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Adam Wegert

Uniwersytet Jagielloński

Algebra of operators affiliated with a finite type I von Neumann algebra

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Opiekun pracy: dr Piotr Niemiec

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

ALGEBRA OF OPERATORS AFFILIATED WITH A FINITE TYPE I VON NEUMANN ALGEBRA

ADAM WEGERT

Abstract. A closed densely defined operator T is said to be *affiliated* with a von Neumann algebra $\mathfrak{A}$ if T commutes with every unitary operator in $\mathfrak{A}'$. We construct the mapping defined on the algebra of all operators affiliated to a finite type I von Neumann algebra which satisfies all the usual axioms of the trace and show the uniqueness of such a mapping. We also prove that finite type I von Neumamann algebras are the only one in which such a construction can be performed.

1. Introduction

With every von Neumann algebra $\mathfrak{A}$ one can associate the set of operators (unbounded, in general), to be denoted by $\mathrm{Aff}(\mathfrak{A})$, which are *affiliated* with $\mathfrak{A}$. In [10] the authors discovered that surprisingly $\mathrm{Aff}(\mathfrak{A})$ turns out to be a $*$-algebra when $\mathfrak{A}$ is finite. This was in fact the first example of a rich set of *unbounded* operators in which one can define algebraic binary operations in a natural manner. This concept was later adapted by Segal [15, 16] who distinguished a certain class of unbounded operators (namely, measurable with respect to a fixed normal faithful semi-finite trace) affiliated with an *arbitrary* semi-finite von Neumann algebra and equipped it with a structure of a $*$-algebra (for an alternative proof see e.g. [11] or §2 in Chapter IX of [19]). A more detailed investigations of algebras of the form $\mathrm{Aff}(\mathfrak{A})$ were initiated by a pioneering work by Stone [17] who described their models for commutative $\mathfrak{A}$ in terms of unbounded continuous functions densely defined on the Gelfand spectrum $\mathfrak{X}$ of $\mathfrak{A}$. Much later Kadison [6] studied this one-to-one correspondence between operators in $\mathrm{Aff}(\mathfrak{A})$ and functions on $\mathfrak{X}$. Very recently Liu [8] established an interesting property of $\mathrm{Aff}(\mathfrak{A})$ concerning Heisenberg uncertainty principle. Namely, she showed that the canonical commutation relation, which has the form $AB - BA = I$, fails to hold for any $A, B \in \mathrm{Aff}(\mathfrak{A})$ provided $\mathfrak{A}$ is finite. We shall obtain her result for finite *type I* algebras as a simple corollary of our results. The main purpose of the paper is to show that whenever $\mathfrak{A}$ is a finite type I von Neumann algebra, then $\mathrm{Aff}(\mathfrak{A})$ has a uniquely determined center-valued trace. Our second goal is to show that if it is possible to construct such a mapping then $\mathfrak{A}$ must be finite type I von Neumann algebra.

The paper is organized as follows. After recalling all the necessary definitions and results we use measure theoretic arguments to obtain a model for a finite type I von Neumann algebra $\mathfrak{A}$. Then we exhibit the form of elements in $\mathrm{Aff}(\mathfrak{A})$. This enables us to define the trace in a natural manner, writing its formula explicitly. We also prove the uniqueness of this mapping and show that finite type I von Neumann algebras are the only von Neumann algebras for which the trace on the algebra of affiliated operators can be defined.

Notation and terminology. Von Neumann algebras will be *usually* denoted by $\mathfrak{A}$; for a given von Neumann algebra $\mathfrak{A}$, by $\mathfrak{Z}(\mathfrak{A})$ we will denote its center and by $\mathrm{Aff}(\mathfrak{A})$—the set of all operators affiliated to $\mathfrak{A}$. For a finite $\mathfrak{A}$ we think about $\mathrm{Aff}(\mathfrak{A})$ as of a $*$-algebra. The center of this algebra in denoted by $\mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$. The algebraic operations in $\mathrm{Aff}(\mathfrak{A})$ are denoted in the same manner as those in $\mathfrak{A}$ (therefore they do not coincide in general with the usual algebraic operations for unbounded operators) but we hope that it will not lead to confusion. For an operator T acting in some Hilbert space we denote: by $\mathrm{dom}(T)$ its domain, by $\mathcal{N}(T)$ its kernel and by $\mathcal{R}(T)$ its range. The trace in a von Neumann algebra is denoted by $\mathrm{tr}_{\mathfrak{A}}$ or simply tr when it is clear from the context on which von Neumann algebra it is defined. The trace on the algebra of affiliated operators will be denoted by $\mathrm{tr}_{\mathrm{Aff}}$. All vector spaces are assumed to be over a field of complex numbers $\mathbb{C}$. When two von Neumann algebras $\mathfrak{A}_1, \mathfrak{A}_2$ are $*$-isomorphic, we denote this fact by $\mathfrak{A}_1 \cong \mathfrak{A}_2$. By I_H or simply I we will denote the identity operator in H.

2. Preliminaries

There is a big difference between bounded and unbounded operators. In the first case, the existence of fine structure is apparent—linear bounded operators form a great variety of von Neumann algebras, which are widely developed. However, in case of unbounded operators the situation is much more delicate and one comes up against several problems: first is the lack of suitable norm and hence there is a problem of defining a natural topology. Second obstacle is even more serious: there are almost always problems with the domain, for example: it is possible that A and B are densely defined but $\mathrm{dom}(A+B) = \mathrm{dom}(A) \cap \mathrm{dom}(B) = \{0\}$ (we will deal only with densely defined and closable operators, passing to the closure when necessary). Therefore in case of unbounded operators even the linear structure is absent. However, in special cases, there is a nice algebraic structure which contains plenty of unbounded operators.

We start with an instructive example:

2.1. **Example.** Let (X, Σ, μ) be a σ-finite measure space and $L^0 := L^0(X, \Sigma, \mu)$ be the linear space of all (classes of) measurable functions finite valued almost everywhere. With every $f \in L^0$ we can associate a linear operator $M_f : \mathrm{dom}(M_f) \to L^2(X, \Sigma, \mu)$ defined by the formula:

$$(M_f g)(x) := f(x)g(x), \quad \mathrm{dom}(M_f) := \{g \in L^2(X, \Sigma, \mu) : fg \in L^2(X, \Sigma, \mu)\}.$$

One shows that M_f is densely defined and closed. It is bounded precisely when $f \in L^\infty(X, \Sigma, \mu)$ (equivalently, if M_f is everywhere defined). Note that in general $M_f + M_g \subsetneq M_{f+g}$ (e.g. for $g = -f$) but after passing to the closures we get equality. In the similar manner, one can define multiplication and involution. Thus it is possible to represent L^0 as the $*$-algebra of (unbounded) operators.

Making the set of all operators affiliated with a von Neumann algebra a $*$-algebra is a far generalisation of the above idea which has its origin in the work of Murray and von Neumann [10]. It appears that the key property for the possibility of doing so is the *finiteness* of the von Neumann algebra.

In order to fix notations, we recall some basic definitions.

