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Silne symplektyczne foldy i ich zastosowania w topologii kontaktowej

Agata Magdalena Chorowska

promotor: dr hab. Bogusław Hajduk

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PhD thesis

Strong symplectic folds and their applications to contact topology

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supervisor: Prof. Dr. Bogusław Hajduk

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Streszczenie

Prezentowana rozprawa dotyczy struktur silnych symplektycznych foldów (SSF) na rozmaitościach. Silne symplektyczne foldy to rozkłady rozmaitości na sumę dokładnych rozmaitości symplektycznych o wypukłych brzegach, takich że struktury kontaktowe indukowane na brzegach się zgadzają.

Rozważamy zagadnienie istnienia struktur SSF na rozmaitościach. Prezentujemy dwie metody konstrukcji struktur SSF: używając argumentu chirurgicznego oraz przy pomocy wypukłych hiperpowierzchni w rozmaitościach kontaktowych.

Co więcej, rozważamy możliwe modyfikacje definicji strutury SSF. Badamy struktury SSF na rozmaitościach z brzegiem, a następnie używamy tych struktur, aby zdefiniować i zbadać SSF kobordyzmy między rozmaitościami kontaktowymi. Ponadto rozważamy struktury SSF, w których rozkładzie rozmaitości symplektyczne mogą mieć zarówno wypukłe jak i wklęsłe brzegi. Dowodzimy istnienia takich struktur na zamkniętych orientowalnych rozmaitościach wymiaru 4.

Wreszcie badamy własności struktur SSF kontaktowych, czyli struktur kontaktowych indukowanych na produkcie rozmaitości SSF z okręgiem. Klasyfikujemy takie struktury w wymiarze 3 z dokładnością do klasy homotopii odpowiadającej im 2-dystrybucji i dowodzimy, że każda klasa homotopii S^1-niezmienniczej 2-dystrybucji zawiera strukturę SSF kontaktową. Ponadto badamy wypełnialność struktur SSF kontaktowych, podając przykłady zarówno wypełnialych jak i niewypełnialnych klas takich struktur.

Abstract

In this thesis, we study strong symplectic fold (SSF) structures on manifolds, i.e. decompositions of manifolds into exact symplectic pieces convex along boundaries such that the contact structures induced on the boundaries agree.

We study the existence question for SSF structures. We present two ways of constructing an SSF structure: using surgery technique and via the notion of convex hypersurfaces in contact manifolds.

What is more, we consider possible modifications of the definition of SSF structures. We examine SSF structures on manifolds with boundary and we use this notion to define and study SSF cobordisms between contact manifolds. Moreover, we consider SSF structures, where symplectic pieces are allowed to have both convex and concave boundaries. We prove the existence of such structures in dimension 4.

Finally, we examine properties of SSF-contact structures - contact structures induced on products of SSF manifolds with S^1. We classify SSF-contact structures in dimension 3 up to homotopy of the corresponding 2-distributions and we prove that for a given oriented surface Σ there exists an SSF-contact structure on $\Sigma \times S^1$ in every homotopy type of S^1-invariant cooriented 2-distributions. Moreover, we study fillability of SSF-contact structures. We give examples of both fillable and non-fillable classes of SSF-contact manifolds.

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Contents

Chapter 1

Introduction

On closed oriented manifolds of even dimension one can consider structures of strong symplectic folds (SSF structures). By an SSF structure we mean a decomposition $M = M_0 \cup_N \overline{M_1}$ of a manifold M into two pieces M_0 and $\overline{M_1}$ admitting exact symplectic structures and glued together along their common boundary N. The symplectic pieces are convex along N and the contact forms induced on N are required to be equal.

SSF structures belong to a larger family of folded symplectic structures studied by Cannas da Silva. She proved in [4] that all stably almost complex manifolds of even dimension admit folded symplectic structures.

There are two reasons to study SSF structures on manifolds. First of all, they allow a decomposition of a manifold into exact symplectic pieces, which are well understood and tractable. Moreover, the SSF structures have applications in contact topology. Indeed, they can be used to construct examples of contact manifolds. In [19] Geiges and Stipsicz present a construction of contact structures on manifolds of the form $M \times S^1$, where M admits an SSF structure. We call such contact structures SSF-contact structures. A simple form of SSF-contact structures allows one to examine their properties.

The existence question for SSF structures in dimension 4 was studied by Baykur. In [1] he uses the notion of Lefschetz fibrations to prove the existence of SSF structures on all 4-dimensional orientable manifolds.

In this thesis, we study the existence question for SSF structures in an arbitrary dimension, possible modifications of the definition of SSF structures and properties of SSF-contact structures.

We organize the thesis in the following way.

In Chapter 2, we introduce basic notions of contact and symplectic topology including contact and symplectic manifolds, symplectic cobordisms and contact surgery, open book decompositions and their applications in contact topology.

In Chapter 3, we study relations on the set of contact forms on a given manifold. We examine properties of the relations and dependencies between them. This chapter can be treated as a warm-up. All of the facts stated there are well known.

Chapter 4 gives a definition of SSF structures and examples of SSF-manifolds. Moreover, we sketch Baykur's proof of the existence result in dimension 4. We consider two ways of constructing an SSF structure in an arbitrary dimension. The first way uses the notion of surgery. More precisely, we introduce a notion of surgery on SSF manifolds and we proof that the existence of an SSF structure is invariant under a surgery of index up to the half of the dimension of the manifold. The second way to obtain an SSF structure is via convex hypersurfaces in contact manifolds. We prove that any such hypersurface admits an SSF structure. We discuss applications of this fact. What is more, we present a new construction of a controllable contact form on the product of an almost complex manifold with the real line, which allows us to find on this manifold a structure related to an SSF structure. More precisely, we decompose the manifold into a sum of two exact symplectic pieces. This can be treated as the first step in a proof of the existence of SSF structures on almost complex manifolds. Finally, we present the construction of Geiges and Stipsicz of a contact form on $M \times S^1$, where M admits an SSF structure.

In Chapter 5, we discuss possible modifications of the definition of an SSF structure. First of all, we examine SSF structures on manifolds with boundary. Similarly as in Chapter 4, we present a surgery construction of such manifolds. Moreover, we use SSF structures with boundary to introduce and study a concept of an SSF cobordism between contact manifolds. Another modification that we consider are SSF structures, where symplectic pieces are allowed to have both convex and concave boundaries. We show the existence of such structures on 4-dimensional closed oriented manifolds significantly simplifying the Baykur's proof.

Finally, in Chapter 6, we study properties of SSF-contact structures. In particular, we classify SSF-contact structures in dimension 3 up to homotopy of the corresponding 2-distributions and we prove that for a given oriented surface Σ there exists an SSF-contact structure on $\Sigma \times S^1$ in every homotopy type of S^1-invariant cooriented 2-distributions. Moreover, we study fillability of SSF-contact structures. We give examples of both fillable and non-fillable classes of SSF-contact manifolds. Our non-fillable examples are 3-dimensional manifolds with planar torsion as defined in [40].

Chapter 2

Preliminaries

In this chapter, we are going to introduce basic notions of contact and symplectic topology, which will be extensively used throughout the thesis. All the results in this chapter are standard and well known.

In Section 2.1, we define contact and symplectic manifolds and we discuss their basic properties. In Section 2.2, we explain the notions of contact surgery and symplectic cobordism. Section 2.3 dedicates itself to the special case of 3-dimensional contact topology. Finally, in Section 2.4, we define open book decompositions of manifolds and we discuss their applications in contact topology.

2.1 Contact and symplectic manifolds

In this section, we are going to discuss basic concepts regarding contact and symplectic manifolds. We will start, in Subsection 2.1.1, by considering symplectic structures. Later, in Subsection 2.1.2, we will introduce the notion of contact manifolds.

2.1.1 Symplectic manifolds

Definition 2.1.1. Consider a smooth manifold M. A 2-form ω on M is called **symplectic** if the following conditions are fulfilled

- ω is *closed*, i.e. $\mathrm{d}\omega = 0$,
- ω is *nondegenerate*, which means that

$$\big(\forall x \in M\big)\,\big(\forall v \in \mathrm{T}_x M\big)\,\Big(v \neq 0 \Rightarrow \big(\exists w \in \mathrm{T}_x M\big)\,\omega(v,w) \neq 0\Big).$$

A symplectic form is also often called a symplectic structure. A pair (M,ω) is called a **symplectic manifold**.

Note that non-degeneracy of a 2-form on M implies that M is a manifold of an even dimension. If the dimension of M is equal to $2n$, then the non-degeneracy condition can be equivalently formulated as $\omega^n \neq 0$ (meaning that the form ω^n vanishes nowhere).

It follows that a symplectic manifold (M^{2n},ω) is orientable and its orientation can be given by the form ω^n. If the orientation of M is chosen, a symplectic form ω is called **positive** if $\omega^n > 0$ and **negative** if $\omega^n < 0$. Unless specified otherwise, by a *symplectic form* on an oriented manifold we will mean a positive symplectic form.

We say that a symplectic manifold (M, ω) is **exact** if the form ω is exact, i.e. if there exists a 1-form μ on M such that $\mathrm{d}\mu = \omega$. Note that from the Stokes theorem it follows that an exact symplectic manifold cannot be closed.

For symplectic manifolds (M_0, ω_0) and (M_1, ω_1) by **symplectomorphism** we mean a diffeomorphism $f : M_0 \to M_1$ such that $f^*\omega_1 = \omega_0$.

Let us consider an example of a symplectic manifold.

Example 2.1.2. $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ is a symplectic manifold, where ω_{st} is given by

$$\omega_{\mathrm{st}} = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n,$$

for the coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{R}^{2n}$.

The form ω_{st} corresponds to the orientation

$$\omega_{\mathrm{st}}^n = n!\, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n,$$

so it is a positive symplectic form (where the orientation of $\mathbb{R}^{2n}$ is given by the choice of the coordinates).

Remark 2.1.3. We will often omit the '$\wedge$' sign, when writing formulas for forms, in order to keep the notation clear.

The following theorem by Darboux implies that the simple example $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ of a symplectic manifold is actually the general local form of a symplectic structure.

Theorem 2.1.4 (Darboux)**.** *Let (M, ω) be a symplectic $2n$-manifold. For any point $p \in M$ there exists a neighbourhood of p symplectomorphic to an open subset in $(\mathbb{R}^{2n}, \omega_{st})$.*

A symplectic structure on a manifold M induces a complex structure on its tangent bundle $\mathrm{T}M$. Such a complex structure is called an almost complex structure of the manifold. The following definition specifies more precisely what an almost complex structure is and how it can related to a symplectic structure.

Definition 2.1.5. Let M be a manifold of an even dimension. An **almost-complex structure** on M is an automorphism J of the tangent bundle $\mathrm{T}M$ such that $J^2 = -\mathrm{Id}$.

If (M, ω) is a symplectic manifold, then an almost-complex structure J is said to be **ω-compatible**, if for all $x \in M$ and $v, w \in \mathrm{T}_x M$

$$\omega(Jv, Jw) = \omega(v, w) \quad \text{and} \quad v \neq 0 \Rightarrow \omega(v, Jv) > 0.$$

The space of smooth ω-compatible almost-complex structures is contractible and nonempty (see Proposition 4.1, [29]).

Given a submanifold $L \subset (M^{2n}, \omega)$ of a symplectic manifold we define an orthogonal complement $(\mathrm{T}L)^\perp \subset \mathrm{T}M$ of its tangent bundle as

$$(\mathrm{T}L)_x^\perp := \{v \in \mathrm{T}_x M \mid \big(\forall w \in \mathrm{T}_x L\big)\ \omega(v, w) = 0\}.$$

The so defined $(\mathrm{T}L)^\perp$ is a vector bundle of dimension $2n - \dim L$.

We say that L is **isotropic** if $\mathrm{T}L \subset (\mathrm{T}L)^\perp$. Note that an isotropic submanifold has dimension at most n. An isotropic submanifold of the maximal dimension is called Lagrangian.

2.1.2 Contact manifolds

Let us now proceed by considering contact manifolds. The relation between symplectic and contact manifolds is very strong. In fact, contact manifolds can be thought of as odd-dimensional analogues of symplectic manifolds.

Contact structures are defined in terms of codimension 1 distributions (i.e. codimension 1 subbundles of a tangent bundle). In this thesis, we will be only interested in coorientable contact structures, i.e. such that the corresponding distribution is a kernel of a 1-form. For simplicity, we will usually omit the word coorientable and refer to *coorientable contact manifolds* by simply *contact manifolds*.

Definition 2.1.6. Let N be a smooth manifold of odd dimension $2n-1$. A **contact structure** $\xi = \ker \eta$ on N is a maximally non-integrable codimension 1 distribution. Maximal non-integrability of ξ means that the defining form η satisfies

$$\eta \wedge (\mathrm{d}\eta)^{n-1} \neq 0.$$

A pair (M, ξ) is called a **contact manifold** and the form η is called a **contact form**.

For the sake of convenience, we will use a term *contact manifold* to refer to a pair (N, η) for a contact form η on N. Such a simplification is commonly used in the literature.

Remark 2.1.7. In particular, we treat $(S^1, \mathrm{d}\theta)$ as a contact manifold. Here, θ is a parameter of S^1.

If η_0 and η_1 are two contact forms defining the same coorientable contact structure, then they differ by multiplication by a non-zero function.

Note that a contact manifold (N^{2n-1}, η) is orientable and its orientation can be defined by the form $\eta \wedge (\mathrm{d}\eta)^{n-1}$. If the orientation of N is chosen, a contact form η is called **positive** if $\eta \wedge (\mathrm{d}\eta)^{n-1} > 0$ and **negative** if $\eta \wedge (\mathrm{d}\eta)^{n-1} < 0$. Unless specified otherwise, by a *contact form* on an oriented manifold we will mean a positive contact form.

Observe that the contact condition $\eta \wedge (\mathrm{d}\eta)^{n-1} > 0$ implies that the form $\mathrm{d}\eta$ defines a symplectic structure on $\xi = \ker \eta \subset \mathrm{T}N$. The bundle $(\xi, \mathrm{d}\eta)$ is a symplectic bundle. All the concepts defined in the previous Subsection for symplectic manifolds, like orthogonal complements or isotropic submanifolds, can be defined analogously for symplectic bundles.

For contact manifolds (N_0, ξ_0) and (N_1, ξ_1) a diffeomorphism $f : N_0 \to N_1$ is called a **contactomorphism** if $\mathrm{T}f(\xi_0) = \xi_1$. If $\xi_0 = \ker \eta_0$ and $\xi_1 = \ker \eta_1$, this condition can be equivalently formulated in terms of contact forms as $f^*\eta_1 = \lambda \eta_0$ for a function $\lambda : N_0 \to \mathbb{R}\backslash\{0\}$. A contactomorphism is **strict** if $f^*\eta_1 = \eta_0$.

Let us consider an example of a contact form on $\mathbb{R}^{2n-1}$.

Example 2.1.8. $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}} = \ker \eta_{\mathrm{st}})$ is a contact manifold with a contact form η_{st} given by

$$\eta_{\mathrm{st}} = \mathrm{d}z + \sum_{i=1}^{n-1} x_i \wedge \mathrm{d}y_i,$$

for the coordinates $(z, x_1, y_1, \dots, x_{n-1}, y_{n-1})$ on $\mathbb{R}^{2n-1}$.

The form η_{st} corresponds to an orientation

$$\eta_{\mathrm{st}} \wedge (\eta_{\mathrm{st}})^{n-1} = (n-1)!\, \mathrm{d}z \wedge \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_{n-1},$$

so it is a positive contact form.

Similarly as in the symplectic case, there is a Darboux theorem for contact manifolds implying that all contact manifolds are locally the contactomorphic.

Theorem 2.1.9 (Darboux). *Let (N, ξ) be a coorientable contact $(2n-1)$ - manifold. For any point $p \in N$ there exists a neighbourhood of p contactomorphic to $(\mathbb{R}^{2n-1}, \xi_{st})$.*

A recent result by Borman, Eliashberg and Murphy (see [2]) solves the long-standing open problem of existence of contact structures on manifolds. They prove that a manifold admits a contact structure whenever a trivial necessary condition is fulfilled, i.e. when there exists an almost contact structure.

Another fundamental theorem in contact topology is the Gray stability theorem saying that there are no non-trivial deformations of contact structures on closed manifolds.

Theorem 2.1.10 (Gray stability theorem). *Consider a smooth path of contact structures $\xi_t, t \in [0,1]$, on a closed manifold N. There exists an isotopy ψ_t of N with $\psi_0 = Id_N$ such that*

$$\left(T\psi_t\right)\xi_0 = \xi_t.$$

Contact structures connected by a path of contact structures will be called **isotopic**, while contact structures connected by a path of hyperplane fields will be called **homotopic**.

Contact manifolds naturally appear as hypersurfaces in symplectic manifolds. To formalize that statement let us introduce the notion of a Liouville vector field on a symplectic manifold.

Definition 2.1.11. A **Liouville vector field** Y on a symplectic manifold (M, ω) is a vector field satisfying the equation $\mathscr{L}_Y \omega = \omega$.

If N is a hypersurface in (M^{2n}, ω) and there exists a Liouville vector field Y defined in a tubular neighbourhood of N and transverse to N, then the form $\eta := (\iota_Y \omega)\,|_{TN}$ is a contact form on N. Indeed, the Liouville condition implies $\omega = \mathscr{L}_Y \omega = \mathrm{d}\iota_Y \omega = \mathrm{d}\eta$ and

$$\eta \wedge (\mathrm{d}\eta)^{n-1} = \iota_Y \omega \wedge \omega^{n-1} = \frac{1}{n}\iota_Y(\omega^n).$$

Since Y is transverse to N, non-degeneracy of ω implies that $\iota_Y(\omega^n) \neq 0$. The submanifold N is called a **Liouville type submanifold**.

The above fact can be used to construct a contact form the sphere S^{2n-1}.

Example 2.1.12. Consider the sphere S^{2n-1} as the unit sphere in the symplectic manifold $(R^{2n}, \omega_{\mathrm{st}})$ endowed with the coordinates $(x_1, y_1, \ldots, x_n, y_n)$. The vector field $Y = \sum_{i=1}^n \left(x_i \partial_{x_i} + y_i \partial_{y_i}\right)$ is a Liouville vector field for ω_{st} (up to a constant). Indeed,

$$\mathscr{L}_Y \omega_{\mathrm{st}} = \mathrm{d}(\iota_Y \omega_{\mathrm{st}}) = \mathrm{d}\left(\sum_{i=1}^n \left(x_i \mathrm{d}y_i - y_i \mathrm{d}x_i\right)\right) = 2\omega_{\mathrm{st}}.$$

It follows that $\eta_{\mathrm{st}} := \iota_Y \omega_{\mathrm{st}} = \sum_{i=1}^n \left(x_i \mathrm{d}y_i - y_i \mathrm{d}x_i\right)$ is a contact form on S^{2n-1}.

Let us now discuss some special types of submanifolds of a contact manifold $(N, \xi = \ker \eta)$.

We will start with introducing isotropic submanifolds, which will be used to define surgery on contact manifolds. Let $L \subset N^{2n-1}$ be a submanifold of N. If $\mathrm{T}L \subset \xi$, then the manifold L is called an **isotropic submanifold** on N. Since $\eta \mid_{\mathrm{T}L} \equiv 0$, we see that $\mathrm{T}L \subset \left(\mathrm{T}L\right)^{\perp}$, where $(\cdot)^{\perp}$ is an orthogonal complement with respect to the symplectic form $\mathrm{d}\eta$ on ξ. This implies that $\dim L \leq \frac{1}{2} \dim \xi = n - 1$. An isotropic submanifold of the maximal dimension $n - 1$ is called a **Legendrian submanifold**.

Now, let us discuss another class of submanifolds of $(N, \xi = \ker \eta)$, namely convex hypersurfaces. This will require a definition of a contact vector field. Let X be a vector field on N. We say that X is a **contact vector field** if the contact structure ξ is invariant under the flow of X, i.e. it fulfills

$$\mathscr{L}_X \eta = \lambda \eta$$

for some function λ on N.

