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Surgery on Strong Sympletic Fold

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Abstract. We recall the theory of contact surgery ([4]). We present the notion of strong symplectic fold (SSF) structure on evendimensional manifolds. We use the theory of contact surgery to prove that the existence of SSF is invariant under surgeries up to half of the dimension. We describe the notion of complex cobordism and prove that generators of the complex cobordism group in dimension 4 admit SSF structures.

1 Introduction

On an evendimensional manifold we consider a structure of strong symplectic fold (SSF). This notion can be used to construct examples of contact manifolds. Indeed, it is shown in [5] that for a manifold M with an SSF structure the product $M \times S^1$ admits a contact form. The result is extended in [3] on all principal S^1-bundles over an SSF.

In Section 1 we recall the theory of contact surgery (see [4]) from both symplectic cobordism and cutting and gluing view point. We recall, moreover, a construction of symplectic handle (see [4]).

In Section 3 we present the structure of strong symplectic fold (SSF) on evendimensional manifolds. We use the theory of contact surgery to prove that the existence of SSF is invariant under surgeries of indices up to the half of the dimension.

We describe the notion of complex cobordism in Section **??**. We prove that generators of the complex cobordism group in dimension 4 admit SSF structures.

Finally, we close in Section 5 by mentioning directions of planed future work.

2 Contact surgery

We introduce notion of contact surgery. This section is based on [4].

2.1 Handle attachment viewpoint

In this subsection we will define a k-contact surgery on a contact manifold (M^{2n-1}, η) along an isotropic sphere $S := S^{k-1} \subset M$ with trivial symplectic normal bundle $\mathrm{SN}_M(S)$. We will consider the trivial symplectic cobordism

$(M \times [0,1], \mathrm{d}(e^t\eta))$ and we will glue in a symplectic handle along $S \subset M \times \{1\}$. The new contact manifold appearing as a convex boundary component of the cobordism will be the result of the contact surgery.

Neighbourhood results We begin be recalling the concept of an isotropic submanifold and stating a neighbourhood result.

Let (M^{2n-1}, η) be a contact manifold. The submanifold L of M is called **isotropic** if $TL \subset \xi = \ker \eta$. By $(TL)^{\perp}$ we denote the symplectic orthogonal complement of TL in the symplectic bundle $(\xi, \mathrm{d}\eta)$. Note that $TL \subset (TL)^{\perp}$ and $\dim L \leq n-1$.

Definition 1. *Let L be an isotropic submanifold of a contact manifold (M, η). The **symplectic normal bundle** of L in M is the quotient bundle*

$$SN_M(L) := (TL)^{\perp}/TL$$

with the symplectic bundle structure induced by $d\eta$.

A symplectic normal bundle of an isotropic submanifold determines a contact structure in its neighbourhood. Indeed, the following theorem is stated in [4] (Theorem 6.2.2). This allows us to identify a neighbourhood of S in $M \times \{1\}$ with a neighbourhood of an isotropic sphere $S^k \subset H$ in a symplectic handle.

Theorem 1. *Let (M_0, η_0) and (M_1, η_1) be contact manifolds with closed isotropic submanifolds L_0 and L_1 respectively. Suppose there exists an isomorphism of symplectic normal bundles $\Phi : SN_{M_0}(L_0) \to SN_{M_1}(L_1)$ covering a diffeomorphism $\phi : L_0 \to L_1$. Then ϕ extends to a contactomorphism $\psi : N(L_0) \to N(L_1)$ of some neighbourhoods $N(L_i)$ of L_i such that $T\psi \mid_{SN_{M_0}(L_0)} = \Phi$.*

Let $S := S^{k-1} \subset (M, \eta)$ be an isotropic sphere. Note that

$$NS = \langle R \rangle \oplus J(TS) \oplus \mathrm{SN}_M(S),$$

where $\langle R \rangle$ is a trivial line bundle spanned by the Reeb vector field R and J is an almost complex structure on ξ compatible with the symplectic structure $\mathrm{d}\eta$. The natural trivialization of $TS^{k-1} \oplus \epsilon$ for a trivial line bundle ϵ determines a trivialization of $\langle R \rangle \oplus J(TS)$. Thus, the trivialization of the symplectic normal bundle $\mathrm{SN}_M(S)$ induces a natural trivialization of NS. We call this trivialization **the natural framing** determined by the trivialization of $\mathrm{SN}_M(S)$.

