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Contact structures on $M \times S^1$

Praca semestralna nr 2

(semestr zimowy 2011/12)

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KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

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Abstract. We recall the theory of punctured pseudoholomorphic curves ([12]) and Symplectic Field Theory ([4]). We present the notion of the algebraic torsion ([15]) and the planar torsion ([18]). We use the torsions to prove properties of relations on the set of contact forms determined by paths of forms, concordances and cobordisms. We apply the relations to examine contact structures on manifolds of the form $W \times S^1$.

1 Introduction

On the set of contact forms on a closed manifold we consider relations determined by paths of forms, concordances and cobordisms. We use the algebraic torsion in Symplectic Field Theory to prove some properties of the relations. The main purpose of this paper is to use the relations to examine contact structures on manifolds of the form $W \times S^1$ extending the results from [11].

In Section 2 we present basic ideas of symplectic cobordisms between contact manifolds ([10]). We define symplectic concordances as a special case of topologically trivial symplectic cobordisms. Further, we recall the notion of pseudoholomorphic curves in almost complex manifolds. We focus on the case of symplectisations of contact manifolds ([12]).

Then we briefly present Symplectic Field Theory defined first in [4]. We introduce the notion of algebraic torsion in Symplectic Field Theory ([15]) and the planar torsion ([18]). These methods are used in Section 4 to prove our results. We strengthen a result from [15] in the special case of concordance (Theorem 3).

In section 4 we recall definitions of eight relations on the set of contact forms: presymplectic homotopy, isotopy, concordance, exact concordance, direct linking, linking, cobordism and exact cobordism described in a previous paper by the author [1]. We also extend here the results of the paper [1] concerning the properties of the relations.

In section 5 we show how to apply the defined relation to study the contact structures on the manifolds of the form $W \times S^1$.

Finally, we close in Section 6 by mentioning directions of planed future work.

2 Preliminaries

2.1 Symplectic cobordism

In this section we recall some basic concepts regarding symplectic cobordisms and concordances.

Consider a symplectic manifold (W, ω).

Definition 1. *A vector field Y satisfying the equation $\mathscr{L}_Y \omega = \omega$ is called a* ***Liouville vector field****.*

Remark 1. Let M be a manifold with a contact form η and determined contact structure ξ. We call both pairs, (M, ξ) and (M, η), a contact manifold.

Proposition 1. *If (W, ω) is a symplectic manifold with a Liouville vector field Y, then the 1-form $\eta := \iota_Y \omega$ is a contact form on any hypersurface transverse to Y.*

Proof. Since ω is a closed form, from Cartan formula for Lie derivative we have

$$\omega = \mathscr{L}_Y \omega = \mathrm{d}(\iota_Y \omega) + \iota_Y(\mathrm{d}\omega) = \mathrm{d}(\iota_Y \omega).$$

Let $2n$ be the dimension of W. We compute

$$\eta \wedge (\mathrm{d}\eta)^{n-1} = \iota_Y \omega \wedge (\mathrm{d}(\iota_Y \omega))^{n-1} = \iota_Y \omega \wedge \omega^{n-1} = \frac{1}{n-1} \iota_Y(\omega^n).$$

Since ω is symplectic, ω^n is a volume form. Thus $\eta \wedge (\mathrm{d}\eta)^{n-1}$ is a volume form on every hypersurface transverse to Y. □

Definition 2. *Let (M_-, η_-) and (M_+, η_+) be contact manifolds of dimension $2n-1$. A* ***symplectic cobordism*** *from (M_-, η_-) to (M_+, η_+) is a compact $2n$-dimensional symplectic manifold (W, ω) with orientation determined by ω such that the following conditions are fulfilled.*

(1) The manifold W is an oriented topological cobordism between M_- and M_+, which means that $\partial W = M_+ \sqcup \overline{M_-}$, where $\overline{M_-}$ stands for M_- with the opposite orientation.
(2) In a neighborhood of ∂W, there exists a Louiville vector field Y for ω which is transverse to the boundary and it points outwards along M_+ and inwards along M_-.
(3) $\eta_+ = \iota_Y \omega \mid_{TM_+}$ and $\eta_- = \iota_Y \omega \mid_{TM_-}$.

The manifold (M_-, η_-) is called a ***concave*** *boundary component of W while (M_+, η_+) is called a* ***convex*** *boundary component of W.*

Remark 2. In (W, ω) there exists a collar neighborhood of (M_-, η_-) of the form

$$([0, \epsilon) \times M_-, d(e^t \eta_-))$$

and a collar neighborhood of (M_+, η_+) of the form

$$((-\epsilon, 0] \times M_+, d(e^t \eta_+)).$$

Here t denotes the flow parameter of the flow lines of a Liouville vector field Y.

Definition 3. *A symplectic cobordism (W, ω) from (M_-, η_-) to (M_+, η_+) is called **exact** if ω is an exact form i.e. there exists a 1-form η on W such that $d\eta = \omega$.*

Remark 3. Equivalently one can say that a symplectic cobordism (W, ω) from (M_-, η_-) to (M_+, η_+) is exact if and only if there exists a Liouville vector field for ω defined on whole manifold W. Then a 1-form η such that $\mathrm{d}\eta = \omega$ can be defined by the equation $\eta := \iota_Y \omega$.

