_φ cannot be a representation on a separable Hilbert space. Now, it remains to appeal to Theorem 7.4.

8. Appendix

We will now gather some facts concerning Borel and analytic spaces: all of these results are independent from the previous considerations. Suppose that a Polish space P is given. Then there is the natural σ-algebra on P consisting of Borel sets, i.e. σ-algebra $\mathcal{B}$ generated by open (eqv. closed) sets. Being a σ-algebra, $\mathcal{B}$ is closed under countable unions and intersections, however, it turns out that the continuous image of a Borel set may fail to be Borel.

8.1. **Definition.**
(1) $A \subset P$ is called *analytic* if there exist a Polish space Q and a continuous function $f\colon Q \to P$ such that $A = f(Q)$.
(2) $C \subset P$ is called *coanalytic* if $C = P \setminus A$ where A is analytic.
(3) $E \subset P$ is called *absolutely measurable* iff for every finite Borel measure μ there exist Borel sets B_1, B_2 such that $B_1 \subset E \subset B_2$ and $\mu(B_2 \setminus B_1) = 0$.

(1) Each Borel set is analytic: in fact, for a Borel set $B \subset P$ one can find a Polish space Q and a continuous *injective* function $f\colon Q \to P$ such that $B = f(Q)$.
(2) Among Polish spaces, the space $\mathbb{N}^\infty$ with the product topology plays a special role: it is universal in the sense that for each Polish space P there is a continuous and open mapping $f\colon \mathbb{N}^\infty \to P$ which is surjective. It is worth mentioning that the space $\mathbb{N}^\infty$ is homemorphic to the space of all irrational numbers. In the class of compact metric spaces the same universal property is shared by the Cantor set.
(3) There exist analytic subsets of a Polish space which are not Borel; see, e.g., [24].
(4) As a consequence of the so-called *separation theorem* one can show that every analytic set A such that $P \setminus A$ is also analytic must be Borel. Therefore, each analytic set which is not Borel has the property that $P \setminus A$ is *not* analytic. This justifies the need of introducing a definition of coanalytic sets. In particular, we see that analytic sets do not form a σ-algebra.

(5) Every analytic set is absolutely measurable and since absolutely measurable sets form a σ-algebra, every coanalytic set is also absolutely measurable.

All the above definitions are *internal* in the sense that they determine certain classes of subsets of any given Polish space. However, it is a common situation that we have to work with a certain abstract space, not requiring that it is subset of a Polish space. So, suppose that we have a set X together with an (abstract!) σ-algebra $\mathcal{B}$. We will call $(X, \mathcal{B})$ a *Borel space*. A Borel space is called *standard* whenever it is isomorphic to the family of all Borel subsets of some Polish space; it is called *analytic* whenever it is isomorphic to the family of all analytic subsets of some Polish space. A Borel space $(X, \mathcal{B})$ is called *countably separated* if there is a sequence $E_1, E_2, \ldots$ of Borel sets such that the family $\{\chi_{E_n}\}_{n \in \mathbb{N}}$ separates points of X. If, moreover, such a sequence generates $\mathcal{B}$ as a σ-algebra, then X is called *countably generated*. We will now describe how certain classes of sets (or spaces) behave under continuous and Borel functions.

(1) We already know that a continuous image of a Polish space may fail to be Borel. However, if P, Q are Polish spaces and $f\colon P \to Q$ is a continuous and *injective* mapping, then $f(P)$ is a Borel subset of Q. It follows that every such function maps Borel sets to Borel sets.
(2) The previous result may be extended to Borel maps as follows: if $f\colon X \to Q$ is an injective Borel map from the standard Borel space X into a Polish space Q, then $f(X)$ is a Borel subset of Q. In particular, X is isomorphic to the Borel set $f(X)$.
(3) If X is an analytic Borel space, Q is a Polish space and $f\colon X \to Q$ is a Borel mapping, then $f(X)$ is analytic.
(4) A slightly more general fact is also true: if X is an analytic Borel space, Y is a countably separated Borel space and $f\colon X \to Y$ is a Borel map *onto* Y, then Y is also an analytic Borel space.
(5) Suppose that X is a subset of a Polish space P. Then X is a Borel set in P iff X is the standard Borel space (in its relative structure). The same is true if the word 'Borel' is replaced by 'analytic'. Therefore, having constructed an analytic subset $A \subset P$ which is not Borel, we can regard A as an analytic Borel space which is not standard.
(6) A Borel space X is a subspace of a Polish space P if and only if it is countably generated.
(7) There is only one uncountable standard Borel space up to isomorphism (the interval $[0, 1]$ is a good representative), however, it is not true for analytic Borel spaces. In particular, each set B of cardinality less than $\mathfrak{c}$ cannot be Borel.