2.2. **Definition.** Let $\mathfrak{A}$ be a von Neumann algebra and $P, Q \in \mathfrak{A}$ be two projections.
(1) We say that P is *Murray-von Neumann equivalent* to Q iff there is a partial isometry $V \in \mathfrak{A}$ such that $P = V^*V$ and $Q = VV^*$. This is denoted by $P \sim Q$.
(2) We write $P \preceq Q$ iff there is a subprojection $Q_0 \in \mathfrak{A}$ of Q such that $P \sim Q_0$.
(3) We say that P is *finite* iff there is no proper subprojection $P_0 \in \mathfrak{A}$ of P equivalent to P. If the identity I is finite we say that $\mathfrak{A}$ is a *finite* von Neumann algebra. If P is such that for any nonzero central projection $Z \in \mathfrak{A}$ the projection PZ is *infinite* (i.e. not finite) we say that P is *properly infinite*.

(4) We say that P is *abelian* iff $P\mathfrak{A}P$ is commutative (as a von Neumann algebra acting in $\mathfrak{R}(P)$).

The Murray-von Neumann equivalance is indeed an equivalence relation and $\preceq$ defines a partial ordering in the set of all equivalence classes of projections. Note that in the case $\mathfrak{A} = B(H)$ two projections P, Q are equivalent iff $\dim P(H) = \dim Q(H)$. Therefore Murray-von Neumann order is a far generalisation of the concept of dimension. With the help of this ordering one can distinguish between various types of von Neumann algebras:

2.3. **Definition.** Let $\mathfrak{A}$ be a von Neumann algebra.

(1) $\mathfrak{A}$ is *type* I if $\mathfrak{A}$ possesses an abelian projection P with central carrier I; that is, I is a unique central projection $Z \in \mathfrak{A}$ such that $P \leq Z$. If I is the sum of α (mutually orthogonal) mutually equivalent abelian projections (where α is a nonzero cardinal) we say that $\mathfrak{A}$ is *type* I_α. In particular, $\mathfrak{A}$ is type I_1 iff $\mathfrak{A}$ is commutative.
(2) $\mathfrak{A}$ is called *type* II iff $\mathfrak{A}$ has no abelian projection but has a finite projection whose central carrier is I. If, in addition, I is finite, we say that $\mathfrak{A}$ is *type* II_1 and if I is properly infinite, $\mathfrak{A}$ is *type* II_∞.
(3) If there is no finite projection in $\mathfrak{A}$, then $\mathfrak{A}$ is *type* III.

3. Finite von Neumann algebras

We will be interested (mostly) in finite type I von Neumann algebras. Type I algebras are characterised as follows (see Chapters 6 and 9 in [5]):

3.1. **Theorem.** *Let n be a positive integer and M_n be the space of all (complex) $n \times n$ matrices.*

(1) *If $\mathfrak{A}$ is a type* I_n *von Neumann algebra, then $\mathfrak{A}$ is $*$-isomorphic to the algebra $M_n \otimes \mathfrak{Z}(\mathfrak{A}) \cong M_n(\mathfrak{Z}(\mathfrak{A}))$.*
(2) *If $\mathfrak{A}$ is a type* I *von Neumann algebra acting on a Hilbert space H, then $\mathfrak{A}$ is $*$-isomorphic to $\bigoplus_\alpha \mathfrak{A}_\alpha$ where each $\mathfrak{A}_\alpha$ is a type* I_α *von Neumann algebra or $\{0\}$ and α runs over all positive cardinals not exceeding* $\dim(H)$.

3.2. *Remarks.* Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space H.

(1) As $\mathfrak{Z}(\mathfrak{A})$ is a commutative von Neumann algebra, therefore it is $*$-isomorphic to $L^\infty(X)$ for suitable measure space (X, Σ, μ), see Proposition 1.18.1 in [14]. This fact is of great importance for us.
(2) When one is interested in the 'spatial' structure of $\mathfrak{A}$ (namely, how $\mathfrak{A}$ is placed in $B(H)$), one may also ask for the type of the commutant $\mathfrak{A}'$ of $\mathfrak{A}$. In the distinction between types I, II, III the commutant does not change the type: if $\mathfrak{A}$ is of type ε where $\varepsilon \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, so is $\mathfrak{A}'$. However, this is no longer true when one consider more detailed types. Moreover, not every $*$-isomorphism is *spatial* (i.e. implemented by a unitary operator): for instance, consider the mapping $A \mapsto A \otimes I_n := \underbrace{A \oplus ... \oplus A}_{n-\text{times}}$. It turns out that every type I_m von Neumann algebra with commutant of type I_n is *spatially* isomorphic to $M_m(\mathfrak{Z}(\mathfrak{A}) \otimes I_n)$ (consult Theorem 9.3.2 in [5]).

A powerful tool in finite von Neumann algebras are *traces*. A trace is a linear mapping $\mathrm{tr} : \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ with the following properties:

(Tr1) $\mathrm{tr}(A) \geq 0$ for $A \geq 0$,
(Tr2) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for $A, B \in \mathfrak{A}$,
(Tr3) $\mathrm{tr}(C) = C$ for each $C \in \mathfrak{Z}(\mathfrak{A})$.

What is more, in a finite von Neumann algebra $\mathfrak{A}$ conditions (Tr1)–(Tr3) uniquely determine the trace and tr has additional properties:

(Tr4) $\operatorname{tr}(AC) = \operatorname{tr}(A)C$ for any $A \in \mathfrak{A}$ and $C \in \mathfrak{Z}(\mathfrak{A})$,
(Tr5) $\operatorname{tr}(A) \neq 0$ for any nonzero nonnegative (selfadjoint) $A \in \mathfrak{A}$,
(Tr6) if $\{T_i\}_{i\in\mathbb{I}} \subset \mathfrak{A}$ is a directed set which is bounded from above and $T \in \mathfrak{A}$ is its least upper bound, then the operators $\operatorname{tr}(T_i)$ converge in the strong operator topology to $\operatorname{tr}(T)$.

4. A model

The following simple lemma will prove useful for us:

4.1. **Lemma.** *If $A \in \mathfrak{A}$ is nonnegative, then $\mathcal{N}(A) = \{0\}$ iff the mapping $\mathfrak{A} \ni T \mapsto AT \in \mathfrak{A}$ is one-to-one.*

Proof. If $\mathcal{N}(A) = \{0\}$ and $AT = 0$, then $\mathcal{R}(T) \subset \mathcal{N}(A) = \{0\}$ and hence $T = 0$.

