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# Relations on the set of contact forms

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Opiekun pracy: Bogusław Hajduk


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# Relations on the set of contact forms

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**Abstract.** On the set of contact forms defined on a closed manifold we consider relations determined by paths of forms, concordances and cobordisms. We examine properties of these relations as well as dependences between them. We use the notion of algebraic torsion defined via Symplectic Field Theory and contact homology. Methods of contact surgery are also used.

## 1 Introduction

Consider the set of contact forms defined on a closed manifold. We analyze several relations on this set determined by paths of forms, concordances and cobordisms. The main purpose of our work is to examine the concept of symplectic concordance that is not well recognized yet.

This project is related to intensively developing tools in contact and symplectic topology such as the theory of pseudoholomorphic curves ([7]), Symplectic Field Theory ([1]) and the method of algebraic torsion in contact homologies ([4]).

In section 2 we present basic ideas of contact and symplectic topology. We give definitions of contact and symplectic manifolds. We also introduce concepts of symplectic cobordism and concordance. This part is based on [6] and [3].

Then we give a brief introduction to the tools we use to obtain our results. We present the methods of contact surgery in section 3 (based on [3]) and the notion of algebraic and planar torsion in section 4 (based on [4], [9], [8]).

In section 5 we define seven relations on the set of contact forms: presymplectic homotopy, isotopy, concordance, exact concordance, linking, cobordism and exact cobordism.

In section 6 we present our results. We describe properties of the presented relations and dependences between them.

Finally, we close in 7 by mentioning a few directions of possible future work.

## 2 Preliminaries

### 2.1 Contact and symplectic manifolds

**Definition 1.** *Consider a compact manifold $W$. A 2-form $\omega$ on $W$ is called* ***symplectic*** *if the following conditions are fulfilled.*

- $\omega$ *is* closed *which means that* $d\omega = 0$.
- $\omega$ *is* nondegenerate *which means that*

$$\forall x \in W \, \forall v \in T_x W \;\; v \neq 0 \Rightarrow \exists w \in T_x W \, \omega(v, w) \neq 0.$$

*A pair* $(W, \omega)$ *is then called a* ***symplectic manifold****.*

Note that the nondegeneracy of a form on $W$ implies that $W$ is a manifold of an even dimension. If $W$ is a $2n$-manifold then a 2-form is nondegenerate if and only if $\omega^n \neq 0$ (meaning that it vanishes nowhere).

*Remark 1.* A symplectic manifold $(W^{2n}, \omega)$ is orientable. Its orientation is determined by the volume form $\omega^n$.

**Definition 2.** *A* ***Liouville vector field*** $Y$ *on a symplectic manifold* $(W, \omega)$ *is a vector field satisfying the equation* $\mathscr{L}_Y \omega = \omega$.

**Definition 3.** *Let* $W$ *be a manifold of an even dimension. An* ***almost-complex structure*** *on* $W$ *is an automorphism* $J$ *of the tangent bundle* $TW$ *such that* $J^2 = -Id$*. If* $(W, \omega)$ *is a closed symplectic manifold then an almost-complex structure* $J$ *is said to be* $\omega$***-compatible*** *if for all* $x \in W$ *and* $v, w \in T_x W$

$$\omega(Jv, Jw) = \omega(v, w) \quad \textit{and} \quad v \neq 0 \Rightarrow \omega(v, Jv) > 0.$$

**Definition 4.** *Let* $M$ *be a closed manifold of odd dimension* $2n+1$*. A* $1$*-form* $\eta$ *on* $M$ *is called* ***contact*** *if*

$$\eta \wedge (d\eta)^n \neq 0.$$

*A hyperplane field* $\xi = \ker \eta \subset TM$ *is then called a* ***contact structure*** *on* $M$*. A pair* $(M, \eta)$ *(or a pair* $(M, \xi)$*) is called a* ***contact manifold****.*

*Remark 2.* Two contact forms $\eta_0$ and $\eta_1$ on a manifold $M$ define the same contact structure $\ker \eta_0 = \xi = \ker \eta_1$ if and only if $\eta_0 = f\eta_1$ for a function $f : M \to \mathbb{R}/\{0\}$.