Most of constructions known from topology, may be adapted in the measure-theoretic framework. One particular example of this is the definition of the Borel structure on a quotient space. Suppose that $\sim \subset X \times X$ is an equivalence relation in a Borel space X. Borel sets in $X/\sim$ are precisely those sets E for which the inverse image $\pi^{-1}(E)$ is a Borel set in X. As in the case of topological spaces, some good properties of Borel spaces may be spoiled by taking the quotient. However, there are situations when the quotient $X/\sim$ *is* as good as the Borel space X:

8.2. **Theorem.** *Suppose that X is an analytic Borel space and $\sim$ is an equivalence relation $\sim \subset X \times X$ with the following property: there is a sequence $\{f_n\}_{n \in \mathbb{N}}$ of (complex valued) functions, such that for any $x, y \in X$ one has $x \sim y \iff f_n(x) = f_n(y)$ for any $n \in \mathbb{N}$. Then $X/\sim$ is also an analytic Borel space.*

There are convenient conditions upon Borel spaces which guarantee being regular or not being expressed in terms of the existence of the so-called Borel cross-sections:

8.3. **Definition.** If $f\colon X \to Y$ is onto, then $g\colon Y \to X$ is called a *cross section* for f provided that $f \circ g = id_Y$.

Since f is onto, the inverse image $f^{-1}(\{y\})$ is non-empty for each $y \in Y$ and, with the aid of the axiom of choice, one can define $g(y)$ to be any point in $f^{-1}(\{y\})$. Therefore, the existence of cross sections follows from the axiom of choice. However, we shall want to get cross sections that are regular in some sense (continuous, Borel, absolutely measurable etc.) Now, we will list some sufficient conditions for the existence of certain cross sections and explain how this problem is related to regularity of the spaces considered.

(1) The following is a sufficent condition for the existence of a Borel cross section:

    8.4. **Theorem.** *Suppose that $f\colon P \to Y$ is onto, where P is a Polish space and Y is a Borel space, and satisfies the following conditions:*
    (a) for ony open set $U \subset P$ its image $f(U)$ is Borel in Y;
    (b) for every $y \in Y$ its inverse image $f^{-1}(\{y\})$ is closed in P.
    Then f admits a Borel cross section.

    Note that any Borel *bijection* between P nad Y satisfies the conditions of the preceding theorem (in particular, if Y is also a topological space and f is continuous).

(2) The existence of cross section for a mapping $f\colon X \to Y$ implies some kind of regularity of Y.

    8.5. **Theorem.** *Let X, Y be Borel spaces, where X is a standard Borel space, and let $f\colon X \to Y$ be a Borel surjection. Suppose that there exists a Borel cross section $g\colon Y \to X$ for f. Then, Y is also a standard Borel space.*

    In particular, when Y is an analytic Borel space which is not standard, then any surjection $f\colon X \to Y$ does *not* admit any Borel cross section.

(3) The following question arises naturally: if X, Y are analytic Borel spaces and $f\colon X \to Y$ is a Borel surjection, must f admit any Borel cross section? It occurs that the answer is negative even if we assume X and Y to be standard Borel spaces; a counterexample is known for which X is a Borel subset of $[0,1]^2$ and $f\colon X \to [0,1]$ is given by $f(x,y) = x$.

(4) When considering *absolutely measurable* cross section, one finds out that in many situations such a function does exist.

    8.6. **Theorem.** *Suppose that two Borel spaces X and Y are given, where X is analytic and Y is countably separated. Let $f\colon X \to Y$ be a Borel map onto Y. Then f has an absolutely measurable cross section*

    Therefore, if a projection $\pi\colon X \to X/\sim$, defined on an analytic Borel space X, has no Borel cross sections, then $X/\sim$ is not even countably separated. Such a situation takes place for $X = \mathbb{T}$ and $\sim$ being the equivalence relation defined by saying that $z_1 \sim z_2$ if and only if $z_1 = e^{ik\alpha}$ for some $k \in \mathbb{Z}$, where α is a fixed angle not being a rational multiple of π.

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