For the converse, let $E : \mathcal{B}(\mathbb{R}) \to B(H)$ be the spectral measure of A defined on the σ-algebra of Borel subsets of $\mathbb{R}$. Then $E(\sigma) \in \mathfrak{A}$ for any Borel set $\sigma \subset \mathbb{R}$. Since $AE(\{0\}) = 0$, we conclude that $E(\{0\}) = 0$ and thus $\mathcal{N}(A) = \{0\}$. □

Let $\mathfrak{A}$ be a finite type I von Neumann algebra. By Theorem 3.1

$$\mathfrak{A} \cong \bigoplus_{n=1}^{\infty} \mathfrak{A}_n \cong \bigoplus_{n=1}^{\infty} M_n \otimes L^\infty(X_n, \mu_n) \cong \bigoplus_{n=1}^{\infty} L^\infty(X_n, \mu_n; M_n).$$

Therefore $\mathfrak{A} \cong L^\infty(X, \mu; M_*)$ where the right-hand side is understood in the following sense: (X, μ) is the disjoint union of measure spaces (X_n, μ_n) (in particular, $X = \bigsqcup_{n=1}^{\infty} X_n$ and $\mu(A) = \sum_{n=1}^{\infty} \mu_n(A \cap X_n)$ for any measurable set $A \subset X$) and $L^\infty(X, \mu; M_*)$ consists of all functions f defined on the set X such that $f(x) \in M_n$ for any $x \in X_n$, each entry of $f|_{X_n}$ is a measurable complex valued function and the functions $f|_{X_n}$ are uniformly (essentially) bounded. We can further decompose X as $X = \bigsqcup_{s\in\mathbb{S}} X_s$ where $\mu = \sum_{s\in\mathbb{S}} \mu_s$ where $\mu_s(X_s) < \infty$ and for $f \in L^\infty(X, \mu; M_*)$ and $x \in X_s$ we have $f(x) \in M_{n(s)}$. As (X_s, μ_s) is a finite measure space we conclude from Maharam's theory [9] that there is some collection of cardinals $(\alpha_j)_{j\in\mathbb{J}_s}$ such that:

$$L^\infty(X_s, \mu_s) \cong \bigoplus_{j\in\mathbb{J}_s} L^\infty(\{0,1\}^{\alpha_j}, \lambda_j) \tag{1}$$

where λ_j are usual product measures on the product σ-algebra. From (1) we get

$$L^\infty(X_s, \mu_s; M_{n(s)}) \cong \bigoplus_{j\in\mathbb{J}_s} L^\infty(\{0,1\}^{\alpha_j}, \lambda_j; M_{n(s)}) \tag{2}$$

But $\mathfrak{A} \cong \bigoplus_{s\in\mathbb{S}} L^\infty(X_s, \mu_s; M_{n(s)})$—combining with (2) we get

$$\mathfrak{A} \cong \bigoplus_{j\in\mathbb{J}} L^\infty\Big(\{0,1\}^{\alpha_j}, \lambda_{\alpha_j}; M_{n(j)}\Big) \cong L^\infty(\Omega, \lambda, M_*) \tag{3}$$

where $\lambda = \sum_{j\in\mathbb{J}} \lambda_{\alpha_j}$ and $\Omega = \bigsqcup_{j\in\mathbb{J}} \{0,1\}^{\alpha_j}$. From now on we will identify $\mathfrak{A}$ with the algebra described above (denoted by $\mathfrak{L}$) via a $*$-isomorphism Φ. Throughout the proof of the next theorem we will use the following notation concerning *cylindrical* sets: suppose that $\mathbb{J} \subset \mathbb{A}$ and $G \subset \{0,1\}^{\mathbb{J}}$. By $\operatorname{Cyl}(G)$ we will mean the set $\{x \in \{0,1\}^{\mathbb{A}} : x|_{\mathbb{J}} \in G\}$

4.2. **Theorem.** *Let $A \in \mathfrak{A}$ and $f := \Phi(A)$. The following conditions are equivalent:*

(1) $\|A\xi\| < \|\xi\|$ provided $\xi \neq 0$,
(2) The set $\{x \in X : \|f(x)\| < 1\}$ is of full λ-measure.

Proof. Assume first that (2) holds. Observe that $1 - f^*f \geq 0$. We will prove that $g - f^*fg \neq 0$ for $g \in \mathfrak{L} \setminus \{0\}$. Take $g \neq 0$ and restrict it to the set of full measure $\{x \in \Omega : \|f(x)\| < 1\}$. Then still $g \neq 0$ and we have (for x such that $f(x) \in M_n$):

$$g(x) - f^*(x)f(x)g(x) = \big(I_n - f^*(x)f(x)\big)g(x).$$

Since $\|f(x)\| < 1$, hence $I_n - f^*(x)f(x)$ is invertible and on the set of positive measure $\{x \in \Omega : g(x) \neq 0\}$ we get:

$$0 \neq g(x) = \big(I_{n(x)} - f^*(x)f(x)\big)^{-1}\big(I_{n(x)} - f^*(x)f(x)\big)g(x).$$

Therefore for almost all $x \in \Omega$:

$$\big(I_{n(x)} - f^*(x)f(x)\big)g(x) \neq 0$$

thus $(1 - f^*f)g \neq 0$ and $(I - A^*A)B \neq 0$ for $B \in \mathfrak{A}$, $B \neq 0$. Lemma 4.1 ensures us that $I - A^*A$ is one-to-one: it is also positive (as $1 - f^*f$ is) and this implies $\|A\xi\| < \|\xi\|$ for $\xi \neq 0$.

Now assume that $\|A\xi\| < \|\xi\|$ or, equivalently, that $I - A^*A \geq 0$ and $B \mapsto (I - A^*A)B$ is one to one. This means that $1 - f^*f \geq 0$ and $(1 - f^*f)g \neq 0$ for $g \in \mathfrak{L}, g \neq 0$. Suppose on the contrary that the set $\{x \in \Omega : \|f(x)\| < 1\}$ is not of full measure. Since the space Ω is localizable, we can restrict our arguments to some subset of finite measure, say $\{0,1\}^{\mathbb{A}}$ where $|\mathbb{A}| = \alpha$ and still $\lambda\big(\{0,1\}^{\mathbb{A}} \setminus \{x \in \Omega : \|f(x)\| < 1\}\big) > 0$. Put $f_0 = f|_{\{0,1\}^{\mathbb{A}}} : \{0,1\}^{\mathbb{A}} \to M_n$. To get a contradiction, it is enough to find measurable $g_0 : \{0,1\}^{\mathbb{A}} \to M_n$ nonzero on some set of positive measure and such that $(1 - f_0^*f_0)g = 0$—then by putting $g = g_0$ on $\{0,1\}^{\mathbb{A}}$ and 0 outside $\{0,1\}^{\mathbb{A}}$ we obtain a contradiction. We can find a countable set $\mathbb{J} \subset \mathbb{A}$ of "variables" such that $f_0(x) = f_1(x|_{\mathbb{J}})$ where $f_1 : \{0,1\}^{\mathbb{J}} \to M_n$ is measurable. We will find measurable $g_1 : \{0,1\}^{\mathbb{J}} \to M_n$ such that $g_0(x) := g_1(x|_{\mathbb{J}})$ will have the desired property. Once g_1 is constructed we have:

$$\begin{aligned}(4)\qquad \lambda_\alpha(\{x \in \{0,1\}^{\mathbb{A}} : g(x) \neq 0\}) &= \lambda_\alpha(\mathrm{Cyl}\{x \in \{0,1\}^{\mathbb{J}} : g_1(x) \neq 0\}) = \\ &= \lambda_{\mathbb{J}}(\{x \in \{0,1\}^{\mathbb{J}} : g_1(x) \neq 0\}).\end{aligned}$$