Let Σ be a hypersurface in N. We say that Σ is a **convex hypersurface**, if there exists a contact vector field defined in a tubular neighbourhood of Σ, which is transverse to Σ. For a convex hypersurface Σ and a transverse contact vector field X we define a dividing set $\Gamma \subset \Sigma$ of Σ as

$$\Gamma = \{x \in \Sigma \mid X_x \in \xi\}.$$

In Section 4.5, we will use convex surfaces and their dividing sets to construct SSF manifolds.

Finally, let us introduce another fundamental concept of contact topology which will prove to be useful in the following parts of the thesis.

Definition 2.1.13. Let (N, η) be a contact manifold. A vector field R_η, uniquely defined by the equations

$$\eta(R_\eta) = 1 \quad \text{and} \quad \iota_{R_\eta} \mathrm{d}\eta \equiv 0,$$

is called the **Reeb vector field**.

2.2 Symplectic cobordism and contact surgery

In this section, we introduce the notion of contact surgery as well as symplectic cobordism between contact manifolds. We will start, in Subsection 2.2.1, by recalling some basic concepts regarding oriented topological cobordisms and surgery. In Subsection 2.2.2, we will define a symplectic cobordism, while, in Subsection 2.2.3, we will give a description of a contact surgery.

2.2.1 Topological cobordism and surgery

Let N_0 and N_1 be closed oriented n-manifolds. An (oriented) **cobordism** from N_0 to N_1 is a compact $(n+1)$-dimensional oriented manifold M with boundary $\partial M = \overline{N_0} \sqcup N_1$. By $\overline{N_0}$ we mean the manifold N_0 with the reversed orientation.

It is often convenient to consider a handle decomposition of the cobordism M, since it allows us to view M as a trace of a sequence of surgeries used to obtain N_1 from N_0. Any cobordism M can be decomposed into handles. Such a decomposition can, in particular, be given in terms of a Morse function $f : M \to [0,1]$ such that $f|_{N_0} \equiv 0$ and $f|_{N_1} \equiv 1$ (see [32] for details). An $(n+1)$-dimensional **handle of index** k (also called a k-handle) is defined as the disc $H = D^k \times D^{n+1-k}$ with the boundary decomposed in the following way

$$\partial H = (\partial D^k \times D^{n+1-k}) \cup (D^k \times \partial D^{n+1-k}) = C \cup G.$$

Given a cobordism $(M, \overline{N_0} \sqcup N_1)$ and an embedding $\phi : C \hookrightarrow N_1$ one can attach the handle H to M along $\phi(C)$ to produce a new cobordism $M' = M \cup_\phi H$ between N_0 and N_1', where $N_1' = \Big(N_1 \backslash \mathrm{Int}\big(\phi(C)\big)\Big) \cup_{\phi|_{\partial C = \partial G}} G$.

Remark 2.2.1. The cobordism M', as defined above, is not a smooth manifold. If we are interested in smooth cobordisms, we can modify M' by a standard procedure of "smoothing the corners". We are not going to get into details here.

The manifold N_1' is the result of the k**-surgery** on N_1 along the framed $(k$–$1)$-sphere $\phi(C)$. N_1' is obtained from N_1 by cutting out the tubular neighbourhood $\phi(C)$ of the sphere $\phi(S^{k-1} \times \{0\})$ and gluing in G along the common boundary $G \cap C = \partial D^k \times \partial D^{n+1-k}$ of C and G.

A surgery on manifolds can also be defined without using the notion of cobordisms. Instead, a so called cut-and-paste approach can be used. Let N be a closed oriented manifold of dimension n. Let $\phi : S^{k-1} \times D^{n-k+1} \hookrightarrow N$ be an embedded sphere with a trivialized normal bundle. We will define a manifold N' by

$$N' = N \setminus \mathrm{Int}\big(\phi(S^{k-1} \times D^{n-k+1})\big) \cup_{\phi|_{S^{k-1} \times S^{n-k}}} D^k \times S^{n-k}.$$

Then, N' is said to be obtained by the k-surgery along the sphere $\phi(S^{k-1} \times \{0\})$ with respect to the framing ϕ.

2.2.2 Symplectic cobordism

The notions of cobordism and surgery admit an extension into the world of symplectic and contact topology.

Let us start by discussing symplectic manifolds with boundary. Let (M, ω) be a symplectic manifold and let N be a connected component of the boundary ∂M of M. Assume that there exists a Liouville vector field Y of ω defined in a neighbourhood of N and transverse to N. We know from Subsection 2.1.2 that such a structure ω induces a contact form $\eta = \iota_Y \omega$ on N. If Y points outwards from M along N, we call (N, η) a **convex boundary component** of (M, ω), while, if it points inwards, (N, η) is a **concave boundary component** of (M, ω).

The symplectic structure ω close to the boundary component N is different depending whether N is convex or concave. Indeed, if (N, η) is a convex boundary component, then there exists a collar neighbourhood of N symplectomorphic to

$$((-\epsilon, 0] \times N, d(e^t \eta)),$$

where $t \in (-\epsilon, 0]$ parameterizes the flow lines of the Liouville vector field Y. Similarly, if (N, η) is a concave boundary component, then there exists a collar neighbourhood of N symplectomorphic to

$$([0, \epsilon) \times N, d(e^t \eta)).$$

We can substitute e^t by any other increasing positive function.

We are now ready to define a symplectic cobordism between contact manifolds.

Definition 2.2.2. Let (N_0, η_0) and (N_1, η_1) be contact manifolds of dimension $2n - 1$. A **symplectic cobordism** from (N_0, η_0) to (N_1, η_1) is a topological cobordism M from N_0 to N_1 with a symplectic structure ω such that (N_0, η_0) is a concave boundary component and (N_1, η_1) is a convex boundary component of (M, ω).

A symplectic cobordism from (N_0, ξ_0) to (N_1, ξ_1) is a symplectic cobordism from (N_0, η_0) to (N_1, η_1) for some contact forms η_0 and η_1 defining ξ_0 and ξ_1.

Example 2.2.3. Let (N^{2n-1}, η) be a contact manifold. Then the pair

$$\big([0, 1] \times N, \mathrm{d}(e^t \eta)\big),$$

where t is a parameter of $[0, 1]$, is a symplectic cobordism from (N, η) to $(N, e{\cdot}\eta)$. Indeed, the easy calculation below shows that the form $\mathrm{d}(e^t \eta)$ is symplectic.

$$(\mathrm{d}(e^t \eta))^n = (e^t \mathrm{d}t \wedge \eta + e^t \mathrm{d}\eta)^n = n e^{nt} \mathrm{d}t \wedge \eta \wedge (\mathrm{d}\eta)^{n-1} \neq 0$$

The vector field ∂_t is a Liouville vector field for $\mathrm{d}(e^t \eta)$.

Such a cobordism is called the **trivial symplectic cobordism** or the **symplectization** of (N, η).

Similarly as in the topological case, contact surgery can be defined in terms of symplectic cobordisms. However, not every symplectic cobordism can be decomposed into symplectic handles and, consequently, viewed as a trace of a sequence of contact surgeries. The symplectic cobordisms that can be decomposed in such a way are called **Weinstein cobordisms**. Weinstein cobordisms from the empty set are called **Weinstein manifolds**. It will follow from Subsection 2.2.3 that a Weinstein manifold is build out of handles of indices up to the half of its dimension. A manifold is Weinstein if and only if it admits a Stein structure (see [5] for details). A **Stein manifold** is a complex manifold which admits a proper holomorphic embedding into some $\mathbb{C}^K$.

A symplectic cobordism (M, ω) from (N_0, η_0) to (N_1, η_1) is called **exact** if ω is an exact form. Note that, equivalently, one can say that (M, ω) is exact, if there exists a Liouville vector field for ω defined on whole manifold M. Then a 1-form η such that $\mathrm{d}\eta = \omega$ can be defined as $\eta := \iota_Y \omega$.

Symplectic concordance is a special type of symplectic cobordism. Let N be a manifold with contact forms η_1 and η_2. A **symplectic concordance** from η_0 to η_1 is a symplectic cobordism from (N, η_0) to (N, η_1) diffeomorphic to $N \times [0, 1]$.

Let (N, η) be a contact manifold. A symplectic cobordism from an empty set to (N, η) is called a **symplectic filling** of (N, η). A contact manifold is called **symplectically fillable** (or simply *fillable*), if it admits a symplectic filling. Similarly one can define exact, Stein or Weinstein fillings.

Remark 2.2.4. Symplectic fillings as described above are usually called *strong symplectic fillings*. Since in this thesis we are only interested in such fillings we will be omitting the word *strong*.

Example 2.2.5. The contact manifold $(S^{2n-1}, \eta_{\text{st}})$ as described in Example 2.1.12 is symplectically fillable by $(B^{2n}, \omega_{\text{st}}) \subset (\mathbb{R}^{2n}, \omega_{\text{st}})$. In fact, this is an exact and even Stein/Weinstein filling of $(S^{2n-1}, \eta_{\text{st}})$.

2.2.3 Contact surgery

In this subsection we are going to introduce the notion of contact surgery. The subsection is mostly based on [18]. We are going to start by looking at contact surgery as a result of a handle attachment to a symplectic cobordism. Then we will briefly mention how it can be seen from the cut-and-paste perspective.

Handle attachment viewpoint

Let us define a k-contact surgery on a contact manifold (N^{2n-1}, η) along an embedded isotropic sphere $S := S^{k-1} \subset N$ with a trivial symplectic normal bundle $\text{SN}_N(S)$. The strategy here will be similar to the one used to define topological surgery. We will consider the trivial symplectic cobordism $(N \times [0,1], \text{d}(e^t\eta))$ and we will attach to it a symplectic handle along the sphere $S \subset N \times \{1\}$. We will use the trivialization of the symplectic normal bundle to define the framing of S and, in consequence, to determine the way of gluing the handle H. The new contact manifold appearing as the convex boundary component of the cobordism will be the result of the contact surgery.

Neighbourhood results We begin by stating a neighbourhood result for isotropic submanifolds.

Let L be an isotropic submanifold of (N, η). The symplectic normal bundle of L can be defined as follows.

Definition 2.2.6. The **symplectic normal bundle** of L in N is the quotient bundle

$$\text{SN}_N(L) := (\text{T}L)^{\perp}/\text{T}L$$

with the symplectic bundle structure induced by $d\eta$. Here $(\cdot)^{\perp}$ is an orthogonal complement with respect to the symplectic form $\text{d}\eta$ on $\xi = \ker \eta$.

A symplectic normal bundle of an isotropic submanifold determines a contact structure in its neighbourhood. This fact will allow us to glue the symplectic handle H to $N \times [0,1]$ after identifying a neighbourhood of S in $N \times \{1\}$ with a neighbourhood of an isotropic sphere $S^k \subset H$ in a symplectic handle.

Theorem 2.2.7 (Theorem 6.2.2, [18])**.** *Let (N_0, η_0) and (N_1, η_1) be contact manifolds with closed isotropic submanifolds L_0 and L_1 respectively. Suppose that there exists an isomorphism of symplectic normal bundles $\Phi : SN_{N_0}(L_0) \to SN_{N_1}(L_1)$ covering a diffeomorphism $\phi : L_0 \to L_1$. Then ϕ extends to a contactomorphism $\psi : N(L_0) \to N(L_1)$ of tubular neighbourhoods $N(L_i)$ of L_i such that $T\psi \mid_{SN_{N_0}(L_0)} = \Phi$.*

Note that the normal bundle $\mathrm{N}_N S$ of S in N decomposes as follows

$$\mathrm{N}_N S = \langle R\rangle \oplus J(\mathrm{T}S) \oplus \mathrm{SN}_N(S), \tag{2.1}$$

where $\langle R\rangle$ is a trivial line bundle spanned by the Reeb vector field R and J is an almost complex structure on ξ compatible with the symplectic structure $\mathrm{d}\eta$. The bundle $\langle R\rangle \oplus J(\mathrm{T}S)$ can be trivialized using the natural trivialization of $\mathrm{T}S^{k-1} \oplus \epsilon$ for a trivial line bundle ϵ. Thus, the trivialization of the symplectic normal bundle $\mathrm{SN}_N(S)$ induces a natural trivialization of $\mathrm{N}S$. We call this trivialization the **natural framing** determined by the trivialization of $\mathrm{SN}_N(S)$.

Symplectic handle In this paragraph, we will describe a construction of k-symplectic handle H of dimension $2n$. See Figure 2.1a for an example of the 2-dimensional symplectic 1-handle. Consider $\mathbb{R}^{2n} = \mathbb{R}^k \times \mathbb{R}^{2n-k}$ with coordinates

$$(\mathbf{q},\mathbf{p}) = (q_1,\dots,q_n,p_1,\dots,p_n),$$
$$(q_1,\dots,q_k)\in\mathbb{R}^k,\quad (q_{k+1},\dots,q_n,p_1,\dots,p_n)\in\mathbb{R}^{2n-k}.$$

Let ω_{st} be the standard symplectic form on $\mathbb{R}^{2n}$

$$\omega_{\mathrm{st}} = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

Define a Liouville vector field Y for ω_{st} by the formula

$$Y = \sum_{i=1}^{k}(-q_i\partial_{q_i} + 2p_i\partial_{p_i}) + \frac{1}{2}\sum_{i=k+1}^{n}(q_i\partial_{q_i} + p_i\partial_{p_i}).$$

Note that Y is the gradient vector field of the function

$$g(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{k}(-\frac{1}{2}q_i^2 + p_i^2) + \frac{1}{4}\sum_{i=k+1}^{n}(q_i^2+p_i^2).$$

Consider the following $(k-1)$-dimensional embedded sphere S_H^{k-1} in $\mathbb{R}^{2n}$

$$S_H^{k-1} = \{(\mathbf{q},\mathbf{p}) \mid \sum_{i=1}^{k} q_i^2 = 2,\ q_{k+1} = \ldots = q_n = p_1 = \ldots = p_n = 0\}.$$

Define the **lower boundary** $L_H \overset{\psi}{\cong} S^{k-1} \times \mathrm{Int}(D^{2n-k})$ of the handle H as an open neighbourhood of S_H^{k-1} in $g^{-1}(-1) \subset \mathbb{R}^{2n}$.

The **symplectic handle** H is the set of points $(\mathbf{q},\mathbf{p})$ lying on the Y flow lines through L_H and satisfying $-1 \le g(\mathbf{q},\mathbf{p}) \le 1$. The **upper boundary** is defined by

$$U_H = \{(\mathbf{q},\mathbf{p}) \in H \mid g(\mathbf{q},\mathbf{p}) = 1\}.$$

Since Y is the gradient vector field of g, it is transverse to its level sets and thus the form

$$\alpha = \iota_Y\omega_{\mathrm{st}} = \sum_{i=1}^{k}(q_i\mathrm{d}p_i + 2p_i\mathrm{d}q_i) + \frac{1}{2}\sum_{i=k+1}^{n}(-q_i\mathrm{d}p_i + p_i\mathrm{d}q_i).$$

is a contact form on the lower and the upper boundary.

The symplectic normal bundle $\mathrm{SN}_{\partial H}(S_H^{k-1})$ can be trivialized by $\partial_{q_i}, \partial_{p_i}$ for $j = k+1, \ldots, n$. Indeed, since $\mathrm{d}\alpha = \omega_{\mathrm{st}}$, the standard almost complex structure J on R^{2n} given by $J(\partial_{p_i}) = \partial_{q_i}$ is compatible with the symplectic structure $\mathrm{d}\alpha$ on $\xi = \ker \alpha \subset \mathrm{T}L_H$. Moreover, the Reeb vector field R of α on L_H is given by

$$R = \frac{1}{2} \sum_{i=1}^{n} \left(q_i \partial_{p_i} - p_i \partial_{q_i} \right).$$

Along the sphere S_H^{k-1} the formula reduces to $R\,|_{S_H^{k-1}} = \frac{1}{2} \sum_{i=1}^{k} q_i \partial_{p_i}$. It follows that $\langle R \rangle \oplus J(\mathrm{T}S_H^{k-1})$ is spanned by $\partial_{p_1}, \ldots, \partial_{p_k}$. Using the decomposition (see Equation (2.1))

$$\mathrm{N}_{L_H} S_H^{k-1} = \langle R \rangle \oplus J(\mathrm{T}S_H^{k-1}) \oplus \mathrm{SN}_{L_H}(S_H^{k-1}),$$

we get that $\mathrm{SN}_{L_H}(S_H^{k-1})$ is spanned by $\partial_{q_i}, \partial_{p_i}$ for $j = k+1, \ldots, n$.

Remark 2.2.8. The sets L_H and U_H correspond to $\mathrm{Int}(C)$ and $\mathrm{Int}(G)$ respectively in the notation from Subsection 2.2.1.

Attaching the handle We are now ready to describe the process of attaching the symplectic handle H.

Let N be a contact manifold and let ϕ denote an embedding of an isotropic sphere together with its natural framing determined by a trivialization of the symplectic normal bundle

$$\phi : S^{k-1} \times D^{2n-k} \hookrightarrow N.$$

Let M be a trivial symplectic cobordism with the handle H attached along ϕ. More explicitly, let M be defined as the quotient of

$$[-1,1] \times \Big(N \backslash \phi\big(S^{k-1} \times \{0\}\big) \Big) \quad \cup \quad H$$

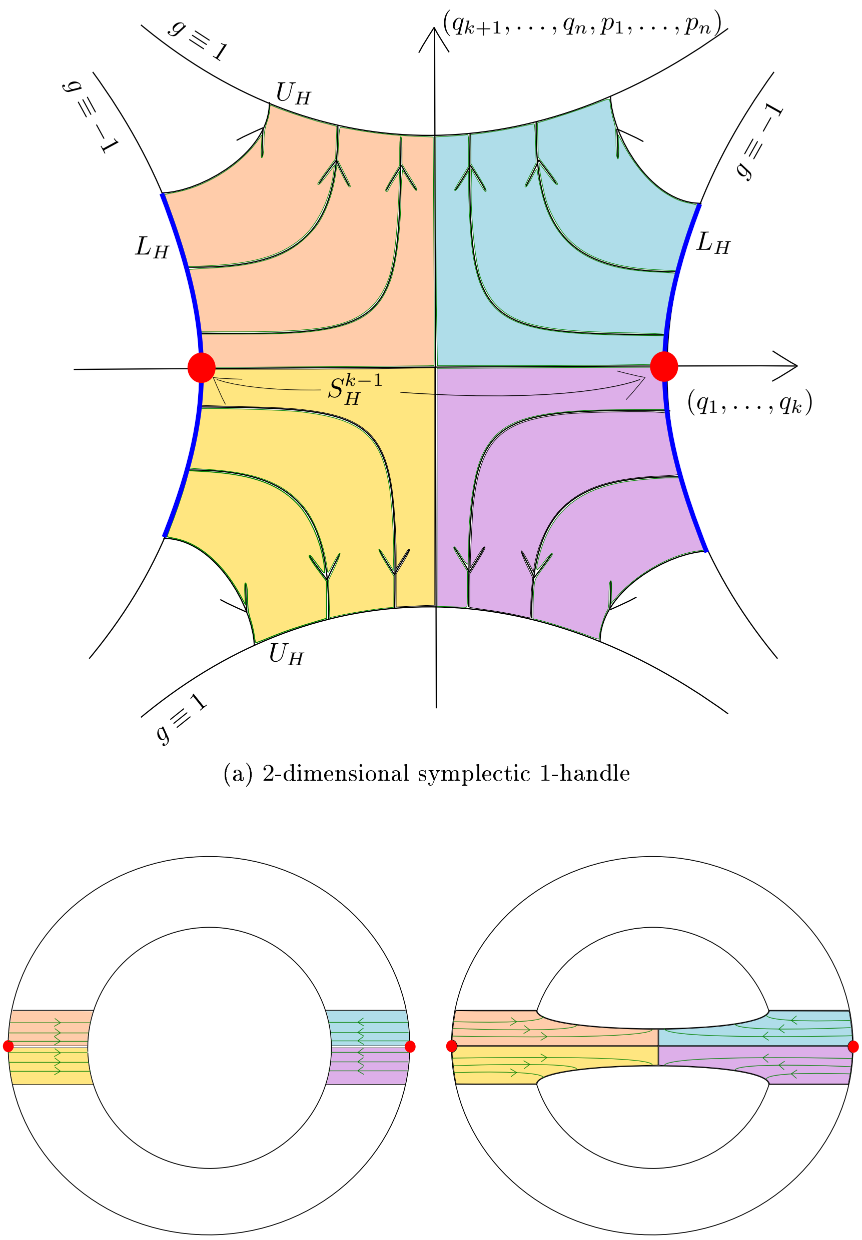
by the equivalence relation defined as follows. Let $u \in S^{k-1}$, $v \in S^{2n-k-1}$, $0 < r < 1$ and $c \in [-1,1]$. Then we identify

$$\big(c, \phi(u, rv)\big) \in [-1,1] \times \Big(N \backslash \phi\big(S^{k-1} \times \{0\}\big) \Big)$$

with such $(\mathbf{q}, \mathbf{p}) \in H$ that $g(\mathbf{q}, \mathbf{p}) = c$ and $(\mathbf{q}, \mathbf{p})$ lies on the trajectory of the flow of Y running through $\psi(u, rv)$ (recall that ψ denotes a diffeomorphism between $S^{k-1} \times D^{2n-k}$ and L_H). The identification and the resulting cobordism are shown on Figures 2.1b, 2.1c.

