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Weak approximation of the fractional Wick-Itô stochastic differential equation with time-dependent coefficients

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Abstract

We study approximation of linear SDEs with time-dependent coefficients driven by a fractional Brownian motion in the Wick-Itô sense. In particular we show that the Euler scheme based on discrete Wick product converges weakly to the solution of corresponding Wick-Itô SDE.

1 Introduction

A fractional Brownian motion B^H is a continuous centered Gaussian process with the covariance function given by

$$\mathbb{E}B_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} + |t-s|^{2H}), \quad t, s \in \mathbb{R}, \quad H \in (0,1).$$

Since fractional Brownian motion with $H \neq 1/2$ is not a semimartingale the integration theory with respect to the B^H interested many authors. In recent years several approaches has been developed: pathwise integration, Wick integration, Skorokhod integration and some others (see e.g. monograph [6]). In 2003 Bender [1] gives a motivation for and definition of Wick-Itô integral based on so-called $\mathcal{S}$-transform witch is an extension of well known Itô integral. In section 2. we give briefly overview for this approach.

The aim of this paper is to approximate solutions of the stochastic differential equations of the form

$$S_t = s_0 + \int_0^t b_s S_s ds + \int_0^t f_s S_s d^\diamond B_s^H, \quad t \in [0,T], \tag{1.1}$$

in terms of Wick-Itô integral.

In 2010 Bender and Parczewski [2] considered similar equation with constant coefficients and $H > 1/2$. Their idea was to use Sottinen's [9] approximation of fractional the Brownian motion by so-called disturbed binary random walk, witch is analogue of Donsker's theorem. Next they constructed a Euler scheme with discrete Wick product instead of ordinary multiplication and showed that the piecewise constant interpolation of the solution of this scheme converges weakly in the Skorokhod space to the solution of corresponding Wick-Itô SDE.

In this paper we extend Sottinen's result to the approximation of fractional Wiener integrals $\int_0^t f_s dB_s^H$ by integrals with respect to the disturbed binary random walk $B^{H,n}$. Next we show that the discrete Wick exponent of $\int_0^t f_s dB_s^{H,n}$ converges weakly to the continuous Wick exponent of $\int_0^t f_s dB_s^H$ in Skorokhod space. Using this result we construct approximating sequences of the solution of equation (1.1). We also prove that solution of the discrete Wick difference equation of the form

$$\tilde{S}_k^n = \tilde{S}_{k-1}^n \diamond_n \left(1 + f_{k/n}\left(B_{k/n}^{H,n} - B_{(k-1)/n}^{H,n}\right)\right), \quad k = 1, \dots, n, \quad \tilde{S}_0^n = 1, \tag{1.2}$$

converges weakly to the geometric fractional Brownian motion in the Wick sense.

2 Fractional Wick-Itô integral and fractional SDE

We recall a construction of fBm starting from a Wiener process. Let $(B)_{t\in\mathbb{R}}$ be the two-sided Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$.

Let $0 < \alpha < 1$ and define the Riemann-Liouville fractional integral on $\mathbb{R}$ by

$$(I_+^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1} dt$$

and

$$(I_-^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1} dt$$

if the integrals exists for almost all $x \in \mathbb{R}$. Here Γ is the Gamma function. We also define Riemann-Liouville fractional derivative i.e.

$$(D_\pm^\alpha f) := \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon\to 0^+} \int_\varepsilon^\infty \frac{f(x) - f(x \mp t)}{t^{1+\alpha}} dt$$

if the limit exists in $L^p(\mathbb{R})$ for some $p > 1$. The general indicator function is given by

$$\mathbf{1}_{(a,b)}(t) = \begin{cases} 1, & \text{if } a \leqslant t < b \\ -1, & \text{if } b \leqslant t < a \\ 0, & \text{otherwise .} \end{cases}$$

Set $K_H := \Gamma(H+1/2)(\int_0^\infty((1+s)^{H-1/2}) - s^{H-1/2} ds + (2H)^{-1})^{-1/2}$ and let

$$M_\pm^H f := \begin{cases} K_H D_\pm^{-(H-1/2)} f, & 0 < H < \frac{1}{2} \\ f, & H = \frac{1}{2} \\ K_H I_\pm^{H-1/2} f, & \frac{1}{2} < H < 1. \end{cases}$$

Denote by $I(f)$ the Wiener integral $\int_{\mathbb{R}} f_s dB_s$, where $f \in L^2(\mathbb{R})$. $|f|_0$ is the usual $L^2(\mathbb{R})$-norm. With above notations we have following result.

Theorem 2.1 (Bender [1]). *Let $0 < H < 1$. Then $M_-^H \mathbf{1}_{(0,t)} \in L^2(\mathbb{R})$ and a fractional Brownian motion B^H is given by a continuous version of the process $(I(M_-^H \mathbf{1}_{(0,t)}))_{t\in\mathbb{R}}$.*

In the case $H > 1/2$ it follows from Theorem 1.1.1 in [6] that, for every $f \in L^{1/H}(\mathbb{R})$ we have that $M_-^H f \in L^2(\mathbb{R})$. Therefore we can define the *fractional Wiener integral* by

$$\int_0^t f_s dB_s^H := I(M_-^H(\mathbf{1}_{(0,t)} f)), \quad t \in \mathbb{R}_+.$$

Now we can introduce the Wick-Itô integral with respect to the fBm. We will follow approach sugested by Bender in [1], which is based on so called $\mathcal{S}$-transform. Consider the σ-field $\mathcal{G}$ generated by $\{I(f) : f \in L^2(\mathbb{R})\}$ and let $L^2 := L^2(\Omega, \mathcal{G}, P)$.

We define the $\mathcal{S}$-transform:

Definition 2.2. For $\Phi \in L^2$ the $\mathcal{S}$*-transform* is defined by

$$\mathcal{S}\Phi(\eta) := E\left[\Phi \exp^\diamond(I(\eta))\right], \quad \eta \in S(\mathbb{R}),$$

where $\exp^\diamond(I(\eta)) := \exp(I(\eta) - 1/2|\eta|_0^2)$ is the *Wick exponential* of $I(\eta)$ and $S(\mathbb{R})$ denotes the Schwartz space of real-valued, smooth rapidly decreasing functions.

The essential property of $\mathcal{S}$-transform is that it's injective:

Theorem 2.3 (Bender [1]). *If* $\mathcal{S}\Phi(\eta) = \mathcal{S}\Psi(\eta)$ *for all* $\eta \in S(\mathbb{R})$*, then* $\Phi = \Psi$.

Using this result we can define the *fractional Wick-Ito integral* in terms of the $\mathcal{S}$-transform.

Definition 2.4 (Bender [1]). Let $B \subset \mathbb{R}$ be a Borel set, $X : M \to L^2$. Then X is said to have a *fractional Wick-Ito integral* if $\mathcal{S}X_\cdot(\eta)(M_+^H\eta)(\cdot) \in L^1(M)$ for any $\eta \in S(\mathbb{R})$ and there is a $\Phi \in L^2$ such that, for all $\eta \in S(\mathbb{R})$,

$$\mathcal{S}\Phi(\eta) = \int_M \mathcal{S}X_t(\eta)(M_+^H\eta)(t)dt.$$

Note that by Theorem 2.3 Φ in definition 2.4 is uniquely determined (if exists), therefore we denote it by $\int_M X_t d^\diamond B_t^H$.

