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# Financial markets modelling by integrals driven by a fractional Brownian motion

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# Financial markets modelling by integrals driven by fractional Brownian motion

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**Abstract**

We prove Girsanov type theorem for Wiener integrals with respect to the fractional Brownian motion. We apply this result to the option pricing problem in fractional Black-Scholes model with time depend volatility.

## 1 Introduction

A fractional Brownian motion (fBm) $(B_t^H)_{t\in[0,T]}$ with Hurst parameter $H \in (1/2, 1)$ is a continuous zero-mean Gaussian process with covariance

$$EB_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

defined on some probability space $(\Omega, \mathcal{F}, P)$. The long-range dependency property makes fBm suitable model in mathematical finance. In [9] Sottinen and Valkeila considered so called *fractional Black-Scholes model* where the stock price is given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. In this paper we consider more general model. In above equation we replace the constants $\mu, \sigma$ with deterministic function, so the dynamics for the stock price $S$ becomes

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dB_t^H.$$

We assume that there is also the bond asset $B_t = \exp(rt)$, where $r > 0$. Since $B^H$ is not a semimartingale and martingale measures do not exist, we cannot apply usual method for option prising. We use introduced in [9] *average risk neutral measure* i.e. measure $Q$, equivalent to the measure $P$, such that $E_Q(S_t/B_t) = S_0$. The existence and uniqueness of $Q$ follows from Girsanov Theorem given in section 4. We define the *fair price* $C(F_T)$ of option $F_T$ by

$$C(F_T) = E_Q \frac{F_T}{B_T}.$$

Note that if $H = 1/2$, $B^H$ becomes ordinary Brownian motion and above model coincides with the well known Black-Scholes model. Also $Q$ becomes the martingale measure and $C(F_T)$ coincides with the classical one as well. For options depending only on the price of $S$ at the expiration time $T$, i.e. $F_T = F(S_T)$, we obtain following explicit formula

$$C(F_T) = e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F\left( S_0 \exp\left( \sqrt{\langle \sigma, \sigma \rangle_T^H} y + rT - \frac{1}{2} \langle \sigma, \sigma \rangle_T^H \right) \right) e^{y^2/2} dy,$$

where

$$\langle f, g \rangle_T^H = H(2H-1) \int_0^T \int_0^T f(t) g(s) |t-s|^{2H-2} ds dt. \tag{1.1}$$

In particular for European call-option $F(S_T) = (S_T - K)^+$ we obtain analogue of Black-Scholes formula:

$$C\left( (S_T - K)^+ \right) = S_0 \Phi(y_1) - K e^{-rT} \Phi(y_2), \tag{1.2}$$

where

$$y_1 = \frac{\log \frac{S_0}{K} + rT + \frac{1}{2} \langle \sigma, \sigma \rangle_T^H}{\sqrt{\langle \sigma, \sigma \rangle_T^H}}, \quad y_2 = \frac{\log \frac{S_0}{K} + rT - \frac{1}{2} \langle \sigma, \sigma \rangle_T^H}{\sqrt{\langle \sigma, \sigma \rangle_T^H}}.$$

Obtained results coincide with Sottinen and Valkeila's results in [9] for $\sigma(t) \equiv \sigma$.

# 2 Preliminaries

## 2.1 Abel integral equation

We need to start from some remarks on integral equations. Let us consider the *Abel integral equation*, i.e.

$$F(x) = \int_a^x \frac{G(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha \in (0,1), \tag{2.1}$$

where $F$ is a known function and $G$ is an unknown function.

Firstly we show that there is at most one solution of (2.1) in $\mathbb{L}^1_{[a,b]}$. Let $G_1$, $G_2 \in \mathbb{L}^1_{[a,b]}$ be a solutions of (2.1), then $G = G_1 - G_2$ is the solution of

$$\int_a^x \frac{G(t)}{(x-t)^{1-\alpha}} dt = 0.$$

Fix $y \in (a,b)$ and consider function $\phi : T_y \to \mathbb{R}$ given by

$$\phi(x,t) = \frac{G(t)}{(x-t)^{1-\alpha}(y-x)^\alpha},$$

where $T_y = \{(x,t) : a \leqslant t \leqslant x \leqslant y\}$. By Tonelli's Theorem, we obtain

$$\begin{aligned} \int_{T_y} |\phi(x,t)| dx dt &= \int_a^y |G(t)| \int_t^y \frac{dx}{(x-t)^{1-\alpha}(y-x)^\alpha} dt \\ &= \beta(\alpha, 1-\alpha) \int_a^y |G(t)| dt < +\infty, \end{aligned}$$

where $\beta(\alpha, 1-\alpha) = \int_0^1 x^{-\alpha}(1-x)^{\alpha-1}dx$ is Euler beta function. Therefore, by Fubini's Theorem, we have

$$\begin{aligned}\int_a^y G(t)dt &= \int_a^y G(t)\frac{1}{\beta(\alpha,1-\alpha)}\int_t^y \frac{dx}{(x-t)^{1-\alpha}(y-x)^{\alpha}}dt \\ &= \int_a^y \left(\frac{1}{\beta(\alpha,1-\alpha)}\int_a^x \frac{G(t)}{(x-t)^{1-\alpha}}dt\right)\frac{dx}{(y-x)^{\alpha}} = 0.\end{aligned} \tag{2.2}$$

Above equation holds for any $y \in (a, b)$. Thus $G \equiv 0$, which gives uniqueness of the solutions of (2.1) in $\mathbb{L}^1_{[a,b]}$.

