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The super-replication price of European call option in models involving fractional Brownian motion

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Abstract

We survey a financial market model driven by fractional Brownian motion. We show that under suitable restrictions of allowed trading strategies such model has no arbitrage. Then we calculate the super-hedging price for European call option under restricted class of trading strategies.

1 Introduction

Let $(B^H)_{t\in[0,T]}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (1/2, 1)$, i.e. B^H is a centered Gaussian process with covariance $EB^H_t B^H_s = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ defined on some probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by B^H.

It is well known that B^H has stationary increments, is H-self-similar and exhibits long-range dependence property. These properties makes (fBm) suitable model in mathematical finance. Various empirical studies of financial time series shows that the log-returns have this long range-dependence property (see [5], [8]). However pricing models with (fBm) admits arbitrage possibilities with continuous trading ([1],[7]). In this paper we present Cheridito's ([1]) idea of excluding arbitrage from model by introducing a minimal amount of time $h > 0$ that must lie between two consecutive transactions. Next we calculate the super-replication price for European call option in our model.

We consider the financial market which consists of the money market account $(B_t)_{t\in[0,T]}$ and the risky stock $(S_t)_{t\in[0,T]}$. We assume that $B_t \equiv 1$ and S is geometric fractional Brownian motion:

$$S_t = S_0 e^{B^H_t} \quad t \in [0, T]. \tag{1.1}$$

A strategy is a pair of adapted stochastic processes $\theta = (\theta^0_t, \theta^1_t)_{t\in 0,T}$. θ^0_t describes the money in the money market account at time t and θ^1_t describes a number of stock shares held at time t. Hence the wealth process is given by

$$V_t(\theta) = \theta^0_t + \theta^1_t S_t, \quad t \in [0, T].$$

Let $\mathcal{A}$ be some subset of set of all strategies. We interpret $\mathcal{A}$ as the set of all strategies that is at the investor's disposal and call the set of *admissible strategies*. Now we can define *a discounted market model* as 6-tuple:

$$\mathcal{M} = \{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P, S, \mathcal{A}\}.$$

Definition 1.1. (i) A strategy θ is an *arbitrage*, if $V_0(\theta) = 0$, $V_T(\theta) \geqslant 0$ P-almost surely, and $P(V_T(\theta) > 0) > 0$.

(ii) A market $\mathcal{M}$ is *arbitrage free*, if no strategy $\theta \in \mathcal{A}$ is an arbitrage.

It is clear that no arbitrage property depends on chosen class of admissible strategies.

Definition 1.2. (i) A strategy $\theta = (\theta_t^0, \theta_t^1)_{t\in[0,T]}$ is *simple*, if there exist a finite number of stopping times $0 = \tau_1 \leqslant \ldots \leqslant \tau_n = T$, such that strategy is constant on $(\tau_k, \tau_{k+1}]$, i.e.

$$\theta^0 = \theta_0^0 \mathbf{1}_{\{0\}} + \sum_{j=1}^{n-1} \theta_{\tau_j}^0 \mathbf{1}_{(\tau_j, \tau_{j+1}]}, \quad \theta^1 = \theta_0^1 \mathbf{1}_{\{0\}} + \sum_{j=1}^{n-1} \theta_{\tau_j}^1 \mathbf{1}_{(\tau_j, \tau_{j+1}]},$$

where $\theta_0^0, \theta_0^1 \in \mathbb{R}$ and $\theta_{\tau_j}^0, \theta_{\tau_j}^1$ are $\mathcal{F}_{\tau_j}$-measurable random variables, $j = 1, \ldots, n-1$.

(ii) Let $h > 0$, a strategy θ is *h-simple*, if it is simple and $\tau_{k+1} - \tau_k \geqslant h$ for all $k = 1, \ldots, n-1$.

Note that we need no integration theory for (fBm) to define self-financing condition for simple strategy:

Definition 1.3. We call simple strategy $\theta = (\theta_t^0, \theta_t^1)_{t\in[0,T]}$ *self-financing* if for all $k = 1, \ldots, n-1$

$$V_{\tau_{k+1}}(\theta) - V_{\tau_k}(\theta) = \theta_{\tau_k+1}^1 (S_{\tau_{k+1}} - S_{\tau_k})$$

or equivalently

$$V_{\tau_{k+1}}(\theta) = V_0(\theta) + \sum_{j=1}^{k} \theta_{\tau_j}^1 (S_{\tau_{j+1}} - S_{\tau_j}).$$

We define *Cheridito Class* of trading strategies (CC).