Moreover,

$$(5)\qquad \big((1 - f_0^*f_0)g\big)(x) = 0 \iff \big((1 - f_1^*f_1)g_1\big)(x|_{\mathbb{J}}) = 0$$

and finally

$$\begin{aligned}(6)\qquad \lambda_{\mathbb{J}}(\{x \in \{0,1\}^{\mathbb{J}} : \|f_1(x)\| = 1\}) &= \lambda_\alpha(\mathrm{Cyl}\{x \in \{0,1\}^{\mathbb{J}} : \|f_1(x)\| = 1\}) = \\ &= \lambda_\alpha(\{x \in \{0,1\}^{\mathbb{A}} : \|f_0(x)\| = 1\}).\end{aligned}$$

Combining (4), (5) and (6) we see that it is enough to consider the case $\alpha \leq \aleph_0$. In case $\alpha < \aleph_0$ it is easy to construct g_0 so let $\alpha = \aleph_0$, i.e. we deal with the standard Cantor set $\{0,1\}^{\mathbb{N}}$. Since for each $x \in \{0,1\}^{\mathbb{N}}$ we deal with finite matrices, we get $\|f_0(x)\| = 1 \iff \mathcal{N}\big((1 - f_0^*f_0)(x)\big) \neq 0$ and the set $G := \{x \in \{0,1\}^{\mathbb{N}} : \mathcal{N}\big((1 - f_0^*f_0)(x)\big) \neq 0\}$ satisfies $\lambda(G) > 0$. As $(\{0,1\}^{\mathbb{N}}, \lambda)$ is a standard Borel space and f_0 is measurable, therefore we can define a measurable (in the Effros-Borel structure—see [1, 2] or Chapter 8 in [18]) *multifunction*

$$\Psi : \{0,1\}^{\mathbb{N}} \ni x \mapsto \mathcal{N}\big((1 - f_0^*f_0)(x)\big).$$

(Measurability of Ψ is a well-known fact and may easily be deduced e.g. from Proposition 2.4 in [3].) It follows from Effros' theory that we can find a sequence $h_1, h_2, \ldots : \{0,1\}^{\mathbb{N}} \to M_n$ of measurable functions such that $(h_k(x))_{k\in\mathbb{N}} \subset \Psi(x)$ and $(h_k(x))_{k\in\mathbb{N}}$ is dense in $\Psi(x)$ for almost all x, say, on a set D. Then $\lambda(G \cap D) > 0$. We conclude that $\lambda(F_N) > 0$ for some $N \in \mathbb{N}$ where $F_k := \{x \in \{0,1\}^{\mathbb{N}} : h_k(x) \neq 0\}$; indeed, if $\lambda(F_n) = 0$ for every $n \in \mathbb{N}$ then $\lambda(\bigcup_{n=1}^{\infty} F_n) = 0$ and $(D \cap G) \setminus \bigcup_{n=1}^{\infty} F_n \neq \emptyset$ but for $x \in (D \cap G) \setminus \bigcup_{n=1}^{\infty} F_n$ we have $h_n(x) = 0$ for $n \in \mathbb{N}$ and thus $\{h_n(x)\}_{n\in\mathbb{N}} =$

$\{0\}$ is not a dense subset of $\Psi(x)$. We have shown that $h_N \neq 0$ on some subset of $D \cap A$ of positive measure. We define $g_1 : \{0,1\}^{\mathbb{N}} \to M_n$ by the formula

$$g_1(x) = \begin{cases} \frac{h_N(x)}{\|h_N(x)\|}, & x \in D \cap G \text{ and } h_N(x) \neq 0 \\ 0, & x \notin D \cap G \text{ or } h_N(x) = 0 \end{cases}$$

This clearly defines a nonzero, measurable and essentially bounded function, i.e. $0 \neq g_1 \in L^\infty(\{0,1\}^{\mathbb{N}}, \lambda; M_n)$ which satisfies $\big((1 - f_0^* f_0) g_1\big)(x) = 0$. Indeed, if $g_1(x) \neq 0$ then $h_N(x) \in \mathcal{N}\big((1 - f_0^* f_0)(x)\big)$ and $g_1(x) \in \mathcal{N}\big((1 - f_0^* f_0)(x)\big)$.

□

4.3. **Corollary.** *Fix an increasing sequence* $(a_n)_{n=1}^\infty \subset (0,1)$ *with* $\lim_{n\to\infty} a_n = 1$. *For* $A \in \mathfrak{A}$ *the following conditions are equivalent:*

(1) $\|A\xi\| < \|\xi\|$ *for every* $\xi \neq 0$,
(2) there exists a countable central decomposition of identity, namely a sequence $(Z_n)_{n=1}^\infty \subset \mathfrak{Z}(\mathfrak{A})$ *of projections with* $\sum_{n=1}^\infty Z_n = I$, *such that* $\|AZ_n\| \leq a_n$ *for every* $n > 0$.

Proof. Let $f = \Phi(A)$ and $p_m(x) := \chi_{A_m}(x) I_{n(x)}$ where $A_m = \{x \in \Omega : a_{m-1} \leq \|f(x)\| < a_m\}$ (with the agreement that $a_0 := 0$). Then p_m are obviously central projections in $\mathfrak{L}$ and therefore $Z_m := \Phi^{-1}(g_m)$ are central projections in $\mathfrak{A}$. As $A_m \cap A_k = \emptyset$ for $m \neq k$ and $\bigcup_{m=1}^\infty A_m = \Omega$, we have $Z_m \perp Z_k$, $\sum_{m=1}^\infty Z_m = I$ and finally

$$\|AZ_m\| = \|fp_m\| \leq a_m.$$

For the opposite implication, take $\xi = \sum_{m=1}^\infty Z_m \xi \neq 0$. Then $Z_m \xi \neq 0$ for some m, hence:

$$\|A\xi\|^2 = \|\sum_{m=1}^\infty Z_m A\xi\|^2 = \sum_{m=1}^\infty \|Z_m A\xi\|^2 =$$

$$= \sum_{m=1}^\infty \|Z_m A(Z_m\xi)\|^2 < \sum_{m=1}^\infty \|Z_m \xi\|^2 = \xi^2.$$

□

5. The algebra of affiliated operators

5.1. **Definition.** We say that a closed densely defined operator T acting in H is *affiliated* with a von Neumann algebra $\mathfrak{A} \subset B(H)$ when $TU = UT$ for any unitary operator $U \in \mathfrak{A}'$. We denote the set of all such operators by $\text{Aff}(\mathfrak{A})$. (It turns out that in case of $\mathfrak{A}$ being a finite von Neumann algebra the set $\text{Aff}(\mathfrak{A})$ is a $*$-algebra, see [10].)

5.2. *Remarks.* Let $\mathfrak{A} = B(H)$ and T be an arbitrary closed operator (with dense domain). Then $\mathfrak{A}' = \mathbb{C}I$ and each unitary operator U in $\mathfrak{A}'$ satisfies $UT = TU$, therefore any closed densely defined operator is affiliated with $\mathfrak{A}$. Note also that the usual notion of commuting operators (where one is unbounded) is the following: we say that $A \in B(H)$ and T (acting in H) *commute* if $AT \subset TA$—the inclusion is understood as the inclusion of graphs. In the definition of an affiliated operator we demand the exact equality of domains of both sides. However one can weaken this condition by the usual notion of commuting (just multiply by U^{-1} from both sides). The definition is motivated by the following observation: if $T \in \text{Aff}(\mathfrak{A})$ and T is bounded, then $T \in \mathfrak{A}$. So, operators from $\text{Aff}(\mathfrak{A}) \setminus \mathfrak{A}$ do not belong to $\mathfrak{A}$ 'only' due to the lack of boundedness.