**Proposition 2.1.** *Assume that there exist a solution $G \in \mathbb{L}^1_{[a,b]}$ of (2.1), then it is given by*

$$G(x) = \frac{1}{\beta(\alpha,1-\alpha)}\frac{d}{dx}\int_0^x \frac{F(t)}{(x-t)^{\alpha}}dt.$$

*Proof.* Multiplying both sides of (2.1) by $(z-x)^{-\alpha}$ and integrating with respect to $x$ over an interval $(a, z)$ gives

$$\int_a^z \frac{F(x)}{(z-x)^{\alpha}}dx = \int_a^z\int_a^x \frac{G(t)}{(x-t)^{1-\alpha}(z-x)^{\alpha}}dtdx.$$

By (2.2) we have

$$\frac{1}{\beta(\alpha,1-\alpha)}\int_a^z \frac{F(x)}{(z-x)^{\alpha}}dx = \int_a^z G(t)dt.$$

Thus

$$\frac{1}{\beta(\alpha,1-\alpha)}\frac{d}{dz}\int_a^z \frac{F(x)}{(z-x)^{\alpha}}dx = G(z),$$

which is our claim. □

Next Theorem gives sufficient condition for existence of solutions.

**Theorem 2.2.** *If $F(t) = (t-a)^{-\alpha}F^*(t)$, where $F^*$ is absolute continuous function on $[a, b]$, then there exist $g \in \mathbb{L}^1_{[a,b]}$ such that (2.1) holds.*

*Proof.* See Samko et al. [8] Theorem 14.8. □

## 2.2 Integration with respect to fBm

As we mention in introduction $B^H$ is not a semimartingale, thus classical Itô's integration theory cannot be used. However if we restrict ourselves to deterministic integrands $f : [0, T] \to \mathbb{R}$ satisfying

$$\|f\|_{\mathbb{L}^{1/H}_{[0,T]}} = \left(\int_0^T |f(s)|^{1/H}ds\right)^H < +\infty,$$

there is simple way to define integral (usually called *Wiener integral*). Denote by $\mathcal{E}$ the set of all step functions i.e.

$$f(t) = \sum_{i=1}^n f_i \mathbf{1}_{(t_{i-1}-t_i]}(t),$$

where $0 = t_0 < t_1 < \ldots < t_n = T$, $f_i \in \mathbb{R}$. For $f \in \mathcal{E}$ we define stochastic integral with respect to the fBm by

$$\int_0^T f(s)dB_s^H := \sum_{i=1}^n f_i(B_{t_i}^H - B_{t_{i-1}}^H).$$

Is easy to see that

$$EB_t^H B_s^H = H(2H-1)\int_0^t \int_0^s |u-v|^{2H-2} du dv,$$

which implies that for any $f, g \in \mathcal{E}$ we have

$$E\left(\int_0^T f(s)dB_s^H \int_0^T g(s)dB_s^H\right) = \langle f, g\rangle_T^H, \tag{2.3}$$

where $\langle f, g\rangle_T^H$ is given by (1.1).

Pipiras et al. [5] proved that $\langle f, g\rangle_T^H$ is an inner product in $\mathbb{L}_{[0,T]}^{1/H}$, moreover by Mémin, Mishura, Valkeila [2] for all $f \in \mathbb{L}_{[0,T]}^{1/H}$ we have

$$\sqrt{\langle f, f\rangle_T^H} \leqslant C_H \|f\|_{\mathbb{L}_{[0,T]}^{1/H}}. \tag{2.4}$$

Now we can define stochastic integral as follows. Let $f \in \mathbb{L}_{[0,T]}^{1/H}$ and $\{f_n\}$ be the sequence of step functions, such that $f_n \to f$ in $\mathbb{L}_{[0,T]}^{1/H}$, then $\{f_n\}$ is Cauchy in $\mathbb{L}_{[0,T]}^{1/H}$. By (2.4) and (2.3), we have that $\{\int_0^T f_n(s)dB_s^H\}$ is Cauchy in $\mathbb{L}^2(\Omega)$. Since $\mathbb{L}^2(\Omega)$ is complete, there is $\int_0^T f(s)dB_s^H \in \mathbb{L}^2(\Omega)$ such that

$$\int_0^T f(s)dB_s^H = \lim_n \int_0^T f_n(s)dB_s^H.$$

To construct stochastic integral with respect to the fBm of non-deterministic integrands we use Hölder continuity of paths of $B^H$. This pathwise approach was introduced in Ruzmaikina [7]. Let $C^\alpha$ be the space of $\alpha$-Hölder continuous functions on $[0, T]$. Since $B^H \in C^\alpha$ for $\alpha < H$ with probability one, it follows that Stieltjes integral $\int_0^T X(s,\omega)dB_s^H(\omega)$ exists for almost all $\omega \in \Omega$ and $X(\cdot,\omega) \in C^\gamma$ with $\gamma > 1 - \alpha$.