We consider topologically trivial symplectic cobordisms.

Definition 4. *Let M be a manifold of dimension $2n-1$ with contact forms η_- and η_+. A **symplectic concordance** from η_- to η_+ is a symplectic cobordism (W, ω) from (M, η_-) to (M, η_+) where the manifold W is diffeomorphic to $M \times [0, 1]$.*

Definition 5. *A symplectic concordance is called **exact** if it is represented by an exact symplectic cobordism.*

In [1] it was observed that two contact forms η_- and η_+ on a closed contact manifold M are concordant if and only if they are exactly concordant.

2.2 Pseudoholomorphic curves

We recall the notion of pseudoholomorphic curves in almost complex manifolds. We focus on curves in symplectisations of contact manifolds. This section is based on [12].

Consider a Riemann surface (Σ, j) and an almost complex manifold (W, J).

Definition 6. *A map $u : \Sigma \to W$ is called (j, J)**-holomorphic** (or J**-holomorphic**, or **pseudoholomorphic**) if*

$$d\,u \circ j = J \circ d\,u. \tag{1}$$

Let $\Gamma \subset \Sigma$ be a finite set of points. Consider the symplectisation $(W, \omega) = (\mathbb{R} \times M, \mathrm{d}(\eta e^t))$ of a contact manifold (M, η). Fix a cylindrical almost complex structure J on W compatible with η.

Definition 7. *Define the **energy** E of a curve $u : \Sigma \backslash \Gamma \to W$ by the formula*

$$E(u) = \sup\{\int_{\Sigma \backslash \Gamma} u^* d\,n_\phi \mid \phi : \mathbb{R} \to [0, 1], \phi' \geq 0\},$$

where η_ϕ is the 1-form on $\mathbb{R} \times M$ defined by

$$\eta_\phi(s, x)(t, v) = \phi(s)\eta(x)v.$$

*A solution $u : \Sigma \backslash \Gamma \to W$ of (1) satisfying $0 < E(u) < \infty$ is called **(finite energy) punctured pseudoholomorphic curve**.*

Let $u : \Sigma \backslash \Gamma \to W$ be a finite energy punctured pseudoholomorphic curve. We assume that all punctures $x \in \Gamma$ are nonremovable. Then Γ can be divided into two sets, $\Gamma = \Gamma^- \cup \Gamma^+$, such that $\mathbb{R}$-component near $x \in \Gamma^-$ is bounded from above while near $x \in \Gamma^+$ it is bounded from below. Γ^- and Γ^+ are sets of **negative** and **positive punctures** respectively.

Remark 4. From the Gromov's removable singularity theorem, the singularity $x \in \Gamma$ can be removed if and only if $\mathbb{R}$-component of u is bounded near $x \in \Gamma$.

3 Torsions

We introduce the notion of algebraic torsion which is a useful tool when deciding whether two contact manifolds are exactly symplectically cobordant. The planar torsion, which is also presented here, can be used in constructing examples in dimension 3. It becomes especially interesting in context of [19] where it is proved that all manifolds with planar torsion are symplectically cobordant.

Before introducing algebraic torsion we present basic concepts of the Symplectic Field Theory (SFT). See [4], [14], [15] for details.

3.1 Symplectic Field Theory

Contact manifold Let (M, ξ) be a closed contact manifold of a dimension $2n-1$ with a cooriented contact structure. Fix a contact form η compatible with ξ. Denote by R_η the Reeb vector field for η, i.e. the field satisfying

$$\iota_{R_\eta} \mathrm{d}\eta \equiv 0, \quad \eta(R_\eta) \equiv 1.$$

Definition 8. *A closed Reeb orbit (orbit of the Reeb field) is called* ***nondegenerate*** *if its linearized Poincare return map has no eigenvalues equal to* 1*. A contact form is* ***nondegenerate*** *if all its closed Reeb orbits are nondegenerate.*

Let u be a finite energy punctured pseudoholomorphic curve in a symplectisation of (M, η). If η is nondegenerate, then u converges exponentially near punctures to closed Reeb orbits.

A generic contact form η for ξ is nondegenerate. Note that then there are only countably many closed Reeb orbits.

Capping surfaces For simplicity let us assume that $H_1(M)$ is torsion-free. Choose curves $C_1, \ldots, C_k$ representing a basis of $H_1(M)$ and symplectic trivializations of bundles $\xi \mid_{C_i}$ with the symplectic form $\mathrm{d}\eta$. Then for every periodic orbit γ let S_γ be a surface such that $[\partial S_\gamma] = [\gamma] - \sum_{i=1}^{k} n_i C_i$. The coefficients n_i are uniquely determined since $H_1(M)$ is torsion-free. The surface S_γ is called a **capping surface** of the orbit γ.

Conley-Zehnder index Given a nondegenerate closed Reeb orbit γ, choose a homotopically unique symplectic trivialization of $\xi \mid_\gamma$ which can be extended to a trivialization of $\xi \mid_{S_\gamma}$ that agrees with previously fixed trivializations of $\xi \mid_{C_i}$. Then the flow of R_η along γ defines a path in Sp$(2n-2, \mathbb{R})$ beginning at the identity and ending at a matrix with eigenvalues different from 1. The Maslov index μ_{CZ} of that path defines the Conley-Zehnder index of the orbit γ denoted by CZ(γ).