This identification is a symplectomorphism. Indeed, a contactomorphism between Liouville type submanifolds of symplectic manifolds extended to a diffeomorphism of their tubular neighbourhoods by sending Liouville flow lines to Liouville flow lines is a symplectomorphism (see Lemma 5.2.4 in [18]).

Note that the concave boundary component of M is contactomorphic to (N, η). The convex boundary component is the result of the contact surgery on (N, η).



(a) 2-dimensional symplectic 1-handle

(b) Identifications in $N \times [0, 1]$

(c) Symplectic handle attached

Figure 2.1: Symplectic handle attachment

Contact surgery theorem The above considerations lead to the following contact surgery theorem.

Theorem 2.2.9 (Theorem 6.2.5, [18])**.** *Let (N, η) be a contact manifold. Consider an isotropic sphere S in N with trivialized symplectic normal bundle $SN_N(S)$. The manifold N' obtained by the topological surgery along S with re-*

spect to the natural framing admits a contact form η'. The form η' coincides with η outside from the surged region. Moreover, there exists a symplectic cobordism from (N, η) to (N', η').

It turns out that for $k \leq n$ and $n \neq 2$ a symplectic structure on a $2n$-dimensional cobordism can always be extended over a k-handle attached along a convex boundary component provided that an ω-compatible almost complex structure extends.

Theorem 2.2.10 (Theorem 6.3.1, [18])**.** *Let (N, ξ) be a $(2n-1)$-dimensional contact manifold, $n \neq 2$, containing an embedded sphere S^{k-1} with a trivialization of its normal bundle defined by an embedding*

$$\phi : S^{k-1} \times D^{2n-k} \to N.$$

Assume that $1 \leq k \leq n$. Let (M, ω) be a symplectic manifold with (N, ξ) as a convex boundary component. Let $M' = M \cup_\phi H$ be the manifold obtained from M by attaching a $k-$handle H along ϕ and let N' be the new boundary component (the result of the surgery on N). If an $\omega-$compatible almost complex structure J (unique up to isotopy) extends to an almost complex structure on M', then M' carries a symplectic form ω' with an $\omega'-$compatible almost complex structure homotopic to J. Moreover, N' is a convex boundary component of (M', ω'). The induced contact structure ξ' on N' is the result of performing contact surgery along an isotropic sphere topologically isotopic to $\phi(S^{k-1} \times \{0\})$.

Theorem 2.2.10 does not hold for contact manifolds of dimension 3. In this case some additional assumptions on the framing of a 2-surgery are needed. Indeed, we need to assume that a contact framing of a Legendrian knot is -1 in order to perform a contact surgery along that knot in a way described above. However, in the case of dimension 3, one can also perform a different kind of contact 2-surgery by attaching a symplectic handle to a concave boundary component of a cobordism. In that case, we need to assume that a contact framing of a knot is $+1$.

Note that if (M, ω) is exact then (M', ω') is also exact. Indeed, a Liouville vector field on M can be extended over the handle H. This implies, in particular, that Weinstein cobordisms are exact.

Cut-and-paste view point

A contact surgery can also be seen from the cut-and-paste view point rather then by constructing a symplectic cobordism. Consider an embedded isotropic $(k-1)$-sphere together with its natural framing represented by an embedding $phi : S^{k-1} \times D^{2n-k} \hookrightarrow N$. We start by cutting out $S^{k-1} \times D^{2n-k}$. This set corresponds to the neighbourhood L_H of S_H^{k-1} in $g^{-1}(-1) \subset H$. The neighbourhoods ϕ and L_H are chosen small enough so that the collar neighbourhoods of the boundaries of the manifolds

$$N \backslash \mathrm{Int}\big(\phi(S^{k-1} \times D^{2n-k})\big) \quad \text{and} \quad g^{-1}(-1) \backslash L_H$$

are contactomorphic. It is possible by Theorem 2.2.7. Moreover, the collar neighbourhood in $g^{-1}(-1)$ can be contactomorphically identified, along the flow lines of Y, with the corresponding subset of $g^{-1}(1)$. It allows us to glue together the upper boundary $U_H \cong D^k \times S^{2n-k-1}$ of the handle H and the manifold $N \backslash \mathrm{Int}(S^{k-1} \times D^{2n-k})$ so that the contact forms glue smoothly. As a result we obtain a contact form on the surgered manifold

$$N' \cong \Big(N \backslash \mathrm{Int}\big(\phi(S^{k-1} \times D^{2n-k})\big)\Big) \cup_{\phi|_{S^{k-1} \times S^{2n-k-1}}} D^k \times S^{2n-k-1}.$$

2.3 Contact topology in dimension 3

In this Section we will focus on contact manifolds of dimension 3. The case of dimension 3 in contact topology is quite special and in many aspects it is much better recognized then the case of the higher dimensions.

A contact structure ξ on a 3-manifold N is of one of the two types: overtwisted or tight. We say that a contact structure is **overtwisted** if it contains an overtwisted disc. A contact structure is **tight** if it is not overtwisted. By an overtwisted disc we mean an embedded disc $\Delta \subset N$ such that its boundary $\partial\Delta$ is a Legendrian knot whose contact framing coincides with the surface framing.

These definitions may seem a bit strange at the first sight but their implications are very strong. One of the implications is the following theorem by Eliashberg (Theorem 1.6.1., [10]) showing that there exists a one-to-one correspondence between homotopy classes of 2-plane fields and isotopy classes of overtwisted contact structures.

Theorem 2.3.1 (Eliashberg)**.** *There exists an overtwisted contact structure in any homotopy class of* 2*-plane fields on a closed oriented* 3*-manifold. Moreover, if two overtwisted contact structures are homotopic as* 2*-plane fields, they are also isotopic as overtwisted contact structures.*

The recent result from [2], already mentioned in Subsection 2.1.2, generalizes the definition of overtwisted contact structures as well as the result above to the higher dimensions.

We are now going to recall some properties of overtwisted contact structures related with symplectic cobordisms. First of all, overtwisted contact manifolds are never fillable (see Theorem 6.5.6, [18]).

Theorem 2.3.2 (Gromov, Eliashberg)**.** *Any symplectically fillable contact structure is tight.*

Note that a tight contact manifold does not necessarily needs to be fillable neither. In fact, there are example of tight non-fillable manifolds (see [17]).

Another important fact about overtwisted manifolds is that they can be connected by Stein cobordisms with all contact manifolds (see Theorem 1.2, [16]).

Theorem 2.3.3 (Etnyre, Honda)**.** *Let (N, ξ_{ot}) be an overtwisted contact manifold. For any contact manifold (N', η') there exists a Stein cobordism from (N, ξ_{ot}) to (N', ξ').*

2.4 Open book decomposition

Open book decompositions play a crucial role in contact topology after the work by Giroux who proved a close relation between contact structures on a manifold and its open book decompositions.

In this section we are going to define open book decompositions and describe how they are related to contact structures. Moreover, we will define a modification of open books called the positive stabilization. Finally, we will describe contact surgery in terms of open books.

Definitions of open books

Let us start with defining an open book decomposition of a manifold.

Definition 2.4.1. An **open book decomposition** of a manifold N is a pair (B, π), where B is a codimension 2 submanifold of N, called the **binding** of the open book, and $\pi : N \backslash B \to S^1$ is a fibration. The compact hypersurface $P := P_\varphi := \overline{\pi^{-1}(\varphi)}$, the closure of a fibre of π, is called the **page** of the open book.

It is frequently more convenient to use the following, closely related, notion of an abstract open book.

Definition 2.4.2. An **abstract open book** is a pair (P, ϕ), where P is a compact manifold with nonempty boundary and $\phi : P \to P$ is a diffeomorphism equal to the identity close to the boundary of P. The manifold P is called the **page** of the abstract open book, its boundary $B_\phi = \partial P$ is the **binding**, while the diffeomorphism ϕ is called the **monodromy**.

Given an abstract open book (P, ϕ) one can define a mapping torus of ϕ

$$P_\phi = P \times [0,1]/\sim$$

for the equivalence relation $\sim$ given by $\big(\phi(x), 0\big) \sim (x, 1)$. Since ϕ is assumed to be equal to the identity close to the boundary of P, the boundary of the mapping torus P_ϕ is diffeomorphic to $\partial P \times S^1$. It allows us to define a manifold N_ϕ as

$$N_\phi = P_\phi \cup_{\mathrm{Id}} \big(\partial P \times D^2\big).$$

As already mentioned, there is a natural correspondence between open book decompositions and abstract open books. Indeed, for an open book decomposition (B, π) of a manifold N there exists an abstract open book (P, ϕ) such that N is diffeomorphic to N_ϕ and B is diffeomorphic to B_ϕ. Conversely, an abstract open book (P, ϕ) determines the manifold N_ϕ as well as its open book decomposition (B, π) uniquely up to a diffeomorphism (see [14] for details). In what follows we will use the two notions interchangeably.

It was proved by Quinn (see Theorem 1.1 in [34]) that every odd dimesional closed manifold admits an open book decomposition.

Contact open books

Let us now define a correspondence between contact structures on a manifold and its open book decompositions.

Definition 2.4.3. Let (N, ξ) be a contact manifold. We say that the contact structure ξ is supported by an open book decomposition (B, π), if it is defined by a contact form η such that

(i) $\mathrm{d}\eta$ induces a positive symplectic structure on each page of the open book,

(ii) η induces a positive contact form on the binding B.

Let (P, ω) be a compact Weinstein manifold and let ϕ be a symplectomorphism of (P, ω) equal to identity close to the boundary (any compact Weinstein manifold, being exact, has a nonempty boundary). Then, one can construct by an explicit formula a contact form on the manifold N_ϕ such that it is supported by the open book (P, ϕ) (see for example Section 2.2 in [38]).

Giroux proved that the converse statement is also true.

Theorem 2.4.4 (Giroux, [21])**.** *Any contact structure on a closed manifold is supported by an open book decomposition with a Weinstein page.*

Let us denote by $\mathrm{Open}(P, \phi)$ the contact manifold determined by the open book (P, ϕ) as described above.

Contact surgery in terms of open books

The operation of contact surgery introduced in Subsection 2.2.3 can be described in terms of supporting open books. The description differs depending whether the surgery is critical (of the maximal index) or subcritical.

Lemma 2.4.5. *Let P be a Weinstein manifold of dimension $2n-2$ and let ϕ be a symplectomorphism of P equal to identity close to the boundary. Consider the contact manifold Open(P, ϕ).*

(a) Let $k < n$ and let S be an isotropic $(k-1)$-sphere contained in the binding $B = \partial P$ of the open book with a trivialized symplectic normal bundle. Then the result of the contact surgery on N along S is contactomorphic to Open$(P \cup_S k$-handle, $\phi \cup_S Id)$, i.e. the open book with a k-handle attached to its pages along S. Note that here we view S as an isotropic submanifold of N with the induced trivialization of the symplectic normal bundle.

(b) Let S be a Legendrian sphere in Open(P, ϕ) and a Lagrangian sphere in P. Then the result of the contact surgery on N along S is contactomorphic to Open$(P, \phi \circ \tau_S)$, where τ_S is a right-handed Dehn twist of P along S.

By Lemma 4.2. in [38], the position of S, as in the assumptions of the lemma above, is generic.

Note that Dehn twists are usually being defined for surfaces. However, this notion can be extended and we can define Dehn twists on any symplectic manifold along an embedded Lagrangian sphere. The detailed description of this extension together with the proof of Lemma 2.4.5 can be found in [38].

Observe that from Lemma 2.4.5 it follows that a contact manifold is Weinstein (and, in consequence, Stein) fillable if and only if it is supported by an open book with a Weinstein page and monodromy being a composition of right-handed Dehn twists.

Positive stabilizations of open books

Open books supporting a given contact structure are not unique. Indeed, in particular, the contact structure is invariant under the modification of open books called a positive stabilization. Let us now define the operation of a positive stabilization of an open book.

Let P be a Weinstein manifold of dimension $2n-2$ with an embedded Lagrangian disc L, whose boundary is a Legendrian sphere in ∂P. Let ϕ be a symplectomorphism of P equal to identity close to the boundary. Consider the abstract open book (P, ϕ). A **positive stabilization** of (P, ϕ) is the abstract open book $(P', \phi' \circ \tau_{L'})$, where $P' = P \cup_{\partial L} n$-handle, $\phi' = \phi \cup_{\partial L} \mathrm{Id}$ and L' is a union of L and the core of the n-handle.

The fact that $\mathrm{Open}(P, \phi)$ is contactomorphic with $\mathrm{Open}(P', \phi' \circ \tau_{L'})$ is due to Giroux [21]. It can be easily proved using Lemma 2.4.5 (see [38]).

Giroux showed, moreover, that in the case of dimension 3 positive stabilizations suffice to define a one-to-one correspondence between contact structures and open books.

Theorem 2.4.6 (Giroux, [21])**.** *On a* 3*-dimensional oriented manifold there exists a one-to-one correspondence between contact structures up to an isotopy and open book decompositions up to a positive stabilization.*

Chapter 3

Relations on the set of contact forms

In this chapter we are going to study relations on the set of contact forms on a given manifold. We are going to examine properties of the relations as well as dependencies between them. All the facts stated in this chapter are well known and most of them can be found in the literature.

We are going to organize the chapter in the following way.

In Section 3.1 we introduce the relations. Some of the relations, like isotopy, cobordism or exact cobordism are already well known and examined.

In Section 3.2 we examine properties of the defined relations, while in Section 3.3 we study dependencies between them.

3.1 Definitions of relations on the set of contact forms

Let N be a closed contact manifold of dimension $2n-1$ and let η_0 and η_1 be contact forms on N. Consider the following relations between that forms. We say that η_0 and η_1 are:

- *presymplectically homotopic*, if there exists a path of presymplectic 1-forms $(\eta_t)_{t\in[0,1]}$ between η_0 and η_1, i.e. a path of forms such that $(\mathrm{d}\eta_t)^{n-1} \neq 0$ for every $t \in [0,1]$;
- *isotopic*, if there exists a path of contact forms $(\eta_t)_{t\in[0,1]}$ between η_0 and η_1;
- *concordant*, if there exists a symplectic concordance from $(N, \ker\eta_0)$ to $(N, \ker\eta_1)$ (see Subsection 2.2.2 for a definition of symplectic concordance);
- *exactly concordant*, if there exists an exact symplectic concordance from $(N, \ker\eta_0)$ to $(N, \ker\eta_1)$;
- *linked*, if there exists a contact form η_3 on N such that η_1 and η_2 are both exactly concordant to η_3;

- *cobordant*, if there exists a symplectic cobordism from $(N, \ker \eta_0)$ to $(N, \ker \eta_1)$;
- *exactly cobordant*, if there exists an exact symplectic cobordism from $(N, \ker \eta_0)$ to $(N, \ker \eta_1)$.

In Subsection 5.1.2, we additionally study the relation of an SSF cobordism.

3.2 Properties of the relations

Below we examine basic properties of the relations introduced in the previous section. More specifically, we ask whether the relations are symmetric or transitive or if they are equivalence relations.

Proposition 3.2.1.

(a) Presymplectic homotopy and isotopy are equivalence relations.

(b) Concordance and exact concordance are reflexive and transitive.

(c) Cobordism and exact cobordism are reflexive and transitive but they are not symmetric.

(d) Linking is a symmetric and reflexive relation.

Proof.

(a) The relations of presymplectic homotopy and isotopy are defined in terms of paths and the relation of being in the same connected component of a given topological space is an equivalence relation.

(b) The relations are reflexive since the trivial symplectic cobordism of a contact manifold (N, η) is an exact concordance.

Transitivity follows from the well known fact that there exists a structure of a symplectic cobordism on a composition of symplectic cobordisms (see Proposition 5.2.5 in [18]). We will recall the proof of that fact to make sure that it works in the case of concordances.

Let us consider the symplectic concordances $(M_0 \cong [0,1] \times N, \omega_0)$ and $(M_1 \cong [0,1] \times N, \omega_1)$ form (N, η_0) to (N, η) and from (N, η') to (N, η_1) such that $\ker \eta = \ker \eta'$. After possibly multiplying ω_0 by a positive constant, we can find a positive function $f : N \to \mathbb{R}$ such that $\eta' = e^f \eta$. Since (N, η) is a convex component boundary of (M_0, ω_0), there exists a collar neighbourhood of (N, η) in (M_0, ω_0) of the form

$$\Big((-\epsilon, 0] \times N, \mathrm{d}(e^t \eta)\Big).$$

Similarly, there exists a collar neighbourhood of the concave boundary (N, η') of (M_1, ω_1) having the form

$$\Big([0, \epsilon) \times N, \mathrm{d}(e^t \eta') = \mathrm{d}(e^{t+f} \eta)\Big).$$

Consider a manifold $M = \{(t, x) \mid 0 \le t \le f(x)\}$ with a symplectic form $\mathrm{d}(e^t \eta)$. Then $M_0 \cup M \cup M_1$ is a symplectic cobordism from (N, η_0) to

(N, η_1). The manifolds M_0, M and M_1 are glued along their boundary components via

$$M_0 \ni (0, x) \sim (0, x) \in M \quad \text{and} \quad M \ni (f(x), x) \sim (0, x) \in M_1.$$

Note that $(M, \mathrm{d}(e^t\eta))$ is actually an exact symplectic concordance. It follows that if M_0 and M_1 are (exact) symplectic concordances, then $M_0 \cup M \cup M_1$ is also a symplectic concordance (or an exact symplectic concordance).

(c) Analogously as in the case of concordances, cobordism and exact cobordism are reflexive and transitive relations.

Now, we will show that the relation of (exact) cobordism is not symmetric. Consider any 3-dimensional overtwisted contact manifold (N, η_{ot}). Then there exists an exact cobordism from (N, η_{ot}) to $(S^3, \eta_{\mathrm{st}})$ (see Theorem 2.3.3). On the other hand, since (N, η_{ot}) is not fillable (see Theorem 2.3.2), there cannot exist a cobordism from $(S^3, \eta_{\mathrm{st}})$ to (N, η_{ot}). Indeed, by composing such cobordism with a filling of $(S^3, \eta_{\mathrm{st}})$, we would get a filling of (N, η_{ot}).

(d) It is obvious that linking is a symmetric relation. Since the relation of exact concordance is reflexive, linking is also reflexive.

□

3.3 Dependences between the relations

We are now going to examine dependencies between the introduced relations. Let us start by showing that two contact forms are concordant if and only if they are exactly concordant.

Proposition 3.3.1. *Let N be a closed manifold. Contact forms η_0 and η_1 on M are concordant if and only if they are exactly concordant.*

Proof. It is obvious that exactly concordant forms are concordant.