Definition 1.4. We say $\theta = (\theta_t^0, \theta_t^1)_{t\in[0,T]} \in CC$ if it is h-simple for some $h > 0$ and self-financing.

Now we can state the main results of [1].

Theorem 1.5 (Cheridito [1])**.** *Let S be a geometric (fBm) given by (1.1). If $\mathcal{A} = CC$, then the market $\mathcal{M}$ is arbitrage free.*

Consider the claim X, i.e. $\mathcal{F}_T$-measurable random variable. We define *super-replication price*:

$$p(X) = \inf\{x : V_0(\theta) = x, V_T(\theta) \geqslant X \text{ a.s. for some } \theta \in \mathcal{A}\}.$$

Theorem 1.6. *The super-replication price for European call option $X = (S_T - K)^+$ in market model $\mathcal{M}$ with Cheridito Class of admissible strategies is given by:*

$$p(X) = S_0.$$

In next two sections we give proofs of Theorems 1.5 and 1.6.

2 No arbitrate in Cheridito Class

In the rest of this paper, without loosing generality, we consider specified probability space. Namely, let $\Omega = C_0(\mathbb{R})$ be the set of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$. There exist a probability measure P on $(\Omega, \mathcal{G})$, where $\mathcal{G}$ is σ-field generated by cylinder sets, such that on the $(\Omega, \mathcal{G}, P)$ the process $(W_t)_{t\in\mathbb{R}}$ given by

$$W_t(\omega) = \omega(t), \quad \omega \in \Omega$$

is the two-sided Wiener process.

By Levy's modulus of continuity for all $n \in \mathbb{N}$ there exists measurable $\Omega_n \subset \Omega$ with $P[\Omega_n] = 1$ such that for all $\omega \in \Omega_n$ and $t \in [-n, n]$

$$\lim_{s\to t} \frac{\omega(t) - \omega(s)}{\sqrt{|t-s| \log\left(\frac{1}{|t-s|}\right)}} = 0.$$

It's clear that $P[\hat{\Omega}] = 1$ for $\hat{\Omega} = \bigcap_{n=1}^{\infty} \Omega_n$. Let $\mathcal{F} = \{A \cap \hat{\Omega} : A \in \mathcal{G}\}$. By Proposition 1.3 in [2] we may assume, without loss of generality, that (fBm) $(B_t^H)_{t\in[0,T]}$ is defined on $(\hat{\Omega}, \mathcal{F}, P)$ by improper Riemann-Stieltjes integrals:

$$B_t^H(\omega) = \int_{-\infty}^{t} \left[(t-s)^{H-1/2} - \mathbf{1}_{\{s\leqslant 0\}}(-s)^{H-1/2}\right] d\omega(s). \tag{2.1}$$

Note that existence of above integral is the consequence of the following proposition.

Proposition 2.1. *For $H \in (1/2, 1)$, $\omega \in \hat{\Omega}$ and $t > 0$ the improper Riemann-Stieltjes integral $\int_0^t (t-s)^{H-1/2} d\omega(s)$ exist and equals*

$$(H - \frac{1}{2}) \int_0^t \omega(s)(t-s)^{H-3/2} ds.$$

Moreover it's continuous in t.

Proof. It follows from Riemann-Stieltjes integration by parts rule (Theorem 2.21 of [10]) that for all $x \in (0, t)$ and $\omega \in \hat{\Omega}$

$$\int_0^x (t-s)^{H-1/2} d\omega(s) = (H - \frac{1}{2}) \int_0^x \omega(s)(t-s)^{H-3/2} ds + (t-x)^{H-1/2}\omega(x).$$

Since $\lim_{x\nearrow t}(t-x)^{H-1/2}\omega(x) = 0$ and

$$\lim_{x\nearrow t} \int_0^x \omega(s)(t-s)^{H-3/2} ds = \int_0^t \omega(s)(t-s)^{H-3/2} ds,$$

the first assertion follows.