In order to define the trace on $\mathrm{Aff}(\mathfrak{A})$ we will prove that every operator in $\mathrm{Aff}(\mathfrak{A})$ has a special form $\sum_{i\in\mathbb{I}} S_i Z_i$ where the set of indices $\mathbb{I}$ is countable and $(Z_i)_{i\in\mathbb{I}}$ is a countable family of central projections with $\sum_{i\in\mathbb{I}} Z_i = I$. The operator $T := \sum_{i\in\mathbb{I}} S_i Z_i$ is understood as follows: its domain is the set of all $\xi \in H$ such that the series $\sum_{i\in\mathbb{I}} \|S_n Z_n \xi\|^2$ converges and $T\xi = \sum_{i\in\mathbb{I}} S_i Z_i \xi$. The lemma below will ensure us that this operator is densely defined and closed. We will be using this fact without comments.

5.3. **Lemma.** *Let $(Z_i)_{i\in\mathbb{I}} \subset \mathfrak{Z}(\mathfrak{A})$ be a countable central decomposition of identity. Denote by H_i the range of Z_i. Then there exists a unitary operator $U : H \to \bigoplus_{i\in\mathbb{I}} H_i$ such that for each countable family $(S_i)_{i\in\mathbb{I}} \subset \mathfrak{A}$:*

$$U\Big(\sum_{i\in\mathbb{I}} S_i Z_i\Big)U^{-1} = \mathrm{diag}\Big((S_i|_{H_i})\Big).$$

Proof. It is enough to define $U\xi = (Z_i\xi)_{i\in\mathbb{I}}$. □

Furthermore, from some technical reasons we will be interested in using the transforms defined on the set of unbounded operators and mapping them to bounded ones.

In the existing literature there are at least two such transformations. The first was studied e.g. by Kaufman [7] and it associates with every closed densely defined operator T the operator $T(I+T^*T)^{-\frac{1}{2}}$. The second, quite similar to the first, is the so-called $\mathfrak{b}$-transform introduced by Niemiec [12] and given by $\mathfrak{b}(T) = T(I+|T|)^{-1}$ where $|T| = (T^*T)^{\frac{1}{2}}$. We will use the following properties of his transform:

5.4. **Theorem.** *Let T be a closed densely defined operator in H and $\mathfrak{A} \subset B(H)$ a von Neumann algebra. Then:*

(1) $\mathfrak{b}(T)$ is everywhere defined and bounded,
(2) T is bounded $\iff$ $\|\mathfrak{b}(T)\| < 1$; conversely, if $\|S\| < 1$, then there is $T \in B(H)$ such that $S = \mathfrak{b}(T)$,
(3) $T \in \mathrm{Aff}(\mathfrak{A}) \iff \mathfrak{b}(T) \in \mathfrak{A}$,
(4) a closed linear subspace K of H reduces $T \iff K$ reduces $\mathfrak{b}(T)$,
(5) the mapping $\mathfrak{b}$ establishes a one-to-one correspondence between the set of all densely defined closed operators and the set $\{S \in B(H) : \|T\xi\| < \|\xi\|,\ \xi \neq 0\}$; the inverse transform is given by $S \mapsto \mathfrak{u}(S) := S(I-|S|)^{-1}$.

5.5. **Theorem.** *Let $(c_n)_{n=1}^{\infty} \subset (0,\infty)$ be an increasing sequence with $\lim_{n\to\infty} c_n = \infty$ and $T : \mathrm{dom}(T) \to H$ be a closed densely defined operator. The following conditions are equivalent:*

(1) $T \in \mathrm{Aff}(\mathfrak{A})$,
(2) there exist a sequence $(Z_n)_{n=1}^{\infty} \subset \mathfrak{Z}(\mathfrak{A})$ of projections with $\sum_{n=1}^{\infty} Z_n = I$ and $S \in \mathfrak{A}$ such that $T = \sum_{n=1}^{\infty} c_n S Z_n$.

Proof. First assume that $T = \sum_{n=1}^{\infty} c_n S_n Z_n$. Then for a unitary $U \in \mathfrak{A}'$, as $\|US_nZ_n\xi\| = \|S_nZ_n\xi\|$, we have:

$$\mathrm{dom}\Big(U\sum_{n=1}^{\infty} c_n S_n Z_n\Big) = \mathrm{dom}\Big(\sum_{n=1}^{\infty} c_n S_n Z_n\Big) = \mathrm{dom}\Big(\big(\sum_{n=1}^{\infty} c_n S_n Z_n U\big)\Big)$$

and

$$\big(\sum_{n=1}^{\infty} c_n S_n Z_n\big)U\xi = \sum_{n=1}^{\infty} c_n S_n Z_n U\xi = \sum_{n=1}^{\infty} c_n U S_n Z_n \xi = U\big(\sum_{n=1}^{\infty} c_n S_n Z_n\big)\xi.$$

Now assume that $T \in \mathrm{Aff}(\mathfrak{A})$. Then $\mathfrak{b}(T) \in \mathfrak{A}$ and $\|\mathfrak{b}(T)\xi\| < \|\xi\|$ for $\xi \neq 0$. Using Corollary 4.3 with $a_n := \frac{c_n}{1+c_n}$ we obtain central projections $(Z_n)_{n=1}^{\infty}$ with

$\sum_{n=1}^{\infty} Z_n = I$ such that $\|\mathfrak{b}(T)Z_n\| \leq a_n < 1$. Therefore, there exists $(S_n)_{n=1}^{\infty} \subset B(H)$ such that $\mathfrak{b}(S_n) = \mathfrak{b}(T)Z_n$. We can express S_n directly as $S_n = \mathfrak{b}(T)Z_n\big(I - |\mathfrak{b}(T)Z_n|\big)^{-1}$ and this formula clearly implies $S_n \in \mathfrak{A}$. We have:

$$S_n Z_n = \mathfrak{b}(T)Z_n^2\big(I - |\mathfrak{b}(T)Z_n|\big)^{-1} = \mathfrak{b}(T)Z_n\big(I - |\mathfrak{b}(T)Z_n|\big)^{-1} \tag{7}$$

and

$$\|S_n\| = \|\mathfrak{b}(T)Z_n\big(I - |\mathfrak{b}(T)Z_n|\big)^{-1}\| = \Big\||\mathfrak{b}(T)Z_n|\big(I - |\mathfrak{b}(T)Z_n|\big)^{-1}\Big\| = \tag{8}$$

$$= \big\|\mathfrak{u}(|b(T)Z_n|)\big\| = u\Big(\big\||b(T)Z_n|\big\|\Big) \leq u(a_n) = c_n$$