# 3 Integral representations of fBm

In this section we give some useful representations of fBm and related processes. These results was originated in Kleptsyna et al. [1] and Norros et al. [4].

Consider so-called *fundamental martingale* i.e.

$$M_t^H = \int_0^t k(t,s)dB_s^H, \tag{3.1}$$

where $k(t,s) = \kappa_H^{-1} s^{1/2-H}(t-s)^{1/2-H}$, $\kappa_H = 2H\beta(3/2 - H, H + 1/2)$. Its obvious that $M^H$ is centered Gaussian process.

Let

$$w_t^H = \lambda_H^{-1} t^{2-2H},$$

where $\lambda_H = (2H\Gamma(3-2H)\Gamma(H+1/2))/(\Gamma(3/2-H))$.

**Lemma 3.1.** *$M^H$ has independent increments and variance function $E(M_t^H)^2 = w^H$. In particular, $M^H$ is martingale.*

*Proof.* See [4] Theorem 3.1. □

By Levy's Theorem there is the Brownian motion $W$, such that

$$M_t^H = \left(\frac{2-2H}{\lambda_H}\right)^{1/2} \int_0^t s^{1/2-H} dW_s, \quad t \in [0,T]. \tag{3.2}$$

**Theorem 3.2** (Norros et al. [4])**.** *The fBm has the following integral representation in terms of $W$.*

$$B_t^H = \int_0^t z(t,s) dW_s,$$

*where* $z(t,s) = \mathbf{1}_{\{t>s\}} c_H s^{1/2-H} \int_s^t u^{H-1/2}(u-s)^{H-3/2} du$, $c_H = \left(\frac{H(2H-1)}{\beta(H-1/2, 2-2H)}\right)^{1/2}$.

Now we can show that if $f$ is deterministic, bounded function on $[0,T]$, then the stochastic integral $\int_0^{\cdot} f(s) dB_s^H$ can be represented in terms of $M^H$. Consider deterministic kernel

$$K_H^f(t,s) = H(2H-1) \int_s^t f(r) r^{H-1/2} (r-s)^{H-3/2} dr, \quad 0 \leqslant s \leqslant t \leqslant T.$$

Since $f$ is bounded and $H > 1/2$, it follows that above integral is well defined.

**Theorem 3.3** (Kleptsyna et al. [1])**.** *If $f$ is deterministic, bounded function on $[0,T]$, then we have representation*

$$\int_0^t f(s) dB_s^H = \int_0^t K_H^f(t,s) dM_s^H.$$

*Proof.* In proof we will use following Lemma.

**Lemma 3.4.** *If $f$ is bounded function and $\int_0^T |f_n(s) - f(s)|^p ds \xrightarrow[n\to+\infty]{} 0$ for any $p > 0$, then*

$$\int_0^t \left(K_H^{f_n}(t,s) - K_H^f(t,s)\right)^2 ds \xrightarrow[n\to+\infty]{} 0, \quad t \in [0,T].$$

*Proof.* Set $q > 1$ such that $(H - 3/2)q > -1$ and $p = q/(q-1)$. By Hölder inequality

$$\begin{aligned}
&\int_0^t \left(K_H^{f_n}(t,s) - K_H^f(t,s)\right)^2 ds \\
&\qquad\leqslant k_H \int_0^t \left(\int_s^t |f^n(r) - f(r)| r^{H-1/2}(r-s)^{H-3/2} dr\right)^2 ds \\
&\qquad\leqslant k_H \int_0^t \left[\left(\int_s^t |f^n(r) - f(r)|^p dr\right)^{1/p}\right. \\
&\qquad\qquad \left.\left(\int_s^t \left(r^{H-1/2}(r-s)^{H-3/2}\right)^q dr\right)^{1/q}\right]^2 ds,
\end{aligned}$$

where $k_H = (H(2H-1))^2$. Note that function $\phi : [0,t] \to \mathbb{R}$ given by

$$\phi(s) = \int_s^t \left(r^{H-1/2}(r-s)^{H-3/2}\right)^q dr$$

is continuous. Let $M = \sup_{s \in [0,t]} \phi(s)$. We have

$$\begin{aligned}
&\int_0^t \left(K_H^{f_n}(t,s) - K_H^f(t,s)\right)^2 ds \\
&\qquad\leqslant k_H \int_0^t \left[\left(\int_0^t |f^n(r) - f(r)|^p dr\right)^{1/p} M^{1/q}\right]^2 ds \\
&\qquad\leqslant k_H M^{2/q} \left[\left(\int_0^t |f^n(r) - f(r)|^p dr\right)^{1/p}\right]^2 t \xrightarrow[n \to +\infty]{} 0,
\end{aligned}$$

which completes proof of Lemma. □

Assume that $f$ is continuously differentiable on $[0,T]$, then using integration by parts rule and Theorem 3.2 we obtain

$$\begin{aligned}
\int_0^t f(s) B_s^H &= B_t^H f(t) - \int_0^t B^H s f'(s) ds \\
&= \int_0^t z(t,s) f(t) dW_s - \int_0^t f'(s) \left(\int_0^s z(s,u) dW_u\right) ds.
\end{aligned}$$