Now we introduce the Wick product:

Definition 2.5. Let Φ, $\Psi \in L^2$ and assume there is an element $\Phi \diamond \Psi \in L^2$, that satisfies $\mathcal{S}(\Phi \diamond \Psi)(\eta) = \mathcal{S}\Phi(\eta)\mathcal{S}\Psi(\eta)$ for all $\eta \in S(\mathbb{R})$. Then $\Phi \diamond \Psi$ is called the *wick product* of Φ and Ψ.

There is a connection between the fractional Wick-Ito integral and the Wick product. For "sufficiently good" process $(X_t)_{t\in[0,T]}$, the fractional Wick-Ito integral can be defined as limit in L^2 of Wick-Riemann sums:

$$\int_0^T X_t d^\diamond B_t^H := \lim_{n\to\infty} \sum_{t_i \in \pi_n} X_{t_{i-1}} \diamond (B_{t_i}^H - B_{t_{i-1}}^H),$$

where $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is sequence of partitions of $[0, T]$ such that, $\mathrm{diam}(\pi_n) \to 0$ with $n \to \infty$ (for more details see e.g. [6]).

Consider stochastic differential equation 1.1. By Theorem 3.3.2 in [6] and Corollary 5.5. in [1] we have that if f, b are continuous, then the unique solution $S : [0,T] \to L^2$ of the equation (1.1) is given by

$$\begin{aligned} S_t &= s_0 \exp\left(\int_0^t b_s ds\right) \exp^{\diamond}\left(\int_0^t f_s dB_s^H\right) \\ &= s_0 \exp\left(\int_0^t b_s ds - \frac{1}{2}|M_-^H(\mathbf{1}_{(0,t)}f)|_0^2 + \int_0^t f_s dB_s^H\right). \end{aligned} \tag{2.1}$$

3 Main results

In this section we give construction of approximating sequences and state the convergence results. Following [2] we use discrete Wick product and Sottinen's ([9]) approximation of the fractional Brownian motion by so called disturbed binary random walks. From this moment we restrict ourselves to the case $H > 1/2$. For simplicity we assume also that fractional Brownian is defined on interval $[0,1]$, but all results can be easily extended to any interval $[0,T]$.

Consider deterministic kernel of the form:

$$z_H(t,s) = \begin{cases} C_H s^{1/2-H} \int_s^t u^{H-1/2}(u-s)^{H-3/2} du, & t > s \\ 0, & t \leqslant s, \end{cases}$$

where $C_H = (H-\frac{1}{2})(2H\Gamma(3/2-H))^{1/2}(\Gamma(H+1/2)\Gamma(2-2H))^{-1/2}$. Decreusefond and Üstünel ([4]) showed that for $H > 1/2$ and $t \geqslant 0$ fractional Brownian motion admits representation

$$B_t^H = \int_0^t z_H(t,s) dB_s.$$

Let $(\Omega, \mathcal{F}, P)$ be some probability space and, for all $n \in \mathbb{N}$, $(\xi_i^n)_{i=1,\dots,n}$ be the i.i.d. sequence with $P(\xi_i^n = 1) = P(\xi_i^n = -1) = \frac{1}{2}$.

Set

$$b_{t,i}^n := \sqrt{n} \int_{(i-1)/n}^{i/n} z_H\left(\frac{[nt]}{n}, s\right) ds$$

and define the *disturbed binary random walk*:

$$B_t^{H,n} = \sum_{i=1}^{[nt]} b_{t,i}^n \xi_i^n. \tag{3.1}$$

Then by Sottinen ([9]) $B^{H,n}$ converges weakly to the fractional Brownian motion.

In the next section we prove the following extension of above result

Theorem 3.1. *Let f be a continuous function and*

$$\int_0^t f_s dB_s^{H,n} := \sum_{k=1}^{[nt]} f_{k/n}\left(B_{k/n}^{H,n} - B_{(k-1)/n}^{H,n}\right) = \int_0^{[nt]/n} f_s^{(n)} dB_s^{H,n},$$

where $f^{(n)}_t = f_{k/n}$ *for* $t \in (\frac{k-1}{n}, \frac{k}{n}]$, *then*

$$\int_0^{\cdot} f_s dB^{H,n}_s \xrightarrow{\mathcal{D}} \int_0^{\cdot} f_s dB^H_s.$$

Here $\xrightarrow{\mathcal{D}}$ *denotes weak convergence in Skorokhod space* $D([0,1],\mathbb{R})$.

Now we define the *discrete Wick products.*

Definition 3.2. For any $n \in \mathbb{N}$ and A, $B \subset \{1,\dots,n\}$ the *discrete Wick product* is defined as

$$\prod_{i\in A} \xi^n_i \diamond_n \prod_{i\in B} \xi^n_i := \begin{cases} \prod_{i\in A\cup B} \xi^n_i, & \text{if} \quad A\cap B = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}_n := \sigma(\xi^n_1, \xi^n_2, \dots, \xi^n_n)$. The discrete Wick product can be extended to any random variables $X, Y \in L^2(\Omega, \mathcal{F}_n, P)$. Define

$$\Xi_A := \prod_{i\in A} \xi^n_i.$$

The family of functions $\{\Xi_A : A \subseteq \{1,\dots,n\}\}$ constitutes a basis in $L^2(\Omega,\mathcal{F}_n,P)$ (since it's orthonormal set with cardinality equal to dimension of $L^2(\Omega,\mathcal{F}_n,P)$). Therefore, any $X \in L^2(\Omega,\mathcal{F}_n,P)$ has a unique expansion, called *Walsh decomposition*:

$$X = \sum_{A\subseteq\{1,\dots,n\}} x^n_A \Xi^n_A, \quad x^n_A \in \mathbb{R}.$$

Now it's easy to verify that, for $X = \sum_{A\subseteq\{1,\dots,n\}} x^n_A \Xi^n_A$ and $Y = \sum_{B\subseteq\{1,\dots,n\}} y^n_B \Xi^n_B$, we have that

$$X \diamond_n Y = \sum_{C\subseteq\{1,\dots,n\}} \left(\sum_{\substack{A\cup B = C \\ A\cap B=\emptyset}} x^n_A y^n_B \right) \Xi^n_C.$$

Note that representation (3.1) is the Walsh decomposition of $B^{H,n}_t \in L^2(\Omega,\mathcal{F}_n,P)$.

For $X \in L^2(\Omega,\mathcal{F}_n,P)$ we can consider the *discrete Wick powers* $X^{\diamond_n k}$ and the *discrete Wick exponential*,

$$\exp^{\diamond_n}(X) := \sum_{k=1}^{\infty} \frac{1}{k!} X^{\diamond_n k}.$$

Now we able to state the main results of this paper.

Theorem 3.3. *If* $f : [0,1] \to \mathbb{R}$ *is continuous, then the discrete Wick exponential* $\exp^{\diamond_n}\left(\int_0^{\cdot} f_s dB^{H,n}_s\right)$ *converges weakly to the Wick exponential* $\exp^{\diamond}\left(\int_0^{\cdot} f_s dB^H_s\right)$ *in the Skorokhod space* $D([0,1],\mathbb{R})$.