Assume that the forms η_0 and η_1 are concordant. Let (M, ω) be a symplectic concordance between them. The form ω is exact in a collar neighbourhood $\mathcal{N}(\partial M)$ of the boundary ∂M of M. Take any $\sigma \in H_2(M, \mathbb{R})$. Since M is diffeomorphic to $N \times [0, 1]$, we can choose a representation s of σ, $[s] = \sigma$, such that $s \subset \mathcal{N}(\partial M)$. It follows that

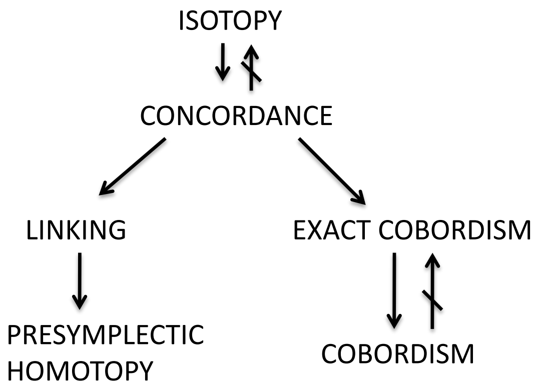
$$[\omega](\sigma) = \int_s \omega = 0.$$

Thus, $[\omega] = 0$ as an element of the de Rham cohomology group $H^2_{dR}(M, \mathbb{R})$. It follows that $\omega = \mathrm{d}\lambda$ for some 1-form λ and (M, ω) is an exact concordance. □

Since terms concordance and exact concordance are equivalent from now on we will use the shorter one - concordance.

Now, let us examine the dependencies between the remaining relations.

Proposition 3.3.2. *The implications indicated in the figure below hold.*



Proof. The implications

- concordance ⇒ linking
- concordance ⇒ exact cobordism
- exact cobordism ⇒ cobordism

are obvious.

The proof of the remaining implications will be divided into lemmas.

Lemma 3.3.3. *Let N^{2n-1} be a manifold with contact forms η_0, η_1. If η_0, η_1 are isotopic then they are also concordant.*

Proof. Let (η_t) be an isotopy between η_0 and η_1. We are going to define a symplectic concordance $(N\times[0,\frac{1}{\epsilon}], \mathrm{d}\lambda)$ from (N, η_0) to $(N, e^{\frac{1}{\epsilon}}\eta_1)$ for a small constant ϵ. Let

$$\lambda = e^t \eta_{\epsilon t}.$$

Then

$$\mathrm{d}\lambda = e^t\Big(\mathrm{d}t \wedge \eta_{\epsilon t} + \epsilon\, \mathrm{d}t \wedge \big(\frac{\mathrm{d}}{\mathrm{d}s}\eta_s \mid_{s=\epsilon t}\big) + \mathrm{d}\eta_{\epsilon t}\Big).$$

For ϵ sufficiently small the second summand is insignificant. Thus, one can approximate

$$(\mathrm{d}\lambda)^n \cong n e^{nt} \mathrm{d}t \wedge \eta_{\epsilon t} \wedge (\mathrm{d}\eta_{\epsilon t})^{n-1}.$$

Since the forms η_t are contact, we have that $\eta_{\epsilon t} \wedge (\mathrm{d}\eta_{\epsilon t})^{n-1} \neq 0$. Therefore, $(\mathrm{d}\lambda)^n \neq 0$. We can assume that the isotopy (η_t) is constant for t close to 0 and 1. As a result, ∂_t is a Liouville vector field close to the boundary of $N \times [0, \frac{1}{\epsilon}]$.

In consequence, $(N \times [0, \frac{1}{\epsilon}], \mathrm{d}\lambda)$ is a symplectic concordance from (N, η_0) to $(N, e^{\frac{1}{\epsilon}}\eta_1)$. □

Lemma 3.3.4. *Let N be a manifold with contact forms η_0, η_1. If η_0, η_1 are linked, then they are presymplectically homotopic.*

Proof. Since presymplectic homotopy is an equivalence relation, it is sufficient to show that a concordance between two forms implies presymplectic homotopy between them. Let $(N \times [0, 1], \omega)$ be a symplectic concordance from η_0 to η_1.

From Proposition 3.3.1 we have that ω is exact. Let λ be a 1-form on $N \times [0,1]$ such that $\omega = \mathrm{d}\lambda$. We define a path of forms η_t on N between η_0 and η_1 by

$$\eta_t = \lambda \mid_{T(N\times\{t\})} .$$

Since $\mathrm{d}\lambda$ is a symplectic form on $N \times [0,1]$, the forms $\mathrm{d}\eta_t$ have the maximal rank on N and the path $(\eta_t)_{t\in[0,1]}$ is a presymplectic homotopy from η_0 to η_1. □

Lemma 3.3.5. *There exists a manifold N and contact forms η_0 and η_1 on it such that contact manifolds (N, η_0) and (N, η_1) are symplectically cobordant but there exists no exact symplectic cobordism from (N, η_0) to (N, η_1).*

Proof. The lemma can be proved using notions of the planar and the algebraic torsions (see [40] and [26] respectively).

In [26], the authors use symplectic field theory to introduce the notion of algebraic torsion (AT) of a contact manifold. They show that, if there exists an exact symplectic cobordism from (N_0, η_0) to (N_1, η_1), then

$$\mathrm{AT}(N_0, \eta_0) \leq \mathrm{AT}(N_1, \eta_1).$$

Moreover, for any integer $k \geq 1$ (and $g > k$) they construct a contact manifold (V_g, ξ_k) with $\mathrm{AT}(V_g, \xi_k) = k - 1$.

Planar torsion is introduced in [40] by means of blown up summed open books (a generalization of open book decompositions also defined in [40]). In Subsection 6.3.3, we will describe planar torsion in more detail. The author shows that any two contact manifolds with finite planar torsion are symplectically cobordant. The manifolds (V_g, ξ_k) have planar torsion $k - 1$.

Let $0 < k_1 < k_0 < g$ be natural numbers. It follows that the manifolds (V_g, ξ_{k_0}) and (V_g, ξ_{k_1}) are symplectically cobordant but there exists no exact symplectic cobordism from (V_g, ξ_{k_0}) to (V_g, ξ_{k_1}). □

Lemma 3.3.6. *There exists a manifold N and non-isotopic contact forms η_0 and η_1 on it such that contact manifolds (N, η_0) and (N, η_1) are concordant.*

Proof. McLean in [31] has constructed infinitely many non-symplectomorphic Stein structures on $\mathbb{R}^{2k}$ for $k > 3$. He points out that some of the structures induce non-contactomorphic (and hence non-isotopic) contact structures on the boundary of the unit ball (see the discussion at page 10 of [31]). Let us consider the unit ball (B^{2k}, ω) with a Stein symplectic structure ω such that the contact structure ξ induced on the boundary S^{2k-1} is not contactomorphic to the standard contact structure on the sphere. Let us consider an open ball $\tilde{B}$ around the origin small enough so that $(\tilde{B}, \omega \mid_{\tilde{B}})$ is symplectomorphic to the standard symplectic ball. Such a ball can be found by the Darboux theorem (see Theorem 2.1.9). Then $B \backslash \tilde{B}$ is a concordance from the standard structure ξ_{st} to the exotic structure ξ. □

□

Chapter 4

Classical strong symplectic folds

In this chapter, we define strong symplectic fold (SSF) structures and we give examples of SSF manifolds. Moreover, we consider the existence question for SSF structures and we present two constructions of SSF manifolds: using surgery technique and via the notion of convex hypersurfaces in contact manifolds. Finally, we indicate applications of SSF structures in contact topology.

We organize the chapter in the following way.

In Section 4.1, we define a strong symplectic fold structure on a closed oriented manifold and, in Section 4.2, we give examples of SSF manifolds: oriented surfaces and doubles of exact symplectic manifolds.

In Section 4.3, we present the existence results in the 4-dimensional case (see Theorem 5.2 in [1] and Theorem 4 in [19]).

Section 4.4 describes surgery construction of SSF-manifolds. We consider the notion of cobordism between manifolds with boundary and a decomposition of such cobordisms into handles and half-handles. These concepts were introduced in [25]. The theory of handle decompositions of manifolds with boundary was further explored and applied in [23]. Here we use the notation proposed in [3]. We extend the presented ideas to the case of symplectic manifolds with contact boundaries. Finally, we use the introduced tools to examine surgeries on SSF manifolds and prove the invariance of the existence of SSF structures on surgeries of indices up to the half of the dimension of a manifold.

In Section 4.5, we prove that a convex hypersurface in a contact manifold admits an SSF structure. We note that this implies the existence of an SSF structure on boundaries of tubular neighbourhoods of Legendrian submanifolds.

In Section 4.6, we present a new construction of a controllable contact form on the product of an almost complex manifold with the real line. Then, a similar arguments as used in the case of convex hypersurfaces in Section 4.5, allow us to find on this manifold a structure related to an SSF structure. More preciselly, we decompose the manifold into a sum of two exact symplectic pieces. This can be treated as the first step in a proof of the existence of SSF structures on almost complex manifolds.

Finally, in Section 4.7, we show a construction of contact manifolds using SSF structures as proposed in [19].

4.1 Definition of strong symplectic folds

In the following section, we will state a definition of strong symplectic fold structures as well as some related notations which will be used throughout the proceeding chapters.

Definition 4.1.1. A **strong symplectic fold (SSF) structure** on a closed oriented manifold M^{2n} is a tuple $(M_0, M_1, \omega_0, \omega_1)$ defining a decomposition of M into a sum of two manifolds with a common boundary $\partial M_0 = \partial M_1 = N$

$$M = M_0 \cup_N \overline{M_1}.$$

The forms ω_0, ω_1 are exact symplectic forms on M_0 and M_1 respectively such that (M_0, ω_0) and (M_1, ω_1) are symplectic fillings of the same contact manifold (N, η).

If contact and symplectic structures are understood, we call the decomposition $M = M_0 \cup_N \overline{M_1}$ an SSF structure on M.

The contact manifold (N, η), as in the definition above, is the **separating hypersurface** of the SSF structure, while the exact symplectic manifolds (M_0, ω_0) and (M_1, ω_1) are the **symplectic parts of the SSF**. The decomposition $M = M_0 \cup_N \overline{M_1}$ is an **SSF decomposition**. By an **SSF manifold** we mean a manifold admitting an SSF structure.

Note that we do not require the symplectic parts of an SSF structure to be connected.

Let us now define a version of an SSF structure, where we make an additional assumption restricting the Morse type of the symplectic parts of the decomposition.

Definition 4.1.2. We say that a manifold with boundary M has the **Morse type** k if there exists a Morse function $f : M \to [0, 1]$ with $f^{-1}(1) = \partial M$ such that f does not have any critical points of indices higher than k.

An SSF structure $(M_0, M_1, \omega_0, \omega_1)$ of a $2n$-dimensional manifold such that its parts M_0 and M_1 have the Morse type n will be called a **restricted Morse type SSF structure**.

One of the goals of this thesis is to study the existence question for the SSF structures. An obvious necessary condition is the existence of a stably almost complex structure.

Proposition 4.1.3. *Any SSF manifold admits a stably almost complex structure.*

Proof. Let $(M_0, M_1, \omega_0, \omega_1)$ be an SSF structure on a manifold M. Then the symplectic structures ω_0 and ω_1 induce almost complex structures J_0 and J_1 on M_0 and $\overline{M}_1$ respectively. As the manifolds (M_0, ω_0) and (M_1, ω_1) are symplectic fillings of the same contact manifold (N, η), one can choose J_0 and J_1 such that they agree on $\xi = \ker \eta$ and $J_0(R) = -J_1(R)$ for the Reeb vector field of η. Then, the complex structures $J_0 \times i$ and $J_1 \times (-i)$, understood as structures on $\mathrm{T}(N \times \mathbb{R}) \times \mathbb{C}$, are isotopic. Indeed, we can define an isotopy between them as $J_t \mid_\xi := J_0 \mid_\xi = J_1 \mid_\xi$ and

$$J_t(R) := N \cos(t\pi) + i \sin(t\pi), \quad J_t(1) := i \cos(t\pi) - N \sin(t\pi)$$

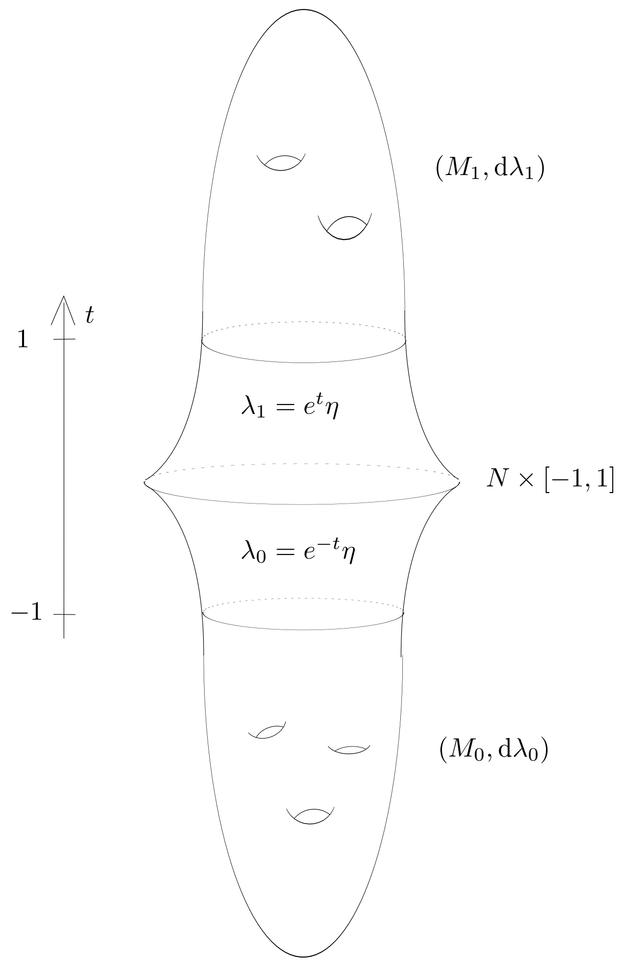


Figure 4.1: Strong symplectic fold

for $N = J_0(R) = -J_1(R)$. Let us identify

$$\mathrm{T}M \times \mathbb{C} \cong \big(\mathrm{T}M_0 \times \mathbb{C}\big) \cup \big(\mathrm{T}(N \times [0,1]) \times \mathbb{C}\big) \cup \big(\mathrm{T}\overline{M}_1 \times \mathbb{C}\big).$$

We can define a complex structure J on $\mathrm{T}M \times \mathbb{C}$ as

$$J = \begin{cases} J_0 \times i & \text{on } \mathrm{T}M_0 \times \mathbb{C}, \\ J_t & \text{on } \mathrm{T}(N \times [0,1]) \times \mathbb{C}, \\ J_1 \times (-i) & \text{on } \mathrm{T}\overline{M}_1 \times \mathbb{C}, \end{cases}$$

where t is a parameter of $[0,1]$. J is a stably almost complex structure on M. □

A structure J as described above will be called a **stably almost complex structure compatible with the SSF structure**.

The following theorem says that every stably almost complex manifold of even dimension equal at least 6 admits a similar but weaker decomposition.

Theorem 4.1.4. *Suppose that (M, J) is a closed stably almost complex manifold of dimension $2n$ for $n > 2$. Then there exist exact symplectic fillings (M_0, ω_0) and (M_1, ω_1) of contact manifolds (N, η_0) and (N, η_1) respectively such that M is diffeomorphic to $M_0 \cup_N \overline{M_1}$ and the forms η_0 and η_1 are presymplectically homotopic.*

Proof. Let $f : M \to [-1, 1]$ be a Morse function such that all critical points of order less then n are in $f^{-1}([-1, 0))$, while all critical points of order at least n are in $f^{-1}((0, 1])$. Let us define

$$M_0 = f^{-1}([-1, 0]), \quad \overline{M_1} = f^{-1}([0, 1]).$$

Stably almost complex structure J on M induces almost complex structures J_0 and J_1 on manifolds M_0 and M_1 respectively. By Theorem 2.2.10, an almost complex manifold M_i decomposable into handles of indices at most n admits an exact symplectic form ω_i compatible with an almost complex structure homotopic to J_i. To simplify the notation, let us keep denoting the compatible almost complex structure by J_i. The symplectic form ω_i induces a coorientable contact structure $\xi_i = \ker \eta_i$ on $N = \partial M_i = f^{-1}(0)$.

Note that $\bar{J_0} = J_0|_N, \bar{J_1} = J_1|_N$ are homotopic complex structures on $TN \oplus \varepsilon^1$, where ε^1 denotes the trivial 1-dimensional vector bundle over N. Let $\overline{J}_t$ be a homotopy between $\bar{J_0}$ and $\bar{J_1}$. It corresponds to a homotopy $\bar{\omega}_t$ of nondegenerate 2−forms on $TN \oplus \varepsilon^1$ between $\bar{\omega}_0 = \omega_0|_N$ and $\bar{\omega}_1 = \omega_1|_N$. Forms $\bar{\omega}_t$ remain nondegenerate, when restricted to the tangent bundle TN of N. By [30] (see also Theorem 10.4.1 in [11]) the space of all nondegenerate 2−forms on an odd-dimensional manifold is homotopically equivalent to the space of closed nondegenerate 2−forms representing a given homology class. Thus, we can choose a homotopy between $\bar{\omega}_0$ and $\bar{\omega}_1$ in such a way that $\bar{\omega}_t = \mathrm{d}\eta_t$ for a homotopy of 1-forms η_t. The path (η_t) is a presymplectic homotopy from η_0 to η_1. □

4.2 Examples of SSF manifolds

Surfaces

Let Σ be an oriented surface and let

$$\Sigma = \Sigma_0 \cup_\Gamma \overline{\Sigma_1}$$

be any decomposition of Σ into a union of two surfaces with a common boundary. Note that $\Gamma \subset \Sigma$ is a union of disjoint loops in Σ.

The following simple fact shows that there are many SSF structures in dimension 2. In fact, any decomposition such as above defines an SSF structure.

Proposition 4.2.1. *Let Σ be a closed oriented surface decomposed into a union $\Sigma_0 \cup_\Gamma \overline{\Sigma}_1$ of surfaces with boundary. If each connected component of Σ_0 and Σ_1 has non-empty boundary, then there exist exact symplectic structures on Σ_0 and Σ_1 such that $\Sigma = \Sigma_0 \cup_\Gamma \overline{\Sigma_1}$ is an SSF structure on Σ.*

Proof. Consider a handle decomposition of the surface Σ_i consisting of handles of indices 0 and 1. Since the indices of the handles are at most equal to the half of the dimension of Σ_i, the handles can be endowed with the symplectic handles structure (see Theorem 2.2.10). It follows that there exists an exact symplectic structure ω_i on Σ_i convex on the boundary Γ. Since there exists a unique up to isotopy contact form on Γ compatible with a given orientation, the contact structures induced on the boundaries of Σ_0 and Σ_1 can be assumed equal and, consequently, $(\Sigma_0, \Sigma_1, \omega_0, \omega_1)$ is an SSF structure on Σ. $\square$

Doubles

Let us start by defining a double of a manifold.

Definition 4.2.2. Let Q be a compact manifold with boundary. A **double** of Q is the manifold

$$Q \times \{0, 1\}/\sim,$$

where $(x, 0) \sim (x, 1)$ for $x \in \partial Q$. We denote the double of Q by $Q \cup \overline{Q}$.

We can, similarly, define a twisted double of a manifold with boundary.

Definition 4.2.3. Consider a compact manifold Q with boundary and a diffeomorphism $\alpha : \partial Q \to \partial Q$. A **twisted double** of Q is the manifold

$$Q \times \{0, 1\}/\sim,$$

where $(x, 0) \sim (\alpha(x), 1)$ for $x \in \partial Q$. We denote such a twisted double of Q by $Q \cup_\alpha \overline{Q}$.

Doubles are natural candidates for strong symplectic folds. Let M^{2n} be a double $M = Q \cup \overline{Q}$ of an exact symplectic filling Q (e.g. an almost complex manifold of the Morse type n). Denote by η the contact form induced on the boundary. Since M is a union of two copies of Q glued together in a trivial way, the contact structures on boundaries of those copies agree. It follows that $M = Q \cup \overline{Q}$ is an SSF structure on M.

Similarly, if M is a twisted double $M = Q \cup_\alpha \overline{Q}$ and α is a contactomorphism of $(\partial Q, \eta)$, then M admits an SSF structure.

4.3 Existence results in dimension 4

The existence of SSF structures on 4-dimensional manifolds was first proved by Baykur in [1]. In fact his result is even stronger as the symplectic parts of the SSF structures that he constructs are Stein manifolds.