By Fubini's Theorem for stochastic integral and integration by parts rule we have

$$\begin{aligned}\int_0^t f(s)dB_s^H &= \int_0^t \left( z(t,u)f(t) - \int_u^t f'(s)z(s,u)ds \right) dW_u \\ &= \int_0^t \left( \int_u^t f(s)\frac{d}{ds}z(s,u)ds \right) dW_u \\ &= c_H \int_0^t u^{1/2-H} \left( \int_u^t f(s)s^{H-1/2}(s-u)^{H-3/2}ds \right) dW_u \\ &= \frac{c_H}{H(2H-1)} \int_0^t u^{1/2-H} K_H^f(t,u)dW_u \\ &= \int_0^t K_H^f(t,s)dM_s^H.\end{aligned}$$

Let now $f$ be a bounded function. Is well-known that, there exists the sequence $\{f_n\}$ of continuously differentiable functions, such that

$$\int_0^T |f_n(s) - f(s)|^p ds \xrightarrow[n\to+\infty]{} 0$$

for any $p > 0$. Directly from definition of stochastic integral we have for all $t \in [0,T]$

$$\int_0^t f_n(s)dB_s^H \xrightarrow[n\to+\infty]{} \int_0^t f(s)dB_s^H$$

in $\mathbb{L}^2(\Omega)$. To complete the proof we need to show that for all $t \in [0,T]$

$$\int_0^t K_H^{f_n}(t,s)dM_s^H \xrightarrow[n\to+\infty]{} \int_0^t K_H^f(t,s)dM_s^H$$

in $\mathbb{L}^2(\Omega)$. Since

$$\begin{aligned}&E\left( \int_0^t \left( K_H^{f_n}(t,s) - K_H^f(t,s) \right)^2 dM_s^H \right)^2 \\ &\qquad = \int_0^t \left( K_H^{f_n}(t,s) - K_H^f(t,s) \right)^2 dw_s^H \\ &\qquad = \lambda_H \int_0^t \left( K_H^{f_n}(t,s) - K_H^f(t,s) \right)^2 s^{2-2H}ds \\ &\qquad \leqslant \lambda_H \int_0^t \left( K_H^{f_n}(t,s) - K_H^f(t,s) \right)^2 t^{2-2H}ds,\end{aligned}$$

the result follows from Lemma 3.4. □

# 4 Girsanov Theorem

Recall that fBm is defined o some fixed probability space $(\Omega, \mathcal{F}, P)$.

**Lemma 4.1.** *Let $M$ be a continuous, centered, Gaussian martingale with independent increments and*

$$\Lambda_T = \exp\left(-\int_0^T \rho(s)dM_s - \frac{1}{2}\int_0^T \rho^2(s)d[M]_s\right).$$

*where $\rho$ is some deterministic function, such that $\int_0^T \rho^2(s)d[M]_s < +\infty$. If we define a measure $Q$ by $\frac{dQ}{dP} = \Lambda_T$, then $M + \int_0^\cdot \rho(s)d[M]_s$ under $Q$ has the same finite dimensional distribution as $M$ under $P$.*

*Proof.* Fix $t_i \in [0,T]$, $i = 1,\ldots,n$ and put

$$\tau^\top = \left(\int_0^{t_1} \rho(s)d[M]_s, \ldots, \int_0^{t_n} \rho(s)d[M]_s\right).$$

We need to show that

$$(M_{t_1},\ldots,M_{t_n})^\top \overset{Q}{\sim} \mathcal{N}(-\tau, V),$$

where $V = \mathrm{Cov}_P(M_{t_i}, M_{t_j})_{i,j=1}^n$. Let $\alpha^\top = (\alpha_1,\ldots,\alpha_n)$ and note that

$$E_Q \exp\left(\sum_{i=1}^n \alpha_i M_{t_i}\right) = E_P \exp\left(\sum_{i=1}^n \alpha_i M_{t_i} - \int_0^T \rho(s)dM_s - \frac{1}{2}\int_0^T \rho^2(s)d[M]_s\right).$$

Consider random variable $U = \sum_{i=1}^n \alpha_i M_{t_i} - \int_0^T \rho(s)dM_s$. With respect to the measure $P$ it's Gaussian with mean zero and variance

$$\begin{aligned} E_P(U^2) = &\int_0^T \rho^2(s)d[M]_s + E_P\left(\sum_{i=1}^n \alpha_i M_{t_i}\right)^2 \\ &- 2\sum_{i=1}^n \mathrm{cov}_P\left(\int_0^T \rho(s)dM_s, \alpha_i M_{t_i}\right). \end{aligned}$$

Clearly

$$\mathrm{cov}_P\left(\int_0^T \rho(s)dM_s, \alpha_i M_{t_i}\right) = \alpha_i \int_0^{t_i} \rho(s)d[M]_s.$$