Consider discrete Wick multiplicative expression:

$$\tilde{S}^n_k = \prod_{i=0}^{k}{}^{\diamond_n} \left(1 + f_{i/n}\left(B^{H,n}_{i/n} - B^{H,n}_{(i-1)/n}\right)\right), \quad k = 0,\dots,n,$$

which is the solution of Wick difference equation (1.2). This is discrete analogue of Wick differential equation

$$S_t = 1 + \int_0^t f_s S_s d^\diamond B_s^H, \quad t \in [0,1]$$

which solution is given by $\exp^\diamond(\int_0^t f_s dB_s^H)$. Note that, in generality $\tilde{S}_k^n \neq \exp^{\diamond_n}\left(\int_0^{k/n} f_s dB_s^{H,n}\right)$, since for $f \equiv 1$ we have that

$$\exp^{\diamond_n}(B_{2/n}^{H,n}) = (1 + b_{2/n,1}^n \xi_1^n)(1 + b_{2/n,2}^n \xi_2^n)$$

and

$$\tilde{S}_2^n = \left(1 + B_{1/n}^{H,n}\right) \diamond_n \left(1 + B_{2/n}^{H,n} - B_{1/n}^{H,n}\right) = 1 + b_{2/n,1}^n \xi_1^n + b_{2/n,2}^n \xi_2^n + b_{1/n,1}^n b_{2/n,2}^n \xi_1^n \xi_2^n.$$

However we have following convergence result:

Theorem 3.4. *Assume that f is continuous and let $S_t^n = \tilde{S}_k^n$ for $t \in [k/n, (k+1)/n)$, $k = 0, \ldots, n$. Then S converges weakly to the Wick exponential $\exp^\diamond\left(\int_0^\cdot f_s dB_s^H\right)$ in the Skorokhod space $D([0,1], \mathbb{R})$.*

Let us consider deterministic expression:

$$\tilde{X}_k^n = \prod_{i=0}^k \left(1 + \frac{b_{i/n}}{n}\right), \quad k = 0, \ldots, n,$$

where $b : [0,1] \to \mathbb{R}$ is continuous.

It's well known that $X_t^n := \tilde{X}_k^n$ for $t \in [k/n, (k+1)/n)$ converges to the $\exp\left(\int_0^\cdot b_s ds\right)$ in sup-norm on $[0,1]$. Therefore we obtain following extention of Theorem 3.3.

Corollary 3.5. *Suppose that $f, b : [0,1] \to \mathbb{R}$ are continuous. Then $S_t^n := X_t^n \exp^{\diamond_n}\left(\int_0^t f_s dB_s^{H,n}\right)$, $t \in [0,1]$, converges weakly to the solution of equation (1.1) in Skorokhod space $D([0,1], \mathbb{R})$.*

Proof. It follows from Theorem 3.3 and Theorem 4.4 in [3] that

$$X_\cdot^n \exp^{\diamond_n}\left(\int_0^\cdot f_s dB_s^{H,n}\right) \xrightarrow{\mathcal{D}} s_0 \exp\left(\int_0^\cdot b_s ds\right) \exp^\diamond\left(\int_0^\cdot f_s dB_s^H\right),$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in Skorokhod space $D([0,1], \mathbb{R})$. Since $s_0 \exp\left(\int_0^\cdot b_s ds\right) \exp^\diamond\left(\int_0^\cdot f_s dB_s^H\right)$ is unique solution of equation (1.1), the proof is complete. □

4 Convergence to the fractional Wiener integral

In this section we use Sottinen's disturbed binary random walk $B^{H,n}$ given by (3.1) to approximate Wiener integrals with respect to the fractional Brownian motion.

Let

$$\int_0^t f_s dB_s^{H,n} = \sum_{k=1}^{[nt]} f_{k/n}\left(B_{k/n}^{H,n} - B_{(k-1)/n}^{H,n}\right) = \int_0^{[nt]/n} f_s^{(n)} dB_s^{H,n},$$

where $f_t^{(n)} = f_{k/n}$ for $t \in (\frac{k-1}{n}, \frac{k}{n}]$.

Set

$$d_{k,i}^n = b_{k/n,i}^n - b_{(k-1)/n,i}^n, \quad \gamma_{t,i}^n(f) = \mathbf{1}_{\{i \leqslant [nt]\}} \sum_{k=i}^{[nt]} f_{k/n} d_{k,i}^n$$

and note that increments of $B^{H,n}$ satisfies

$$B_{k/n}^{H,n} - B_{(k-1)/n}^{H,n} = \sum_{i=1}^{k} d_{k,i}^n \xi_i^n.$$

Moreover for $0 \leqslant t_1 \leqslant t_2 \leqslant 1$ we have that

$$\begin{aligned} \int_{t_1}^{t_2} f_s dB_s^{H,n} &= \sum_{k=[nt_1]+1}^{[nt_2]} f_{k/n} \sum_{i=1}^{k} d_{k,i}^n \xi_i^n \\ &= \sum_{i=1}^{[nt_1]} \xi_i^n \gamma_{t_2,[nt_1]+1}^n(f) + \sum_{i=[nt_1]+1}^{[nt_2]} \xi_i^n \gamma_{t_2,i}^n(f). \end{aligned} \tag{4.1}$$

Lemma 4.1. *Let f be continuous function and $p > 1$, then for any $0 \leqslant t_1 \leqslant t_2 \leqslant 1$ there exists constant C_p such that*

$$\mathbb{E}\left|\int_{t_1}^{t_2} f_s dB_s^{H,n}\right|^p \leqslant C_p \sup_{t \leqslant T} |f|^p \left|\frac{[nt_2]}{n} - \frac{[nt_1]}{n}\right|^{pH}.$$

Proof. It follows from (4.1) and Burkholder inequality that

$$\begin{aligned} \mathbb{E}\left|\int_{t_1}^{t_2} f_s dB_s^{H,n}\right|^p &= \mathbb{E}\left|\sum_{i=1}^{[nt_1]} \xi_i^n \gamma_{t_2,[nt_1]+1}^n(f) + \sum_{i=[nt_1]+1}^{[nt_2]} \xi_i^n \gamma_{t_2,i}^n(f)\right|^p \\ &\leqslant C_p \left(\sum_{i=1}^{[nt_1]} \left(\gamma_{t_2,[nt_1]+1}^n(f)\right)^2 + \sum_{i=[nt_1]+1}^{[nt_2]} \left(\gamma_{t_2,i}^n(f)\right)^2\right)^{p/2}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}\left|\int_{t_1}^{t_2} f_s dB_s^{H,n}\right|^p &\leqslant C_p \sup_{t \leqslant T} |f_t|^p \left(\sum_{i=1}^{[nt_1]} \left(\sum_{k=[nt1]+1}^{[nt_2]} d_{k,i}^n\right)^2 + \sum_{i=[nt_1]+1}^{[nt_2]} \left(\sum_{k=i}^{[nt_2]} d_{k,i}^n\right)^2\right)^{p/2} \\ &= C_p \sup_{t \leqslant T} |f_t|^p \left(\sum_{i=1}^{[nt_1]} \left(b_{[nt_2],i}^n - b_{[nt_1],i}^n\right)^2 + \sum_{i=[nt_1]+1}^{[nt_2]} \left(b_{[nt_2],i}^n - b_{i-1,i}^n\right)^2\right)^{p/2}. \end{aligned}$$