*Remark 3.* For a contact manifold $(M, \eta)$ the form $\mathrm{d}\eta$ defines a symplectic structure on $\xi = \ker \eta$.

*Remark 4.* A contact manifold $(M^{2n+1}, \eta)$ is orientable. Its orientation is determined by the volume form $\eta \wedge (\mathrm{d}\eta)^n$.

**Definition 5.** *Let* $M$ *be a closed manifold of an odd dimension. A* $1$*-form* $\eta$ *on* $M$ *is called* ***presymplectic*** *if its differential* $d\eta$ *has a maximal rank.*

*Remark 5.* A contact form is always presymplectic. Indeed, if $(M, \eta)$ is a contact manifold of dimension $2n+1$ we have that $\eta \wedge (\mathrm{d}\eta)^n \neq 0$. Then, of course, $(\mathrm{d}\eta)^n \neq 0$.

### 2.2 Symplectic cobordism and concordance

**Proposition 1.** *If $(W, \omega)$ is a symplectic manifold with a Liouville vector field $Y$ then the 1-form $\eta := \iota_Y \omega$ is a contact form on any hypersurface transverse to $Y$.*

*Proof.* Since $\omega$ is a closed form, from Cartan formula for Lie derivative we have

$$\omega = \mathscr{L}_Y \omega = \mathrm{d}(\iota_Y \omega) + \iota_Y(\mathrm{d}\omega) = \mathrm{d}(\iota_Y \omega).$$

If $2n + 2$ is a dimension of $W$ then

$$\eta \wedge (\mathrm{d}\eta)^n = \iota_Y \omega \wedge (\mathrm{d}(\iota_Y \omega))^n = \iota_Y \omega \wedge \omega^n = \frac{1}{n} \iota_Y(\omega^{n+1}).$$

Since $\omega$ is symplectic, $\omega^{n+1}$ is a volume form. Thus $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form on every hypersurface transverse to $Y$.

**Definition 6.** *Let $(M_-, \eta_-)$ and $(M_+, \eta_+)$ be contact manifolds of dimension $2n - 1$. A* ***symplectic cobordism*** *from $(M_-, \eta_-)$ to $(M_+, \eta_+)$ is a compact $2n$-dimensional symplectic manifold $(W, \omega)$ with orientation determined by $\omega$ such that the following conditions are fulfilled.*

*(1) The manifold $W$ is an oriented topological cobordism between $M_-$ and $M_+$, which means that $\partial W = M_+ \sqcup \bar{M}_-$, where $\bar{M}_-$ stands for $M_-$ with the opposite orientation.*
*(2) In a neighborhood of $\partial W$, there exists a Louiville vector field $Y$ for $\omega$ which is transverse to the boundary and it points outwards along $M_+$ and inwards along $M_-$.*
*(3) $\eta_+ = \iota_Y \omega \mid_{TM_+}$ and $\eta_- = \iota_Y \omega \mid_{TM_-}$.*

*The manifold $(M_-, \eta_-)$ is called a* ***concave*** *boundary component of $W$ while $(M_+, \eta_+)$ is called a* ***convex*** *boundary component of $W$.*

*Remark 6.* In $(W, \omega)$ there exists a collar neighborhood of $(M_-, \eta_-)$ of the form

$$([0, \epsilon) \times M_-, d(e^t \eta_-))$$

and a collar neighborhood of $(M_+, \eta_+)$ of the form

$$((-\epsilon, 0] \times M_+, d(e^t \eta_+)).$$

Here $t$ denotes the parameter of the flow lines of a Liouville vector field $Y$ as in 6.

**Definition 7.** *A symplectic cobordism $(W, \omega)$ from $(M_-, \eta_-)$ to $(M_+, \eta_+)$ is called* ***exact*** *if $\omega$ is an exact form i.e. there exists a 1-form $\eta$ on $W$ such that $d\eta = \omega$.*

*Remark 7.* Equivalently one can say that a symplectic cobordism $(W, \omega)$ from $(M_-, \eta_-)$ to $(M_+, \eta_+)$ is exact if and only if there exists a Liouville vector field for $\omega$ defined on whole manifold $W$. Then a 1-form $\eta$ such that $\mathrm{d}\eta = \omega$ can be defined by the equation $\eta := \iota_Y \omega$.