Hence

$$E_P(U^2) = \int_0^T \rho^2(s)d[M]_s + \alpha^\top V\alpha - 2\alpha^\top\tau$$

and

$$E_P \exp(U) = \exp\left(\frac{\int_0^T \rho^2(s)d[M]_s + \alpha^\top V\alpha - 2\alpha^\top\tau}{2}\right).$$

As a consequence,

$$E_Q \exp\left(\sum_{i=1}^n \alpha_i M_{t_i}\right) = \exp\left(\frac{\alpha^\top V\alpha}{2} - \alpha^\top\tau\right),$$

which completes the proof. □

**Remark 4.2.** If moreover there exist function $\delta : [0,T] \to \mathbb{R}$, such that

$$W_t = \int_0^t \delta(s) dM_s, \quad t \in [0,T], \tag{4.1}$$

is Brownian motion, then the above measure $Q$ is unique (given that $Q$ is equivalent to $P$).

*Proof.* If $M + \int_0^\cdot \rho(s) d[M]_s$ under $Q$ has the same distribution as $M$ under $P$, the by (4.1) we have that process

$$W_t^\delta = W_t + \int_0^t \delta(s)\rho(s) d[M]_s, \quad t \in [0,T]$$

is Brownian motion under $Q$. Thus assertion follows from uniqueness in usual Girsanov Theorem. □

**Lemma 4.3.** *Let $f$, $c$ be absolutely continuous functions on $[0,T]$ and $f(t) \neq 0$ for all $t \in [0,T]$. Define*

$$\rho_H(t) = r_H^{-1} t^{2H-1} \frac{d}{dt} \int_0^t s^{1/2-H}(t-s)^{1/2-H} \frac{c(s)}{f(s)} ds, \quad t \in [0,T]. \tag{4.2}$$

*where $r_H = (2-2H)\beta(3/2-H, 3/2-H)$. Then the following equation holds:*

$$\int_0^t K_H^f(t,s)\rho_H(s) dw_s^H = \int_0^t c(s) ds, \quad t \in [0,T].$$

*Proof.* Put $\alpha = H - 1/2$ and note that function

$$F(t) = t^{-\alpha} \frac{c(t)}{f(t)}, \quad t \in [0,T],$$

fulfils assumptions of Theorem 2.2. Thus there exist the solution of (2.1) with $F$ defined as above. Since by Proposition 2.1 function

$$t \longmapsto \int_0^t F(x)(t-x)^{-\alpha} dx$$

is differentiable almost everywhere with respect to the Lebesgue measure and $\rho_H$ is well defined. Fubini's Theorem gives

$$\begin{aligned}\int_0^t K_H^f(t,s)\rho_H(s) dw_s^H &= H(2H-1) \int_0^t \int_s^t f(r) r^{H-1/2}(r-s)^{H-3/2} dr \rho_H(s) dw_s^H \\ &= H(2H-1) \int_0^t f(r) r^{H-1/2} \int_0^r \rho_H(s)(r-s)^{H-3/2} dw_s^H dr.\end{aligned}$$

By Proposition 2.1

$$\begin{aligned}&H(2H-1) \int_0^r \rho_H(s)(r-s)^{H-3/2} dw_s^H \\ &= B(H-1/2, 3/2-H) \int_0^r (r-s)^{H-3/2} \frac{d}{ds} \int_0^s u^{1/2-H}(s-u)^{1/2-H} \frac{c(u)}{f(u)} du ds \\ &= r^{1/2-H} \frac{c(r)}{f(r)},\end{aligned}$$

which is our claim. □

We are now able to prove Girsanov-type Theorem for Wiener integrals with respect to the fBm.

**Theorem 4.4.** *Let $f$, $c$ fulfils assumptions of Lemma 4.3 and $M^H$ be the fundamental martingale defined in (3.1). Assume that for $\rho_H$ given by* (4.2) *we have that*

$$\int_0^T \rho_H^2(s) dw_s^H < +\infty. \tag{4.3}$$

*If we define a measure $Q$ by*

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \rho_H(s) dM_s^H - \frac{1}{2}\int_0^T \rho_H^2(s) dw_s^H\right),$$

*then*

(i) *the process $\int_0^\cdot f(s)dB_s^H + \int_0^\cdot c(s)ds$ has the same finite dimensional distribution under $Q$ as $\int_0^\cdot f(s)dB_s^H$ under $P$,*

(ii) *above $Q$ is unique.*

*Proof.* (i) Applying Lemma 4.1 to $M^H$, we obtain that the distribution of the process $M^H + \int_0^\cdot \rho_H(s)dw_s^H$ under $Q$ is the same as that of the process $M^H$ under $P$. Hence, by Theorem 3.3 and Lemma 4.3, (i) holds.

(ii) Is easy to see

$$\int_0^t k(t,s)\frac{c(s)}{f(s)}ds = \int_0^t \rho_H(s)dw_s^H, \quad t \in [0,T].$$

Thus, due to the (3.1), we obtain that there is equivalence between

- $\int_0^\cdot f(s)dB_s^H + \int_0^\cdot c(s)ds$ under $Q$ has the same finite dimensional distribution as $\int_0^\cdot f(s)dB_s^H$ under $P$ and $Q \sim P$,
- $M^H + \int_0^\cdot \rho_H(s)dw_s^H$ under $Q$ has the same finite dimensional distribution as $M^H$ under $P$ and $Q \sim P$.