Furthermore by Cauchy-Schwarz inequality we have that

$$
\begin{aligned}
\mathbb{E}\left|\int_{t_1}^{t_2} f_s dB_s^{H,n}\right|^p &\leqslant C_p \sup_{t\leqslant T}|f_t|^p \left[\sum_{i=1}^{[nt_1]}\left(\sqrt{n}\int_{(i-1)/n}^{i/n} z\left([nt_2]/n,s\right) - z\left([nt_1]/n,s\right) ds\right)^2\right. \\
&\quad \left. + \sum_{i=[nt_1]+1}^{[nt_2]}\left(\sqrt{n}\int_{(i-1)/n}^{i/n} z\left([nt_2]/n,s\right) - z\left((i-1)/n,s\right) ds\right)^2\right]^{p/2} \\
&\leqslant C_p \sup_{t\leqslant T}|f_t|^p \left(\sum_{i=1}^{[nt_2]}\int_{(i-1)/n}^{i/n}\left(z\left([nt_2]/n,s\right) - z\left([nt_1]/n,s\right)\right)^2 ds\right)^{p/2} \\
&= C_p \sup_{t\leqslant T}|f_t|^p \left(\int_0^{[nt_2]/n}\left(z\left([nt_2]/n,s\right) - z\left([nt_1]/n,s\right)\right)^2 ds\right)^{p/2} \\
&= C_p \sup_{t\leqslant T}|f_t|^p \left(\frac{[nt_2]}{n} - \frac{[nt_1]}{n}\right)^{pH},
\end{aligned}
$$

so the proof of lemma is complete. □

Denote by $\|\cdot\|_p$ the L^p norm, we have following estimates:

Lemma 4.2. *Let $a_1,\dots,a_n \in \mathbb{R}$ and $B_1,\dots,B_n$ be the random variables such that $\mathbb{E}B_i = 0$, $\mathbb{E}(B_i)^2 < +\infty$ for $i = 1,\dots,n$. Then*

(i) there exists $1 \leqslant k \leqslant n$ such that

$$\|a_k B_k\|_2 \leqslant \left(\frac{1}{n}\sum_{i=1}^n |a_i|\right)\left(\frac{1}{n}\sum_{i=1}^n \mathbb{E}(B_i)^2\right)^{1/2},$$

(ii) there exists $C \in \mathbb{R}$ with the property that

$$\left\|\sum_{1\leqslant i\leqslant j\leqslant n} a_i B_j\right\|_2 \leqslant C S(a,B),$$

where $S(a,B)$ is the largest value of the sums of the form $(\sum_{k=1}^m |\tilde{a}_k|)(\sum_{k=1}^m |\tilde{B}_k|^2)^{1/2}$, where $\tilde{a}_k = a_{i_k} + a_{i_k+1} + \cdots + a_{i_{k+1}}$, $1 = i_1 < \cdots < i_{m+1} = n$, $m \leqslant n$. $\tilde{B}_k$ is defined similarly.

Proof. Pick k such that

$$\mathbb{E}(a_k B_k)^2 = \min_{1\leqslant i\leqslant n} \mathbb{E}(a_i B_i)^2.$$

Then

$$\begin{aligned}
\|a_k B_k\|_2 &\leqslant (\|a_1 B_1\|_2 \dots \|a_n B_n\|_2)^{1/n} \\
&= \left(\prod_{i=1}^{n} |a_i|\right)^{1/n} \left(\prod_{i=1}^{n} \|B_i\|_2\right)^{1/n} \\
&= \left(\prod_{i=1}^{n} |a_i|\right)^{1/n} \left(\left[\prod_{i=1}^{n} \mathbb{E}\,(B_i)^2\right]^{1/n}\right)^{1/2} \\
&\leqslant \left(\frac{1}{n}\sum_{i=1}^{n} |a_i|\right) \left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\,(B_i)^2\right)^{1/2},
\end{aligned}$$

which ends the proof of (i).

To prove (ii), note that by (i) there exists $1 \leqslant k \leqslant n-1$ such that

$$\|a_{k+1} B_k\|_2 \leqslant \left(\frac{1}{n-1}\sum_{i=1}^{n-1} |a_{i+i}|\right) \left(\frac{1}{n-1}\sum_{i=1}^{n-1} \mathbb{E}\,(B_i)^2\right)^{1/2}.$$

Set $\bar{a}_i = a_i$ for $i = 1, \dots, k-1$, $\bar{a}_k = a_k + a_{k+1}$, $\bar{a}_i = a_{i+1}$ dla $i = k+1, \dots, n-1$. Then

$$\sum_{1 \leqslant i \leqslant j \leqslant n-1} \bar{a}_i \bar{B}_j = a_{k+1} B_k + \sum_{1 \leqslant i \leqslant j \leqslant n} a_i B_j.$$

Thus

$$\begin{aligned}
\left\| \sum_{1 \leqslant i \leqslant j \leqslant n} a_i B_j \right\|_2 &\leqslant (n-1)^{-3/2} \left(\sum_{i=1}^{n-1} |a_{i+1}|\right) \left(\sum_{i=1}^{n-1} \mathbb{E}\,(B_i)^2\right)^{1/2} + \left\| \sum_{1 \leqslant i \leqslant j \leqslant n-1} \bar{a}_i \bar{B}_j \right\|_2 \\
&\leqslant (n-1)^{-3/2} S(a, B) + \left\| \sum_{1 \leqslant i \leqslant j \leqslant n-1} \bar{a}_i \bar{B}_j \right\|_2.
\end{aligned}$$

Similarly

$$\left\| \sum_{1 \leqslant i \leqslant j \leqslant n-1} \bar{a}_i \bar{B}_j \right\|_2 \leqslant (n-2)^{-3/2} S(\bar{a}, \bar{B}) + \left\| \sum_{1 \leqslant i \leqslant j \leqslant n-2} \bar{\bar{a}}_i \bar{\bar{B}}_j \right\|_2.$$

Clearly $S(\bar{a}, \bar{B}) \leqslant S(a, B)$ hence

$$\left\| \sum_{1 \leqslant i \leqslant j \leqslant n} a_i B_j \right\|_2 \leqslant S(a, B) \left[(n-1)^{-3/2} + (n-2)^{-3/2}\right] + \left\| \sum_{1 \leqslant i \leqslant j \leqslant n-2} \bar{\bar{a}}_i \bar{\bar{B}}_j \right\|_2.$$