To show continuity in t we consider for $t > 0$

$$f_\omega^t(s) = \mathbf{1}_{[0,t]}(s)\omega(s)(t-s)^{H-3/2}, \quad s \in \mathbb{R}.$$

Note that for $T > t$

$$\int_0^t \left|\omega(s)(t-s)^{H-3/2}\right| ds \leqslant \sup_{s\in[0,T]} |\omega(s)| \frac{T^{H-1/2}}{H - 1/2}.$$

Therefore the family $(f_\omega^t)_{t\in(0,T)}$ is uniformly integrable with respect to the Lebesgue measure. Hence the continuity in t follows from a generalized version of Lebesgues Dominated Convergence Theorem (see e.g. Theorem II.6.4.b in [9]). □

Lemma 2.2. *Let $(Z_t)_{t\geqslant 0}$ be the process given by improper Riemann-Stieltjes integral $\left(\int_0^t (t-s)^{H-1/2} d\omega(s)\right)_{t\geqslant 0}$. Then for all $c \geqslant 0$ and $0 < h \leqslant T$,*

$$P\left[\inf_{t\in[h,T]} Z_t \geqslant c\right] = P\left[\sup_{t\in[h,T]} Z_t \leqslant -c\right] > 0.$$

Proof. Let $c \geqslant 0$ and $0 < h \leqslant T$. Since $(-Z_t)_{t\geqslant 0}$ has the same distribution as $(Z_t)_{t\geqslant 0}$,

$$P\left[\inf_{t\in[h,T]} Z_t \geqslant c\right] = P\left[\sup_{t\in[h,T]} Z_t \leqslant -c\right].$$

We have that

$$P\left[\inf_{t\in[h,T]} Z_t \geqslant c\right] = P\left[\omega \in \hat{\Omega} : \inf_{t\in[h,T]} \int_0^t (t-s)^{H-1/2} d\omega(s) \geqslant c\right].$$

Fix $m = \frac{H+1/2}{h^{H+1/2}}(c + T^{H-1/2})$ and define set

$$A_m = \{\omega \in \hat{\Omega} : \sup_{t\in[0,T]} |\omega_m(t)| \leqslant 1\},$$

where $\omega_m(t) = \omega(t) - mt$ for $t \in [0,T]$. It follows from Girsanov's Theorem that there exists a probability measure Q on $\hat{\Omega}$ equivalent to P such that $(\omega_m(t))_{t\in[0,T]}$ is a Wiener process under Q. It is well known that $Q(A_m) > 0$. Therefore by equivalence Q and P

$$P(A_m) > 0. \tag{2.2}$$

Note that by Proposition 2.1 for all $\omega \in \hat{\Omega}$ and $t \in [h,T]$ we have that

$$\begin{aligned}
\int_0^t (t-s)^{H-1/2} d\omega(s) &= (H - \frac{1}{2}) \int_0^t \omega(s)(t-s)^{H-3/2} ds \\
&= (H - \frac{1}{2}) \left[\int_0^t \omega_m(s)(t-s)^{H-3/2} ds + m \int_0^t s(t-s)^{H-3/2} ds\right] \\
&= (H - \frac{1}{2}) \int_0^t \omega_m(s)(t-s)^{H-3/2} ds + m\frac{t^{H+1/2}}{H+1/2}.
\end{aligned}$$

Since for every $\omega \in A_m$ and $t \in [h,T]$

$$(H - \frac{1}{2}) \int_0^t \omega_m(s)(t-s)^{H-3/2} ds \geqslant -(H - \frac{1}{2}) \int_0^t (t-s)^{H-3/2} ds \geqslant -T^{H-1/2}$$

and

$$m\frac{t^{H+1/2}}{H+1/2} = \left(\frac{t}{h}\right)^{H+1/2} (c + T^{H-1/2}) \geqslant c + T^{H-1/2}$$

it follows that

$$\int_o^t (t-s)^{H-1/2} d\omega(s) \geqslant -TH - 1/2 + c + T^{H-1/2} = c.$$

Hence

$$A_m \subset \left\{\omega \in \hat{\Omega} : \inf_{t\in[h,T]} \int_0^t (t-s)^{H-1/2} d\omega(s) \geqslant c\right\}$$

which together with (2.2) completes the proof of Lemma. □

Lemma 2.3. *If $\mathcal{R}$ is the π-system on Ω and $\omega_1 \in C \Leftrightarrow \omega_2 \in C$ for all $C \in \mathcal{R}$ then $\omega_1 \in A \Leftrightarrow \omega_2 \in A$ for all $A \in \sigma(\mathcal{R})$.*