The following proposition presents properties of the Maslov index.

Proposition 2. *Let Ψ be a nondegenerate path in Sp$(2n)$. The Maslov index μ_{CZ} has the following properties.*

naturality *For any path Φ in Sp$(2n)$ $\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi)$.*

homotopy *If Ψ' is homopotic to Ψ, then $\mu_{CZ}(\Psi) = \mu_{CZ}(\Psi')$.*

zero *If $\Psi(s)$ has no eigenvalues equal to 1 for all $s > 0$, then $\mu_{CZ}(\Psi) = 0$.*

product *If $n' + n'' = n$, Ψ' and Ψ'' are nondegenerate paths in $Sp(2n')$ and $Sp(2n'')$ respectively, then $\mu_{CZ}(\Psi' \oplus \Psi'') = \mu_{CZ}(\Psi') + \mu_{CZ}(\Psi'')$, where we identify $Sp2(n') \oplus Sp(2n'')$ with a subgroup of $Sp(2n)$.*

loop *Let Φ be a loop in $Sp(2n)$ with $\Phi(0) = \Phi(1) = Id$. Then $\mu_{CZ}(\Phi\Psi) = 2\mu(\Phi) + \mu_{CZ}(\Psi)$, where μ denotes the Maslov index of a loop.*

signature *If $S \in \mathbb{R}^{2n\times 2n}$ is a symmetric matrix, $|S| < 2\pi$ and $\Psi(s) = \exp(JSs)$ then $\mu_{CZ}(\Psi) = \frac{1}{2} sign(S)$.*

determinant $(-1)^{n-\mu_{CZ}(\Psi)} = sign(\det(Id - \Psi(1)))$.

inverse $\mu_{CZ}(\Psi^T) = \mu_{CZ}(\Psi^{-1}) = -\mu_{CZ}(\Psi)$.

For technical reasons it is convenient to exclude from the consideration so called **bad Reeb orbits**, i.e. k-th iterates γ^k of closed Reeb orbits γ such that $\mathrm{CZ}(\gamma) \neq \mathrm{CZ}(\gamma^k) \mod 2$. All the other closed Reeb orbits are called **good**.

Weyl algebra $\mathscr{W}$ For a linear subspace $\mathscr{R} \subset H_2(M;\mathbb{R})$ consider the group ring $R_{\mathscr{R}} := \mathbb{R}[H_2(M;\mathbb{R})/\mathscr{R}]$. For every good Reeb orbit γ consider formal variables p_γ and q_γ with a degree given by shifted Conley-Zehnder index

$$|p_\gamma| := n - 3 - \mathrm{CZ}(\gamma) \quad |q_\gamma| := n - 3 + \mathrm{CZ}(\gamma).$$

Moreover, let $\hbar$ be an additional variable of a degree

$$|\hbar| = 2(n-3).$$

Let $\mathscr{W}$ be a graded Weyl algebra over $R_{\mathscr{R}}$ of power series in the variables $\hbar$, p_γ with coefficients being polynomials in q_γ.

Let $\star$ be a associative product on $\mathscr{W}$ commutative for all variables except p_γ and q_γ for which the following commutation relation is fulfilled

$$p_\gamma \star q_\gamma - (-1)^{|p_\gamma||q_\gamma|} q_\gamma \star p_\gamma = \kappa_\gamma \hbar.$$

Pseudoholomorphic curves Choose an almost complex structure J on the symplectization $(W,\omega) = (M \times \mathbb{R}, \mathrm{d}(\eta e^t))$ of (M,ξ). Assume that J is cylindrical and adjusted to η.

Consider a number $g \in \mathbb{N}$. Let Γ^-, Γ^+ be ordered collections of good Reeb orbits with $s^\pm = |\Gamma^\pm|$. By $\mathscr{M}_g(\Gamma^-,\Gamma^+)$ denote the space of J-holomorphic curves in W with genus g, s^- negative punctures asymptotic to the orbits from Γ^- and s^+ positive punctures asymptotic to the orbits from Γ^+.

Theorem 1. *The dimension of the moduli space $\mathscr{M}_g(\Gamma^-,\Gamma^+)$ is given by the following formula.*

$$\dim \mathscr{M}_g(\Gamma^-,\Gamma^+) = (n-3)(2-2g-s^- - s^+) + \sum_{\gamma\in\Gamma^+} CZ(\gamma) - \sum_{\gamma\in\Gamma^-} CZ(\gamma).$$

Hamiltonian The moduli spaces $\mathscr{M}_g(\Gamma^-,\Gamma^+)$ defined above are $\mathbb{R}$-symmetric in the sense that if u is a pseudoholomorphic curve then all its shifts in the $\mathbb{R}$-direction in $M\times\mathbb{R}$ are also pseudoholomorphic.