Since (3.2), (ii) follows from Remark 4.2. □

If we analyse conditions of Theorem 4.4, we see that condition (4.3) may cause some difficulties. Now we shall establish that in one particular, but important for us case this condition holds.

**Remark 4.5.** If $f$, $c$ are absolutely continuous on $[0,T]$ and for all $t \in (0,T)$

$$\left|\frac{d}{dt}\left(\frac{c(t)}{f(t)}\right)\right| \leqslant Ct^{2H-2},$$

for some $C \in \mathbb{R}$, then (4.3) holds.

*Proof.* Since

$$
\begin{aligned}
\rho_H(t) &= r_H^{-1} t^{2H-1} \frac{d}{dt} \int_0^t s^{1/2-H}(t-s)^{1/2-H} \frac{c(s)}{f(s)} ds \\
&= r_H^{-1} t^{2H-1} \frac{d}{dt} \left( t^{2-2H} \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{c(ut)}{f(ut)} du \right) \\
&= r_H^{-1} \left( (2-2H) \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{c(ut)}{f(ut)} du \right. \\
&\quad \left. + t \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{d}{dt} \left( \frac{c(ut)}{f(ut)} \right) du \right),
\end{aligned}
$$

by inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we obtain

$$
\begin{aligned}
&\int_0^T \rho_H^2(s) dw_s^H \\
&\quad \leqslant 2 r_H^{-2} \lambda_H^{-1} \left[ (2-2H)^2 \int_0^T t^{1-2H} \left( \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{c(ut)}{f(ut)} du \right)^2 dt \right. \\
&\qquad \left. + \int_0^T t^{3-2H} \left( \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \frac{d}{dt} \left( \frac{c(ut)}{f(ut)} \right) du \right)^2 dt \right] \\
&\quad = I_1 + I_2.
\end{aligned}
$$

Clearly $I_1 \leqslant C_1 \int_0^T t^{1-2H} dt = (C_2/(2-2H)) T^{2-2H}$, for some $C_1 \in \mathbb{R}$. To estimate $I_2$ note that

$$
\begin{aligned}
I_2 &\leqslant 2 r_H^{-2} \lambda_H^{-1} \int_0^T t^{3-2H} \left( \int_0^1 u^{1/2-H}(1-u)^{1/2-H} \left| \frac{d}{dt} \left( \frac{c(ut)}{f(ut)} \right) \right| du \right)^2 dt \\
&\leqslant 2 r_H^{-2} \lambda_H^{-1} C^2 \int_0^T t^{3-2H} \left( \int_0^1 u^{1/2-H}(1-u)^{1/2-H} u^{2H-1} t^{2H-2} du \right)^2 dt \\
&= C_2 \int_0^T t^{2H-1} dt = (C_2/2H) T^{2H},
\end{aligned}
$$

which completes the proof. □

# 5 Application to finance

We consider a continuous market model with two assets.

The riskless bond $B$, which dynamics are

$$B_t = \exp(rt), \tag{5.1}$$

where $r > 0$.

The risky stock denoted by $S$, with dynamics

$$dS_t = \mu(t) S_t dt + \sigma(t) S_t dB_t^H. \tag{5.2}$$

where $\sigma$, $\mu \in C^1[0,T]$ and $\sigma(t) \neq 0$ for $t \in [0,T]$. As usual we assume that there are no dividends, and no transaction costs.

By Ruzmaikina [7] Lemma 3.1 and Theorem 4 the unique pathwise solution of (5.2) is given by

$$S_t = S_0 \exp\left(\int_0^t \mu(s)ds + \int_0^t \sigma(s)dB_s^H\right).$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $B^H$. Denote by $F_T$ an option i.e. non-negative $\mathcal{F}_T$ measurable random variable. In classical Black-Scholes model a *fair price* of option is the discounted expectation of $F_T$ with respect to the *martingale measure* i.e. the measure $Q$ such that $(S_t/B_t)_{t\in[0,T]}$ is $Q$-martingale. In particular, the average growth of the stock in classical Black-Scholes model is the growth of the bond under $Q$:

$$E_Q(S_t/B_t) = S_0. \tag{5.3}$$

It was shown in Valkeila [9] that in fractional model martingale measure do not exist. However we can prove that, there exist unique measure $Q$ (called in [9] *average risk neutral measure*) such that (5.3) holds. To do this we need two technical lemmas.