Proof. Consider Dirac measures δ_{ω_1}, δ_{ω_2}. Since $\delta_{\omega_1}(C) = \delta_{\omega_2}(C)$ for all $C \in \mathcal{R}$ and δ_{ω_1}, δ_{ω_2} are probability measures on $(\Omega, \sigma(\mathcal{R}))$ it follows that $\delta_{\omega_1}(A) = \delta_{\omega_2}(A)$ for all $A \in \sigma(\mathcal{R})$, which completes the proof. □

Lemma 2.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$. Denote by X $\mathcal{G}$-measurable, $(E_1, \mathcal{E}_1)$ valued random variable and by Y, $(E_2, \mathcal{E}_2)$ valued, random variable independent of $\mathcal{G}$, where $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ are some measurable spaces. Then for every Borel, nonnegative (or bounded) function $\Phi : (E_1 \times E_2, \mathcal{E}_1 \times \mathcal{E}_2) \to \mathbb{R}$ the function ϕ given by*

$$\phi(x) = E(\Phi(x, Y)), \quad x \in E_1$$

is Borel on $(E_1, \mathcal{E}_1)$ and

$$E(\Phi(X, Y)|\mathcal{G}) = \phi(X), \quad a.s.$$

Proof. See Lamberton, Lapeyre [4]. □

Proof of Theorem 1.5. Without loss of generality we assume that $S_t = e^{B_t^H}$. Suppose that there exist an arbitrage strategy $\theta = (\theta_t^0, \theta_t^1)_{t\in[0,T]} \in CC$ with

$$\theta^1 = \theta_0^1 \mathbf{1}_{\{0\}} + \sum_{j=1}^{n-1} \theta_{\tau_j}^1 \mathbf{1}_{(\tau_j, \tau_{j+1}]}$$

and $\tau_{k+1} - \tau_k \geqslant h$ for all $k = 1, \ldots, n-1$.

By definition of arbitrage strategies

$$V_T(\theta) = \sum_{j=1}^{n-1} \theta_{\tau_j}^1 (e^{B_{\tau_{j+1}}^H} - e^{B_{\tau_j}^H}) \geqslant 0 \quad \text{a.s.}$$

and $P(V_T(\theta) > 0) > 0$. If $\theta_{\tau_1}^1 = \cdots = \theta_{\tau_{n-1}}^1 = 0$ a.s. then $V_T(\theta) = 0$ a.s. Hence there must exists a $l \in \{1, \ldots, n-1\}$ with $P[\theta_{\tau_l}^1 \neq 0] > 0$ and $\sum_{j=1}^{l} \theta_{\tau_j}^1 (e^{B_{\tau_{j+1}}^H} - e^{B_{\tau_j}^H}) \geqslant 0$ a. s. Let

$$k = \min \left\{ l : P[\theta_{\tau_l}^1 \neq 0] \text{ and } \sum_{j=1}^{l} \theta_{\tau_j}^1 (e^{B_{\tau_{j+1}}^H} - e^{B_{\tau_j}^H}) \geqslant 0 \right\}.$$

Then either $\theta_{\tau_1}^1 = \cdots = \theta_{\tau_k - 1}^1 = 0$ a.s. or

$$P\left[\sum_{j=1}^{k-1} \theta_{\tau_j}^1 (e^{B_{\tau_{j+1}}^H} - e^{B_{\tau_j}^H}) < 0 \right] > 0.$$

Thus $P[C] > 0$ for $C := \{\sum_{j=1}^{k-1} \theta_{\tau_j}^1 (e^{B_{\tau_{j+1}}^H} - e^{B_{\tau_j}^H}) \leqslant 0\}$.