Repeating the above arguments we get

$$\left\| \sum_{1 \leqslant i \leqslant j \leqslant n} a_i B_j \right\|_2 \leqslant \left(\sum_{n=1}^{\infty} n^{-3/2}\right) S(a, B),$$

which is our claim. □

Lemma 4.3. *Assume f is a function of finite variation, then for any partitions $\pi = \{0 = t_0 < t_1 <, \dots, t_m = 1\}$, $\pi' = \{0 = t'_0 < t'_1 <, \dots, t'_k = 1\}$ there exists $C \in \mathbb{R}$ such that*

$$\left\| \sum_{\substack{t_i \in \pi \\ t_i \leqslant t}} f^{(n)}_{t_i} \left(B^{H,n}_{t_i} - B^{H,n}_{t_{i-1}} \right) - \sum_{\substack{t'_i \in \pi' \\ t'_i \leqslant t}} f^{(n)}_{t'_i} \left(B^{H,n}_{t'_i} - B^{H,n}_{t'_{i-1}} \right) \right\|_2 \leqslant C\, V(f)_{[0,1]} \epsilon(\pi, \pi', n),$$

where $V(f)_{[0,1]}$ denotes variation of f over $[0,1]$ and

$$\epsilon(\pi, \pi', n) = \max \left\{ \sup_{t_i \in \pi} \left(\frac{[nt_i]}{n} - \frac{[nt_{i-1}]}{n} \right)^H, \sup_{t'_i \in \pi'} \left(\frac{[nt'_i]}{n} - \frac{[nt'_{i-1}]}{n} \right)^H \right\}.$$

Proof. Note that

$$\sum_{i=1}^m f^{(n)}_{t_i} \left(B^{H,n}_{t_i} - B^{H,n}_{t_{i-1}} \right) = f^{(n)}_{t_0} \left(B^{H,n}_{t_m} - B^{H,n}_{t_0} \right) - \sum_{1 \leqslant i \leqslant j \leqslant m} \left(f^{(n)}_{t_i} - f^{(n)}_{t_{i-1}} \right) \left(B^{H,n}_{t_j} - B^{H,n}_{t_{j-1}} \right).$$

Hence by lemma 4.2 (ii) we have

$$\left\| \sum_{i=1}^m f^{(n)}_{t_i} \left(B^{H,n}_{t_i} - B^{H,n}_{t_{i-1}} \right) - f^{(n)}_{t_0} \left(B^{H,n}_{t_m} - B^{H,n}_{t_0} \right) \right\|_2 \leqslant C V(f)_{[0,1]} \tilde{V}_2(B^{H,n})_{[0,1]}. \tag{4.2}$$

Here $\tilde{V}_2(X)_{[0,1]} = \sup_\pi \left(\sum_{t_i \in \pi} \mathbb{E}|X_{t_i} - X_{t_{i-1}}|^2 \right)^{1/2}$ where supremum is taken over all partitions $\pi = \{0 = t_0 < t_1 <, \dots, t_n = 1\}$. Moreover

$$\begin{aligned}
&\left\| \sum_{\substack{t_i \in \pi \\ t_i \leqslant t}} f^{(n)}_{t_i} \left(B^{H,n}_{t_i} - B^{H,n}_{t_{i-1}} \right) - \sum_{\substack{t'_i \in \pi' \\ t'_i \leqslant t}} f^{(n)}_{t'_i} \left(B^{H,n}_{t'_i} - B^{H,n}_{t'_{i-1}} \right) \right\|_2 \\
&\qquad \leqslant \left\| \sum_{\substack{t_i \in \pi \\ t_i \leqslant t}} f^{(n)}_{t_i} \left(B^{H,n}_{t_i} - B^{H,n}_{t_{i-1}} \right) - \sum_{\substack{t_j \in \pi \cup \pi' \\ t_j \leqslant t}} f^{(n)}_{t_j} \left(B^{H,n}_{t_j} - B^{H,n}_{t_{j-1}} \right) \right\|_2 \\
&\qquad\quad + \left\| \sum_{\substack{t'_i \in \pi' \\ t'_i \leqslant t}} f^{(n)}_{t'_i} \left(B^{H,n}_{t'_i} - B^{H,n}_{t'_{i-1}} \right) - \sum_{\substack{t_j \in \pi \cup \pi' \\ t_j \leqslant t}} f^{(n)}_{t_j} \left(B^{H,n}_{t_j} - B^{H,n}_{t_{j-1}} \right) \right\|_2 \\
&\qquad = I_1 + I_2.
\end{aligned}$$

By (4.2) and lemma 4.1,

$$\begin{aligned}
I_1 &\leqslant \sum_{i=1}^{m} \left\| f_{t_i}^{(n)} \left(B_{t_i}^{H,n} - B_{t_{i-1}}^{H,n} \right) - \sum_{\substack{t_j \in \pi \cup \pi' \\ t_{i-1} \leqslant t_j \leqslant t_i}} f_{t_j}^{(n)} \left(B_{t_j}^{H,n} - B_{t_{j-1}}^{H,n} \right) \right\|_2 \\
&\leqslant C \sum_{i=1}^{m} V(f)_{[t_{i-1},t_i]} \tilde{V}_2(B^{H,n})_{[t_{i-1},t_i]} \\
&\leqslant C \sup_{t_i \in \pi} \left(\tilde{V}_2(B^{H,n})_{[t_{i-1},t_i]} \right) \sum_{i=1}^{m} V(f)_{[t_{i-1},t_i]} \\
&\leqslant C \sup_{t_i \in \pi} \left(\frac{[nt_i]}{n} - \frac{[nt_{i-1}]}{n} \right)^H V(f)_{[0,1]},
\end{aligned}$$

and similarly

$$I_2 \leqslant C \sup_{t'_i \in \pi} \left(\frac{[nt'_i]}{n} - \frac{[nt'_{i-1}]}{n} \right)^H V(f)_{[0,1]},$$

which completes the proof. □

Lemma 4.4. *Assume f is a function of finite variation, then for any sequence of partitions $\pi_n = \{0 = t_{n,0} < t_{n,1} <, \dots, t_{n,m_n} = 1\}$ such that diam$(\pi_n) \to 0$ and $p \geqslant 1$*

$$\sup_{t \leqslant T} \left\| \sum_{t_{n,i} \leqslant t} f_{t_{n,i}} \left(B_{t_{n,i}}^H - B_{t_{n,i-1}}^H \right) - \int_0^t f_s dB_s^H \right\|_p \xrightarrow[k\to\infty]{} 0.$$

Proof. See Ziemkiewicz [12] theorem 5.11. □

Now we able to prove convergence to the fractional Wiener integral:

Proof of Theorem 3.1. Firstly we show that above convergence holds in finite-dimensional distributions

$$\int_0^{\cdot} f_s dB_s^{H,n} \xrightarrow{\mathcal{D}_f} \int_0^{\cdot} f_s dB_s^H. \tag{4.3}$$

Let $\pi_k = \{0 = t_{k,0} < t_{k,1} <, \dots, t_{k,m_k} = 1\}$ be a partition with diam$(\pi_k) \to 0$ and

$$f_t^h = \int_t^{t+h \wedge T} f_s ds, \quad h > 0,\, T \in [0,T]$$

be the Stieklov functions. Since for any $h > 0$, f^h is function of finite variation, it follows from lemma 4.3 that

$$\lim_{k\to\infty} \sup_{n \geqslant k} \sup_{t \leqslant T} \left\| \int_0^t f_s^h dB_s^{H,n} - \sum_{t_{k,i} \leqslant t} f_{t_{k,i}}^{h,(n)} \left(B_{t_{k,i}}^{H,n} - B_{t_{k,i-1}}^{H,n} \right) \right\|_2 = 0.$$