Symplectic handle We describe a construction of k-symplectic handle H which is used in a definition of k-contact surgery. Consider $\mathbb{R}^{2n} = \mathbb{R}^k \times \mathbb{R}^{2n-k}$ with coordinates

$$(\mathbf{q}, \mathbf{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n),$$

$$(q_1, \ldots, q_k) \in \mathbb{R}^k, \quad (q_{k+1}, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n-k}.$$

Let ω_0 be the standard symplectic form on $\mathbb{R}^{2n}$

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

Define a Liouville vector field Y for ω_0 by the formula

$$Y = \sum_{i=1}^{k}(-q_i\partial_{q_i} + 2p_i\partial_{p_i}) + \frac{1}{2}\sum_{i=k+1}^{n}(q_i\partial_{q_i} + p_i\partial_{p_i}).$$

Y is the gradient vector field of the function

$$g(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{k}(-\frac{1}{2}q_i^2 + p_i^2) + \frac{1}{4}\sum_{i=k+1}^{n}(q_i^2 + p_i^2).$$

Consider a $(k-1)$ - dimensional embedded sphere S_H^{k-1} in $\mathbb{R}^{2n}$

$$S_H^{k-1} = \{(\mathbf{q},\mathbf{p}) \mid \sum_{i=1}^{k} q_i^2 = 2,\ q_{k+1} = \ldots = q_n = p_1 = \ldots = p_n = 0\}.$$

Define the **lower boundary** $L_H \cong S^{k-1} \times \mathrm{Int}(D^{2n-k})$ of the handle H as an open neighbourhood of S_H^{k-1} in $g^{-1}(-1) \subset \mathbb{R}^{2n}$.

The **symplectic handle** H is the set of points $(\mathbf{q},\mathbf{p})$ lying on a Y flow line through L_H and satisfying $-1 \leq g(\mathbf{q},\mathbf{p}) \leq 1$. The **upper boundary** is defined by

$$U_H = \{(\mathbf{q},\mathbf{p}) \mid g(\mathbf{q},\mathbf{p}) = 1\}.$$

Since Y is the gradient vector field of g, it is transverse to its level sets and thus the form

$$\alpha = \iota_Y\omega_0 = \sum_{i=1}^{k}(q_i\mathrm{d}p_i + 2p_i\mathrm{d}q_i) + \frac{1}{2}\sum_{i=k+1}^{n}(-q_i\mathrm{d}p_i + p_i\mathrm{d}q_i).$$

is a contact form on the lower and upper boundary.

Attaching the handle Below we describe the process of attaching the symplectic handle H.

Let M be a contact manifold. Consider an isotropic sphere in M together with its natural framing determined by a trivialization of the symplectic normal bundle