Recall the definition of (fBm) in (2.1) and define the filtration $(\mathcal{F}_t^{\hat{\Omega}})_{t\in[0,T]}$ by

$$\mathcal{F}_t^{\hat{\Omega}} := \sigma(\mathcal{R}_t), \tag{2.3}$$

where

$$\mathcal{R}_t = \{\{\omega \in \hat{\Omega} : (\omega(t_1), \ldots, \omega(t_n)) \in V\} : -\infty < t_1 < \cdots < t_n \leqslant t, V \in \mathcal{B}(\mathbb{R}^n)\}.$$

It's clear that $\mathcal{F}_t^{B^H} \subset \mathcal{F}_t^{\hat{\Omega}}$ for $t \in [0,T]$, where $(\mathcal{F}_t^{B^H})_{t\in[0,T]}$ is the filtration generated by (fBm). Therefore every $(\mathcal{F}_t^{B^H})_{t\in[0,T]}$ stopping τ time is also $(\mathcal{F}_t^{\hat{\Omega}})_{t\in[0,T]}$ stopping time. We split each function $\omega \in \hat{\Omega}$ at the time point $\tau_k(\omega)$. Set

$$\pi_1\omega(s) = \omega(s)\mathbf{1}_{(-\infty,\tau_k(\omega)]}(s) + \omega(\tau_k(\omega))\mathbf{1}_{(\tau_k(\omega),+\infty)}(s), \quad s \in \mathbb{R}, \tag{2.4}$$

$$\pi_2\omega(s) = \omega(\tau_k(\omega) + s) - \omega(\tau_k(\omega)), \quad s \geqslant 0. \tag{2.5}$$

Let

$$\Omega_1 := \left\{\pi_1\omega : \omega \in \hat{\Omega}\right\},$$

and $\mathcal{B}_1$ be the σ-field of subsets of Ω_1 generated by cylinder sets. Similarly we put

$$\Omega_2 := \left\{\pi_2\omega : \omega \in \hat{\Omega}\right\}$$

and denote by $\mathcal{B}_2$ the σ-field of subsets of Ω_2 generated by cylinder sets. Clearly the mapping $\pi_1 : (\hat{\Omega}, \mathcal{B}) \to (\Omega_1, \mathcal{B}_1)$ is $\mathcal{F}_{\tau_k}^{\hat{\Omega}}$ measurable. On the other hand it follows from Theorem I.32 in Protter [6] that $(\pi_2\omega(s))_{s\geqslant 0}$ is the Wiener process independent of $\mathcal{F}_{\tau_k}^{\hat{\Omega}}$.

Clearly $\mathcal{R}_t$ is π-system for all $t \in [0,T]$. Fix $\omega \in \hat{\Omega}$ and note that $\omega \in A \Leftrightarrow \pi_1\omega \in A$ for all $A \in \mathcal{R}_{\tau_k(\omega)}$. Therefore by Lemma 2.3 we have that $\omega \in \hat{A} \Leftrightarrow \pi_1\omega \in \hat{A}$ for all $\hat{A} \in \mathcal{F}_{\tau_k(\omega)}$. Since $\{\tilde{\omega} : \tau_k(\tilde{\omega}) = \tau_k(\omega)\} \in \mathcal{F}_{\tau_k(\omega)}$ it follows from above that $\tau_k(\pi_1\omega) = \tau_k(\omega)$. Using similar arguments one can obtain that $X(\pi_1\omega) = X(\omega)$ for every $\mathcal{F}_{\tau_k}$ measurable random variable X and $\omega \in \hat{\Omega}$. Hence it follows from (2.1) that for all $\omega \in \hat{\Omega}$ and $t \in [h,T]$,

$$\begin{aligned}
&\left[\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}})\right](\omega) \\
&\quad= \left[\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \theta^1_{\tau_k} e^{B^H_{\tau_k}}(e^{B^H_{\tau_k+t} - B^H_{\tau_k}} - 1)\right](\omega) \\
&\quad= U_t(\pi_1\omega, \pi_2\omega)
\end{aligned}$$

where for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $t \in [h,T]$,

$$U_t(\omega_1,\omega_2) = \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}})(\omega_1) + \theta^1_{\tau_k}(\omega_1) e^{B^H_{\tau_k}(\omega_1)}(e^{U^1_t(\omega_1)+U^2_t(\omega_2)} - 1),$$