Theorem 4.3.1 (Theorem 5.2, [1])**.** *Let M be a closed oriented 4-manifold. Then it admits an SSF structure $M = M_0 \cup \overline{M_1}$ such that M_0 and M_1 are Stein manifolds.*

Sketch of proof. Let $M = M_0' \cup \overline{M_1'}$ be a decomposition of M into a union of two submanifolds with the common boundary $\partial M_0' = \partial M_1'$ such that M_0' and M_1' have the Morse type 2.

It was shown in [24] that any 4-dimensional 2-handlebody admits an allowable achiral Lefschetz fibration over $\mathbb{D}^2$. By using a stabilization procedure described in [15] one can match the open books induced on the boundaries of the Lefschetz fibrations on M_0' and M_1'. It can be done in such a way that all the vanishing cycles introduced by stabilizations are non-separating so the new Lefschetz fibrations are still allowable.

By gluing the two fibrations together we obtain an achiral Lefschetz fibration f over S^2 with closed fibres.

We proceed by decomposing S^2 into a union of two discs D_+ and D_- such that D_+ contains only positive and D_- only negative critical values. Define $M_0 = f^{-1}(D_+)$ and $\overline{M_1} = f^{-1}(D_-)$. It follows that f is a positive allowable Lefschetz fibration on M_0 and M_1. The latter comes from a negative allowable Lefschetz fibration on $\overline{M_1}$. This can be done in such a way that the open books on the boundaries of the Lefschetz fibrations continue to coincide.

By the result of Loi-Piergallini (see [27]) a 4-dimensional manifold is Stein if and only if it admits a positive allowable Lefschetz fibration. This give us Stein structures on M_0 and M_1. As discussed in Section 2.4, since the open books on the boundaries coincide, the induced contact structures are isotopic. □

An alternative proof of the existence of SSF structures in dimension 4 using entirely different techniques was presented by Geiges and Stipsicz in [19]. The proof relies on the Donaldson's result (see [9]) that the Poincaré dual of the cohomology of an appropriately scaled symplectic form can be represented by a 2-dimensional connected symplectic submanifold. This approach allows an additional control of the homotopy type of the corresponding contact structures on products with S^1 (We will define such contact structures in 4.7).

4.4 Surgery construction in high dimensions

Both of the two approaches presented in Section 4.3 to the proof of the existence of SSF structures in dimension 4 use techniques that are specific to this dimension and therefore the proofs does not allow extension to higher dimensions.

In this section, we are going to describe a construction of SSF manifolds using methods of contact surgery and symplectic handle attachment, which can be applied in higher dimensions.

In Subsection 4.4.1, we recall the notions of half-handles and cobordisms between manifolds with boundary in the topological setting. In Subsection 4.4.2, we extend these notions to the world of symplectic manifolds and define half-surgery on symplectic cobordisms. Finally, in Subsection 4.4.3, we define surgery on SSF manifolds and prove that the existence of SSF structures is invariant under surgeries of indices up to the half of the dimension of a manifold.

4.4.1 Cobordism between manifolds with boundary and half-handles

In this subsection, we describe the notion of cobordisms between manifolds with boundaries as well as their handle and half-handle decomposition. These concepts were introduced in [25] and applied in [23]. Here we use the notation from [3].

Cobordism between manifolds with boundary

Let M_0 and M_1 be compact oriented n-manifolds with (possibly empty) boundaries N_0 and N_1 respectively. A **cobordism** from (M_0, N_0) to (M_1, N_1) is a pair (W, Y) such that

- W is a compact oriented $(n+1)$-manifold with $\partial W = \overline{M}_0 \cup Y \cup M_1$,
- $\overline{M}_0 \cap M_1 = \emptyset$,
- Y is a cobordism between closed manifolds from N_0 to N_1.

Note that W could be regarded as a manifold with corners along N_0 and N_1. However, sometimes it will be convenient for us to assume that the corners are smoothed and treat W as a manifold with boundary. We will omit a detailed discussion here.

Half-handles

As described in Chapter 2, a cobordism between closed manifolds can be decomposed into handles. To deal with cobordisms between manifolds with boundaries we additionally need to use a new notion of handles – **half-handles** (see [3]). In order to avoid confusion, in the current Subsection we will call the handles defined in Chapter 2 the **full handles**.

We distinguish two types of half-handles: **right half-handles** and **left half-handles** depending on which disc component of a full handle $H = D^k \times D^{n+1-k}$ is cut in half.

Before giving a definition of a half-handle we introduce the following notation for subsets of the k-dimensional disc $D^k = \{x_1^2 + \ldots + x_k^2 \leq 1\}$. See Figure 4.2 for an example in dimension 2.

$$\begin{aligned} D_+^k &= D^k \cap \{x_1 \geq 0\} \\ D_0^{k-1} &= D^k \cap \{x_1 = 0\} \\ S_+^{k-1} &= \partial D^k \cap \{x_1 \geq 0\} \\ S_0^{k-2} &= \partial D^k \cap \{x_1 = 0\} \end{aligned}$$

We see that $\partial D_+^k = D_0^{k-1} \cup_{S_0^{k-2}} S_+^{k-1}$.

Definition 4.4.1.

(a) For $1 \leq k \leq n$ an $(n+1)$**-dimensional right half-handle of index** k is the manifold $H_{\text{right}} = D^k \times D_+^{n+1-k}$ with boundary $\partial H_{\text{right}} = C \cup h \cup G$, where

$$C = S^{k-1} \times D_+^{n+1-k}, \quad h = D^k \times D_0^{n-k}, \quad G = D^k \times S_+^{n-k}.$$

(b) For $2 \leq k \leq n+1$ an $(n+1)$**-dimensional left half-handle of index** k is the manifold $H_{\text{left}} = D_+^k \times D^{n+1-k}$ with boundary $\partial H_{\text{left}} = C \cup h \cup G$, where

$$C = S_+^{k-1} \times D^{n+1-k}, \quad h = D_0^{k-1} \times D^{n+1-k}, \quad G = D_+^k \times S^{n-k}.$$

We denote

$$c = C \cap h, \quad g = G \cap h.$$

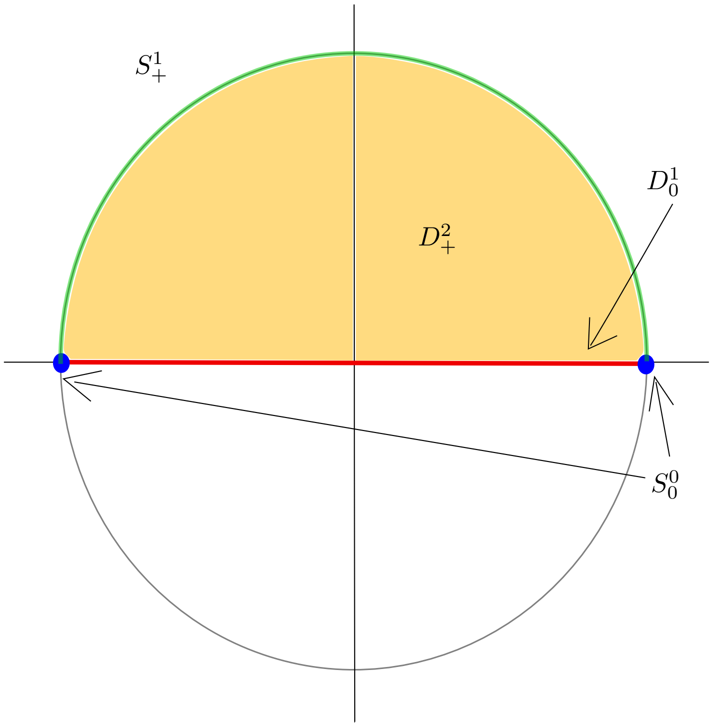


Figure 4.2: Subsets D^2_+, D^1_0, S^1_+ and S^0_0 of 2-dimensional disc D^2

Figure 4.3 depicts examples of right and left half-handles.

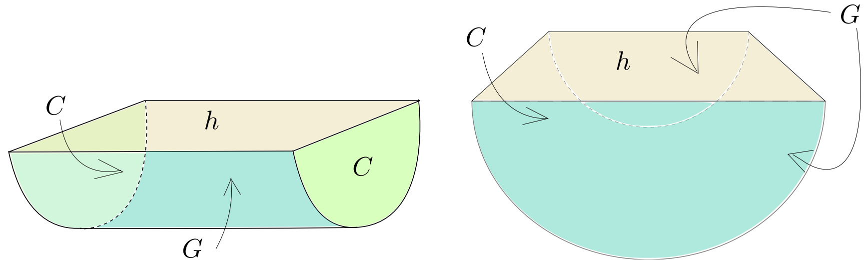
Now let us describe the attachment of half-handles. Let (W, Y) be a cobordism between n-dimensional manifolds with boundary (M_0, N_0) and (M_1, N_1). Consider an $(n+1)$-dimensional (right or left) half k-handle H. Given an embedding $\phi : (C, c) \to (M_1, N_1)$ we define a new cobordism (W', Y') between (M_0, N_0) and (M'_1, N'_1) by

$$W' = W \cup_\phi H, \quad Y' = Y \cup_{\phi|_c} h,$$
$$M'_1 = (M_1 \backslash \phi(C)) \cup_{\phi|_{\partial C = \partial G}} G, \quad N'_1 = (N_1 \backslash \phi(c)) \cup_{\phi|_{\partial c = \partial g}} g.$$

Figure 4.4 depicts examples of attachment of right and left half-handles.

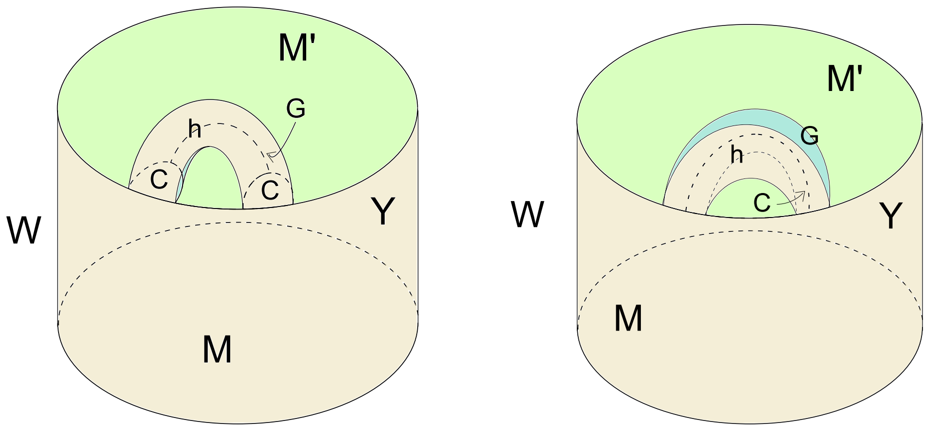
Remark 4.4.2. Observe that for the $(n+1)$-dimensional right half k-handle H, h is a full k-handle of dimension n with boundary $\partial h = c \cup g$. Moreover, if (W', Y') is the result of the attachment of H to a cobordism (W, Y) along $\phi : (C, c) \to (M_1, N_1)$, then Y' is the result of the attachment of h to Y along $\phi\,|_c \colon c \to N_1$. Finally, N'_1 is obtained from N_1 by a k-surgery.

On the other hand, if H is a left half k-handle, h is a full $(k-1)$-handle of dimension n with boundary $\partial h = c \cup g$. Moreover, Y' is the result of the attachment of h to Y along $\phi\,|_c \colon c \to N_1$ and N'_1 is obtained from N_1 by a $(k-1)$-surgery.



(a) 3-dimensional right half 1-handle (b) 3-dimensional left half 2-handle

Figure 4.3: Half-handles in dimension 3



(a) Attachment of 3-dimensional right half 1-handle (b) Attachment of 3-dimensional left half 2-handle

Figure 4.4: Attachment of half-handles in dimension 3

In [25] the authors prove the following fact, which describes an effect of a half-handle attachment from the level of the manifold (M_1, N_1). Here we use the formulation from [3].

Lemma 4.4.3.

(a) Let (W, Y) be a cobordism from (M_0, N_0) to (M_1, N_1). The result (W', Y') of the attachment of a right half k-handle H to (W, Y) along ϕ is a cobordism between (M_0, N_0) and (M_1', N_1'), where M_1' is diffeomorphic to M_1 after the attachment of the k-handle h along $\phi\,|_c$

$$M_1' \cong M_1 \cup_{\phi|_c} h.$$

(b) Let (W, Y) be a cobordism from (M_0, N_0) to (M_1, N_1). The result (W', Y') of the attachment of a left half k-handle H to (W, Y) along ϕ is a cobordism between (M_0, N_0) and (M_1', N_1'), where M_1' is diffeomorphic to M_1 after

cutting out the $(k-1)$*-handle* $\phi(C)$

$$M_1' \cong M_1 \backslash \phi(C).$$

The manifold M_1' is the result of a **right/left half** k**-surgery** on M_1.

4.4.2 Symplectic half-surgery

As mentioned above, we want to apply the technique of surgery to modify symplectic cobordisms. In general, it is not possible to define a symplectic surgery as a result of a full handle attachment. In particular, a connected sum of symplectic manifolds does not necessarily admit a symplectic structure (see Subsection 3.9.5 in [35]). Here we define symplectic half-surgery as a result of attaching a half-handle to a cobordism between symplectic manifolds with boundary. This notion will be applied to construct SSF structures on closed manifolds (in Subsection 4.4.3) and on manifolds with boundary (in Subsection 5.1.1).

Let (M^{2n}, ω), $n > 2$, be a symplectic cobordism from (N_0, η_0) to (N_1, η_1). We will define symplectic half-surgery on (M, ω) addressing the cases of the right and the left half-surgery separately.

Symplectic right half-surgery

We will define symplectic right half k-surgery under the assumption that $k \leq n$ (half of the dimension of M) and that the surgery is performed along the convex boundary component N_1 of (M, ω).

Consider an embedding $\phi : (C, c) \to (M, N_1)$, where (C, c) is the subset of the right half k-handle as described in Subsection 4.4.1. Let M' be the result of the right half-surgery on M along ϕ. According to Lemma 4.4.3, the manifold M' is diffeomorphic to M with the k-handle h attached along $\phi\mid_c$

$$M' \cong M \cup_{\phi|_c} h.$$

M' is a cobordism from N_0 to N_1', where N_1' is obtained from N_1 by the k-surgery along $\phi\mid_c$.

Since $k \leq n$, it follows from Theorem 2.2.10, that we can realize the attachment of h as a symplectic handle attachment provided that the almost complex structure on M extends over M'. In consequence, we obtain a symplectic structure ω' on the manifold M' and a contact structure η_1' induced on the convex boundary component N_1'.

We call (M', ω') the result of a **symplectic right half** k**-surgery** on (M, ω).

Symplectic left half-surgery

Now, let us consider the case of symplectic left half-surgery on (M, N). We will assume that the half-surgery is of index k such that $2 \leq k \leq n$ and that it is performed along the concave boundary component N_0 of (M, ω).

Consider an embedding $\phi : (C, c) \to (M, N_1)$, where (C, c) is the subset of the left half k-handle as described in Subsection 4.4.1. Let us recall that $C \cong S_+^{k-1} \times D^{2n-k+1}$, where S_+^{k-1} is a $(k-1)$-dimensional disc. Denote by D the disc $S_+^{k-1} \times \{0\} \subset C$.

Consider the embedding $\hat{\phi} = \phi|_D \colon D \hookrightarrow M$. It follows from the relative version of h-principle that D can be perturbed to an isotropic submanifold of the symplectic manifold (M, ω) transverse to the boundary and such that ∂D is an isotropic submanifold of the contact manifold (N, η).

After possibly perturbing ϕ, we can assume $\phi(C)$ to be a tubular neighbourhood of an isotropic disc in (M, ω) and $\phi(c)$ to be a tubular neighbourhood of an isotropic sphere in (N_0, η_0).

According to the isotropic submanifold neighbourhood theorem (see Isotropic Manifold Theorem on page 24 in [39]) a neighbourhood of an isotropic submanifold of a symplectic manifold is determined by the isomorphism class of its symplectic normal bundle. Since $(D, \partial D)$ is contractible, all bundles over $(D, \partial D)$ are trivial. It follows that after taking $\phi(C)$ thin enough one can assume that it is symplectomorphic to the $2n$-dimensional symplectic handle of index $k-1$ with $\phi(c)$ being its concave boundary.

Denote by M' the result of the left half-surgery on M along ϕ. According to Lemma 4.4.3, the manifold M' is diffeomorphic to M with the $(k-1)$-handle $\phi(C)$ cut out

$$M' \cong M \backslash \phi(C).$$

A symplectic structure ω' on M' can be defined as a restriction of the form ω to $M' \cong M \backslash \phi(C)$

$$\omega' = \omega|_{M'}.$$

Note that M' is concave at the boundary along the cut out $\phi(C)$. Moreover, since $\phi(C)$ is symplectomorphic to a symplectic handle, $N' = \partial M' = \overline{N}_0' \sqcup N_1$, where N_0' is obtained from N_0 by a contact $(k-1)$-surgery along $\phi|_c$.

We call (M', ω') a result of a **symplectic left half k-surgery** on (M, ω).

Remark 4.4.4. Note that both in the case of a right and a left half-surgery, if (M, ω) is exact, then the result of the surgery is also an exact symplectic manifold. The fact that the result of a right half-surgery is exact follows from the discussion after Theorem 2.2.10. The case of the left half-surgery is obvious.

With this discussion we have proved the following proposition.

Proposition 4.4.5. *Let (M^{2n}, ω), $n > 2$, be a symplectic cobordism from (N_0, η_0) to (N_1, η_1).*

(a) If $k \leq n$ and if the result M' of a right half k-surgery on (M^{2n}, ω) along the convex boundary component $\phi : (C, c) \hookrightarrow (M, N_1)$ admits an almost complex structure equal to the almost complex structure on (M, ω) outside the surged area, then it also admits a structure ω' of a symplectic cobordism from (N_0, η_0) to (N_1', η_1'), where (N_1', η_1') is a result of the contact k-surgery on (N_1, η_1) along $\phi|_c$.

(b) If $2 \leq k \leq n$, the result M' of a left half k-surgery on (M^{2n}, ω) along the concave boundary component $\phi : (C, c) \hookrightarrow (M, N_0)$, admits a structure ω' of a symplectic cobordism from (N_0', η_0') to (N_1, η_1), where (N_0', η_0') is a result of the contact $(k-1)$-surgery on (N_0, η_0) along $\phi|_c$.

Moreover, if (M, ω) is exact, then (M', ω') is also exact.

4.4.3 Surgery on SSF

In this subsection, we will introduce surgery on SSF manifolds. Moreover, we are going to prove that the existence of a restricted Morse type SSF structure is invariant under surgeries of indices up to the half of the dimension of a manifold.

Let us start by stating the main result of this subsection. Surgery on SSF manifolds will be defined in the proof of the following theorem.

Theorem 4.4.6. *Let M be a restricted Morse type SSF manifold of dimension $2n$ with a compatible stably almost complex structure J. Consider a manifold M' being the result of a k-surgery on M with $k \leq n$. Assume that M' has a stably almost complex structure equal to J outside the surged area. Then M' admits a restricted Morse type SSF structure.*

Proof. Let $(M_0, M_1, \omega_0, \omega_1)$ be an SSF structure on M and let η be the contact form induced on the common boundary N of M_0 and M_1.

Consider a Morse function $f : M \to \mathbb{R}$ with

$$f^{-1}((-\infty, 0]) = M_0, \quad f^{-1}([0, \infty)) = \overline{M_1}.$$

Since M_0 and M_1 have the Morse type n, we can assume that all critical points of indices at most n belong to M_0, while all critical points of indices at least n belong to $\overline{M_1}$. Consider a handle decomposition of M determined by f.

Let $S^{k-1} \times D^{2n-k+1} =: S \times D \hookrightarrow M$ be the framed sphere in M and consider the surgery along this sphere.

Without loss of generality, one can assume that $S \subset N$. Indeed, since for $k \leq n$ we have $\pi_{k-1}(M_0, N) = 0$ and $\pi_{k-1}(M_0, N) = 0$, the sphere S can be homotoped to a sphere in N. What is more, we can choose the homotopy $S^{k-1} \times [0, 1] \to M$ to be an immersion and thus we can homotope the sphere together with its framing. Moreover, note that since

$$\dim S = k - 1 < \frac{2n - 1}{2} = \frac{\dim N}{2},$$

the homotoped sphere can be made an embedded sphere in N. We continue to call the new sphere S.