$$\psi : S^{k-1} \times D^{2n-k} \to M.$$

Let W be a symplectic cobordism defined by

$$W = ([-1,1] \times (M \backslash \psi(S^{k-1} \times \{0\})) \cup H)/\sim,$$

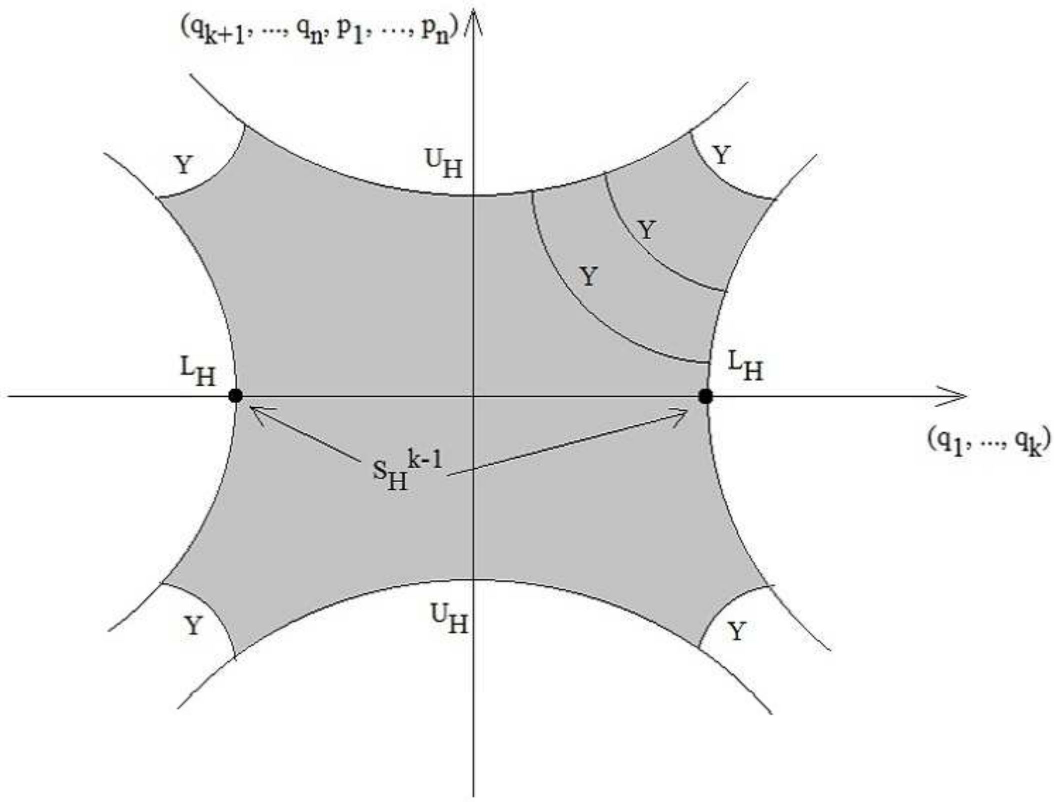

Fig. 1. Symplectic handle

where the equivalence relation $\sim$ is given by

$$[-1,1] \times (M \backslash \psi(S^{k-1} \times \{0\})) \ \ni \quad (c, \psi(u, rv)) \sim (\mathbf{q}, \mathbf{p}) \quad \in \ H,$$

for $u \in S^{k-1}$, $v \in S^{2n-k-1}$, a real number $0 < r < 1$ and $c \in [-1,1]$. Here $(\mathbf{q}, \mathbf{p})$ lies on the trajectory of Y flow through (u, rv) and $g(\mathbf{q}, \mathbf{p}) = c$. A symplectic structure ω on W is given by $\mathrm{d}(e^t\eta)$ on the symplectization $M \times [-1,1]$ of M and the described symplectic structure on the handle H.

Note that the boundary component of W corresponding to the c value -1 is concave (i.e. a Liouville vector field of ω is pointing outwards along it) and contactomorphic to M via the identification of $x \in M$ with

$$(-1, x) \in (M \backslash \psi(S^{k-1} \times \{0\})), \ \text{ if } \quad x \notin \psi(S^{k-1} \times \{0\})$$

or with

$$(u, rv) \in H, \ \text{ if } \quad x = \psi(u, rv).$$

It is a well defined map since for $x \in \psi(S^{k-1} \times \mathrm{Int}(D^{2n-k} \backslash \{0\}))$ the two definitions are identified by $\sim$. It is a contactomorphism by the neighbourhood theorem stated above (Theorem 1).

The result M' of the surgery is defined by

$$M' = (M \backslash \psi(S^{k-1} \times \{0\}) + D^k \times S^{2n-k-1}) / \sim$$

where

$$M \ni \psi(u, rv) \sim (ru, v) \in (D^k \times S^{2n-k-1}).$$

It is contactomorphic with the boundary component of W corresponding to the c value 1 via identification of $x \in M'$ with

$$(1, x) \in (M \backslash \psi(S^{k-1} \times \{0\})), \quad \text{if} \quad x \in M \backslash \psi(S^{k-1} \times \{0\})$$

or with

$$(u, rv), \quad \text{if} \quad x = (ru, v) \in (D^k \times S^{2n-k-1}).$$

Contact surgery theorem The above considerations lead to the following contact surgery theorem stated in [4] (Theorem 6.2.5).

Theorem 2. *Let (M, η) be a contact manifold. Consider an isotropic sphere S in M with trivialized symplectic normal bundle $SN_M(S)$. The manifold M' obtained by the topological surgery along S with respect to the natural framing admits a contact form η'. The form η' coincides with η outside from the surged region.*

It turns out that every topological k-surgery on (M^{2n-1}, η) for $k \leq n$ can be actualized as a contact surgery provided that an almost complex structure can be extended over the glued handle. The following theorem (Theorem 6.3.1 from [4]) stating this fact is used in next sections.