$$U^1_t(\omega_1) = \int_{-\infty}^{\tau_k(\omega_1)} [(\tau_k(\omega_1) + t - s)^{H-1/2} - (\tau_k(\omega_1) - s)^{H-1/2}] d\omega_1(s), \tag{2.6}$$

$$U^2_t(\omega_2) = \int_0^t (t-s)^{H-1/2} d\omega_2(s). \tag{2.7}$$

Let

$$A = \left\{(\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 : \sup_{t\in[h,T]} U_t(\omega_1,\omega_2) < 0\right\}.$$

Note that $(U_t)_{t\in[h,T]}$ is continuous stochastic process thus A is $\mathcal{B}_1 \times \mathcal{B}_2$ measurable and by Lemma 2.4 for almost all $\omega \in \hat{\Omega}$ we have that

$$E(\mathbf{1}_A(\pi_1, \pi_2)|\mathcal{F}^{\hat{\Omega}}_{\tau_k}) = \phi(\pi_1\omega),$$

where $\phi : \Omega_1 \to \mathbb{R}$ is given by

$$\phi(\omega_1) = E(\mathbf{1}_A(\omega_1, \pi_2)), \quad \omega_1 \in \Omega_1.$$

Since $t \mapsto U^1_t(\omega_1)$ is for all $\omega_1 \in \Omega_1$ continuous and $(\pi_2\omega(t))_{t\geqslant 0}$ is Wiener process it follows from Lemma 2.2 that for almost all $\omega \in C$

$$\begin{aligned} P&\left[\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \sup_{t\in[h,T]} \theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}) < 0 \middle| \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right](\omega) \\ &= P[\sup_{t\in[h,T]} U_t(\pi_1\omega, \pi_2) < 0] \\ &= P\left[\sum_{j=1}^{k-1} \theta^1_{\tau_j}(\pi_1\omega)(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}})(\pi_1\omega)\right. \\ &\qquad \left. + \sup_{t\in[h,T]} \theta^1_{\tau_k}(\pi_1\omega) e^{B^H_{\tau_j}(\pi_1\omega)}(e^{U^1_t(\pi_1\omega)+U^2_t(\pi_2)} - 1) < 0\right] > 0. \end{aligned}$$

Therefore

$$\begin{aligned} P&\left[\sum_{j=1}^{k} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) < 0\right] \\ &\geqslant P\left[\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \sup_{t\in[h,T]} \theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}) < 0\right] \\ &= E\left[P\left[\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \sup_{t\in[h,T]} \theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}) < 0 \middle| \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right] \\ &\geqslant E\left[\mathbf{1}_C P\left[\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \sup_{t\in[h,T]} \theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}) < 0 \middle| \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right] > 0, \end{aligned}$$

which contradicts with choice of k, so the proof is complete. □

3 Super-hedging price

In this section we prove Theorem 1.6.

Proof of Theorem 1.6. Note that if $V_0(\theta) = S_0$ then the investor can buy a stock and hold it until expiration time T. It's obvious that this is super-hedging strategy and belongs to class CC. Thus to complete the proof we need to show that if $\theta \in CC$ and

$$V_T(\theta) = V_0(\theta) + \sum_{j=1}^{n-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) \geqslant (S_T - K)^+,$$

then $V_0 \geqslant S_0$. Without loss of generality we may assume that $S_0 = 1$.

Suppose that there exists $k \in \{1, \dots, n-1\}$ such that

$$P\left[V_0(\theta) + \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) < e^{B^H_{\tau_k}}\right] > 0. \tag{3.1}$$

From above it follows that there exists $\varepsilon > 0$ such that $P[C] > 0$ with

$$C := \left\{V_0(\theta) + \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) < e^{B^H_{\tau_k}}(1-\varepsilon)\right\}.$$