**Definition 8.** *Let $M$ be a manifold of dimension $2n-1$ with contact forms $\eta_1$ and $\eta_2$. A* ***symplectic concordance*** *from $\eta_-$ to $\eta_+$ is a symplectic cobordism $(W, \omega)$ from $(M, \eta_-)$ to $(M, \eta_+)$ where the manifold $W$ is diffeomorphic to $M \times [0, 1]$.*

**Definition 9.** *A symplectic concordance is called* ***exact*** *if it is an exact symplectic cobordism.*

# 3 Contact surgery

We introduce notion of contact surgery. This section is based on [3]. We start with recalling basic concepts of the topological surgery.

## 3.1 Topological surgery

Let $M$ be a manifold of dimension $n$. Assume that $S^k \subset M$ is an embedded sphere with a trivial normal bundle. This induces an embedding $S^k \times D^{n-k} \subset M$.

**Definition 10.** *Let $S^k \times D^{n-k} \hookrightarrow M$ be an embedding. The manifold $M'$ defined by*

$$M' = M \setminus (S^k \times \mathit{Int} D^{n-k}) \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1}$$

*where gluing is made by an obvious identification*

$$S^k \times S^{n-k-1} = \partial(M \setminus (S^k \times \mathit{Int} D^{n-k})) = \partial(D^{k+1} \times S^{n-k-1})$$

*is said to be obtained by **surgery along** $S^k \subset M$.*

The surgery can be alternatively described in terms of attaching handles.
For the manifold $M$ and $S^k \subset M$ as above consider the trivial cobordism $[-1,1] \times M$. Having an embedding $S^k \times D^{n-k} \hookrightarrow M \cong M \times \{1\}$ one can define

$$W = ([-1,1] \times M) \cup_{S^k \times D^{n-k}} D^{k+1} \times D^{n-k}.$$

We say that $W$ is obtained from $[-1,1]\times M$ by attaching $(k+1)$**-handle** to the boundary component $\{1\} \times M$.

*Remark 8.* The space $W$, as defined above, is not a smooth manifold. However, it can be transformed to such a manifold by a process of "smoothing the corners".

The boundary of $W$ is a disjoint union of the manifold $M$ and the result $M'$ of performing a surgery along $S^k \subset M$. $W$ is a topological cobordism between $M$ and $M'$.

## 3.2 Contact surgery

Let $(M, \xi)$ be a contact manifold. The submanifold $L$ of $M$ is called **isotropic** if $TL \subset \xi$. By $(TL)^\perp$ we denote the symplectically orthogonal complement of $TL$ in a symplectic bundle $\xi$.

**Definition 11.** *Let $L$ be an isotropic submanifold of a contact manifold $(M, \eta)$. The **symplectic normal bundle** of $L$ in $M$ is the quotient bundle*

$$SN_M(L) := (TL)^\perp / TL$$

*with the symplectic bundle structure induced by $d\eta$.*

The contact surgery can be performed along isotropic spheres with trivial normal bundles. In [3] it is defined via symplectic cobordisms.

**Theorem 1.** *Let $(M, \xi)$ be a $(2n-1)$-dimensional contact manifold containing an embedded sphere $S^{k-1}$ with trivialisation of its normal bundle defined by an embedding*

$$f : S^{k-1} \times D^{2n-k} \to M.$$

*Assume that $n > 2$ and $1 \leq k \leq n$. Let $(W, \omega)$ be a symplectic manifold with $(M, \xi)$ as a convex boundary component. Let $W' = W \cup_f H$ be the manifold obtained from $W$ by attaching a $k-$handle $H$ and $M'$ the new boundary component (the result of surgery on M). If the $\omega-$compatible almost complex structure $J$ (it is unique up to homotopy [6]) extends to an almost complex structure on $W'$, then $W'$ carries a symplectic form $\omega'$ and an $\omega'-$compatible almost complex structure homotopic to $J$ such that $M'$ is a convex boundary component of $(W, \omega)$. The induced contact structure $\xi'$ on $M'$ is the result of performing contact surgery along an isotropic sphere topologically isotopic to $f(S^{k-1} \times \{0\})$.*

## 4 Torsions

We introduce the notion of algebraic torsion which is a useful tool when deciding whether two contact manifolds are exactly symplectically cobordant. The planar torsion, which is also presented here, can be conveniently used in constructing examples in dimension 3. It becomes especially interesting in context of [9] where it is proved that all manifolds with planar torsion are cobordant.