Theorem 3. *Let (M, ξ) be a $(2n{-}1)$ - dimensional contact manifold containing an embedded sphere S^{k-1} with trivialization of its normal bundle defined by an embedding*

$$f : S^{k-1} \times D^{2n-k} \to M.$$

Assume $1 \leq k \leq n$. Let (W, ω) be a symplectic manifold with (M, ξ) as a convex boundary component. Let $W' = W \cup_f H$ be the manifold obtained from W by attaching a $k-$handle H and M' the new boundary component (the result of surgery on M). If the $\omega-$compatible almost complex structure J (it is unique up to homotopy [7]) extends to an almost complex structure on W', then W' carries a symplectic form ω' and an $\omega'-$compatible almost complex structure homotopic to J such that M' is a convex boundary component of (W, ω). The induced contact structure ξ' on M' is the result of performing contact surgery along an isotropic sphere topologically isotopic to $f(S^{k-1} \times \{0\})$.

2.2 Cutting and gluing view point

A contact surgery can be seen also from the cutting and gluing view point rather then by constructing a symplectic cobordism. Here we also use the symplectic handle defined in Subsection 2.1 above. We describe the k-surgery along an embedded isotropic sphere $S^{k-1} \in M$ with the natural framing represented by the embedding $S^{k-1} \times D^{2n-k} \in M$. We first cut out $S^{k-1} \times D^{2n-k}$. It corresponds to a neighbourhood L_H of S_H^{k-1} in $g^{-1}(-1)$. The neighbourhoods are chosen small enough for the collar neighbourhoods of the boundaries of the manifolds

$$M \backslash \mathrm{Int}(S^{k-1} \times D^{2n-k}) \quad \text{and} \quad g^{-1} \backslash L_H$$

to be contactomorphic. Then, they can be contactomorphically identified along the Y flow lines with the corresponding subset of $g^{-1}(1)$. It allows us to glue in the upper boundary $U_H \cong D^k \times S^{2n-k-1}$ of the handle H. We obtain as a result a contact form on the surgered manifold

$$M' \cong (M \backslash S^{k-1} \times D^{2n-k}) \cup_{S^{k-1} \times S^{2n-k-1}} D^k \times S^{2n-k-1}.$$

3 Surgery on SSF

Definition 2. *Let M be a manifold with boundary. We say that M has a* ***Morse type*** *n if there exists a Morse function $f : M \to [0,1]$ with $f^{-1}(1) = \partial M$ such that f has only critical points of indices at most n.*

Definition 3. *A* ***strong symplectic fold (SSF) structure*** *on a manifold M^{2n} is a decomposition $M = M_0 \cup_N M_1$, where M_0 and M_1 are of Morse type n and are glued along the common boundary $\partial M_0 = \partial M_1 = N$. Moreover, they admit symplectic forms $\omega_0,\ \omega_1$ such that $(M_0, \omega_0), (M_1, \omega_1)$ are strong symplectic fillings of the same contact manifold (N, η).*

In this section we are interested in the question of existence on an SSF structure on an evendimensional manifold. In [1] we show the following result proving the existence of a weaker structure.

Proposition 1. *Suppose that (M^{2n}, J) is a closed almost complex manifold. Then there exist symplectic fillings (M_0, ω_0) and (M_1, ω_1) of contact manifolds $(N, \xi_0 = \ker \eta_0)$ and $(N, \xi_1 = \ker \eta_1)$ respectively such that M is diffeomorphic to $M_0 \cup_N M_1$. Moreover, the forms η_0 and η_1 are presymplectically homotopic, i.e. there exists a homotopy η_t between them such that $(d\eta_t)^{n-1} \neq 0$.*

The existence of an SSF structure is invariant under surgeries of indices up to half of dimension.

Proposition 2. *Let M be a manifold of dimension $2n$ with an SSF structure. Consider a manifold M' being a result of a k-surgery on M with $k \leq n$. Assume that M' is almost complex. Then M' admits an SSF structure.*

Remark 1. If a manifold M admits an SSF structure, M is almost complex. Indeed, symplectic structures ω_0 and ω_1 induces almost complex structures J_0 and J_1 on M_0 and M_1 respectively. As $(M_0, \omega_0), (M_1, \omega_1)$ are strong symplectic fillings of the same contact manifold (N, η), one can choose J_0 and J_1 to be equal on N. Then, we define almost complex structure J on M as J_0 on M_0 and J_1 on M_1 after possibly smoothing it on N.