Let $C_1 := C \cap \{\theta^1_{\tau_k} < 1\}$ and $C_2 := C \cap \{\theta^1_{\tau_k} \geqslant 1\}$, then either

$$P[C_1] > 0 \quad \text{or} \quad P[C_2] > 0.$$

1. Consider the case $P[C_1] > 0$.

   Recall definition of in (fBm) (2.1). Let $(\mathcal{F}^{\hat{\Omega}}_t)_{t\in[0,T]}$ be the filtration defined by (2.3) and π_1 ,π_2 be the mappings given in (2.4), (2.5) resp. By similar arguments to the ones in proof of Theorem 1.5 we obtain that for almost all $\omega \in C$ and $t \in [h, T]$

$$\begin{aligned}
&\left[V_0(\theta) + \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}) - (e^{B^H_{\tau_k+t}} - K)^+\right](\omega) \\
&\quad < \left[e^{B^H_{\tau_k}}\left(1 + \theta^1_{\tau_k}(e^{B^H_{\tau_k+t} - B^H_{\tau_k}} - 1)\right) - (e^{B^H_{\tau_k+t}} - K)^+\right](\omega) \\
&\quad \leqslant \left[e^{B^H_{\tau_k}}\left(1 + \theta^1_{\tau_k}(e^{B^H_{\tau_k+t} - B^H_{\tau_k}} - 1)\right) - e^{B^H_{\tau_k+t}} + K\right](\omega) \\
&\quad = \left[e^{B^H_{\tau_k}}(\theta^1_{\tau_k} - 1)(e^{B^H_{\tau_k+t} - B^H_{\tau_k}} - 1) + K\right](\omega) \\
&\quad = U_t(\pi_1\omega, \pi_2\omega),
\end{aligned}$$

   where for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $t \in [h, T]$,

$$U_t(\omega_1, \omega_2) = e^{B^H_{\tau_k}(\omega_1)}(\theta^1_{\tau_k}(\omega_1) - 1)(e^{U^1_t(\omega_1) + U^2_t(\omega_2)} - 1) + K,$$

   with U^1, U^2 given by (2.6), (2.7) resp.

   Hence by Lemmas 2.4 and 2.2 we have that for almost all $\omega \in C_1$

$$\begin{aligned}
&P\Bigg[V_0(\theta) + \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) \\
&\qquad + \sup_{t\in[h,T]}\left\{\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}) - (e^{B^H_{\tau_k+t}} - K)^+ < 0\right\}\Bigg|\mathcal{F}^{\hat{\Omega}}_{\tau_k}\Bigg](\omega) \\
&\quad \geqslant P\left[\sup_{t\in[h,T]}\left(e^{B^H_{\tau_k}}(\theta^1_{\tau_k} - 1)(e^{B^H_{\tau_k+t} - B^H_{\tau_k}} - 1) + K\right) < 0\middle|\mathcal{F}^{\hat{\Omega}}_{\tau_k}\right](\omega) \\
&\quad = P[\sup_{t\in[h,T]} U_t(\pi_1\omega, \pi_2) < 0] \\
&\quad = P\left[e^{B^H_{\tau_k}(\pi_1\omega)}(\theta^1_{\tau_k}(\pi_1\omega) - 1)\inf_{t\in[h,T]}(e^{U^1_t(\pi_1\omega) + U^2_t(\pi_2)} - 1) + K < 0\right] > 0.
\end{aligned}$$

Therefore

$$
\begin{aligned}
P & \left[V_0(\theta)+\sum_{j=1}^{k} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}}-e^{B^H_{\tau_j}})-(e^{B^H_{\tau_{k+1}}}-K)^+<0\right] \\
& \geqslant P\left[V_0(\theta)+\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}}-e^{B^H_{\tau_j}})\right. \\
& \qquad \left. +\sup_{t\in[h,T]}\left\{\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}}-e^{B^H_{\tau_k}})-(e^{B^H_{\tau_k+t}}-K)^+\right\}<0\right] \\
& = E\left[P\left[V_0(\theta)+\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}}-e^{B^H_{\tau_j}})\right.\right. \\
& \qquad \left.\left. +\sup_{t\in[h,T]}\left\{\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}}-e^{B^H_{\tau_k}})-(e^{B^H_{\tau_k+t}}-K)^+\right\}<0\middle|\mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right] \\
& \geqslant E\left[\mathbf{1}_{C_1}P\left[V_0(\theta)+\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}}-e^{B^H_{\tau_j}})\right.\right. \\
& \qquad \left.\left. +\sup_{t\in[h,T]}\left\{\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}}-e^{B^H_{\tau_k}})-(e^{B^H_{\tau_k+t}}-K)^+\right\}<0\middle|\mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right]>0.
\end{aligned}
$$