Clearly

$$\sum_{t_{k,i}\leqslant \cdot} f^{h,(n)}_{t_{k,i}}\left(B^{H,n}_{t_{k,i}} - B^{H,n}_{t_{k,i-1}}\right) \xrightarrow{\mathcal{D}_f} \sum_{t_{k,i}\leqslant \cdot} f^{h}_{t_{k,i}}\left(B^{H}_{t_{k,i}} - B^{H}_{t_{k,i-1}}\right).$$

Combining this with lemma 4.4 we obtain convergence in finite-dimensional distributions for f^h. It's easy to verify that $\sup_{t\leqslant T}\left|f^h_t - f_t\right| \to 0$ with $h \to 0$, therefore by lemma 4.1 we have

$$\lim_{h\to 0}\sup_n \sup_{t\leqslant T}\left\|\int_0^t f_s dB^{H,n}_s - \int_0^t f^h_s dB^{H,n}_s\right\|_2 = 0,$$

witch together with convergence in finite-dimensional distributions for f^h gives (4.3). Since by lemma 4.1 we have that sequence $\int_0^\cdot f_s dB^{H,n}_s$ is tight, the proof is complete. □

5 Proofs of Theorems 3.3 and 3.4

Proof of Theorem 3.3. It follows from (4.1) that

$$\begin{aligned}
\exp^{\diamond_n}\left(\int_0^t f_s dB^{H,n}_s\right) &= \sum_{l=0}^{\infty}\frac{1}{l!}\left(\int_0^t f_s dB^{H,n}_s\right)^{\diamond_n l} \\
&= \sum_{l=0}^{\infty}\frac{1}{l!}\left(\sum_{i=1}^{[nt]}\xi^n_i \gamma^n_{t,i}(f)\right)^{\diamond_n l} \\
&= \sum_{l=0}^{\infty}\frac{1}{l!}\left(\sum_{\substack{i_1,i_2,\dots,i_l=1\\ \text{pairwise distinct}}}^{[nt]}\left[\prod_{j=1}^{l}\gamma^n_{t,i_j}(f)\prod_{j=1}^{l}\xi^n_{i_j}\right]\right) \qquad (5.1)\\
&= \sum_{l=0}^{[nt]}\left(\sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\ |C|=l}}\left[\prod_{p\in C}\gamma^n_{t,p}(f)\prod_{p\in C}\xi^n_p\right]\right) \\
&= \prod_{i=1}^{[nt]}\left(1+\gamma^n_{t,i}(f)\xi^n_i\right).
\end{aligned}$$

Moreover by Cauchy-Schwarz inequality we have for all $i \leqslant [nt]$

$$\begin{aligned}|\gamma^n_{t,i}(f)| &= \left|\sum_{k=i}^{[nt]} f_{k/n}\left(b^n_{k/n,i} - b^n_{(k-1)/n,i}\right)\right| \\ &\leqslant \sup_{t\leqslant T}|f_t| \sum_{k=i}^{[nt]} \sqrt{n}\int_{(i-1)/n}^{i/n} z\left(k/n,s\right) - z\left((k-1)/n,s\right) ds \\ &= \sup_{t\leqslant T}|f_t|\sqrt{n}\int_{(i-1)/n}^{i/n} z\left([nt]/n,s\right) - z\left((i-1)/n,s\right) ds \qquad (5.2)\\ &\leqslant \sup_{t\leqslant T}|f_t|\left(\int_{(i-1)/n}^{i/n} z^2\left([nt]/n,s\right) ds\right)^{1/2} \\ &\leqslant \sup_{t\leqslant T}|f_t|\left(\int_0^{1/n} z^2\left(T,s\right) ds\right)^{1/2} \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

Therefore for sufficient large n expression $\log\left(1+\gamma^n_{t,i}(f)\xi^n_i\right)$ is well defined and according to Taylor formula

$$\begin{aligned}&\exp^{\diamond_n}\left(\int_0^t f_s dB^{H,n}_s\right) \\ &\quad= \prod_{i=1}^{[nt]}\left(1+\gamma^n_{t,i}(f)\xi^n_i\right) \\ &\quad= \exp\left(\sum_{i=1}^{[nt]}\gamma^n_{t,i}(f)\xi^n_i - \frac{1}{2}\sum_{i=1}^{[nt]}\left(\gamma^n_{t,i}(f)\right)^2 + \sum_{i=1}^{[nt]} r\left(\gamma^n_{t,i}(f)\xi^n_i\right)\left(\gamma^n_{t,i}(f)\right)^2\right) \qquad (5.3)\\ &\quad= \exp\left(\int_0^t f_s dB^{H,n}_s - \frac{1}{2}\mathbb{E}\left(\int_0^t f_s dB^{H,n}_s\right)^2 + \sum_{i=1}^{[nt]} r\left(\gamma^n_{t,i}(f)\xi^n_i\right)\left(\gamma^n_{t,i}(f)\right)^2\right),\end{aligned}$$

where $r(x) \to 0$ with $x \to 0$.

By Lemma 4.1 we have also

$$\sup_{t\leqslant T}\mathbb{E}\left|\int_0^t f_s dB^{H,n}_s\right|^{2+\varepsilon} < +\infty,$$

for all $\varepsilon > 0$.

Combining this with Theorem 3.1 we obtain

$$\sup_{t\leqslant T}\left|\mathbb{E}\left(\int_0^t f_s dB^{H,n}_s\right)^2 - \mathbb{E}\left(\int_0^t f_s dB^H_s\right)^2\right| \xrightarrow[n\to\infty]{} 0. \qquad (5.4)$$

By (5.2), we thus get the following convergence:

$$\begin{aligned}\sup_{t\leqslant T}\mathbb{E}\left|\sum_{i=1}^{[nt]} r\left(\gamma_{t,i}^n(f)\xi_i^n\right)\left(\gamma_{t,i}^n(f)\right)^2\right| &\leqslant \sup_{t\leqslant T}\left[\max_i\left\{r\left(\gamma_{t,i}^n(f)\right)\right\}\sum_{i=1}^{[nt]}\left(\gamma_{t,i}^n(f)\right)^2\right]\\ &= \sup_{t\leqslant T}\left[\max_i\left\{r\left(\gamma_{t,i}^n(f)\right)\right\}\mathbb{E}\left(\int_0^t f_s dB_s^{H,n}\right)^2\right]\xrightarrow[n\to\infty]{} 0.\end{aligned} \tag{5.5}$$

Our claim is implied by (5.3), (5.4), (5.5) and Theorem 3.1. □

Before we prove Theorem 3.4, we need some auxiliary results. Recall the notations:

$$\Xi_A := \prod_{i\in A}\xi_i^n, \quad d_{k,i}^n = b_{k/n,i}^n - b_{(k-1)/n,i}^n, \quad \gamma_{t,i}^n(f) = \mathbf{1}_{\{i\leqslant[nt]\}}\sum_{k=i}^{[nt]} f_{k/n}d_{k,i}^n.$$