Define $S := \sum_{n=1}^{\infty} \frac{1}{c_n} S_n$. From (7) we have that the series converges—hence, $S \in \mathfrak{A}$. Moreover, $c_n S Z_n = S_n Z_n$. In order to prove $T = \sum_{n=1} c_n S Z_n$ it is enough to show that $\mathfrak{b}(T) = \mathfrak{b}(\sum_{n=1} c_n S Z_n)$. Using Lemma 5.3 we get

$$\mathfrak{b}\big(U \sum_{n=1} c_n S Z_n U^{-1}\big) = U\mathfrak{b}\big(\sum_{n=1} c_n S Z_n\big)U^{-1} = \mathfrak{b}\Big(\bigoplus_{n=1}^{\infty} c_n S|_{H_n}\Big) =$$

$$= \bigoplus_{n=1}^{\infty} \mathfrak{b}\big(c_n S|_{H_n}\big) = \bigoplus_{n=1}^{\infty} \mathfrak{b}(c_n S)|_{H_n} = \bigoplus_{n=1}^{\infty} \mathfrak{b}(T)|_{H_n} = \mathfrak{b}(T).$$

□

5.6. **Corollary.** *If $(c_n)_{n=1}^{\infty}$ is as above then for a finite collection $T_1, ..., T_k \in \mathrm{Aff}(\mathfrak{A})$ we can find a family $(Z_{n_1,n_2,...,n_k})_{n_1,n_2,...,n_k=1}^{\infty} \subset \mathfrak{Z}(\mathfrak{A})$ of central projections with $\sum_{n_1,...,n_k=1}^{\infty} Z_{n_1,...,n_k} = I$ and $S_1, ..., S_k \in \mathfrak{A}$ such that*

$$T_j = \sum_{n_1,...,n_k=1}^{\infty} c_{n_j} S_j Z_{n_1,...,n_k}.$$

Proof. Write each T_j as $\sum_{n=1}^{\infty} c_n S_j Z_n^{(j)}$ and define $Z_{n_1,...,n_k} = Z_{n_1}^{(1)} Z_{n_2}^{(2)} ... Z_{n_k}^{(k)}$. □

5.7. *Remark.* Given a finite collection $T_1, ..., T_k \in \mathrm{Aff}(\mathfrak{A})$ express them as above. We can find joint *core* $D \subset \mathrm{dom}(T_1) \cap ... \cap \mathrm{dom}(T_k) \cap \mathrm{dom}(T_1^*) \cap ... \cap \mathrm{dom}(T_k^*)$ for each of operators $T_1, ..., T_k, T_1^*, ..., T_k^*$: just put $D = \mathrm{lin} \bigcup_{n_1,...,n_k=1}^{\infty} \mathcal{R}(Z_{n_1,...,n_k})$. This is no longer true in case of infinite families of operators $T_1, T_2, ... \in \mathrm{Aff}(\mathfrak{A})$.
We can also prove directly in our setting the following fact: if $T_1, T_2 \in \mathrm{Aff}(\mathfrak{A})$ and $T_1 \subset T_2$, then $T_1 = T_2$. In fact, we have for appropriate sequence $(Z_{n_1,n_2})_{n_1,n_2=1}^{\infty}$ that $n_1 S_1 Z_{n_1,n_2} = T_1 Z_{n_1,n_2} \subset T_2 Z_{n_1,n_2} = n_2 S_2 Z_{n_1,n_2}$ but both sides are in $B(H)$ which yields equallity and thus $T_1 = T_2$.

Slight modification of the above proof shows that $\sum_{i\in\mathbb{I}} S_i Z_i \in \mathrm{Aff}(\mathfrak{A})$ for a countable decomposition of the identity $(Z_i)_{i\in\mathbb{I}}$ and $(S_i)_{i\in\mathbb{I}} \subset \mathfrak{A}$. The theorem above allows us to define the trace in the following manner:

5.8. **Definition.** For $T = \sum_{i\in\mathbb{I}} S_i Z_i \in \mathrm{Aff}(\mathfrak{A})$ where $(Z_i)_{i\in\mathbb{I}}$ is some countable decomposition of identity and $(S_i)_{i\in\mathbb{I}} \subset \mathfrak{A}$ we define

$$\mathrm{tr}_{\mathrm{Aff}}(T) := \sum_{i\in\mathbb{I}} \mathrm{tr}(S_i) Z_i.$$

In particular, for $T = \sum_{n=1}^{\infty} c_n S Z_n$ we have $\mathrm{tr}_{\mathrm{Aff}}(T) = \sum_{n=1}^{\infty} c_n \mathrm{tr}(S) Z_n$.

The theorem above together with the definition of the trace imply that $\mathrm{tr}_{\mathrm{Aff}}(T) \in \mathrm{Aff}\big(\mathfrak{Z}(\mathfrak{A})\big)$, but it is not clear at the moment, that $\mathrm{tr}_{\mathrm{Aff}}(T) \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$. However we have the following:

5.9. **Theorem.** $\mathrm{Aff}(\mathfrak{Z}(\mathfrak{A})) = \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$

Proof. Take $T \in \mathrm{Aff}(\mathfrak{Z}(\mathfrak{A}))$ of the form $T = \sum_{n=1}^{\infty} nSZ_n$ with $S \in \mathfrak{Z}(\mathfrak{A})$ and $A = \sum_{n=1}^{\infty} nBZ'_n$ with $B \in \mathfrak{A}$. Take a joint decomposition of the identity $(Z_{n,k})_{n,k=1}^{\infty}$ such that $T = \sum_{n,k=1}^{\infty} nSZ_{n,k}$ and $A = \sum_{n,k=1}^{\infty} kBZ_{n,k}$ and compute:

$$TA = \sum_{n,k=1}^{\infty} nkSBZ_{n,k} = \sum_{n,k=1}^{\infty} nkBSZ_{n,k} = AT.$$

Conversely take $T \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$ of the form $T = \sum_{n=1}^{\infty} nSZ_n$. Then $\frac{1}{n}TZ_n = SZ_n \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A})) \cap \mathfrak{A} \subset \mathfrak{Z}(\mathfrak{A})$ thus $S = \sum_{n=1}^{\infty} SZ_n \in \mathfrak{A}$. Therefore $T \in \mathrm{Aff}(\mathfrak{Z}(\mathfrak{A}))$. □

Straight from the definition we see that $\mathrm{tr}_{\mathrm{Aff}}$ is linear, positive and satisfies $\mathrm{tr}_{\mathrm{Aff}}(AB) = \mathrm{tr}_{\mathrm{Aff}}(BA)$ as well $\mathrm{tr}_{\mathrm{Aff}}(C) = C$ for $C \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$. It remains to prove that the definition is correct, namely that it does not depend from the choice of sequences $(S_i)_{i\in\mathbb{I}}, (Z_i)_{i\in\mathbb{I}}$. Indeed, suppose that $T = \sum_{i\in\mathbb{I}} S_i Z_i = \sum_{j\in\mathbb{J}} S'_j Z'_j$. Then

$$TZ_iZ'_j = S_iZ_iZ'_j = S'_jZiZ'_j$$

and therefore

$$\mathrm{tr}(S_i)Z_iZ'_j = \mathrm{tr}(S_iZ_iZ'_j) = \mathrm{tr}(S'_jZ_iZ'_j) = \mathrm{tr}(S'_j)Z_iZ'_j.$$