Consider $g,\ \Gamma^-,\ \Gamma^+$ such that

$$\dim \mathscr{M}_g(\Gamma^-,\Gamma^+) = 1.$$

Denote by $n_g(\Gamma^-,\Gamma^+)\in\mathbb{Z}$ the algebraic count of points in $\mathscr{M}_g(\Gamma^-,\Gamma^+)/\mathbb{R}$. Define the correlator

$$\text{cor}(s^-,s^+,g) := \sum_{|\Gamma^\pm|=s^\pm} n_g(\Gamma^-,\Gamma^+) q^{\Gamma^-} p^{\Gamma^+},$$

where $q^\Gamma=\prod_{\gamma\in\Gamma} q_\gamma, p^\Gamma=\prod_{\gamma\in\Gamma} p_\gamma$, and

$$\mathbb{H}_g := \sum_{s^-,s^+} \frac{1}{s^-!s^+!}\text{cor}(s^-,s^+,g).$$

Consider the element $\mathbb{H}\in\mathscr{W}$ collecting the information about all J-holomorphic curves in W

$$\mathbb{H} := \frac{1}{\hbar}\sum_{g=0}^{\infty}\mathbb{H}_g\hbar^g.$$

Note that $\mathbb{H}$ is a homogenous element of $\mathscr{W}$ of the degree -1.

SFT Homology Consider the graded commutative subalgebra $\mathscr{A}$ in $\mathscr{W}$ consisting of polynomials in the variables q_γ. Then elements of $\mathscr{W}$ act on $\mathscr{A}[[\hbar]]$ as differential operators via the replacements

$$p_\gamma \mapsto \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}.$$

Let us define a differential operator D_{SFT} on $\mathscr{A}[[\hbar]]$ by the formula

$$\text{D}_{\text{SFT}} := \overrightarrow{\mathbb{H}} : \mathscr{A}[[\hbar]] \to \mathscr{A}[[\hbar]].$$

It turns out that D_{SFT} has the following properties which make $(\mathscr{A},\text{D}_{\text{SFT}})$ be a BV_∞-algebra.

- $\text{D}_{\text{SFT}}\text{D}_{\text{SFT}}=0$;
- $\text{D}_{\text{SFT}}=\frac{1}{\hbar}\sum_{k=1}^{\infty} D^k\hbar^k$, where $D^k:\mathscr{A}\to\mathscr{A}$ are differential operators on $\mathscr{A}$ of order at most k;
- $D(1)=0$.

The homology of defined the pair $(\mathscr{A}[[\hbar]],\text{D}_{\text{SFT}})$ is called the homology of Symplectic Field Theory. We denote it by $H_*^{SFT}(M,\xi;\mathscr{R})$.

Remark 5. For special choices of $\mathscr{R}$ we use the following shorter notation.

- If $\mathscr{R}=H_2(M;\mathbb{R})$, we write $H_*^{SFT}(M,\xi) := H_*^{SFT}(M,\xi;\mathscr{R})$.
- If Ω is a 2-form on M and $\mathscr{R}=\ker\Omega$, we write $H_*^{SFT}(M,\xi,\Omega) := H_*^{SFT}(M,\xi;\mathscr{R})$.

Torsions and Symplectic Cobordisms Consider contact manifolds (M_-, ξ_-), (M_+, ξ_+) and a symplectic cobordism (W, ω) between them. Let $\mathscr{R}_\pm \subset H_2(M_\pm; \mathbb{R})$. Assume that $\mathscr{R} \subset \ker \omega$ is a linear subspace of $H_2(W, \omega)$ such that $\mathscr{R}_\pm$ are mapped into $\mathscr{R}$ by the maps $H_2(M_\pm; \mathbb{R}) \to H_2(W; \mathbb{R})$. As above define

$$R_{\mathscr{R}_\pm} = \mathbb{R}[H_2(M; \mathbb{R})/\mathscr{R}_\pm], \quad R_{\mathscr{R}} = \mathbb{R}[H_2(M; \mathbb{R})/\mathscr{R}].$$

Let $(\mathscr{A}_\pm, \mathrm{D}^\pm_{\mathrm{SFT}})$ be corresponding BV_∞-algebras. Consider also the algebra $\mathscr{A}$ with coefficients in $R_{\mathscr{R}}$ generated by the variables q_γ for Reeb orbits γ in (M_-, ξ_-).

Let $A \in \hbar^{-1}\mathscr{A}[[\hbar]]$ be the element counting all pseudoholomorphic curves with no positive punctures, i.e. $\mathrm{D}^-_{\mathrm{SFT}}(e^A) = 0$. Defining

$$D(Q) = e^{-A}\mathrm{D}^-_{\mathrm{SFT}}(e^A Q)$$

we obtain a twisted version of SFT for (M_-, ξ_-). Then there exists a chain map

$$\Phi : (\mathscr{A}_+, \mathrm{D}^+_{\mathrm{SFT}}) \to (\mathscr{A}, D).$$

Note that, from the Stokes theorem, in an exact cobordism there are no pseudoholomorphic curves without positive ends. Thus, in the exact case we obtain a chain map between nontwisted SFT

$$\Phi : (\mathscr{A}_+, \mathrm{D}^+_{\mathrm{SFT}}) \to (\mathscr{A}_-, \mathrm{D}^-_{\mathrm{SFT}}). \tag{2}$$

3.2 Algebraic torsion

The definition of algebraic torsion [15] is based on the Symplectic Field Theory.