Proof. Let $M = M_0 \cup_N M_1$ be an SSF structure on M. Let ω_0 and ω_1 be symplectic forms on M_0 and M_1 respectively and let η be the contact form induced on their common boundary N.

Consider a Morse function $f : M \to \mathbb{R}$ with

$$f^{-1}((-\infty, 0]) = M_0, \quad f^{-1}([0, \infty)) = M_1.$$

Assume that all critical points of indices at most n belong to M_0 while all critical points of indices at least n belong to M_1. Then the function f determines a handle decomposition of M such that M_0 consists of handles of indices at most n and M_1 consists of handles of indices at least n.

Let $S^{k-1} \times D^{2n-k+1} =: S \times D \hookrightarrow M$ be an embedded sphere in M together with its framing. We consider the surgery along this sphere.

Without loss of generality, one can assume that $S \subset N$. Indeed, as S has the dimension $k-1$ it can be homotoped to a sphere in $(k-1)$ - skeleton of M determined by the handle decomposition. Note that this skeleton is fully contained in $f^{-1}((-\infty, 0]) = M_0$. Similarly, considering now the function $-f$ we conclude that S can be homotoped to a sphere in N. We continue to call the new sphere S. Since $k \leq n$, we can choose the homotopy $S^{k-1} \times [0, 1] \to M$ to be an immersion and thus we homotope the sphere together with its framing. Note, moreover, that since

$$k - 1 \leq n - 1 < \frac{2n - 1}{2},$$

we can pick S to be an embedded sphere in N.

Now let us show that we can assume the framing of S to have one vector pointing outward from M_0. For this reason consider the following fibration

$$\mathrm{SO}(m-1) \to \mathrm{SO}(m) \to S^m$$

and the induced homotopy long exact sequence

$$\ldots \to \pi_{i+1}(S^m) \to \pi_i(\mathrm{SO}(m-1)) \to \pi_i(\mathrm{SO}(m)) \to \pi_i(S^m) \to \pi_{i-1}(\mathrm{SO}(m-1)) \to \ldots$$

For $i + 1 < m$ we obtain

$$\pi_i(\mathrm{SO}(m-1)) \cong \pi_i(\mathrm{SO}(m)).$$

Thus, since $k \leq n$, we have

$$\pi_{k-1}(\mathrm{SO}(2n-k)) \cong \pi_{k-1}(\mathrm{SO}(2n-k+1)).$$

Therefore, one can homotope the framing of S such that it has a vector pointing outward from M_0.

The surgery can be decomposed into two steps: cutting out $S \times D$ and gluing in its place $D^k \times S^{2n-k}$. Formally,

$$M' = (M \setminus S \times D) \cup_{S^{k-1} \times S^{2n-k}} D^k \times S^{2n-k}.$$

We claim that $M' = M'_0 \cup M'_1$ with

$$M'_0 = (M_0 \setminus S \times D_{\leq 0}) \cup_{S^{k-1} \times S^{2n-k}_{\leq 0}} D^k \times S^{2n-k}_{\leq 0}$$

and

$$M'_1 = (M_1 \setminus S \times D_{\geq 0}) \cup_{S^{k-1} \times S^{2n-k}_{\geq 0}} D^k \times S^{2n-k}_{\geq 0}$$

is a SSF structure on M'. The subscript ≤ 0 (≥ 0) means that only that part of the set is considered where the first coordinate is nonnegative (nonpositive).

Let us now restrict our attention to M'_0. Note that $M_0 \setminus S \times D_{\leq 0}$ is diffeomorphic to M_0 and cutting out $S \times D_{\leq 0}$ can be treated as merely choosing a framing F of S in N. This framing is the same as the framing $S \times D$ of S in M while omitting the vector pointing outward from M_0. Then, since $S^{2n-k}_{\leq 0}$ is a $(2n-k)$ - dimensional disc, M'_0 is a result of gluing a k-handle to M_0 along S with respect to the framing F. From Theorem 3 M'_0 admits a symplectic from convex on the boundary N'. The induced contact form η' on N' is the result of performing contact surgery along an isotropic sphere isotopic to S.

Similarly, M'_1 is a result of gluing a k-symplectic handle to M_1 along S with respect to the framing F.