Summing up such equalities yields:

$$\sum_{i\in\mathbb{I}} \mathrm{tr}(S_i)Z_i = \sum_{i\in\mathbb{I}}\sum_{j\in\mathbb{J}} \mathrm{tr}(S_i)Z_iZ'_j = \sum_{i\in\mathbb{I}}\sum_{j\in\mathbb{J}} \mathrm{tr}(S'_j)Z_iZ'_j =$$

$$= \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}} \mathrm{tr}(S'_j)Z_iZ'_j = \sum_{j\in\mathbb{J}} \mathrm{tr}(S'_j)Z'_j$$

Therefore we have proven the following:

5.10. **Theorem.** *Let $\mathfrak{A}$ be a finite type* I *von Neumann algebra. There exists a linear mapping* $\mathrm{tr}_{\mathrm{Aff}} : \mathrm{Aff}(\mathfrak{A}) \to \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$ *with the following properties:*

(Tr'1) $\mathrm{tr}_{\mathrm{Aff}}(A) \geq 0$ *for* $A \geq 0$,
(Tr'2) $\mathrm{tr}_{\mathrm{Aff}}(TS) = \mathrm{tr}_{\mathrm{Aff}}(ST)$ *for* $T, S \in \mathrm{Aff}(\mathfrak{A})$,
(Tr'3) $\mathrm{tr}_{\mathrm{Aff}}(Z) = Z$ *for* $Z \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$.

5.11. *Remark.* Take a positive element $A \in \mathfrak{A}$ and $c > 0$ such that $A < cI$. Then $\mathrm{tr}_{\mathrm{Aff}}(A) \geq 0$ and $\mathrm{tr}_{\mathrm{Aff}}(cI - A) \geq 0$ hence:

$$0 \leq \mathrm{tr}_{\mathrm{Aff}}(A) \leq \mathrm{tr}_{\mathrm{Aff}}(cI) = cI.$$

Therefore $\mathrm{tr}_{\mathrm{Aff}}(A) \in \mathfrak{A}$. Each $S \in \mathfrak{A}$ is a linear combination of (at least four) positive elements, therefore $\mathrm{tr}_{\mathrm{Aff}}(S) \in \mathfrak{A}$ for $S \in \mathfrak{A}$. From the uniqueness of the trace, we have $\mathrm{tr}_{\mathrm{Aff}}|_{\mathfrak{A}} = \mathrm{tr}$.

6. The uniqueness of the trace

The uniqueness of the trace is established by the following

6.1. **Theorem.** *Suppose that there exists another linear mapping* $\mathrm{tr}' : \mathrm{Aff}(\mathfrak{A}) \to \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$ *which satisfies above (Tr'1-Tr'3) axioms. Then* $\mathrm{tr}' = \mathrm{tr}_{\mathrm{Aff}}$.

Proof. First observe that for a central decomposition $(Z_n)_{n=1}^{\infty}$ of I the operator $\sum_{n=1}^{\infty} nZ_n$ is invertible with $(\sum_{n=1}^{\infty} nZ_n)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n} Z_n \in \mathfrak{A}$. Consider the map

$$f : \mathfrak{A} \ni S \mapsto \mathrm{tr}'(\sum_{n=1}^{\infty} nSZ_n)(\sum_{n=1}^{\infty} nZ_n)^{-1} \in \mathfrak{Z}(\mathfrak{A})$$

Then f is linear and

$$\begin{aligned} f(S_1S_2) &= \mathrm{tr}'(\sum_{n=1}^{\infty} nS_1S_2Z_n)(\sum_{n=1}^{\infty} nZ_n)^{-1} = \\ &= \mathrm{tr}'\Big((\sum_{n=1}^{\infty} \sqrt{n}S_1Z_n)(\sum_{n=1}^{\infty} \sqrt{n}S_2Z_n)\Big)(\sum_{n=1}^{\infty} nZ_n)^{-1} = \\ &= \mathrm{tr}'\Big((\sum_{n=1}^{\infty} \sqrt{n}S_2Z_n)(\sum_{n=1}^{\infty} \sqrt{n}S_1Z_n)\Big)(\sum_{n=1}^{\infty} nZ_n)^{-1} = \\ &= \mathrm{tr}'(\sum_{n=1}^{\infty} nS_2S_1Z_n)(\sum_{n=1}^{\infty} nZ_n)^{-1} = f(S_2S_1). \end{aligned}$$

Further, if $S \geq 0$ then $T := \sum_{n=1}^{\infty} nSZ_n \geq 0$ so also $\mathrm{tr}'(T) \geq 0$ and $f(S) \geq 0$. Finally, for $C \in \mathfrak{Z}(\mathfrak{A})$ we have that $\sum_{n=1}^{\infty} nCZ_n \in \mathrm{Aff}(\mathfrak{Z}(\mathfrak{A})) = \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$ thus

$$\begin{aligned} f(C) = \mathrm{tr}'(\sum_{n=1}^{\infty} nCZ_n)(\sum_{n=1}^{\infty} nZ_n)^{-1} &= (\sum_{n=1}^{\infty} nCZ_n)(\sum_{n=1}^{\infty} nZ_n)^{-1} = \\ &= C(\sum_{n=1}^{\infty} nZ_n)(\sum_{n=1}^{\infty} nZ_n))^{-1} = C. \end{aligned}$$

As f satisfies all properties of the usual trace in $\mathfrak{A}$ we have:

$$\mathrm{tr}'(\sum_{n=1}^{\infty} nSZ_n)(\sum_{n=1}^{\infty} nZ_n)^{-1} = \mathrm{tr}(S)$$

and therefore:

$$\mathrm{tr}'(\sum_{n=1}^{\infty} nSZ_n) = \mathrm{tr}(S)(\sum_{n=1}^{\infty} nZ_n) = \sum_{n=1}^{\infty} n\mathrm{tr}(S)Z_n = \mathrm{tr}_{\mathrm{Aff}}(\sum_{n=1}^{\infty} nSZ_n).$$

□

7. The existence of trace and type of a von Neumann algebra

It is natural to ask whether it is still possible to define the trace on $\mathrm{Aff}(\mathfrak{A})$ where $\mathfrak{A}$ is not necesseraily of type I. It appears that the existence of the trace on $\mathrm{Aff}(\mathfrak{A})$ implies that $\mathfrak{A}$ must be a finite type I von Neumann algebra:

7.1. **Theorem.** *Let $\mathfrak{A}$ be an arbitrary von Neumann algebra. Suppose that there exists another mapping $\varphi : \mathrm{Aff}(\mathfrak{A}) \to \mathrm{Aff}(\mathfrak{A})$ satisfying the following axioms:*