### 4.1 Algebraic torsion

The definition of algebraic torsion [4] is based on the Symplectic Field Theory (SFT) which was outlined in [1].

**Definition 12.** *Let $(M, \xi)$ be a closed contact manifold. For any natural number $k$ we say that $(M, \xi)$ has **algebraic torsion of order** $k$ (or **algebraic** $k$**-torsion**) if $[\hbar^k] = 0$ in $H_*^{SFT}(M, \xi)$.*

For a contact manifold $(M, \xi)$ we define the minimal algebraic torsion $\mathbf{AT}(M, \xi) \in \mathbb{N} \cup \{\infty\}$

$$\mathrm{AT}(M, \xi) \quad = \quad \inf\{k \geq 0 \ | \ [\hbar^k] = 0 \in H_*^{SFT}(M, \xi)\}.$$

The notion of algebraic torsion is used in [4] to construct a subtle obstruction to exact cobordisms.

**Theorem 2.** *Let $(M_0, \xi_0)$ and $(M_1, \xi_1)$ be contact manifolds. Suppose that there is a symplectic exact cobordism from $(M_0, \xi_0)$ to $(M_1, \xi_1)$. Then AT$(M_0, \xi_0) \leq$ AT$(M_1, \xi_1)$.*

*Remark 9.* It is proved in [2] that if $\mathrm{AT}(M_0, \xi_0) = 0$ then for any contact manifold $(M_1, \xi_1)$ there exists an exact symplectic cobordism from $(M_0, \xi_0)$ to $(M_1, \xi_1)$. However, in general it is not known if an exact cobordism from $(M_0, \xi_0)$ to $(M_1, \xi_1)$ exists for contact manifolds $(M_0, \xi_0)$ and $(M_1, \xi_1)$ with $\mathrm{AT}(M_0, \xi_0) \leq \mathrm{AT}(M_1, \xi_1)$.

### 4.2 Planar torsion

Planar torsion is introduced in [8] by means of blown up summed open books (a generalization of open book decompositions also defined in [8]).

**Definition 13.** *Let $M$ be a compact 3-manifold. A* ***blown up summed open book*** *on $M$ is a fibration*

$$\pi : M \setminus (B \cup \mathscr{I}) \to S^1,$$

*where $B \subset M \setminus \partial M$ (the* ***binding****) is an oriented link and $\mathscr{I} \subset M \setminus \partial M$ (the* ***interface****) is a set of disjoint 2-tori.*

*The connected components of the fibers are called* ***pages****.*

*A contact structure on $M$ is said to be* ***supported by*** *$\pi$ if binding components are positively transverse links while interface and boundary components are pre-Lagrangian tori.*

*Remark 10.* For any blown up summed open book on 3-manifold $M$ without closed pages there exists a supported contact structure on $M$ which is unique up to isotopy.

A blown up summed open book is called **irreducible** if the fibers are connected. A general blown up summed open book $\pi$ on a manifold $M$ can be decomposed to a union of **irreducible subdomains**

$$M = M_1 \cup M_2 \cup \ldots M_n,$$

where $M_i$ are manifolds with irreducible blown up summed open book structures $\pi_i$ such that pages of $\pi_i$ are pages of $\pi$ and they are glued along tori from the interface of $\pi$. A blown up summed open book is called **symmetric** if it has no boundary and it contains exactly two irreducible subdomains each of pages of the same topological type and empty binding and interface.

**Definition 14.** *Consider a contact 3-manifold and an integer $k \geq 0$. We say that the manifold has* ***planar*** *$k$**-torsion** if it admits a contact embedding of a connected contact manifold $(M, \xi)$ with a blown up summed open book $\pi$ supporting $\xi$ and fulfilling the following conditions.*

- *There exists a planar irreducible subdomain $M^P \subset M$ with pages of $k + 1$ boundary components.*
- *$M \setminus M^P$ is not empty.*
- *$\pi$ is not symmetric.*

In [9] it is shown that a contact manifold with planar torsion is cobordant to any contact manifold.