Definition 9. *Let (M, ξ) be a closed contact manifold. For any natural number k we say that (M, ξ) has **algebraic torsion of order** k (or **algebraic** k**-torsion**) if $[\hbar^k] = 0$ in $H_*^{SFT}(M, \xi)$.*

*For a 2-form Ω on M we say that (M, ξ) has $\Omega-$**twisted algebraic** k**-torsion** if $[\hbar^k] = 0$ in $H_*^{SFT}(M, \xi, \Omega)$. (M, ξ) has **fully twisted algebraic** k**-torsion** if $[\hbar^k] = 0$ in $H_*^{SFT}(M, \xi; \{0\})$.*

Remark 6. Note that for every contact manifold (M, ξ) and every number $k \in \mathbb{N}$ if (M, ξ) has $\mathscr{R}$-twisted algebraic k-torsion then it has also $\mathscr{R}$-twisted algebraic $k+1$-torsion. Indeed, since $\mathrm{D}_{\mathrm{SFT}}$ commutes with $\hbar$, we have that, if $\mathrm{D}_{\mathrm{SFT}}(Q) = \hbar^k$, then $\mathrm{D}_{\mathrm{SFT}}(\hbar Q) = \hbar^{k+1}$.

For a contact manifold (M, ξ) we define the minimal algebraic torsion $\mathrm{AT}(M, \xi) \in \mathbb{N} \cup \{\infty\}$, minimal $\Omega-$twisted algebraic torsion $\mathrm{AT}_\Omega(M, \xi) \in \mathbb{N} \cup \{\infty\}$ and minimal fully twisted algebraic torsion $\mathrm{AT}_{\{0\}}(M, \xi) \in \mathbb{N} \cup \{\infty\}$ respectively by formulas

$$\begin{aligned} \mathrm{AT}(M, \xi) &= \min\{k \geq 0 \mid [\hbar^k] = 0 \in H_*^{SFT}(M, \xi)\}, \\ \mathrm{AT}_\Omega(M, \xi) &= \min\{k \geq 0 \mid [\hbar^k] = 0 \in H_*^{SFT}(M, \xi, \Omega)\}, \\ \mathrm{AT}_{\{0\}}(M, \xi) &= \min\{k \geq 0 \mid [\hbar^k] = 0 \in H_*^{SFT}(M, \xi; \{0\})\}. \end{aligned}$$

Remark 7. For every contact manifold (M,ξ) and every 2-form Ω on M the following inequalities hold

$$\mathrm{AT}(M,\xi) \leq \mathrm{AT}_{\Omega}(M,\xi) \leq \mathrm{AT}_{\{0\}}(M,\xi).$$

The notion of algebraic torsion is used in [15] to construct a subtle obstruction to exact cobordisms.

Theorem 2. *Let (M_0,ξ_0) and (M_1,ξ_1) be contact manifolds. Suppose that there is a symplectic exact cobordism (W,ω) from (M_0,ξ_0) to (M_1,ξ_1). Then $AT(M_0,\xi_0) \leq AT(M_1,\xi_1)$.*

Proof. Since $\ker\omega = H_2(W;\mathbb{R})$, we can choose $\mathscr{R}_{\pm} = H_2(M_{\pm};\mathbb{R})$ and $\mathscr{R} = H_2(W;\mathbb{R})$. Then the result follows from (2).

Remark 8. However, in general it is not known if an exact cobordism from (M_0,ξ_0) to (M_1,ξ_1) exists for contact manifolds (M_0,ξ_0) and (M_1,ξ_1) with $\mathrm{AT}(M_0,\xi_0) \leq \mathrm{AT}(M_1,\xi_1)$.

If $\mathrm{AT}(M,\xi) = 0$ then the contact homology and SFT homology of the manifold (M,ξ) both vanish (see [15]). In this case (M,ξ) is called algebraically overtwisted (see [8]). There are some reasons to consider it a good generalization of overtwistedness in dimensions greater then 3. First of all, from Theorem 2 one knows that algebraically overtwisted manifolds are not filliable. Moreover, it is proved in [7] that if (M_0,ξ_0) is algebraically overtwisted, then for any contact manifold (M_1,ξ_1) there exists an exact symplectic cobordism from (M_0,ξ_0) to (M_1,ξ_1). Finally, all manifolds containing a plastikstufe (see [17]) are algebraically overtwisted, while from [13] every contact manifold (M,ξ) can be modified using generalized Lutz twist to one containing a plastikstufe and a modification can be performed in such a way that the homotopy class of the distribution ξ is preserved. On the other hand, however, there are no results generalizing topological classification of three dimensional overtwisted manifolds [6].

In case of concordances conclusions of theorem 2 can be strengthen.