We can assume that framing of S has one vector pointing outward from M_0. Indeed, let us consider the following fibration

$$\mathrm{SO}(m - 1) \to \mathrm{SO}(m) \to S^m$$

and the induced homotopy long exact sequence

$$\ldots \to \pi_{i+1}(S^m) \to \pi_i(\mathrm{SO}(m - 1)) \to \pi_i(\mathrm{SO}(m)) \to \pi_i(S^m) \to \ldots .$$

For $i + 1 < m$ we obtain an isomorphism

$$\pi_i(\mathrm{SO}(m - 1)) \cong \pi_i(\mathrm{SO}(m)).$$

Since $k \leq n$, we have $k < 2n - k + 1$ and we can use the above isomorphism to get

$$\pi_{k-1}(\mathrm{SO}(2n - k)) \cong \pi_{k-1}(\mathrm{SO}(2n - k + 1)).$$

It follows that any framing of S can be homotoped to a framing with one vector pointing outwards from M_0. Denote by ν the induced framing of S in N.

The manifold M', being the result of the surgery on M, has the following form

$$M' = (M \setminus S \times D) \cup_{S^{k-1} \times S^{2n-k}} D^k \times S^{2n-k}.$$

Let us define

$$M'_0 = (M_0 \setminus S \times D_+) \cup_{S^{k-1} \times S^{2n-k}_+} D^k \times S^{2n-k}_+$$

and

$$\overline{M'_1} = (\overline{M}_1 \setminus S \times D_-) \cup_{S^{k-1} \times S^{2n-k}_-} D^k \times S^{2n-k}_-.$$

Here, the subscript $+/-$ means that only that part of a set is considered, where the coordinate along the framing vector pointing outward from M_0 is nonnegative/nonpositive.

We claim that $M' = M'_0 \cup \overline{M'_1}$ is an SSF decomposition of M'. Indeed, M'_0 and M'_1 are results of the right half k-surgeries on M_0 and M_1 respectively. Since M' is stably almost complex, the manifolds M'_0 and M'_1 are almost complex. Moreover, we can assume that the almost complex structures agree with those on M_0 and M_1 compatible with ω_0 and ω_1 outside the surged area. Then, from Proposition 4.4.5, it follows that both M'_0 and M'_1 are exact symplectic fillings of a manifold (N', η'), where (N', η') is the result of the contact k-surgery on (N, η) along S with respect to the framing ν. □

The manifold M' is the result of an **SSF-surgery** on M. Note that to define an SSF-surgery along a sphere contained in the dividing hypersurface we do not need to assume that M is a restricted Morse type SSF manifold. This assumption was needed to homotope the sphere to the dividing surface.

4.5 Strong symplectic folds as convex hypersurfaces in contact manifolds

In this section, we are going to study convex hypersurfaces in contact manifolds and their applications in constructing SSF manifolds. For a definition of convex hypersurfaces and their dividing sets see Subsection 2.1.2.

We are going to prove that any convex hypersurface in a contact manifold admits an SSF structure. This fact follows from the proof of Lemma 2.2 in [6]. However, the assertion of Lemma 2.2 in [6] does not involve SSF structures and the authors remark that the proof implies the existence of a folded symplectic structures as defined by Cannas da Silva in [4], which is a much less restrictive definition. Moreover, there apparently is a confusion of notation in the proof of Lemma 2.2, which makes some of the formulas at the beginning of the page 11 of [6] incorrect. We proof of the proposition below follows the lines of the proof of Lemma 2.2 in [6].

Proposition 4.5.1. *A closed convex hypersurface in a cooriented contact manifold admits an SSF structure.*

Proof. Let (W^{2n+1}, ξ) be a cooriented contact manifold and let Σ be a convex hypersurface in W. Lemma 2.9 in [12] says that, since Σ is convex, we can choose a neighbourhood

$$\mathscr{N}(\Sigma) \cong [-\epsilon, \epsilon] \times \Sigma$$

of $\Sigma = \{0\} \times \Sigma$ in W such that there exists a contact form α for ξ, which on $\mathscr{N}(\Sigma)$ is given by

$$\alpha = f\mathrm{d}t + \beta, \tag{4.1}$$

where β is a 1-form on Σ, f is a function $\Sigma \to \mathbb{R}$ and $t \in [-\epsilon, \epsilon]$.

Note that $X = \partial_t$ is a contact vector field for α and the dividing set $\Gamma \subset \Sigma$ with respect to X is given by

$$\Gamma = \{f = 0\}.$$

Let us define

$$\Sigma_+ := \{f \geq 0\}, \quad \overline{\Sigma}_- := \{f \leq 0\}.$$

We are going to find structures of exact symplectic fillings on Σ_+ and Σ_- such that $\Sigma = \Sigma_+ \cup_\Gamma \overline{\Sigma}_-$ is an SSF structure on Σ. More precisely, we will show that $\mathrm{d}\alpha$, after a slight modification in a neighbourhood on Γ, defines such symplectic fillings.

Let us begin by analyzing the contact condition for the form α on $\mathscr{N}(\Sigma)$. By Formula (4.1) the condition takes the following form

$$0 < \alpha \wedge (\mathrm{d}\alpha)^n = f\mathrm{d}t\,(\mathrm{d}\beta)^n + n\,\mathrm{d}f\,\mathrm{d}t\,\beta\,(\mathrm{d}\beta)^{n-1}, \tag{4.2}$$

which reduces to $0 < \mathrm{d}f\,\mathrm{d}t\,\beta\,(\mathrm{d}\beta)^{n-1}$ along Γ. This implies, in particular, that f is nondegenerate along its level set Γ and, consequently, Γ is a codimension 1 submanifold of Σ. Moreover, it follows that the form $\alpha\,|_{\mathrm{T}\Gamma} = \beta\,|_{\mathrm{T}\Gamma}$ is a contact form on Γ. Consequently, the contact structure $\xi = \ker\alpha$ on W is transverse to Γ and $\mathrm{d}\alpha$ is nondegenerate on $\xi \cap \mathrm{T}\Gamma$.

Let us consider the characteristic foliation Σ_ξ of Σ, i.e. the singular 1-dimensional foliation defined by the distribution $(\mathrm{T}\Sigma \cap \xi)^\perp$, where $(\cdot)^\perp$ is an orthogonal complement with respect to the symplectic form $\mathrm{d}\alpha$ on ξ. Since $\xi \pitchfork \Gamma$ and $\mathrm{d}\alpha$ is nondegenerate on $\xi \cap \mathrm{T}\Gamma$, it follows that Σ_ξ is transverse to Γ.

Define

$$\mathscr{N}(\Sigma) \supset \mathscr{N}(\Gamma) \cong [-1, 1] \times [-\epsilon, \epsilon] \times \Gamma \ni (\tau, t, x)$$

to be a sufficiently small neighbourhood of Γ in W such that ∂_τ parameterizes the singular foliation Σ_ξ, the coordinate t agrees with the coordinate t of $\mathscr{N}(\Sigma)$ and β is a contact form for all $\{(\tau, t)\} \times \Gamma$. Let Γ be oriented in such a way that β is a positive contact form. This induces a direction of the vector ∂_τ, which makes the orientation of $\mathscr{N}(\Gamma)$ agree with the orientation of W.

After possibly multiplying α by a positive function we can assume that f is constant outside of $\mathscr{N}(\Gamma)$, while on $\mathscr{N}(\Gamma)$ it depends only on τ and it is equal to ± 1 for $|\tau| \geq \frac{1}{2}$.

We are now going to prove that, for the choice of the direction of ∂_τ we have made, f is actually a increasing function of τ. Indeed, the orientation of W is given by $\alpha \wedge (\mathrm{d}\alpha)^n$, which reduces to $n\mathrm{d}f\mathrm{d}t\beta(\mathrm{d}\beta)^{n-1}$ along Γ. On the other hand, from the form of the neighbourhood $\mathscr{N}(\Gamma)$ in W, we have $\mathrm{d}\tau \wedge \mathrm{d}t \wedge \beta \wedge (\mathrm{d}\beta)^{n-1} > 0$. It follows that $f'(\tau) > 0$.

Observe, that from the form of the neighbourhood $\mathscr{N}(\Sigma)$ in W, the orientation of Σ along Γ is given by $-\mathrm{d}\tau \wedge \beta \wedge (\mathrm{d}\beta)^{n-1} > 0$. It follows that Γ is an oriented boundary of Σ_+. Indeed, $-\partial_\tau$ is the normal vector field to Γ in Σ and it points outwards from Σ_+.

Let us proceed with looking for structures of exact symplectic fillings on Σ_+ and Σ_-. Note that from the contact condition (4.2) for α it follows that, when f is locally constant, then

$$(\mathrm{d}\alpha)^n \mid_{\mathrm{T}\Sigma_+} = (\mathrm{d}\beta)^n \mid_{\mathrm{T}\Sigma_+} > 0, \quad (\mathrm{d}\alpha)^n \mid_{\mathrm{T}\overline{\Sigma}_-} = (\mathrm{d}\beta)^n \mid_{\mathrm{T}\overline{\Sigma}_-} < 0.$$

Thus, outside from $\mathscr{N}(\Gamma)$, the form $\mathrm{d}\alpha$ restricts to a positive exact symplectic form on Σ_+ and Σ_-.

Let us now focus on $\mathscr{N}(\Gamma)$. In this neighbourhood we get the following formula for α.

$$\alpha = f(\tau)\mathrm{d}t + \beta$$

Since $\partial_\tau \in \ker\beta$ we can assume that β does not have any $\mathrm{d}\tau$ terms. Let us calculate the derivative $\mathrm{d}\alpha$ of α.

$$\mathrm{d}\alpha = f'\mathrm{d}\tau\mathrm{d}t + \mathrm{d}\tau\frac{\mathrm{d}\beta}{\mathrm{d}\tau} + \mathrm{d}_\Gamma\beta$$

By $\mathrm{d}_\Gamma\beta$ we denote the derivative of β in the direction of Γ.

Since ∂_τ directs the characteristic foliation Σ_ξ, we have that $\iota_{\partial_\tau}(\mathrm{d}\alpha\mid_\zeta) \equiv 0$, where $\zeta = \mathrm{T}\Sigma\cap\xi$. It follows that $\frac{\mathrm{d}\beta}{\mathrm{d}\tau}(v) = 0$ for $v \in \zeta = \ker\alpha\mid_{\mathrm{T}\Sigma} = \ker\beta$. Equivalently, $\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = a\beta$ for some nowhere-zero function a. This leads to the following form of β on $\{\tau_0\}\times\Gamma$

$$\beta = e^{\int_0^{\tau_0} a(\tau,x)\mathrm{d}\tau}\cdot\beta_0 =: g(\tau_0,x)\beta_0$$

for $\beta_0 = \beta\mid_{\{(0,0)\}\times\Gamma}$.

For $\tau \geq \frac{1}{2}$, we have $f = 1$ and the contact condition (4.2) translates into

$$\begin{aligned}0 < \mathrm{d}t\wedge(\mathrm{d}\beta)^n &= \mathrm{d}t\,(\mathrm{d}g\beta_0 + \mathrm{d}\beta_0)^n\\ &= n\mathrm{d}t\,\mathrm{d}g\,\beta_0\,(\mathrm{d}\beta_0)^{n-1}\\ &= n\mathrm{d}t\,\frac{\partial g}{\partial\tau}\,\mathrm{d}\tau\,\beta_0(\mathrm{d}\beta_0)^{n-1}\\ &= -n\frac{\partial g}{\partial\tau}\,\mathrm{d}\tau\,\mathrm{d}t\,\beta_0(\mathrm{d}\beta_0)^{n-1}.\end{aligned}$$

It follows that $\frac{\partial g}{\partial\tau} < 0$ for $\tau \geq \frac{1}{2}$. Similarly, we can argue that $\frac{\partial g}{\partial\tau} > 0$ for $\tau \leq -\frac{1}{2}$.

Let h be a t-invariant positive function on $\mathscr{N}(\Gamma)$ such that

1. $h = g$ for $\tau \leq -\frac{1}{2}, \tau \geq \frac{1}{2}$,
2. $\frac{\partial h}{\partial\tau} > 0$ for $\tau < 0$,
3. $\frac{\partial h}{\partial\tau} < 0$ for $\tau > 0$.

Let us consider a smooth 1-form $\tilde{\alpha}$ on $\mathscr{N}(\Sigma)$ such that it is equal to α on the complement of $\mathscr{N}(\Gamma)$ and $\tilde{\alpha} = \frac{h}{g}\alpha$ on $\mathscr{N}(\Gamma)$. We claim that $\mathrm{d}\tilde{\alpha}$ gives structures of exact symplectic fillings on manifolds Σ_+ and Σ_-.

We have already observed that $\mathrm{d}\tilde{\alpha}\mid_{\mathrm{T}\Sigma} = \mathrm{d}\alpha\mid_{\mathrm{T}\Sigma}$ is an exact symplectic form outside from $\mathscr{N}(\Gamma)$. On $\mathscr{N}(\Gamma)\cap\Sigma$ we have

$$\tilde{\alpha}\mid_{\mathrm{T}\Sigma} = \frac{h}{g}\alpha\mid_{\mathrm{T}\Sigma} = h\beta_0.$$

Let us calculate

$$[\mathrm{d}(h\beta_0)]^n = (\mathrm{d}h\beta_0 + \mathrm{d}\beta_0)^n = n\mathrm{d}h\beta_0(\mathrm{d}\beta_0)^{n-1} = n\frac{\partial h}{\partial \tau}\mathrm{d}\tau\beta_0(\mathrm{d}\beta_0)^{n-1}.$$

By conditions 2 and 3, we get that the form $\tilde{\alpha} |_{\mathrm{T}\Sigma}$ is a positive exact symplectic form on Σ_+ and Σ_- and it is convex on the boundary Γ. The induced contact form on the boundaries of both Σ_+ and Σ_- is $h |_{\tau=0} \beta_0$. Consequently, the decomposition

$$\Sigma = \Sigma_+ \cup_\Gamma \overline{\Sigma}_-$$

with the described symplectic structures is an SSF structure on Σ. □

The usefulness of Proposition 4.5.1 depends on how common the convex hypersurfaces are. By Giroux' result from [20], in dimension 3 convex surfaces are generic: any surface in a 3-dimensional contact manifold can be C^∞-approximated by a convex surface. However, a similar fact does not hold in higher dimensions (see [33]).

A well known class of examples of convex hypersurfaces are boundaries of tubular neighbourhoods of Legendrian submanifolds.

Proposition 4.5.2. *For a Legendrian submanifold of a contact manifold there exists a tubular neighbourhood, whose boundary is a convex hypersurface.*

Proof. Theorem 2.2.7 implies that diffeomorphic Legendrian submanifolds have contactomorphic neighbourhoods.

Given a smooth manifold L a universal model of its neighbourhood as a Legendrian submanifold can be defined as follows. Consider the 1-jet space $\mathscr{J}^1(L)$ of L. It is diffeomorphic to $\mathbb{R} \times \mathrm{T}^*L$ by a diffeomorphism sending $j^1_y f$ to $(f(y), \mathrm{d}f_y)$. Consider local coordinates $(z, x_1, y_1, \ldots, x_k, y_k)$ on $\mathscr{J}^1(L)$, where z is a $\mathbb{R}$-coordinate, y_i's are local coordinates on L and x_i's are the dual coordinates. The form

$$\alpha = \mathrm{d}z - \sum_{i=1}^{k} x_i \mathrm{d}y_i$$

is a contact form on $\mathscr{J}^1(L)$ and the embedding of L as a zero level in $\mathscr{J}^1(L)$ is Legendrian.

The vector field

$$X = z\partial_z + \sum_{i=1}^{k} x_i \partial_{x_i}$$

is a contact vector field for the contact structure $\ker\alpha$. Indeed, the calculations

$$\mathscr{L}_{z\partial_z}\alpha = \mathrm{d}\iota_{z\partial_z}\mathrm{d}z = \mathrm{d}z, \quad \mathscr{L}_{x_i\partial_{x_i}}\alpha = \iota_{x_i\partial_{x_i}}\mathrm{d}(-x_i\mathrm{d}y_i) = -x_i\mathrm{d}y_i,$$

lead to $\mathscr{L}_X\alpha = \alpha$. The vector field X is transverse to the boundary of the tubular neighbourhood of L, which implies that the boundary is a convex surface. □

From Propositions 4.5.1 and 4.5.2 we get the following corollary.

Corollary 4.5.3. *Given a smooth manifold L, the unit bundle $S(\mathbb{R} \times T^*L)$ admit an SSF structure.*

4.6 Construction of a controllable contact structure

A possible strategy to construct an SSF structure on a manifold Σ is to endow $\Sigma \times \mathbb{R}$ with a controllable contact structure and use a similar reasoning as in the proof of Proposition 4.5.1 to deduce an SSF structure on an embedding of Σ to $\Sigma \times \mathbb{R}$. Let us now describe a construction of such a controllable contact structure and an embedding of the manifold Σ. This construction allows us to decompose Σ into a sum of two exact symplectic manifolds glued along the common boundary - the separating hypersurface Γ. Note that for this to be an SSF, we would additionally need convexity of the exact symplectic structures along the boundary.

The advantage of our construction over the convex hypersurface construction presented in Section 4.5 is its stability. Indeed, the Reeb vector field of our contact form is transverse to the separating hypersurface of Σ, while it is tangent in the convex hypersurface construction.

Let us give an argument showing that transversality of the Reeb vector field to the separating hypersurface Γ allows us to omit some obstructions arising, when we want to make a hypersurface convex.

If Σ is a closed convex hypersurface in a contact manifold W of dimension $2n+1>3$, then the separating set Γ is contact submanifold of W and the Reeb field is tangent to Γ. Since Γ is a closed manifold, the Weinstein conjecture says that the Reeb vector field has closed orbits on Γ. However, it is not difficult to find examples of (open) contact manifolds such that their Reeb fields do not have closed orbits at all. So one can not expect that the Giroux' approximation theorem ([20], see the discussion above Proposition **??**) can be extended to higher dimensions.

The construction given below applies to every almost complex manifold and may be considered as an alternative to the convex hypersurface construction. On the other hand, as we will see in the next section, any strong symplectic fold embeds as a convex hypersurface into $\Sigma \times \mathbb{R}$ endowed with the contact form given by Formula 4.4 (this is a formula for a contact form on $\Sigma \times S^1$ but it works for $\Sigma \times \mathbb{R}$ as well). This rises the question whether a (stably) almost complex manifold can be embedded as a convex hypersurface in a contact manifold. This is considerably weaker than asking if any hypersurface in a fixed contact manifold can be deformed to a convex one (even allowing large deformations). Moreover, it is equivalent to the question if any (stably) almost complex manifold admits a strong symplectic fold structure. Possibly, constructing strong symplectic folds is the most prospective way to attack this question.

Controllable contact forms on $\Sigma \times \mathbb{R}$ if Σ is almost complex

If Σ is an almost complex manifold, then, by Gromov's theory, $\Sigma \times \mathbb{R}$ admits a contact structure. However, there is no way to see, how this contact form behaves near Σ. The main purpose of the argument below is to get a contact structure on $\Sigma \times \mathbb{R}$ with controllable properties near Σ.

Let Σ^{2n} be an almost complex manifold. Then Σ admits a nondegenerate 2-form ω_0. Consider the manifold $\Sigma \times \mathbb{R}$ with the projection $\pi : M \times \mathbb{R} \to M$. We will use the following h-principle theorem about approximation with exact forms (see Theorem 4.7.1 [11]).

Theorem 4.6.1. *Let $\Sigma \subset W$ be a polyhedron of codimension at least 1 and ω a p-form. Then there exists an arbitrarily C^0-small diffeotopy $i_t : W \to W$ such that ω can be C^0-approximated near $\widetilde{M} = i_1(M)$ by an exact p-form $\tilde{\omega} = d\tilde{\alpha}$. Moreover, given a $(p-1)$-form α on W, one can choose $\tilde{\alpha}$ to be C^0-close to α near $\widetilde{M}$.*

If we apply the above theorem to the form $\pi^*\omega_0$, we conclude that there exists a C^0-small diffeotopy i_t of $\Sigma \times \mathbb{R}$ and a C^0-approximation $\tilde{\omega}$ of the form $\pi^*\omega_0$ on a neighbourhood U of $i_1(M)$ such that $\tilde{\omega} = \mathrm{d}\tilde{\alpha}$. Moreover, we can choose $\tilde{\alpha}$ to be C^0-close to $\mathrm{d}\theta$, where θ is a parameter of $\mathbb{R}$.