**Lemma 5.1.** *If $f$ is absolutely continuous function on $[0,T]$ and $H \in (1/2,1)$, then $g : [0,T] \to \mathbb{R}$ given by*

$$g(s) = 2f(s)\int_0^s f(r)(s-r)^{2H-2}dr, \quad s \in [0,T]$$

*is also absolutely continuous and for all $t \in [0,T]$*

$$\int_0^t g(s)ds = \int_0^t \int_0^t f(s)f(r)|s-r|^{2H-2}drds.$$

*Proof.* Since $f$ is absolutely continuous,

$$f(r) = f(0) + \int_0^r f'(u)du.$$

Hence

$$\int_0^s f(r)(s-r)^{2H-2}dr = \frac{f(0)}{2H-1}s^{2H-1} + \int_0^s \left(\int_0^r f'(u)du\right)(s-r)^{2H-2}dr. \tag{5.4}$$

Using Fubini's Theorem, we obtain

$$\begin{aligned}\int_0^s \left(\int_0^r f'(u)du\right)(s-r)^{2H-2}dr &= \int_0^s f'(u)\left(\int_u^s (s-r)^{2H-2}dr\right)du\\ &= \int_0^s f'(u)\left(\int_u^s (r-u)^{2H-2}dr\right)du\\ &= \int_0^s \int_0^r f'(u)(r-u)^{2H-2}dudr,\end{aligned}$$

whitch implies that $g$ is absolutely continuous. Note that, using Fubini's Theorem again we have

$$
\begin{aligned}
2\int_0^t f(s)\left(\int_0^s f(r)(s-r)^{2H-2}dr\right)ds \\
&= \int_0^t f(s)\left(\int_0^s f(r)(s-r)^{2H-2}dr\right)ds \\
&\quad + \int_0^t f(r)\left(\int_0^r f(s)(r-s)^{2H-2}ds\right)dr \\
&= \int_0^t f(s)\left(\int_0^s f(r)(s-r)^{2H-2}dr\right)ds \\
&\quad + \int_0^t f(s)\left(\int_s^t f(r)(r-s)^{2H-2}dr\right)ds \\
&= \int_0^t \int_0^t f(s)f(r)|s-r|^{2H-2}drds,
\end{aligned}
$$

which is our claim. □

**Lemma 5.2.** *If $u, f \in C^1([0,T])$, $f(t) \neq 0$ for $t \in [0,T]$ and $H \in (1/2, 1)$, then there exists $M \in \mathbb{R}^+$ such that for all $t \in (0,T)$*

$$\left|\frac{d}{dt}\left(\frac{h(t)}{f(t)}\right)\right| \leqslant Mt^{2H-2},$$

*where*

$$h(t) = K + u(t) + f(t)\int_0^t f(r)(t-r)^{2H-2}dr, \quad K \in \mathbb{R}, \quad t \in [0,T].$$

*Proof.* Fix $t \in (0,T)$. By Lemma 5.1 function $h$ is absolutely continuous, thus

$$
\begin{aligned}
\left|\frac{d}{dt}\left(\frac{h(t)}{f(t)}\right)\right| &= \left|\frac{h(t)f'(t) - h'(t)f(t)}{f^2(t)}\right| \\
&\leqslant \sup_{t\in[0,T]}\left|\frac{h(t)f'(t)}{f^2(t)}\right| + \left(\sup_{t\in[0,T]}\frac{1}{|f(t)|}\right)|h'(t)|.
\end{aligned}
$$

Note, that

$$
\begin{aligned}
h'(t) &= u'(t) + f'(t)\int_0^t f(r)(t-r)^{2H-2}dr + f(t)\frac{d}{dt}\left(\int_0^t f(r)(t-r)^{2H-2}\right) \\
&\leqslant \sup_{t\in[0,T]}|u'(t)| + \sup_{t\in[0,T]}\left|\frac{f'(t)f(t)}{2H-1}\right|T^{2H-1} \\
&\quad + \left(\sup_{t\in[0,T]}|f(t)|\right)\frac{d}{dt}\left(\int_0^t f(r)(t-r)^{2H-2}\right).
\end{aligned}
$$

Since (5.4), it follows that

$$\frac{d}{dt}\left(\int_0^t f(r)(t-r)^{2H-2}\right) = f(0)t^{2H-2} + \int_0^t f'(u)(t-u)^{2H-2}du$$
$$\leqslant f(0)t^{2H-2} + \sup_{t\in[0,T]}\left|\frac{f'(t)}{2H-1}\right| T^{2H-1}.$$

Therefore

$$|h'(t)| \leqslant 3K_1 + |f(0)|t^{2H-2},$$

where $K_1 = \max(\sup_{t\in[0,T]}|u'(t)|, \sup_{t\in[0,T]}\left|\frac{f'(t)f(t)}{2H-1}\right| T^{2H-1})$.

Thus

$$\left|\frac{d}{dt}\left(\frac{h(t)}{f(t)}\right)\right| \leqslant K_2 + K_3 t^{2H-2} \leqslant \left(\frac{K_2}{T^{2H-2}} + K_3\right)t^{2H-2},$$

where $K_2 = \sup_{t\in[0,T]}\left|\frac{h(t)f'(t)}{f^2(t)}\right| + 3K_1\sup_{t\in[0,T]}\left|\frac{1}{f(t)}\right|$, $K_3 = \sup_{t\in[0,T]}\left|\frac{1}{f(t)}\right| |f(0)|$. □

Now we can prove existence and uniqueness of average risk neutral measure.