Proposition 5.1. *For all k, $n\in\mathbb{N}$ we have the Walsh decompositions:*

(i)

$$\tilde{S}_k^n = \sum_{l=0}^{k}\sum_{\substack{C\subseteq\{1,\dots,k\}\\ |C|=l}}\Xi_C^n\left(\sum_{\substack{m:C\to\{1,\dots,k\}\\ \text{injective}}}\prod_{p\in C} f_{m(p)/n}d_{m(p),p}^n\right)$$

(ii)

$$\exp^{\diamond_n}\left(\int_0^t f_s dB_s^{H,n}\right) = \sum_{l=0}^{[nt]}\sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\ |C|=l}}\Xi_C^n\left(\sum_{m:C\to\{1,\dots,[nt]\}}\prod_{p\in C} f_{m(p)/n}d_{m(p),p}^n\right)$$

(iii)

$$\exp^{\diamond_n}\left(\int_0^{k/n} f_s dB_s^{H,n}\right) - \tilde{S}_k^n = \sum_{l=2}^{k}\sum_{\substack{C\subseteq\{1,\dots,k\}\\ |C|=l}}\Xi_C^n\left(\sum_{\substack{m:C\to\{1,\dots,k\}\\ \text{not injective}}}\prod_{p\in C} f_{m(p)/n}d_{m(p),p}^n\right)$$

Proof. We show equation (i) by induction. Assume following convention: empty product equals 1 and there exists exactly one function from empty set. Therefore (i) holds for $k=0$.

Moreover we have

$$\left(\sum_{\substack{C\subseteq\{1,\dots,k\}\\|C|=l}}\Xi^n_C\left(\sum_{\substack{m:C\to\{1,\dots,k\}\\\text{injective}}}\prod_{p\in C}f_{m(p)/n}d^n_{m(p),p}\right)\right)\diamond_n f_{(k+1)/n}\sum_{i=1}^{k+1}d^n_{k+1,i}\xi^n_i$$

$$=\sum_{\substack{C\subseteq\{1,\dots,k\}\\|C|=l}}\sum_{\substack{i\in\{1,\dots,k+1\}\\i\notin C}}\Xi^n_C\,\xi^n_i\left(\sum_{\substack{m:C\to\{1,\dots,k\}\\\text{injective}}}\prod_{p\in C}f_{m(p)/n}f_{(k+1)/n}d^n_{m(p),p}d^n_{k+1,i}\right)$$

$$=\sum_{\substack{C'\subseteq\{1,\dots,k+1\}\\|C'|=l+1}}\Xi^n_{C'}\left(\sum_{\substack{m:C'\to\{1,\dots,k+1\}\\\text{injective, }\exists q:m(q)=k+1}}\prod_{p\in C'}f_{m(p)/n}d^n_{m(p),p}\right).$$

Thus from induction hypothesis we obtain

$$\begin{aligned}\tilde S^n_{k+1}&=\tilde S^n_k\diamond_n\left(1+f_{(k+1)/n}\left(B^{H,n}_{(k+1)/n}-B^{H,n}_{k/n}\right)\right)\\&=\sum_{l=0}^{k}\sum_{\substack{C\subseteq\{1,\dots,k\}\\|C|=l}}\Xi^n_C\left(\sum_{\substack{m:C\to\{1,\dots,k\}\\\text{injective}}}\prod_{p\in C}f_{m(p)/n}d^n_{m(p),p}\right)\\&\quad+\sum_{l=1}^{k+1}\sum_{\substack{C\subseteq\{1,\dots,k+1\}\\|C|=l}}\Xi^n_C\left(\sum_{\substack{m:C\to\{1,\dots,k\}\\\text{injective, }\exists q:m(q)=k+1}}\prod_{p\in C}f_{m(p)/n}d^n_{m(p),p}\right)\\&=\sum_{l=0}^{k+1}\sum_{\substack{C\subseteq\{1,\dots,k+1\}\\|C|=l}}\Xi^n_C\left(\sum_{\substack{m:C\to\{1,\dots,k\}\\\text{injective}}}\prod_{p\in C}f_{m(p)/n}d^n_{m(p),p}\right).\end{aligned}$$

To show (ii) note that $d^n_{k,i}=0$ for $i>k$. Hence

$$\begin{aligned}\prod_{p\in C}\gamma^n_{t,p}&=\prod_{p\in C}\sum_{m(p)=1}^{[nt]}f_{m(p)/n}d^n_{m(p),p}\\&=\sum_{m:C\to\{1,\dots,[nt]\}}\prod_{p\in C}f_{m(p)/n}d^n_{m(p),p}.\end{aligned}\tag{5.6}$$

Equation (ii) is, thus, implied by (5.1). Since every function from singleton is injective, equation (iii) follows from (i) and (ii). □

Proposition 5.2. *For all k, $n \in \mathbb{N}$:*

$$k!\left(\sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\|C|=k}}\left(\prod_{p\in C}\gamma^n_{t,p}(f)-\prod_{p\in C}\gamma^n_{s,p}(f)\right)^2\right)\leqslant\left[\mathbb{E}\left(\int_0^t f_u dB^{H,n}_u\right)^2\right]^k+\left[\mathbb{E}\left(\int_0^s f_u dB^{H,n}_u\right)^2\right]^k$$
$$-2\left[\mathbb{E}\left(\int_0^t f_u dB^{H,n}_u\int_0^s f_u dB^{H,n}_u\right)\right]^k.$$

Proof. Note that

$$\begin{aligned}\left[\mathbb{E}\left(\int_0^t f_u dB^{H,n}_u\int_0^s f_u dB^{H,n}_u\right)\right]^k&=\left[\mathbb{E}\left(\sum_{i_1,i_2=1}^{[nt]}\gamma^n_{t,i_1}(f)\gamma^n_{s,i_2}(f)\xi^n_{i_1}\xi^n_{i_2}\right)\right]^k\\&=\left[\sum_{i=1}^{[nt]}\gamma^n_{t,i}(f)\gamma^n_{s,i}(f)\right]^k\\&=\sum_{i_1,\dots,i_k=1}^{[tn]}\prod_{j=1}^k\gamma^n_{t,i_j}(f)\gamma^n_{s,i_j}(f)\\&=k!\left(\sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\|C|=k}}\prod_{p\in C}\gamma^n_{t,p}(f)\gamma^n_{s,p}(f)\right)+\sum_{\substack{i_1,\dots,i_k=1\\\exists p,q\ i_p=i_q}}^{[tn]}\prod_{j=1}^k\gamma^n_{t,i_j}(f)\gamma^n_{s,i_j}(f).\end{aligned}$$