**Theorem 3.** *For every closed contact 3-manifolds $(M_0, \xi_0)$ and $(M_1, \xi_1)$ if $(M_0, \xi_0)$ has planar torsion then there exists a symplectic cobordism from $(M_0, \xi_0)$ to $(M_1, \xi_1)$.*

The following theorem relating algebraic and planar torsions is proved in [4].

**Theorem 4.** *Let $(M, \xi)$ be a contact 3-manifold. For any integer $k \geq 0$ if $(M, \xi)$ has planar $k$-torsion then it also has algebraic $k$-torsion.*

### 4.3 Examples

The following example was presented in [4].

For integers $k \geq 1$ and $g > k$ let $\Sigma_\pm$ be compact connected oriented surfaces with $k$ boundary components. Assume that $\Sigma_-$ has genus 0 and $\Sigma_+$ has genus $g' = g - k + 1$. Let $\Sigma = \Sigma_- \cup_\Gamma \Sigma_+$ be the surface obtained by gluing $\Sigma_-$ and $\Sigma_+$ along boundary components $\Gamma$. Then $\Sigma$ is a closed oriented surface of genus $g$. Consider the manifold

$$V_g = \Sigma \times S^1$$

with $S^1$-invariant contact structure $\xi_k$ such that the loops $S^1 \times \{z\}$ are transverse for $z$ in the interiors of $\Sigma_-$ and $\Sigma_+$ (positively and negatively transverse in $\Sigma_-$ and $\Sigma_+$ respectively) and Legendrian for $z \in \partial\Sigma_- = \partial\Sigma_+$. Due to [5] such a structure exists and is unique up to isotopy.

**Theorem 5.**

$$AT(V_g, \xi_k) = k - 1.$$

## 5 Relations

Let $M$ be a closed manifold with contact forms $\eta_0$ and $\eta_1$. Consider the following relations between them. We say that $\eta_0$ and $\eta_1$ are:

- *presymplectically homotopic*, if there exists a path of presymplectic 1-forms $(\eta_t)_{t\in[0,1]}$ between them;
- *isotopic*, if there exists a path of contact forms $(\eta_t)_{t\in[0,1]}$ between them;
- *concordant*, if there exists a symplectic concordance between $(M, \eta_0)$ and $(M, f\eta_1)$ for a function $f : M \to \mathbb{R}/\{0\}$;
- *exactly concordant*, if there exists a symplectic exact concordance between $(M, \eta_0)$ and $(M, f\eta_1)$ for a function $f : M \to \mathbb{R}/\{0\}$;
- *linked*, if there exists a contact form $\eta_3$ on $M$ such that $\eta_1$ and $\eta_2$ are both exactly concordant to $\eta_3$;
- *cobordant*, if there exists a symplectic cobordism $(W, \omega)$ between $(M, \eta_0)$ and $(M, f\eta_1)$ for a function $f : M \to \mathbb{R}/\{0\}$;
- *exactly cobordant*, if there exists a symplectic exact cobordism between $(M, \eta_0)$ and $(M, f\eta_1)$ for a function $f : M \to \mathbb{R}/\{0\}$.

## 6 Results

### 6.1 Properties of relations

**Proposition 2.** *(a) Presymplectic homotopy and isotopy are equivalence relations.*
*(b) Concordance and exact concordance are reflexive and transitive.*
*(c) Cobordism and exact cobordism are reflexive and transitive but they are not symmetric.*
*(d) Linking is a symmetric and reflexive relation.*

*Proof.* (a) The proof of that part is obvious since the relations of presymplectic homotopy and isotopy are defined in terms of paths.

(b) First, we prove that the relations are reflexive. Let $(M^{2n-1}, \eta)$ be a contact manifold. Then the pair $([0,1] \times M, \mathrm{d}(e^t\eta))$, where $t$ is a parameter of $[0,1]$, is an exact concordance between $(M,\eta)$ and $(M, e\eta)$. Indeed, let us compute

$$(\mathrm{d}(e^t\eta))^n = (e^t\mathrm{d}t \wedge \eta + e^t\mathrm{d}\eta)^n = ne^n\mathrm{d}t \wedge \eta \wedge (\mathrm{d}\eta)^n \neq 0.$$

In order to prove transitivity, consider (exact) concordances $(W_0 = [0,1] \times M, \omega_0)$ and $(W_1 = [0,1] \times M, \omega_1)$ form $(M, \eta_0)$ to $(M, f_0\eta)$ and from $(M,\eta)$ to $(M, f_1\eta_1)$ respectively.