Nondegeneracy of ω_0 implies that $(\pi^*\omega_0)^n \neq 0$. Since $\tilde{\omega}$ is C^0-close to $\pi^*\omega_0$, it follows that $\tilde{\omega}^n \neq 0$ and the form $\tilde{\omega}$ is nondegenerate. Similarly, since $\ker(\pi^*\omega_0) = \mathrm{span}(\partial_\theta)$, we have that $\ker(\tilde{\omega})$ is C^0-close to $\mathrm{span}(\partial_\theta)$.

The form $\tilde{\alpha}$ is C^0-close to $\mathrm{d}\theta$ on U and $\ker(\mathrm{d}\tilde{\alpha})$ is C^0-close to $\mathrm{span}(\partial_\theta)$ on U. This implies that $\tilde{\alpha} \wedge (\mathrm{d}\tilde{\alpha})^n \neq 0$ and the form $\tilde{\alpha}$ is contact on U.

The separating hypersurface

Let R be the Reeb vector field of $\tilde{\alpha}$. The embedding i_1 can be perturbed to be transverse to R in the 1-jet space. More specifically, the embedding i_1 induces the embedding

$$\mathrm{D}i_1 : \mathrm{T}M \to \mathrm{T}(M \times \mathbb{R}), \quad v_x \mapsto (\mathrm{D}i_1)v_x.$$

This embedding can be made transverse to $V := \{R_{i(x)}\}_{x \in M} \subset \mathrm{T}(M \times \mathbb{R})$. Let us denote by i the perturbed embedding. Consider the subset Γ of $i(M)$ given by

$$\Gamma = \{x \in M \mid R_{i(x)} \in \mathrm{T}i(M)\} = \{x \in M \mid (V \cap \mathrm{Im}\,\mathrm{D}i)_{i(x)} \neq 0\}.$$

From the transversality as described above, it follows that Γ is a codimension 1 submanifold of Σ and R is transverse to $i(\Gamma)$ in Σ.

Since R is transverse to $i(\Sigma)$ away from $i(\Gamma)$, the contact form $\tilde{\alpha}$ induces an exact symplectic form on $\Sigma \setminus \Gamma$.

A neighbourhood of Γ

Let us now examine the form $\tilde{\alpha}$ close to $i(\Gamma)$. Denote by Ξ the hypersurface in U given by the push of $i(\Gamma)$ along the flow lines of R and consider an R-invariant vector field N transverse to Ξ. This gives us the following tubular neighbourhood of $i(\Gamma)$ in U

$$\mathscr{N}(\Gamma) \cong \Gamma \times [-\epsilon, \epsilon] \times [-\delta, \delta],$$

where $\tau \in [-\epsilon, \epsilon]$ is a parameterization of the Reeb flow lines such that $\partial_\tau = R$ and $s' \in [-\delta, \delta]$ parameterizes the flow of N. Since ∂_τ is the Reeb vector field of $\tilde{\alpha}$, we get the following formula

$$\tilde{\alpha} = \mathrm{d}\tau + \alpha_0, \tag{4.3}$$

where α_0 is a form on $\Delta := \Gamma \times [-\delta, \delta] \subset \mathscr{N}(\Gamma)$. See Figure 4.5 for a picture of the neighbourhood $\mathscr{N}(\Gamma)$.

Let us now find coordinates on Δ better adjusted to the form $\tilde{\alpha}$. The contact condition for $\tilde{\alpha}$ implies that $\mathrm{d}\alpha_0$ is a symplectic form on Δ. It follows that

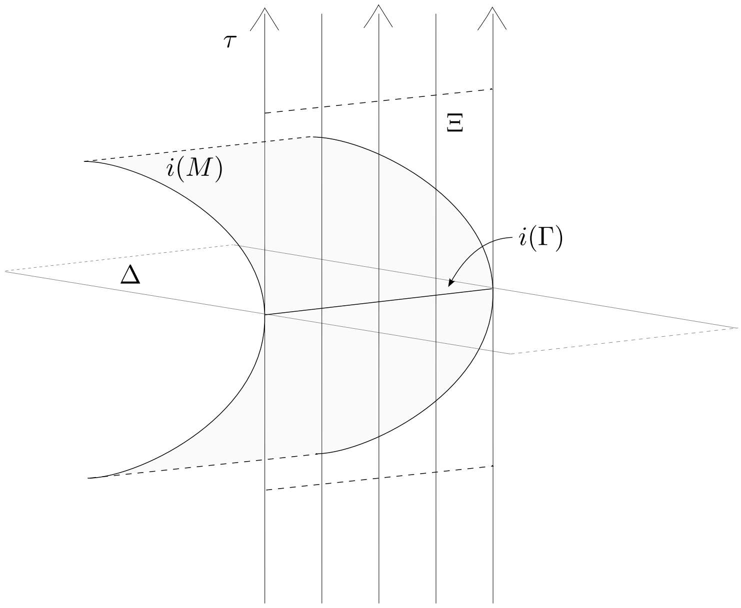


Figure 4.5: The neighbourhood $\mathscr{N}(\Gamma)$ of Γ

$d\alpha_0 \mid_{\mathrm{T}\Gamma}$ is presymplectic (see Chapter 3 for a definition of a presymplectic form). This means, in particular, that $\ker(\mathrm{d}\alpha_0 \mid_{\mathrm{T}\Gamma})$ is 1-dimensional. Let R_Γ be a positive generator of $\ker(\mathrm{d}\alpha_0 \mid_{\mathrm{T}\Gamma})$. Note that R_Γ can be defined on the whole hypersurface Δ by, similarly, considering the presymplectic forms $\alpha_0 \mid_{\mathrm{T}\Gamma\times\{s'\}}$ for $s' \in [-\delta, \delta]$. Let J be a τ-invariant automorphism of $\mathrm{T}\mathscr{N}(\Gamma)$ such that $J(\partial_\tau) = 0$ and $J \mid_{\mathrm{T}\Delta}$ is a compatible almost complex structure on $(\Delta, \mathrm{d}\alpha_0)$. Then, the coordinate on Δ transverse to Γ can be chosen as $\partial_s := JR_\Gamma$. This gives $\mathrm{d}\alpha_0(R_\Gamma, \partial_s) = \mathrm{d}\alpha_0(R_\Gamma, JR_\Gamma) > 0$.

Embeddings of Σ into $\Sigma \times \mathbb{R}$

Such a choice of coordinates on $\mathscr{N}(\Gamma)$ determines coordinates on a neighbourhood of Γ in Σ. Indeed, the flow of s allows us to project the coordinate τ to $i(M)$. Let us denote the induced coordinate by t. The coordinates $\mathbf{x}$ on Γ together with t give coordinates on a neighbourhood of Γ in Σ. In such coordinates, the embedding i can be assumed to be given by

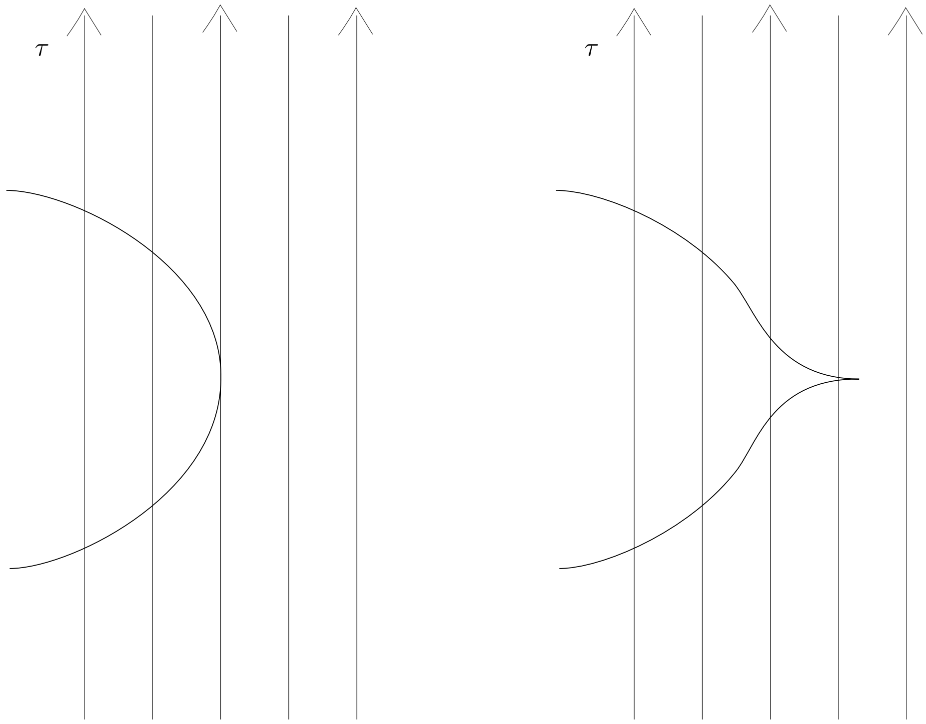
$$i(\mathbf{x}, t) = (\mathbf{x},\, f(x) + b_1(x)t,\, b_2(\mathbf{x})t^2) \in \mathscr{N}(\Gamma).$$

This embedding can be replaced by

$$j(\mathbf{x}, t) = (\mathbf{x},\, f(x) + b_1(x)t,\, b_2(\mathbf{x})e^{\pm t} + b_3(x)),$$

where the $\pm$ sign is chosen to be $+$ for $t \leq 0$ and $-$ for $t \geq 0$. See Figure 4.6 for a picture of the embeddings i and j. An embedding j gives us an exact symplectic structure on closures of both components of $\Sigma \setminus \Gamma$.

It is conceivable that under an appropriate deformation of the data $\{\alpha_0, J\}$ one can obtain the Liouville field in both exact symplectic parts transverse



(a) Embedding i (b) Embedding j

Figure 4.6: Embeddings i and j

to $\Gamma \times \{t_0\}$, where t_0 is close enough to 0. This would imply convexity along the boundary Γ of the two exact symplectic parts.

4.7 Applications of strong symplectic folds to construct contact manifolds

The authors of [19] prove the following fact showing how to apply the notion of SSF structures to construct contact manifolds.

Proposition 4.7.1. *For an SSF-manifold M, there exists a contact form on $M \times S^1$.*

Proof. Assume that M has dimension $2n$. Let $M = M_0 \cup_N \overline{M}_1$ be an SSF structure on M, where the manifolds M_0 and M_1 are endowed with exact symplectic structures $w_0 = \mathrm{d}\lambda_0$ and $w_1 = \mathrm{d}\lambda_1$. Since the symplectic forms are convex along their boundaries, we can assume that in a collar neighbourhood $N \times [-1, 0] \cong U_i \subset M_i$ of the boundary $\partial M_i = N \times \{0\}$ the form λ_i is equal to $e^t\eta$, where η is the contact form induced on the boundary and t is a parameter of the interval $[-1, 0]$. Let us use the coordinate t to parameterize $U_0 \cup \overline{U}_1$ by $N \times [-1, 1]$. We get the following decomposition of M

$$M = M_0 \cup_N M_1 \cong \mathring{M}_0 \cup N \times [-1, 1] \cup \overline{\mathring{M}}_1,$$

where $\mathring{M}_i = M_i \setminus U_i$.

A contact form on $M \times S^1$ can be defined as

$$\alpha = \begin{cases} \lambda_0 + \mathrm{d}\theta, & \text{on} \quad \mathring{M}_0, \\ f(t)\eta + g(t)\mathrm{d}\theta, & \text{on} \quad N \times [-1,1], \\ \lambda_1 - \mathrm{d}\theta, & \text{on} \quad \mathring{M}_1, \end{cases} \tag{4.4}$$

where $f, g : [-1,1] \to \mathbb{R}$ are functions such that

(i) $f = e^{\mp t}$ for t close to ± 1,

(ii) $g = \mp 1$ for t close to ± 1,

(iii) $f > 0$,

(iv) $fg' - f'g > 0$.

Note that the conditions (i) and (ii) imply that the form α is smooth.

First, observe that such functions f and g exist. Indeed, if one considers the curve $(f(t), g(t))$ in $\mathbb{R}^2$, the last condition (iv) translates as follows.

The tangent space to the curve is nondegenerate and the vector from the origin to a point on the curve is never tangent to the curve in that point.

One can easily see that the curve on Figure 4.7 can be parameterized by functions f and g in such a way that the conditions (i)-(iv) are fulfilled.

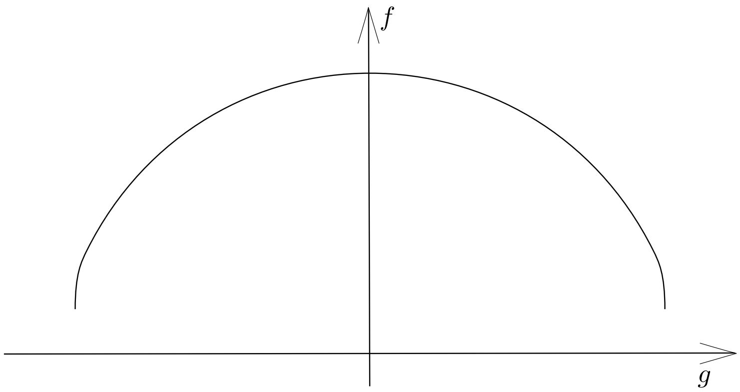


Figure 4.7: The curve parameterized by (f, g)

In what follows we will use functions f and g as given on the graphs on Figure 4.8.

Now, we need to show that the form α is contact. Let us start with calculating its derivative.

$$\mathrm{d}\alpha = \begin{cases} \mathrm{d}\lambda_0, & \text{on} \quad \mathring{M}_0, \\ f(t)\mathrm{d}\eta + f'(t)\mathrm{d}t \wedge \eta + g'(t)\mathrm{d}t \wedge \mathrm{d}\theta, & \text{on} \quad N \times [-1,1], \\ \mathrm{d}\lambda_1, & \text{on} \quad \mathring{M}_1. \end{cases}$$

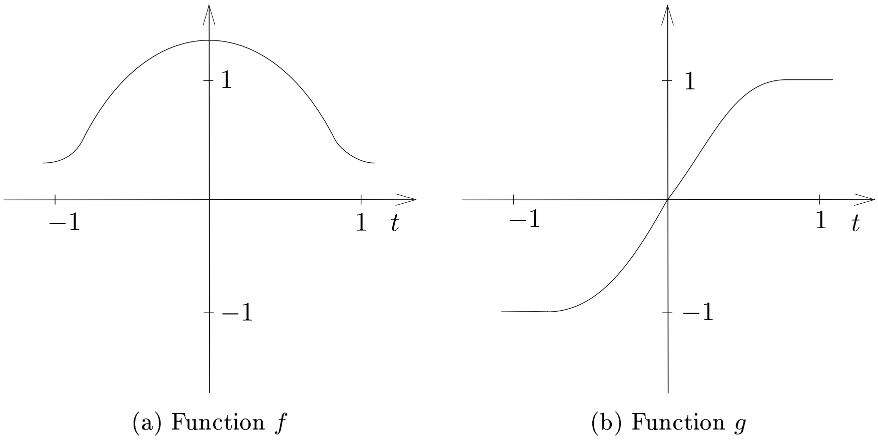


Figure 4.8: Graphs of functions f and g

In order to verify the contact condition we need to consider the following form.

$$\alpha\wedge(\mathrm{d}\alpha)^n = \begin{cases} \mathrm{d}\theta \wedge (\mathrm{d}\lambda_0)^n, & \text{on} \quad \mathring{M}_0, \\ f^{n-1}(t)\big(f(t)g'(t) - f'(t)g(t)\big)\eta\,(\mathrm{d}\eta)^{n-1}\mathrm{d}t\,\mathrm{d}\theta, & \text{on} \quad \overline{N} \times [-1,1], \\ -\mathrm{d}\theta \wedge (\mathrm{d}\lambda_1)^n, & \text{on} \quad \mathring{M}_1. \end{cases}$$

Since the forms $\mathrm{d}\lambda_0$ and $\mathrm{d}\lambda_1$ are symplectic and the form η is contact, the conditions (iii) and (iv) imply that $\alpha \wedge (\mathrm{d}\alpha)^n$ is positive. It follows that α is a contact form on $M \times S^1$. □

Definition 4.7.2. For an SSF manifold M we will call the induced contact structure on $M \times S^1$ defined by Equation 4.4 an **SSF-contact structure**.

Proposition 4.7.1 above together with previous results of this chapter lead to the following conclusion.

Theorem 4.7.3. *For the following classes of manifolds M, the manifold $M \times S^1$ admits a contact structure.*

(a) 4-dimensional oriented manifolds (observed in [19]);

(b) Doubles of exact symplectic manifolds;

(c) Unit bundles $S(\mathbb{R} \times T^*L)$ of smooth manifolds L.

Chapter 5

Modifications of SSF

In this chapter, we are going to introduce some possible modifications of the definition of SSF structures.

In Section 5.1, we examine SSF structures on manifolds with boundary. In particular, we introduce a concept of an SSF cobordism between contact manifolds. In Section 5.2, we consider a more general definition of an SSF structure, where the symplectic pieces can be glued both along convex and concave boundaries.

5.1 SSF manifolds with boundary

In this section, we study strong symplectic fold structures on manifolds with boundary. Let M be an oriented manifold with boundary $\partial M = \overline{Q}_0 \sqcup Q_1$. Here we allow $Q_0 = \emptyset$. An SSF structure on M is a decomposition

$$M = M_0 \cup_N \overline{M_1},$$

where M_0 and M_1 admit exact symplectic structures ω_0, ω_1 such that (M_0, ω_0) is a symplectic cobordism from (Q_0, μ_0) to (N, η) (or a symplectic filling of (N, η) if $Q_0 = \emptyset$) and (M_1, ω_1) is a symplectic cobordism from (Q_1, μ_1) to (N, η). Here (Q_0, μ_0), (Q_1, μ_1) and (N, η) are contact manifolds.

Remark 5.1.1. One could also consider SSF structures convex along some of the boundary components. However, such structures can be treated as a special case of SSF manifolds with boundary as defined above. Indeed, consider for example a decomposition $M = M_0 \cup_N \overline{M_1}$, where (M_0, ω_0) is a symplectic cobordism from (Q_0, μ_0) to (N, η) and (M_1, ω_1) is a symplectic manifold with two convex boundary components (Q_1, μ_1) and (N, η). Let (M_0', ω_0') be a trivial symplectic cobordism of $(Q_1, e^{-1}\mu_1)$. Then

$$M \cong (M_0 \sqcup M_0') \cup_{N \sqcup Q_1} \overline{M}_1$$

is a SSF manifold with boundary structure on M.

5.1.1 Surgery on SSF manifolds with boundary

We will show that the existence of a restricted Morse type SSF structure with boundary on $2n$-manifolds is invariant under left half-surgeries up to index n.

Note that from Proposition 4.4.6 we know that it is invariant under (full) surgeries up to index n.

Proposition 5.1.2. *Let M' be the result of a left half k-surgery on a manifold with boundary M^{2n} for $2 \leq k \leq n$. Assume that there exists a restricted Morse type SSF structure on M (possibly with boundary). Then M' is a restricted Morse type SSF manifold with boundary.*

Proof. Let us use the notation as introduced at the beginning of this section. We can assume without loss of generality that the surgery is being performed along Q_1.

Let $\phi : (C, c) \to (M, Q_1)$ be the attaching embedding . We can modify the embedding ϕ so that $\phi(C)$ is contained in M_1. Indeed, since M_0 is built out of handles of indices up to n, $C = S_+^{k-1} \times D^{2n-k+1}$ can be homotoped into M_1 by the same argument as in the proof of Proposition 4.4.6. We continue denoting the attaching embedding by $\phi : (C, c) \to (M, Q_1)$.