Therefore

$$\begin{aligned}&\left[\mathbb{E}\left(\int_0^t f_u dB^{H,n}_u\right)^2\right]^k+\left[\mathbb{E}\left(\int_0^s f_u dB^{H,n}_u\right)^2\right]^k-2\left[\mathbb{E}\left(\int_0^t f_u dB^{H,n}_u\int_0^s f_u dB^{H,n}_u\right)\right]^k\\&\qquad-k!\left(\sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\|C|=k}}\left(\prod_{p\in C}\gamma^n_{t,p}(f)-\prod_{p\in C}\gamma^n_{s,p}(f)\right)^2\right)\\&=\sum_{\substack{i_1,\dots,i_k=1\\\exists p,q\ i_p=i_q}}^{[tn]}\left(\prod_{j=1}^k\left(\gamma^n_{t,i_j}(f)\right)^2-\prod_{j=1}^k\left(\gamma^n_{s,i_j}(f)\right)^2\right.\\&\qquad\left.-2\prod_{j=1}^k\gamma^n_{t,i_j}(f)\gamma^n_{s,i_j}(f)\right)\\&=\sum_{\substack{i_1,\dots,i_k=1\\\exists p,q\ i_p=i_q}}^{[tn]}\left(\prod_{j=1}^k\gamma^n_{t,i_j}(f)-\prod_{j=1}^k\gamma^n_{s,i_j}(f)\right)^2\geqslant 0,\end{aligned}$$

which is our claim. □

Lemma 5.3 (Bender [2]). *Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space and $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ the corresponding norm on X. Then, for all $x, y \in X$ and $k > 1$,*

$$\|x\|^{2k} + \|y\|^{2k} - 2(\langle x, y\rangle)^k \leqslant 2^{k+1}(\|x\| + \|y\|)^{2(k-1)}\|x-y\|^2.$$

Theorem 5.4. *There exists $L \in \mathbb{R}$ such that for all $t, s \in [0,1]$*

$$\mathbb{E}\,(S_t^n - S_s^n)^2 \leqslant L\left|\frac{[nt]}{n} - \frac{[ns]}{n}\right|^{2H} \tag{5.7}$$

Proof. Assume $s \leqslant t$ and recall that $d_{k,i}^n = 0$ for $k < i$. Therefore:

$$S_s^n = \sum_{k=0}^{[nt]} \sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\ |C|=k}} \Xi_C^n \left(\sum_{\substack{m:C\to\{1,\dots,[ns]\}\\ \text{injective}}} \prod_{p\in C} f_{m(p)/n} d_{m(p),p}^n \right)$$

By (5.6) we have

$$\begin{aligned}
\sum_{\substack{m:C\to\{1,\dots,[nt]\}\\ m:\text{"}1-1\text{"}\ \exists q:m(q)>[ns]}} \prod_{p\in C} f_{m(p)/n} d_{m(p),p}^n &\leqslant \sum_{\substack{m:C\to\{1,\dots,[nt]\}\\ \exists q:m(q)>[ns]}} \prod_{p\in C} \left|f_{m(p)/n}\right| d_{m(p),p}^n \\
&= \sum_{m:C\to\{1,\dots,[nt]\}} \prod_{p\in C} \left|f_{m(p)/n}\right| d_{m(p),p}^n \\
&\quad - \sum_{m:C\to\{1,\dots,[ns]\}} \prod_{p\in C} \left|f_{m(p)/n}\right| d_{m(p),p}^n \\
&= \prod_{p\in C} \gamma_{t,p}^n(|f|) - \prod_{p\in C} \gamma_{s,p}^n(|f|)
\end{aligned}$$

. Since

$$\mathbb{E}\,[\Xi_C^n\,\Xi_{C'}^n] = \mathbf{1}_{\{C=C'\}}, \tag{5.8}$$

it follows that

$$\begin{aligned}
\mathbb{E}\,(S_t^n - S_s^n)^2 &= \sum_{k=0}^{[nt]} \sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\ |C|=k}} \left(\sum_{\substack{m:C\to\{1,\dots,[nt]\}\\ \text{injective, } \exists q:m(q)>[ns]}} \prod_{p\in C} f_{m(p)/n} d_{m(p),p}^n \right)^2 \\
&\leqslant \sum_{k=0}^{[nt]} \sum_{\substack{C\subseteq\{1,\dots,[nt]\}\\ |C|=k}} \left(\prod_{p\in C} \gamma_{t,p}^n(|f|) - \prod_{p\in C} \gamma_{s,p}^n(|f|) \right)^2.
\end{aligned}$$

Thanks to Proposition 5.2 and Lemmas 5.3, 4.1 we obtain

$$
\begin{aligned}
&\sum_{\substack{C\subseteq\{1,\ldots,[nt]\}\\ |C|=k}} \left(\prod_{p\in C}\gamma^n_{t,p}(|f|) - \prod_{p\in C}\gamma^n_{s,p}(|f|)\right)^2 \\
&\quad\leqslant \frac{1}{k!}\left(\left[\mathbb{E}\left(\int_0^t f_u dB_u^{H,n}\right)^2\right]^k + \left[\mathbb{E}\left(\int_0^s f_u dB_u^{H,n}\right)^2\right]^k\right. \\
&\quad\quad \left. - 2\left[\mathbb{E}\left(\int_0^t f_u dB_u^{H,n}\int_0^s f_u dB_u^{H,n}\right)\right]^k\right) \\
&\quad\leqslant \frac{2^{k+1}}{k!}\left(\left\|\int_0^t f_u dB_u^{H,n}\right\|_2 + \left\|\int_0^s f_u dB_u^{H,n}\right\|_2\right)^{2(k-1)} \mathbb{E}\left(\int_s^t f_u dB_u^{H,n}\right)^2 \\
&\quad\leqslant \frac{2^{k+1}}{k!} C\left[\left(\frac{[nt]}{n}\right)^H + \left(\frac{[ns]}{n}\right)^H\right]^{2(k-1)} \left|\frac{[nt]}{n} - \frac{[ns]}{n}\right|^{2H} \\
&\quad\leqslant C'\frac{\left(8T^{2H}\right)^k}{k!}\left|\frac{[nt]}{n} - \frac{[ns]}{n}\right|^{2H}.
\end{aligned}
$$

Therefore (5.7) holds with $L = C'\exp(8T^{2H})$. □

Proof of Theorem 3.4. It follows from (5.8), Lemma 5.1 (iii) and inequality

$$
\sum_{\substack{C\in\{1,\ldots,[nt]\}\\ |C|=k}} \left(\sum_{\substack{m:C\to\{1,\ldots,[nt]\}\\ m:\text{nie"}1-1\text{"}}} \prod_{p\in C} d^n_{m(p),p}\right)^2 \leqslant \frac{(k-1)^3}{(k-1)!} t^{2H(k-1)} n^{1-2H},
$$

witch was established in [2], that

$$
\begin{aligned}
\mathbb{E}\left(\exp^{\diamond_n}\left(\int_0^t f_s dB_s^{H,n}\right) - S_t^n\right)^2 &\leqslant \sum_{k=2}^{[nt]} \sum_{\substack{C\subseteq\{1,\ldots,[nt]\}\\ |C|=k}} \left(\sup_{t\leqslant T}|f_t|^k \sum_{\substack{m:C\to\{1,\ldots,[nt]\}\\ m:\text{nie"}1-1\text{"}}} \prod_{p\in C} d^n_{m(p),p}\right)^2 \\
&\leqslant \left(\sum_{k=2}^{\infty} \sup_{t\leqslant T}|f_t|^{2k} \frac{(k-1)^3}{(k-1)!} t^{2H(k-1)}\right) n^{1-2H}.
\end{aligned}
$$

In view of Theorem 3.3 and Slutsky's theorem, the above inequality implies that convergence holds in finite-dimensional distributions. Since by Theorem 5.4 the sequence S^n is tight, assertion follows. □

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