2. Suppose that $P[C_2]>0$.

   Using the same method as in first part we obtain that for almost all $\omega\in C$ and $t\in[h,T]$

$$
\begin{aligned}
& \left[V_0(\theta)+\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}}-e^{B^H_{\tau_j}})+\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}}-e^{B^H_{\tau_k}})\right](\omega) \\
& \quad < e^{B^H_{\tau_k}}\left(1-\varepsilon+\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}-B^H_{\tau_k}}-1)\right)(\omega) \\
& \quad = U_t(\pi_1\omega,\pi_2\omega),
\end{aligned}
$$

   where for $\omega_1\in\Omega_1$, $\omega_2\in\Omega_2$ and $t\in[h,T]$,

$$
U_t(\omega_1,\omega_2)=e^{B^H_{\tau_k}(\omega_1)}\left(1-\varepsilon+\theta^1_{\tau_k}(\omega_1)(e^{U^1_t(\omega_1)+U^2_t(\omega_2)}-1)\right).
$$

   Therefore it follows from Lemmas 2.4 and 2.2 we have that for almost all $\omega\in C_2$

$$
\begin{aligned}
P & \left[V_0(\theta)+\sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}}-e^{B^H_{\tau_j}})+\sup_{t\in[h,T]}\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}}-e^{B^H_{\tau_k}})<0\middle|\mathcal{F}^{\hat{\Omega}}_{\tau_k}\right](\omega) \\
& \geqslant P\left[\sup_{t\in[h,Y]}\left(e^{B^H_{\tau_k}}\left(1-\varepsilon+\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}-B^H_{\tau_k}}-1)\right)\right)<0\middle|\mathcal{F}^{\hat{\Omega}}_{\tau_k}\right](\omega) \\
& = P[\sup_{t\in[h,T]}U_t(\pi_1\omega,\pi_2)<0] \\
& = P\left[e^{B^H_{\tau_k}(\pi_1\omega)}\left(1-\varepsilon+\theta^1_{\tau_k}(\pi_1\omega)\sup_{t\in[h,T]}(e^{U^1_t(\pi_1\omega)+U^2_t(\pi_2)}-1)\right)<0\right]>0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&P\left[V_0(\theta) + \sum_{j=1}^{k} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) - (e^{B^H_{\tau_{k+1}}} - K)^+ < 0\right] \\
&\quad \geqslant P\left[V_0(\theta) + \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) + \sup_{t\in[h,T]}\left\{\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}})\right\} < 0\right] \\
&\quad = E\left[P\left[V_0(\theta) + \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}})\right.\right. \\
&\qquad\qquad \left.\left. + \sup_{t\in[h,T]}\left\{\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}})\right\} < 0 \middle| \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right] \\
&\quad \geqslant E\left[\mathbf{1}_{C_2} P\left[V_0(\theta) + \sum_{j=1}^{k-1} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}})\right.\right. \\
&\qquad\qquad \left.\left. + \sup_{t\in[h,T]}\left\{\theta^1_{\tau_k}(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}})\right\} < 0 \middle| \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right] > 0.
\end{aligned}
$$

We showed that in both cases if there exists $k \in \{1,\dots,n-1\}$ such that (3.1) holds then

$$
P\left[V_0(\theta) + \sum_{j=1}^{k} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) < (e^{B^H_{\tau_{k+1}}} - K)^+\right] > 0. \tag{3.2}
$$

From above it follows that

$$
P\left[V_0(\theta) + \sum_{j=1}^{k} \theta^1_{\tau_j}(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}) < e^{B^H_{\tau_{k+1}}}\right] > 0.
$$

Therefore if (3.2) holds for some $k \in \{1,\dots,n-1\}$ then it holds for all $l \in \{k,\dots,n-1\}$. To complete the proof of theorem is enough to notice that condition $V_0(\theta) < S_0$ is equivalent to (3.1) with $k = 1$. Hence it follows from above that if $V_0(\theta) < S_0$ then θ cannot be the super-hedging strategy. □

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