Theorem 3. *Let (M_0,ξ_0) and (M_1,ξ_1) be contact manifolds. Suppose that there is a symplectic concordance (X,ω) from (M_0,ξ_0) to (M_1,ξ_1). Then for every linear subspace $\mathscr{R} \subset H_2(M_0) = H_2(M_1)$ the following inequality holds $AT(M_0,\xi_0;\mathscr{R}) \leq AT(M_1,\xi_1;\mathscr{R})$.*

Proof. The proof is very similar to that of theorem 2. The only difference is that now

$$H_2(X) = H_2(M_0) = H_2(M_1).$$

Thus one can choose $\mathscr{R}_+ = \mathscr{R}_- = \mathscr{R}$ and obtain a chain map

$$(\mathscr{A}_+[[\hbar]],\mathrm{D}^+_{\mathrm{SFT}}) \to (\mathscr{A}_-[[\hbar]],\mathrm{D}^-_{\mathrm{SFT}})$$

where $\mathscr{A}_{\pm}$ are algebras over $R_{\mathscr{R}}$.

3.3 Planar torsion

Planar torsion is introduced in [18] by means of blown up summed open books (a generalization of open book decompositions also defined in [18]).

Definition 10. *Let M be a compact 3-manifold. A* ***blown up summed open book*** *on M is a fibration*

$$\pi : M \setminus (B \cup \mathscr{I}) \to S^1,$$

where $B \subset M \setminus \partial M$ (the ***binding****) is an oriented link and $\mathscr{I} \subset M \setminus \partial M$ (the* ***interface****) is a set of disjoint 2-tori.*

The connected components of the fibers are called ***pages****.*

A contact structure on M is said to be ***supported by*** *π if binding components are positively transverse links while interface and boundary components are pre-Lagrangian tori.*

Remark 9. For any blown up summed open book on 3-manifold M without closed pages there exists a supported contact structure on M which is unique up to isotopy.

A blown up summed open book is called **irreducible** if the fibers are connected. A general blown up summed open book π on a manifold M can be decomposed to a union of **irreducible subdomains**

$$M = M_1 \cup M_2 \cup \dots M_n,$$

where M_i are manifolds with irreducible blown up summed open book structures π_i such that pages of π_i are pages of π and they are glued along tori from the interface of π. A blown up summed open book is called **symmetric** if it has no boundary and it contains exactly two irreducible subdomains each of pages of the same topological type and empty binding and interface.

Definition 11. *Consider a contact 3-manifold and an integer $k \geq 0$. We say that the manifold has* ***planar*** *k**-torsion** if it admits a contact embedding of a connected contact manifold (M, ξ) with a blown up summed open book π supporting ξ and fulfilling the following conditions.*

- *There exists a planar irreducible subdomain $M^P \subset M$ with pages of $k+1$ boundary components.*
- *$M \setminus M^P$ is not empty.*
- *π is not symmetric.*

In [19] it is shown that a contact manifold with planar torsion is symplectically cobordant to any contact manifold.

Theorem 4. *For every closed contact 3-manifolds (M_0, ξ_0) and (M_1, ξ_1) if (M_0, ξ_0) has planar torsion then there exists a symplectic cobordism from (M_0, ξ_0) to (M_1, ξ_1).*

The following theorem relating algebraic and planar torsions is proved in [15].

Theorem 5. *Let (M, ξ) be a contact 3-manifold. For any integer $k \geq 0$ if (M, ξ) has planar k-torsion then it also has algebraic k-torsion.*

3.4 Examples

The following example was presented in [15].

For integers $k \geq 1$ and $g > k$ let $\Sigma_{\pm}$ be compact connected oriented surfaces with k boundary components. Assume that Σ_- has genus 0 and Σ_+ has genus $g' = g - k + 1$. Let $\Sigma = \Sigma_- \cup_{\Gamma} \Sigma_+$ be the surface obtained by gluing Σ_- and Σ_+ along boundary components Γ. Then Σ is a closed oriented surface of genus g. Consider the manifold

$$V_g = \Sigma \times S^1$$

with S^1-invariant contact structure ξ_k such that the loops $S^1 \times \{z\}$ are transverse for z in the interiors of Σ_- and Σ_+ (positively and negatively transverse in Σ_- and Σ_+ respectively) and Legendrian for $z \in \partial\Sigma_- = \partial\Sigma_+$. Due to [16] such a structure exists and is unique up to isotopy.

Theorem 6.

$$AT(V_g, \xi_k) = k - 1.$$

4 Relations

Let M be a closed manifold with contact forms η_0 and η_1. Consider the following relations between them which are introduced in [1]. We say that η_0 and η_1 are:

- *presymplectically homotopic*, if there exists a path of presymplectic 1-forms $\left(\eta_t\right)_{t\in[0,1]}$ between them;
- *isotopic*, if there exists a path of contact forms $\left(\eta_t\right)_{t\in[0,1]}$ between them;
- *concordant*, if there exists a symplectic concordance between (M, η_0) and $(M, f\eta_1)$ for a function $f : M \to \mathbb{R}/\{0\}$;
- *directly linked*, if there exists a contact form η_3 on M such that η_1 and η_2 are both exactly concordant to η_3;
- *linked*, if there exists a sequence of contact forms $\eta_1 = \beta_0, \beta_1, \ldots, \beta_n = \eta_2$ on M such that β_{i-1} and β_i are directly linked for all $1 \leq i \leq n$;
- *cobordant*, if there exists a symplectic cobordism (W, ω) between (M, η_0) and $(M, f\eta_1)$ for a function $f : M \to \mathbb{R}/\{0\}$;
- *exactly cobordant*, if there exists a symplectic exact cobordism between (M, η_0) and $(M, f\eta_1)$ for a function $f : M \to \mathbb{R}/\{0\}$.