**Theorem 5.3.** *In the $(B, S)$-market given by (5.1) and (5.2) there exists the unique measure $Q$ equivalent to the real-world measure $P$ such that (5.3) holds.*

*Proof.* Note, that (5.3) holds if and only if the process

$$\int_0^t \sigma(s)dB_s^H + \int_0^t \mu(s)ds - rt + \frac{\langle\sigma,\sigma\rangle_t^H}{2}, \quad t\in[0,T]$$

has the same finite dimensional distribution under $Q$ as $\int_0^\cdot \sigma(s)dB_s^H$ under $P$ (assuming that $Q$ is equivalent to $P$). Let

$$f(t) = \sigma(t), \qquad t\in[0,T],$$
$$c(t) = \mu(t) - r + H(2H-1)\sigma(t)\int_0^t \sigma(s)(t-s)^{2H-2}ds, \qquad t\in[0,T].$$

Using Lemma 5.1 we obtain that $c$ is absolutely continuous and

$$\int_0^t c(s)ds = \int_0^t \mu(s)ds - rt + \frac{\langle\sigma,\sigma\rangle_t^H}{2}.$$

Since, by Lemma 5.2 and Remark 4.5, we have that above $c$ and $f$ satisfies conditions of Theorem 4.4, the statement in the theorem holds. □

Define the *fair price* $C(F_T)$ of option $F_T$ by

$$C(F_T) = E_Q\frac{F_T}{B_T}.$$

**Theorem 5.4.** *If $F_T = F(S_T)$ i.e. the price depend only on value of $S$ at time $T$, we have following formula for fair price:*

$$C(F_T) = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F\left(S_0\exp\left(\sqrt{\langle\sigma,\sigma\rangle_T^H}y + rT - \frac{1}{2}\langle\sigma,\sigma\rangle_T^H\right)\right)e^{y^2/2}dy.$$

*Proof.* Easily follows from fact that the process $\int_0^t \sigma(s)dB_s^H + \int_0^t \mu(s)ds - rt + \frac{\langle\sigma,\sigma\rangle_t^H}{2}$ has the same law under $Q$ as $\int_0^\cdot \sigma(s)dB_s^H$ under $P$. □

In particular we have following analogue of well known Black-Scholes formula.

**Corollary 5.5.** *Fair price of European call option $F(S_T) = (S_T - K)^+$ is given by*

$$C\left((S_T - K)^+\right) = S_0\Phi(y_1) - Ke^{-rT}\Phi(y_2),$$

*where*

$$y_1 = \frac{\log\frac{S_0}{K} + rT + \frac{1}{2}\langle\sigma,\sigma\rangle_T^H}{\sqrt{\langle\sigma,\sigma\rangle_T^H}}, \quad y_2 = \frac{\log\frac{S_0}{K} + rT - \frac{1}{2}\langle\sigma,\sigma\rangle_T^H}{\sqrt{\langle\sigma,\sigma\rangle_T^H}}.$$

*Proof.* Using above Theorem we have

$$\begin{aligned}
&C_T\left((S_T - K)^+\right) \\
&\quad = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(S_0\exp\left(\sqrt{\langle\sigma,\sigma\rangle_T^H}y + rT - \frac{\langle\sigma,\sigma\rangle_T^H}{2}\right) - K\right)^+ e^{-y^2/2}dy \\
&\quad = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{y_0}^{+\infty}\left(S_0\exp\left(\sqrt{\langle\sigma,\sigma\rangle_T^H}y + rT - \frac{\langle\sigma,\sigma\rangle_T^H}{2}\right) - K\right)e^{-y^2/2}dy,
\end{aligned}$$

where

$$y_0 = \frac{\ln\frac{K}{S_0} - rT + \frac{\langle\sigma,\sigma\rangle_T^H}{2}}{\sqrt{\langle\sigma,\sigma\rangle_T^H}}.$$

Hence

$$\begin{aligned}
&C_T\left((S_T - K)^+\right) \\
&\quad = S_0\frac{1}{\sqrt{2\pi}}\int_{y_0}^{+\infty}\exp\left(-\frac{\langle\sigma,\sigma\rangle_T^H}{2} + y\sqrt{\langle\sigma,\sigma\rangle_T^H} - \frac{y^2}{2}\right)dy - e^{-rT}K(1 - \Phi(y_0)) \\
&\quad = S_0\frac{1}{\sqrt{2\pi}}\int_{y_0}^{+\infty}\exp\left(\frac{-\left(y - \sqrt{\langle\sigma,\sigma\rangle_T^H}\right)^2}{2}\right)dy - e^{-rT}K(1 - \Phi(y_0)) \\
&\quad = S_0\frac{1}{\sqrt{2\pi}}\int_{y_0 - \sqrt{\langle\sigma,\sigma\rangle_T^H}}^{+\infty} e^{-y^2/2}dy - e^{-rT}K(1 - \Phi(y_0)) \\
&\quad = S_0\left(1 - \Phi(y_0 - \sqrt{\langle\sigma,\sigma\rangle_T^H})\right) - e^{-rT}K(1 - \Phi(y_0)) \\
&\quad = S_0\Phi\left(\sqrt{\langle\sigma,\sigma\rangle_T^H} - y_0\right) - e^{-rT}K(\Phi(-y_0)) \\
&\quad = S_0\Phi(y_1) - Ke^{-rT}\Phi(y_2),
\end{aligned}$$

which completes the proof. □

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