(1) if $A, B \in \mathfrak{A}$ are such that $\varphi(A) \in \mathfrak{A}$ then $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$ for scalars $\alpha, \beta \in \mathbb{C}$
(2) if $A = A^* \geq 0$ then $\varphi(A) = \varphi(A^*) \geq 0$
(3) if $A = A^* \geq 0$ and $B = B^* \geq 0$ are such that $B \in \mathfrak{A}$ and $\varphi(A) \in \mathfrak{A}$ then $\varphi(A + B) = \varphi(A) + \varphi(B)$,
(4) if $A, B \in \mathfrak{A}$ then $\varphi(AB) = \varphi(BA)$,
(5) if $Z \in \mathfrak{Z}(\mathfrak{A})$ then $\varphi(Z) = Z$,
(6) if $A \in \mathfrak{A}$ and $U \in \mathfrak{A}$ is unitary then $U^*\varphi(A)U = \varphi(A)$

Then $\mathfrak{A}$ is a finite type I *von Neumann algebra.*

Proof. First observe that if $A \in \mathfrak{A}$ then $\varphi(A)$ is bounded and $\varphi(A) \in \mathfrak{A}$. Indeed, it is enough to show this for nonnegative $A \in \mathfrak{A}$. Such A satisfies $\|A\|I - A \geq 0$, therefore $\varphi(\|A\|I - A) \geq 0$ and $\varphi(A) \geq 0$. But

$$\varphi(\|A\|I - A) = \varphi\big(\|A\|I\big) - \varphi(A) = \|A\|I - \varphi(A)$$

so we get $0 \leq \varphi(A) \leq \|A\| I$, hence $\varphi(A)$ is bounded. As $\varphi(A)$ commutes with each unitary operator in $\mathfrak{A}$ then $\varphi(A) \in \mathfrak{Z}(\mathfrak{A})$ for $A \in \mathfrak{A}$. We have shown that $\varphi|_{\mathfrak{A}}$ satisfies all properties of the usual trace, hence $\mathfrak{A}$ is finite.

Suppose that $\mathfrak{A}$ is not of type I—therefore one can find a central projection $Z \in \mathfrak{Z}(\mathfrak{A})$ such that $\mathfrak{A}_0 := \mathfrak{A}Z$ is of type II_1. Denote by K the Hilbert space in which $\mathfrak{A}_0$ acts, i.e. $K = \mathcal{R}(Z)$. Let us observe that $\mathrm{tr}_{\mathfrak{A}}|_{\mathfrak{A}_0}$ is the trace $\mathrm{tr}_{\mathfrak{A}_0}$ of $\mathfrak{A}_0$. Indeed:

$$\mathrm{tr}_{\mathfrak{A}}(XZ) = \mathrm{tr}_{\mathfrak{A}}(X)Z \in \mathfrak{A}_0$$

thus we get

$$\mathrm{tr}_{\mathfrak{A}}(\mathfrak{A}_0) = \mathrm{tr}_{\mathfrak{A}}(\mathfrak{A}Z) \subset \mathfrak{A}_0 \cap \mathfrak{Z}(\mathfrak{A}) = \mathfrak{Z}(\mathfrak{A}_0).$$

and we see that $\mathrm{tr}_{\mathfrak{A}}|_{\mathfrak{A}_0}$ satisfies all the axioms of the trace.
Every type II_1 von Neumann algebra $\mathfrak{M}$ has a following property: for each projection $P \in \mathfrak{M}$ and central projection $C \in \mathfrak{Z}(\mathfrak{M})$ such that $0 \leq C \leq \mathrm{tr}_{\mathfrak{M}}(P)$ one can find a projection $Q \leq P$ in $\mathfrak{M}$ with $\mathrm{tr}_{\mathfrak{M}}(Q) = C$. By induction we define the sequence $(P_n)_{n=1}^{\infty}$ of projections in the following way: for $I_K = Z$ find $P_1 \leq Z$ in $\mathfrak{A}_0$ with $\mathrm{tr}_{\mathfrak{A}_0}(P_1) = \frac{1}{2} I_K$, further for $Z - P_1$ find $P_2 \leq I_K - P_1$ in $\mathfrak{A}_0$ with $\mathrm{tr}_{\mathfrak{A}_0}(P_2) = \frac{1}{4} I_K$ and so on: we can find a projection $P_n \leq I_K - \sum_{k=1}^{n-1} P_k$ in $\mathfrak{A}_0$ such that $\mathrm{tr}_{\mathfrak{A}_0}(P_n) = \frac{1}{2^n} I_K$. Observe that

$$\mathrm{tr}_{\mathfrak{A}_0}(\sum_{n=1}^{\infty} P_n) = \sum_{n=1}^{\infty} \mathrm{tr}_{\mathfrak{A}_0}(P_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} I_K = I_K.$$

As $\mathrm{tr}_{\mathfrak{A}_0}$ is strictly positive we get $\sum_{n=1}^{\infty} P_n = I_K$. Put $T_N := \sum_{k=N+1}^{\infty} 2^k P_k$ (understood as usual). Then T_N is closed, densely defined and $T_N = T_N^* \geq 0$. As each $P_k \in \mathfrak{A}$ then $T_N \in \mathrm{Aff}(\mathfrak{A}_0)$ and

$$\varphi(T_0) = \varphi\big(\sum_{n=1}^{N} 2^n P_n\big) + \varphi(T_N) = \mathrm{tr}\big(\sum_{n=1}^{N} 2^n P_n\big) + \varphi(T_N) = N I_K + \varphi(T_N).$$

Therefore, taking $\xi \in \mathrm{dom}(\varphi(T_0)) = \mathrm{dom}(\varphi(T_N))$ we compute:

$$\langle \varphi(T_0)\xi, \xi \rangle = \langle N\xi + \varphi(T_N)\xi, \xi \rangle = N\|\xi\|^2 + \langle \varphi(T_N)\xi, \xi \rangle \geq N\|\xi\|^2$$

□

7.2. *Remark.* Note that the above proof also implies that conditions (1) and (3) may be simplified to the following: (1') if $A, B \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$ then $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$—in other words, $\varphi|_{\mathfrak{A}}$ is linear, (3') if $A = A^* \in \mathrm{Aff}(\mathfrak{A})$ and $B = B^* \in \mathfrak{A}$ are positive then $\varphi(A + B) = \varphi(A) + \varphi(B)$ but we have formulated required properties in a weakest possible form.

As an immediate consequence of the existence of the trace we get the following:

7.3. **Corollary.** *Suppose that $\mathfrak{A}$ is finite type* I *von Neumann algebra. There are no $A, B \in \mathrm{Aff}(\mathfrak{A})$ such that $AB - BA = I$.*

Proof. Apply the trace for both sides. □

It is well known that the relation $ab - ba = e$ can not be satisfied in any normed unital algebra (where e denotes the unit element), see Theorem 13.6 in [13]. However, it is possible to find selfadjoint *unbounded* operators P, Q such that $PQ - QP = I$. Therefore knowing that $\mathrm{Aff}(\mathfrak{A})$ has a fine algebraic structure, one can ask whether it is possible to find A, B in $\mathrm{Aff}(\mathfrak{A})$ satisfying this relation. The above theorem states that it is impossible for type a finite type I von Neumann algebra. It appears that it is also impossible in any finite von Neumann algebra: this is done in [8]. For the von Neumann algebras which are not finite, the set $\mathrm{Aff}(\mathfrak{A})$ no longer forms a $*$-algebra. We stress that the importance of this relation follows from its connection to quantum mechanics and so called *Heisenberg uncertainty principle*.

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