After possibly multiplying $\omega_1$ by a constant, we may assume that $0 < f_0 < 1$. Let $f : M \to \mathbb{R}^+$ be a function such that $e^f = \frac{1}{f_0}$. By 6 there exists $\epsilon > 0$ and a collar neighborhood of $(M,\eta)$ in $(W_0, \omega_0)$ of the form

$$((-\epsilon, 0] \times M, \mathrm{d}(e^t f_0 \eta) = \mathrm{d}(e^t\eta')).$$

Similarly, there exists a collar neighborhood of $(M, f_0\eta)$ in $(W_1, \omega_1)$ of the form

$$([0,\epsilon) \times M, \mathrm{d}(e^t\eta) = \mathrm{d}(e^{t+f}\eta').$$

Consider a manifold $W = \{(t,x) \mid 0 \leq t \leq f(x)\}$ with a symplectic form $\mathrm{d}(e^t\eta')$. Then $W_0 \cup W \cup W_1$ is a concordance (or an exact concordance) from $(M, \eta_0)$ to $(M, f_1\eta_1)$.

(c) The fact that the relations of cobordism and exact cobordism are reflexive and transitive can be proved using the same methods as in the part (b) above.
Now, we show that exact cobordism is not symmetric. Let $g > 0$ and $1 < k < g$ be natural numbers. Consider $(V_g, \xi_1)$ and $(V_g, \xi_k)$ as described in 4.3. Since $\mathrm{AT}(V_g, \xi_1) = 0$, there exists an exact cobordism from $(V_g, \xi_1)$ to any contact manifold with algebraic torsion. In particular there exists an exact cobordism from $(V_g, \xi_1)$ to $(V_g, \xi_k)$. On the other hand $\mathrm{AT}(V_g, \xi_k) = k - 1 > 0 = \mathrm{AT}(V_g, \xi_1)$, thus there are no exact cobordism from $(V_g, \xi_k)$ to $(V_g, \xi_1)$.

(d) It is obvious that linking is a symmetric relation. Since the relation of exact concordance is reflexive, linking is also reflexive.

### 6.2 Dependences between relations

**Proposition 3.** *Let $M$ be a closed manifold. Contact forms $\eta_0$ and $\eta_1$ on $M$ are concordant if and only if they are exactly concordant.*

*Proof.* It is obvious that exactly concordant forms are concordant.

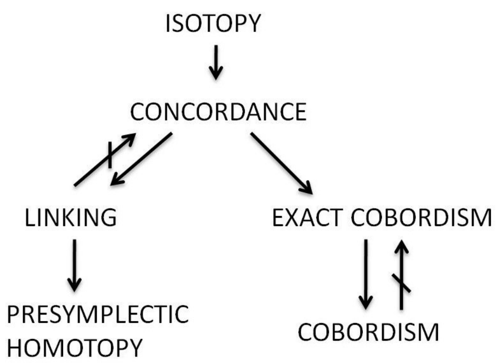
Assume that $\eta_0$ and $\eta_1$ are concordant. Let $(W, \omega)$ be a symplectic concordance between them. From the definition of symplectic cobordism the form $\omega$ is exact in a neighborhood $\mathscr{N}$ of the boundary of $W$. Take any $\sigma \in H_2(W, \mathbb{R})$. Since $W$ is diffeomorphic to $M \times [0,1]$, we can choose $s \in \sigma$ such that $s \subset \mathscr{N}$. It follows that

$$[\omega](\sigma) = \int_s \omega = 0.$$

Thus, $[\omega] = 0$ as an element of $H^2(W, \mathbb{R})$. So $(W, \omega)$ is an exact concordance.

Since terms concordance and exact concordance are equivalent we will use the shorter one - concordance.

**Theorem 6.** *The following implications hold.*




*Proof.* The implications

- concordance $\rightarrow$ linking
- concordance $\rightarrow$ exact cobordism
- exact cobordism $\rightarrow$ cobordism

are obvious.