The relations defined above have the following properties (see [1] for a proof).

Proposition 3. *(a) Presymplectic homotopy, isotopy and linking are equivalence relations.*
(b) Concodrance is reflexive and transitive.
(c) Cobordism and exact cobordism are reflexive and transitive but they are not symmetric.
(d) Direct linking is a symmetric and reflexive relation.

Dependences between the relations are presented in the following theorem.

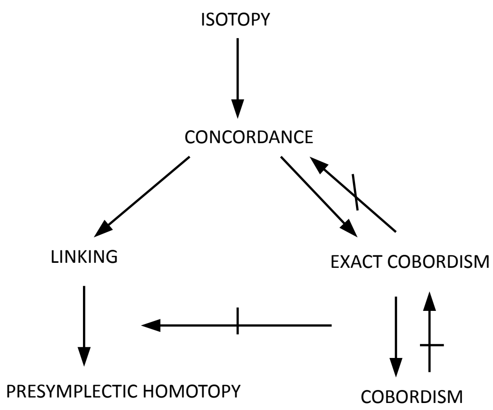

Fig. 1. Theorem 7

Theorem 7. *The implications from Figure 1 hold.*

Proof. Let us prove first the following easy lemma.

Lemma 1. *There exist a manifold M and contact forms η_0, η_1 on it such that η_0 and η_1 are exactly cobordant but there are no concordance between them.*

Proof. The lemma holds due to the fact that concordant forms are always homotopic while a cobordism need not to preserve a homotopy class of contact forms. Consider for instance the following simple example. Let $M = S^3$, η_0 be a standard contact form on S^3 and η_1 - an overtwisted structure non-homotopic to η_0 (on S^3 there exists exactly one tight contact structure, while there is an overtwisted structure in every homotopy class of 2-distributions [10]). There exists an exact cobordism from η_1 to η_0 but the forms are not be concordant.

Observe, moreover, that the relations of exact cobordism and cobordism do not imply the relations of direct linking, linking and homotopy. Indeed, the relations of direct linking, linking and homotopy all preserves a homotopy class of 2-distribution while it is possible that two contact forms of different homotopy classes are exactly cobordant (see Lemma 1).

The remining dependences are proved in [1].

5 Contact structures on $W \times S^1$

We begin with some results concerning decomposition of almost complex manifolds that will be needed in the presented below constructions of contact structures on manifolds of the form $W \times S^1$.

5.1 Decomposition of almost complex manifolds

From the decomposition result stated in [2] it follows that for every closed oriented 4-dimensional manifold W there exist exact symplectic fillings (W_1, ω_1) and (W_2, ω_2) of the same contact manifold (M, ξ) such that $W_1 \cup_M \overline{W_2}$ is diffeomorphic to W, where the manifolds W_1 and W_2 are glued using a contactomorphism between the boundaries $\partial W_1 = \partial W_2 = M$.

In dimensions higher then 4 a decomposition result similar to that from [2] is not known to be true. However, one can prove the following weaker result.

Theorem 8. *Suppose that (M^{2n}, J) is a closed almost complex manifold. Then there exist symplectic fillings (M_0, ω_0) and (M_1, ω_1) of contact manifolds $(N, \xi_0 = \ker \eta_0)$ and $(N, \xi_1 = \ker \eta_1)$ respectively such that M is diffeomorphic to $M_0 \cup N \times [0,1] \cup M_1$. Moreover, the forms η_0 and η_1 are presymplectically homotopic.*

Proof. Let $f : M \to [-1, 1]$ be a Morse function such that all critical points of order less then n are in $f^{-1}([-1, 0))$ while all critical points of order at least n are in $f^{-1}((0, 1])$. Then one can define

$$M_0 = f^{-1}([-1, 0]), \quad M_1 = f^{-1}([0, 1]).$$

Theorem 6.3.1 from [10] implies that for $i \in \{0, 1\}$ one can define a symplectic form ω_i on M_i compatible with an almost complex structure J_i homotopic to J. The symplectic form ω_i induces a contact structure ξ_i on $N = \partial M_i = f^{-1}(0)$ which is determined by a $1-$form η_i such that there is an $\epsilon > 0$ and neighborhoods of N in (M_i, ω_i) of the form

$$([0, \epsilon) \times N, \omega_0 = d(e^t \eta_0)) \quad \text{and} \quad ((-\epsilon, 0] \times N, \omega_1 = d(e^t \eta_1))$$

for $i = 0$ and $i = 1$ respectively.