It follows that both M'_0 and M'_1 are strong symplectic fillings of (N', η') and, consequently,

$$M' = M'_0 \cup_{N'} M'_1$$

is an SSF structure on M'. □

4 Complex cobordism

Complex cobordism theory is the theory of cobordism of stably almost complex manifolds.

Definition 4. *Let M be a manifold. We say that M is* ***stably almost complex*** *if there exists $k \in \mathbb{N}$ such that $TM \oplus \epsilon^k$ is a complex bundle, where ϵ^k is a trivial k-dimensional bundle over M.*

Let Ω^U_n be the set of equivalence classes of n-dimensional stably almost complex manifolds. Two manifolds M and N are equivalent if there exists a stably almost complex cobordism between them. Note that there is a group structure on Ω^U_n given by

$$[M] + [N] = [M \sqcup N],$$
$$-[M] = [\bar{M}],$$
$$0 = \emptyset,$$

where $\bar{M}$ is the manifold M with the opposite orientation and 0 denotes the neutral element of the group.

By Ω_*^U we denote the complex cobordism ring, $\Omega_*^U = \bigcup_n \Omega_n^U$.

Definition 5. *Let $i \leq j \in \mathbb{N}$.* ***Milnor hypersurface*** *$H_{i,j}$ is defined by*

$$H_{i,j} = \{[z_0 : \ldots : z_i] \times [w_0 : \ldots : w_j] \in \mathbb{C}P^i \times \mathbb{C}P^j \mid z_0 w_0 + \ldots z_i w_i = 0\}.$$

Milnor hypersurface $H_{i,j}$ can be considered as the set of pairs (l, h), with l being a line in $\mathbb{C}^{i+1}$ and with h being a hypersurface in $\mathbb{C}^{j+1}$ containing l. Then $H_{i,j}$ is the total space of the bundle

$$\mathbb{C}P^{j-1} \to H_{i,j} \to \mathbb{C}P^i,$$

where the projection function is defined by $H_{i,j} \ni (l, h) \mapsto l \in \mathbb{C}P^i$.

It follows from [8] and [9] that Milnor hypersurfaces generate the complex cobordism ring.

Proposition 3. *Generators of Ω_4^U admit SSF structures.*

Proof. $\Omega_4^U = \mathbb{Z}^2$ is generated by $[\mathbb{C}P^2]$ and $[\mathbb{C}P^1 \times \mathbb{C}P^1]$.

$\mathbb{C}P^1 \times \mathbb{C}P^1 = S^2 \times S^2$ being a double of a manifold $S^2 \times D^2$ of a Morse type up to the half of the dimension obviously admits an SSF structure.

$\mathbb{C}P^2$ can be decomposed into a sum of a disc D^4 and a disc Hopf bundle H

$$D^2 \to H \to \mathbb{C}P^1.$$

Both manifolds have a Morse type at most 2. They are glued together along common boundary S^3. Let η_0 and η_1 be contact structures on S^3 as the boundary of symplectic manifolds H and D^4 respectively. Since there is the unique fillable contact structure on S^3 (see Theorem 4.10.3 in [4]), we conclude that η_0, η_1 are contactomorphic. □

5 Future projects

In future we want to extend Proposition 2 for surgeries of indices greater then the half of dimension. Then we are planning to find an SSF structure on complex projective spaces $\mathbb{C}P^n$ and using this construct an SSF structure on Milnor hypersurfaces. This would provide a proof that all almost complex manifolds admit an SSF structure.

Moreover, we want to use SSF structures to find new examples of contact manifolds generalizing the results from [5] where the authors show the existence of contact structures on product of SSF and S^1. We are planning to consider the existence of contact structures on open book decompositions over SSF manifolds.

Finally, we are planning to use algebraic torsion to classify contact manifolds constructed as mentioned above with respect to symplectic cobordism. This would generalize methods introduced in [6].

6 Conclusion

On an evendimensional manifold we considered a structure of strong symplectic fold.

We recalled the theory of contact surgery from both symplectic cobordism and cutting and gluing view point. We recalled, moreover, a construction of symplectic handle.

We presented the notion of SSF structure on evendimensional manifolds. We proved that the existence of SSF is invariant under surgeries of indices up to the half of the dimension.

Finally, in 5 we indicate some future perspectives of the presented project.

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