**Lemma 1.** *Let $M$ be a manifold with contact forms $\eta_0, \eta_1$. If $\eta_0, \eta_1$ are isotopic then they are also concordant.*

*Proof.* Let $\{\eta_t\}$ be an isotopy between $\eta_0$ and $\eta_1$. We define a symplectic concordance $(M \times [0, \frac{1}{\epsilon}], \mathrm{d}\lambda)$ from $\eta_0$ to $e^{\frac{1}{\epsilon}}\eta_1$ for a small constant $\epsilon$. Let

$$\lambda = e^t \eta_{\epsilon t}.$$

Then

$$\mathrm{d}\lambda = e^t \mathrm{d}t \wedge \eta_{\epsilon t} + e^t \epsilon \frac{\mathrm{d}}{\mathrm{d}s}\eta_s \mid_{s=\epsilon t} + e^t \mathrm{d}\eta_{\epsilon t}.$$

For $\epsilon$ sufficiently small the second summand is insignificant. Thus, one can approximate

$$(\mathrm{d}\lambda)^n = n e^{nt} \mathrm{d}t \wedge \eta_{\epsilon t} \wedge (\mathrm{d}\eta_{\epsilon t})^{n-1},$$

where $M$ has a dimension $2n-1$. Since $\{\eta_t\}$ is a contact isotopy we have that $\eta_{\epsilon t} \wedge (\mathrm{d}\eta_{\epsilon t})^{n-1} \neq 0$. In consequence, we obtain that

$$(\mathrm{d}\lambda)^n \neq 0$$

and $(M \times [0, \frac{1}{\epsilon}], \mathrm{d}\lambda)$ is a symplectic concordance from $\eta_0$ to $e^{\frac{1}{\epsilon}}\eta_1$.

**Lemma 2.** *Let $M$ be a manifold with contact forms $\eta_0, \eta_1$. If $\eta_0, \eta_1$ are linked then they are presymplectically homotopic.*

*Proof.* Since presymplectic homotopy is an equivalence relation, it is sufficient to show that a concordance between two forms implies presymplectic homotopy between them. Let $(M \times [0,1], \omega)$ be a symplectic concordance from $\eta_0$ to $\eta_1$. From 3 we have that $\omega = \mathrm{d}\eta$ for a 1-form $\eta$ on $M \times [0,1]$. Let

$$\eta_t = \eta \mid_{M \times \{t\}} .$$

Since $\mathrm{d}\eta$ is a symplectic form, we obtain that the form $\mathrm{d}\eta_t$ is of the maximal rank on $M$. Hence, the path $\{\eta_t\}$ is a presymplectic homotopy from $\eta_0$ to $\eta_1$.

**Lemma 3.** *There exist a manifold $M$ and contact forms $\eta_0$ and $\eta_1$ on it such that contact manifolds $(M, \eta_0)$ and $(M, \eta_1)$ are symplectically cobordant but there is no exact cobordism from $(M, \eta_0)$ to $(M, \eta_1)$.*

*Proof.* Let $0 < k_1 < k_0 < g$ be natural numbers. Consider $(V_g, \xi_{k_0})$ and $(V_g, \xi_{k_1})$ as described in 4.3. Since both $(V_g, \xi_{k_0})$ and $(V_g, \xi_{k_1})$ have planar torsion, from 3 there exist a cobordism from $(V_g, \xi_{k_0})$ to $(V_g, \xi_{k_1})$. On the other hand $\mathrm{AT}(V_g, \xi_{k_0}) = k_0 - 1 > k_1 - 1 = \mathrm{AT}(V_g, \xi_{k_1})$, thus from 2 there are no exact cobordism from $(V_g, \xi_{k_0})$ to $(V_g, \xi_{k_1})$.

## 7 Conclusion

We defined seven relations on the set of contact forms on a manifold: presymplectic homotopy, isotopy, concordance, exact concordance, linking, cobordism and exact cobordism. We examined properties of the presented relations as well as dependences between them.

There are still questions to be answered in this area. The matter we consider the most interesting is the dependence between the relation of concordance and the relations of isotopy and exact cobordism. We would like to determine the possible obstructions for exactly cobordant forms to be concordant and for concordant forms to be isotopic. We also want to construct nontrivial examples of concordant forms in order to understand better the concept of concordance.

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