It follows that $\bar{J}_0 = J_0|_N$, $\bar{J}_1 = J_1|_N$ are complex structures on $TN \oplus \varepsilon^1$, where ε^1 denotes a trivial vector bundle over N of dimension 1. Let J_t be a homotopy between J_0 and J_1. It corresponds to a homotopy $\bar{\omega}_t$ of nondegenerate $2-$forms $TN \oplus \varepsilon^1$ between $\bar{\omega}_0 = \omega_0|_N$ and $\bar{\omega}_1 = \omega_1|N$. Forms $\bar{\omega}_t$ remain nondegenerate when restricted to the tangent bundle TN of N. In [3] (see also [5]) it is proved that for a manifold of odd dimension the space of all nondegenerate $2-$forms is homotopically equivalent to the space of closed nondegenerate $2-$forms representing a given homology class. Thus we can choose a homotopy between $\bar{\omega}_0$ and $\bar{\omega}_1$ in such a way that $\bar{\omega}_t = \mathrm{d}\eta_t$ for a homotopy η_t between η_0 and η_1. □

5.2 Construction of contact structure on $W \times S^1$

The following theorem is proved in [11].

Theorem 9. *Let $(W_1, d\lambda_1)$ and $(W_2, d\lambda_2)$ be two exact symplectic fillings of the same manifold $(M, \xi = \ker(\eta))$. Then the manifold $(W_1 \cup_M \overline{W_2}) \times S^1$ admits a contact structure.*

This Theorem together with the results recalled in Section 5.1 leads to the following conclusion stated in [11].

Proposition 4. *For every closed oriented 4-dimensional manifold W, the manifold $W \times S^1$ admits a contact structure.*

Note that Theorem 9 admits an obvious extension.

Proposition 5. *Let $(W_1^{2n}, d\lambda_1)$ and $(W_2^{2n}, d\lambda_2)$ be exact symplectic fillings of $(M, \xi_1 = \ker(\eta_1))$ and $(M, \xi_2 = \ker(\eta_2))$ respectively. Assume that forms η_1 and η_2 are directly linked. Then the manifold $(W_1 \cup_M \overline{W_2}) \times S^1$ admits a contact structure.*

Proof. Indeed, as the forms η_1 and η_2 are directly linked, there exists a contact form η on M such that the manifolds (M, ξ_1) and (M, ξ_2) are both exactly concordant to $(M, \xi = \ker(\eta)$. Then the manifolds $(W_1, \mathrm{d}\lambda_1)$ and $(W_2, \mathrm{d}\lambda_2)$ are both exact symplectic fillings of (M, ξ) and the result follows from 9. □

It is conceivable that the construction used in [11] can be modified to the case where $(W_1, \mathrm{d}\lambda_1)$ and $(W_2, \mathrm{d}\lambda_2)$ are two exact symplectic concave fillings of a manifold (M, ξ).

Hypothesis 1 *Let $(W_1^{2n}, d\lambda_1)$ and $(W_2^{2n}, d\lambda_2)$ be exact symplectic fillings or cappings of $(M, \xi_1 = \ker(\eta_1))$ and $(M, \xi_2 = \ker(\eta_2))$ respectively. Assume that forms η_1 and η_2 are linked. Then the manifold $(W_1 \cup_M \overline{W_2}) \times S^1$ admits a contact structure.*

5.3 Contact structures on principal circle bundles

Very interesting results closely connected to the present topic are stated in [9]. The authors generalize the result from [11] (here presented in Theorem 9) to the case of nontrivial bundles proving the following theorem.

Theorem 10. *Let N be a principal S^1-bundle over a closed, connected, oriented manifold W of Euler class e. Assume that there exists a decomposition $W = W_1 \cup_M \overline{W_2}$ and symplectic structures ω_1 and ω_2 on W_1 and W_2 respectively, such that (W_1, ω_1) and (W_2, ω_2) are both weak fillings of the same contact manifold (M, ξ). Assume, moreover, that $-[\omega_1/2\pi] = e\,|_{W_1}$ and $[\omega_2/2\pi] = e\,|_{W_2}$. Then N admits a S^1-invariant contact structure.*

They also present the following necessary condition for the existence of S^1-invariant contact structures on the principle circle bundles.

Theorem 11. *Let N be a principal S^1-bundle over W of the Euler class e. If N admits a S^1-invariant contact structure , then there exist a decomposition $W = W_1 \cup_M \overline{W_2}$ and symplectic structures ω_1 and ω_2 on W_1 and W_2 respectively, such that (W_1, ω_1) and (W_2, ω_2) are both weak fillings of the same contact manifold (M, ξ). Moreover, $-[\omega_1/2\pi] = e\,|_{W_1}$ and $[\omega_2/2\pi] = e\,|_{W_2}$.*

6 Conclusion

We defined eight relations on the set of contact forms on a manifold: presymplectic homotopy, isotopy, concordance, exact concordance, direct linking, linking, cobordism and exact cobordism. We recalled concepts of SFT and algebraic and planar torsions. We used these tools to prove properties of the presented relations extending results from our previous paper [1]. Further we apply the notion of the defined relations to examine contact structures on manifolds of the form $W \times S^1$.

There are still questions to be answered in this area. We are especially interested in finding examples of concordant forms that are not isotopic. Moreover, we want to determine whether the result of Proposition 5 can be extended to the case when forms are only linked (not necessarily directly), see Hypothesis 1. We would also like to find a way to distinguish whether given two forms are (directly) linked or not.

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