An SSF structure on M' can be given by

$$M' = M_0 \cup_N \overline{M_1'},$$

were M_1' is the result of the left half k-surgery on M_1 along ϕ. Indeed, from Proposition 4.4.5 it follows that M_1' has a structure of an exact symplectic cobordism. □

5.1.2 SSF cobordism between contact manifolds

Definition 5.1.3. Let (Q_0, μ_0) and (Q_1, μ_1) be contact manifolds. An **SSF cobordism** from (Q_0, μ_0) to (Q_1, μ_1) is an SSF manifold $M = M_0 \cup_N \overline{M}_1$ with boundary $\partial M = \overline{Q}_0 \sqcup Q_1$, where $\partial M_0 = \overline{Q}_0 \sqcup N$, $\partial M_1 = \overline{Q}_1 \sqcup N$, such that μ_0 and μ_1 are the contact forms induced on the boundary.

In this section, we are going to consider the relation of SSF cobordism between contact manifolds.

It turns out that the relation of SSF cobordism identifies much more elements then the relation of exact symplectic cobordism. Indeed, we will show that all exactly fillable manifolds are SSF cobordant. From this fact it will follow that in dimension 3 the relation of SSF cobordism is trivial.

Proposition 5.1.4. *Let (Q_0^{2n-1}, μ_0) and (Q_1^{2n-1}, μ_1) be exactly fillable manifolds. There exists an SSF cobordism between those manifolds.*

Proof. Let (M_0^0, ω_0^0) be an exact filling of $(Q_0, e\mu_0)$ and let (M_0^1, ω_0^1) be a trivial cobordism of (Q_1, μ_1). The two exact cobordisms can be connected by a symplectic 1-handle. As the result we get an exact symplectic cobordism (M_0, ω_0) from (Q_1, μ_1) to $(Q_0, e\mu_0)\#(Q_1, e\mu_1)$. Similarly, we can get an exact symplectic cobordism (M_1, ω_1) from (Q_0, μ_0) to $(Q_0, e\mu_0)\#(Q_1, e\mu_1)$. The manifold

$$M := M_0 \cup \overline{M}_1$$

is an SSF cobordism from (Q_0, μ_0) to (Q_1, μ_1). □

The result above implies that the relation of SSF cobordism between 3-dimensional contact manifolds trivializes.

Proposition 5.1.5. *There exists an SSF cobordism between any two closed 3-dimensional contact manifolds.*

Proof. Let (Q_0, μ_0) and (Q_1, μ_1) be closed contact manifolds of dimension 3. It was proved in [16] that from any contact 3-manifold there exists a Stein cobordism to a Stein fillable manifold. Let (Q_0', μ_0') and (Q_1', μ_1') be such Stein fillable manifolds for (Q_0, μ_0) and (Q_1, μ_1) respectively and let M_0 and M_1 be the Stein cobordisms. From Proposition 5.1.4 it follows that there exists an SSF cobordism M' from (Q_0', μ_0') to (Q_1', μ_1'). By gluing the Stein cobordisms and the SSF cobordism together we get an SSF cobordism

$$M = M_0 \cup_{Q_0'} M' \cup_{Q_1'} \overline{M_1}$$

between (Q_0, μ_0) and (Q_1, μ_1). □

5.2 Generalized strong symplectic fold structures

In the following section, we study **generalized strong symplectic fold structures**. By a generalized SSF structure we mean a decomposition of a manifold M into pieces admitting structures of exact symplectic cobordisms, i.e.

$$M = M_1 \cup \ldots \cup M_n.$$

The pieces M_i are glued together along common boundaries. The induced contact forms on the boundary components that are glued together are equal. Note that, unlike in the case of classical SSF structures, here we admit gluing both along convex and concave boundaries.

Similarly, we can define generalized SSF structures with boundary.

The main result of this section is the existence of a concave SSF structure on every 4-dimensional manifold. In order to prove this fact we significantly simplify the proof of a stronger result from [1] showing the existence of classical SSF structures on manifolds of dimension 4. On the course of the proof we use classical invariants of Legendrian knots as well as homotopy invariants of plane fields. We introduce the necessary tools in Subsection 5.2.1 and in Subsection 5.2.2 we present the proof of the main result.

5.2.1 Invariants of Legendrian knots and plane fields in dimension 3

The goal of this subsection is to introduce tools which will be used later in order to prove the existence of concave SSF structures on 4-dimensional manifolds. All the facts in this subsection are well known.

We are going to discuss classical invariants of Legendrian knots in contact 3-manifolds and homotopy invariants of plane fields in 3-manifolds.

Classical invariants of Legendrian knots

We start by introducing the classical invariants of Legendrian knots. For more details see [18].

Consider a contact 3-manifold (N, ξ) and a homologically trivial Legendrian knot K in N.

Let us start with the **Thurston-Bennequin invariant** denoted by tb. It is defined as the contact framing of the knot relative to its surface framing. If K is a knot in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ (see Example 2.1.8), the invariant tb can be given explicitly using the formula

$$\mathrm{tb}(K) = \mathrm{writhe}(K_F) - \frac{1}{2}\#\mathrm{cusps}(K_F), \tag{5.1}$$

where by K_F we mean the front projection of K (an image of K by the projection $(x, y, z) \mapsto (y, z)$) and by writhe we denote the number of self crossings of a knot counted together with their signs.

The second classical invariant we are going to introduce is the **rotation**. It counts the number of rotations of the vector tangent to K relative to a trivialization of $\xi|_\Sigma$ as we go once around K. Here by Σ we denote a Seifert surface of K. The rotation depends on the homology class induced by the Seifert surface and the dependence is given by the following formula

$$\mathrm{rot}(K, \Sigma) - \mathrm{rot}(K, \Sigma') = \langle e(\xi), c - c' \rangle,$$

where Σ and Σ' are two Seifert surfaces of K representing classes $c, c' \in H_2(N; \mathbb{Z})$ respectively.

As for the tb invariant, let us consider a special case, when K is a knot in $(\mathbb{R}^3, \xi_{\mathrm{st}})$. Then, since $H_2(\mathbb{R}^3; \mathbb{Z}) = \{0\}$, the rotation does not depend on the choice of the Seifert surface. It this case rot can be given explicitly by the formula

$$\mathrm{rot}(K) = \#\text{down-cusps}(K_F) - \#\text{up-cusps}(K_F). \tag{5.2}$$

Given two Legendrian knots K_0 and K_1, we can define a new Legendrian knot $K_0\#K_1$ - a connected sum of K_0 and K_1. The classical invariants of $K_0\#K_1$ are given by

$$\mathrm{tb}(K_0\#K_1) = \mathrm{tb}(K_0) + \mathrm{tb}(K_1) - 1,$$
$$\mathrm{rot}(K_0\#K_1) = \mathrm{rot}(K_0) + \mathrm{rot}(K_1).$$

Plane field homotopy invariants

We are now going to introduce homotopy invariants of plane fields on a 3-manifold.

Let N be a 3-manifold. To simplify the discussion assume that $H^2(N; \mathbb{Z})$ has no 2-torsion. We will see that such a restricted class of manifolds will be enough for our applications. Denote by $\Xi(N)$ the space of oriented plane fields on N. We are going to describe, how the homotopy type of $\xi \in \Xi(N)$ is uniquely determined by two invariants: $c_1(\xi)$ and $d_3(\xi)$. This fact is well known and was for instance discussed in [8].

We will use the following definition of a spin^c structure. Note that this is not a standard definition and its equivalence with the standard one was proved in [37].

Definition 5.2.1. Two nowhere vanishing vector fields on a 3-manifold N are said to be homologous, if they are homotopic outside of a ball $D^3 \subset N$ via nowhere vanishing vector fields. An equivalence class of this relation is called a **spin^c structure** on N. The set of all spin^c structures on N is denoted by $\mathrm{Spin}^c(N)$.

Let us define a map $p : \pi_0(\Xi(N)) \to \mathrm{Spin}^c(N)$ sending the homotopy class $[\xi]$ of $\xi \in \Xi(N)$ to the spin^c structure of the unit normal vector field of ξ. Note that p is well defined since a spin^c structure is invariant under the homotopy of ξ as directly follows from the definition above.

Let $\xi_1, \xi_2 \in \Xi(N)$ and let ξ_i be considered as a complex line bundle. By [37], if the spin^c structures of ξ_1 and ξ_2 agree, then $c_1(\xi_1) = c_1(\xi_2)$. This allows us to define a first Chern class of a spin^c structure. As it was pointed out in [37], the cohomology group $H^2(N;\mathbb{Z}) \cong H_1(N;\mathbb{Z})$ acts freely and transitively on $\mathrm{Spin}^c(N)$ and

$$c_1(\mathbf{t} \otimes a) = c_1(\mathbf{t}) + 2a,$$

where $\mathbf{t} \in \mathrm{Spin}^c(N)$ and $a \in H^2(N;\mathbb{Z})$. It follows that, since $H^2(N;\mathbb{Z})$ has no 2-torsion, a spin^c structure is uniquely defined by its first Chern class.

Assume now that (N, ξ) is a result of a series of index 2 contact surgeries on (S^3, ξ_{st}) with respect to framings ± 1. Any contact 3-manifold can be obtained in this way (see [7]). Denote by $\mathbb{L}$ the link of knots K_i along which the 2-surgeries are performed. Then the first Chern class $c_1(\xi) \in H^2(N;\mathbb{Z})$ can be understood by looking at its Poincaré dual class $P.D.c_1(\xi) \in H_1(N;\mathbb{Z})$

$$P.D.c_1(\xi) = \sum_{K_i \in \mathbb{L}} \mathrm{rot}(K_i)[N_i], \tag{5.3}$$

where N_i is the cocore of the handle corresponding to the knot K_i.

Now we need a tool to distinguish between different plane fields inducing the same spin^c structure $\mathbf{t}$.

From [22] it follows that the homotopy class of ξ with $p([\xi]) = \mathbf{t}$ is fully determined by a single number. Let us denote this number by $d_3(\xi)$.

Let M be the topological filling of N induced by the surgeries along the knots in the link $\mathbb{L}$. Then

$$d_3(\xi) = \frac{1}{4}(c^2(M) - 3\sigma(M) - 2\chi(M)) + q, \tag{5.4}$$

where σ and χ are the signature and the Euler number respectively, q denotes the number of knots in $\mathbb{L}$ with the contact framing $+1$ and $c(M) \in H^2(M;\mathbb{Z})$ is the cohomology class determined by $c(\Sigma_K) = \mathrm{rot}(K)$ for $K \in \mathbb{L}$. By Σ_K we denote a Seifert surface of K glued together with the core disc of the corresponding handle. Equation (5.4) was proved in [8] for the case, where there were no 1-surgeries. In [15] the authors pointed out that the same argument can be used to prove the general case.

5.2.2 Construction of generalized SSF structures on 4-manifolds

In this subsection, we are going to prove the existence of generalized SSF structures on 4-dimensional oriented manifolds. In [1], the author proves the existence of a classical SSF structure on any 4-dimensional oriented manifold (see Section 4.3 for a sketch of the proof). Our result is weaker but the proof is considerably simpler.

We will use the notion of stabilizations of open book decompositions. Apart from positive stabilizations, as defined in Section 2.4, we will also need negative stabilizations. The negative stabilizations are defined analogously as the positive stabilization but instead of composing the monodromy with a right Dehn twist we use a left Dehn twist. As pointed out in Section 2.4, a positive stabilization does not change the isotopy class of a contact structure. This is not true for negative stabilizations. In fact, a negative stabilization adds an overtwisted disc to a contact structure (see [14]).

The following lemma will be useful in the proof of the main result of this section.

Lemma 5.2.2. *Let (N, η_0) be a contact manifold. Let us negatively stabilize on open book an N supporting η_0. Denote by η_1 the contact structure supported by the result open book. There exists a topologically trivial SSF cobordism $N \times I$ from (N, η_0) to (N, η_1).*

Proof. Consider the following symplectically trivial cobordisms

$$\widetilde{M_0} = (N \times [-1, 0], \mathrm{d}(e^t \eta_0)), \quad \widetilde{M_1} = (N \times [0, 1], \mathrm{d}(e^{-t} \eta_1)).$$

A negative stabilization of the open book supporting η_0 corresponds to two surgeries given by attaching a pair of cancelling handles h_1 and h_2 of indices 1 and 2 to $\widetilde{M_0}$ (see [36]). It gives the following decomposition of the trivial cobordism $N \times I$.

$$N \times I \overset{\text{diff}}{\cong} \widetilde{M_0} \cup h_1 \cup h_2 \cup \widetilde{M_1}.$$

The 2-handle h_2 is being attached to $\widetilde{M_0} \cup h_1$ along a Legendrian knot in the convex boundary component (N', η') of $\widetilde{M_0} \cup h_1$ with respect to the contact framing $+1$. We can, however, look at h_2 as being attached to $\widetilde{M_1}$. It, then, corresponds to a classical contact surgery on (N, η_1), i.e. a surgery with respect to the contact framing -1. Proposition 6.4.5 in [18] implies that the result of this surgery is contactomorphic to (N', η'). It follows that the required SSF structure can be defined by the following decomposition

$$N \times I \cong M_0 \cup_{N'} M_1 := (\widetilde{M_0} \cup h_1) \cup_{N'} (h_2 \cup \widetilde{M_1}).$$

Note that it is actually an SSF cobordism between $(N, e^{-1}\eta_0)$ and $(N, e\eta_1)$. This can be, however, easily modified to cobordism between the required manifolds. □

We will call the SSF cobordism constructed in Lemma 5.2.2 an **overtwisting SSF block**. The form η_1 appearing at the boundary of the block is indeed overtwisted, being the result of a negative stabilization.

Before stating the main theorem of this subsection let us introduce the techniques that will be used to modify the classical invariants of a Legendrian knot. See [18] for details.

Firstly, note from Equation (5.1) that one can easily decrease the Thurston-Bennequin invariant of a knot by adding additional cusps (so called "zig-zags") to a front projection of the knot in an appropriate Darboux map. The operation of adding upward (resp. downward) "zig-zags" is called a **positive** (resp. **negative**) **stabilization** of the knot (not to be confused with a stabilization of an open book). Observe from Equation (5.2) that the positive stabilization results, additionally, in the rotation number being decreased by 1. Similarly, the negative stabilization increases the rotation by 1.

knot's modification	**tb**	**rot**
positive stabilization	-1	-1
negative stabilization	-1	$+1$
positive destabilization	$+1$	$+1$
negative destabilization	$+1$	-1

Table 5.1: Impact of knot's modifications on its classical invariants

In the case of an overtwisted manifold one can also increase the value of the Thurston-Bennequin invariant of a knot. It can be achieved by taking a connected sum of the knot with a boundary of an overtwisted disc. The tb of such a boundary is 0 while rot relative to an overtwisted disc is ± 1 depending on the orientation. Note that such a construction can be performed only if we, additionally, assume that the original knot is loose (i.e. the contact structure is still overtwisted in the complement of the knot). In this case the operation is called the **positive** or **negative destabilization** and it is the inverse of the corresponding operation of stabilization (in the sense that after a destabilization we get a knot which can be stabilized to the original one). The positive (resp. negative) destabilization results in increasing (resp. decreasing) the rotation number by 1. If the knot is non-loose, we need to isotope the knot moving it away from the overtwisted disc before taking a connected sum. Then the modification of the knot is not necessarily an inverse of a stabilization but it results in the same change of the classical invariants as if the knot was loose.

For a summary of the impact of the described modifications on the classical invariants of a Legendrian knot consult Table 5.1.

Theorem 5.2.3. *Each closed oriented* 4*-dimensional manifold admits a generalized SSF structure.*

Proof. Let M be a closed oriented 4-dimensional manifold. Consider a nice handle decomposition of M. Let M_0, P and M_1 be codimension zero submanifolds of M such that M_0 is a sum of 0- and 1-handles, P is a sum of 2-handles while M_1 is a sum of the handles of indices 3 and 4. We obtain the following decomposition of the manifold M

$$M \cong M_0 \cup_{N_0} S_0 \cup_{N_0} P \cup_{N_1} S_1 \cup_{N_1} M_1,$$

where $\partial M_i = N_i$, $\partial P = N_0 \cup N_1$ and $S_i = N_i \times I$.

The manifolds M_0 and $\overline{M_1}$ are built out of handles of indices 0 and 1 so using contact 1-surgery we can construct exact symplectic filling structures on them. Let η_0 and η_1 denote the induced contact structures on the boundaries N_0 and N_1 of M_0 and $\overline{M_1}$ respectively.

Let $\mathbb{L}$ denote the link of the framed knots in N_0 used to attach the 2-handles. We can assume that $\mathbb{L}$, possibly after its isotopic perturbation, consists of Legendrian knots with respect to the contact structure η_0.

In order to get the required structure on M, we need to construct a generalized SSF structure with boundary on $S_0 \cup_{N_0} P \cup_{N_1} S_1$ such that (N_0, η_0) and (N_1, η_1) are the induced contact manifolds on the boundary components of $S_0 \cup_{N_0} P \cup_{N_1} S_1$.

Let us start with the trivial SSF structures on S_i coming from the trivial symplectic cobordisms

$$S_0 = (N_0 \times [-1,0], \mathrm{d}(e^{t+1}\eta_0)), \quad S_1 = (N_1 \times [0,1], \mathrm{d}(e^{-t+1}\eta_1)).$$

Denote by $\widetilde{\eta_i}$ the contact structure induced on $N_i \times \{0\}$ by the SSF structure on S_i. Obviously, at first we have $\widetilde{\eta_i} = e\eta_i$. Let us call $N_i \times \{0\} \subset S_i$ a free boundary of S_i (this indicates the fact that all the modifications of the SSF structure on S_i will be made along this component of the boundary and the other boundary component will stay fixed). Our goal is to modify the generalized SSF structures on S_i so that the following conditions are fulfilled.

(i) The knots in the link $\mathbb{L}$ can be appropriately stabilized or destabilized so that all of them have contact framing -1. We will denote the new link by $\mathbb{L}'$.

(ii) $(N_1, \widetilde{\eta_1})$ is the result of the contact surgery on $(N_0, \widetilde{\eta_0})$ along $\mathbb{L}'$. This gives P a structure of an exact symplectic cobordism from $(N_0, \widetilde{\eta_0})$ to $(N_1, \widetilde{\eta_1})$.

Note that constructing such generalized SSF structures on S_i is enough to prove the theorem. Indeed, a generalized SSF structure on M can then be obtained by gluing together the exact symplectic manifolds M_i, the generalized SSF manifolds S_i and the exact symplectic cobordism P along their common boundaries. Observe that, by the conditions (i) and (ii), the contact structures induced on the boundaries agree.

Let us start by arranging the contact framings of the knots in $\mathbb{L}$. Take $K \in \mathbb{L}$. If its contact framing is greater than -1, then we are able to decrease the Thurston-Bennequin invariant by stabilizing the knot an appropriate number of times. By the definition of tb this results in decreasing the contact framing. Note that since at this stage we are only interested in tb and we do not care about the rotation, it does not matter whether we use positive or negative stabilizations of K.

Let us now assume that the contact framing of K is less than -1. In this case we need to introduce an overtwisted disc to be able to destabilize K and, consequently, increase its tb invariant. To this end let us modify the generalized SSF structure on S_0 by attaching an overtwisting SSF block to its free boundary. In this way, on the free boundary of S_0 we get an overtwisted contact manifold $(N_0, \widetilde{\eta_0})$ with a loose knot K. One can now destabilize the knot K and, consequently, increase its framing.

After repeating the described procedure for all knots in $\mathbb{L}$, we obtain a new link $\mathbb{L}'$ consisting of knots of the contact framing -1.

It is now possible to perform a contact surgery on $(N_0, \widetilde{\eta_0})$ along $\mathbb{L}'$. As a result we obtain a contact manifold (N_1, η_1'). To finish the proof it is enough to make the two contact structures on N_1, η_1' and $\widetilde{\eta_1}$, agree.

We will use Theorem 2.3.1 saying that homotopic overtwisted contact structures are isotopic. By the discussion in Subsection 5.2.1, we know that the homotopy class of an oriented plane field ξ on a 3-manifold is completely determined by the two invariants: $c_1(\xi)$ and $d_3(\xi)$ provided that the second integral cohomology group of the manifold has no 2-torsion. The assumption on the cohomology group is, indeed, fulfilled by the manifold N_1 since it is a result of a sequence of 1-surgeries on S^3. Therefore, if only we ensure that the contact

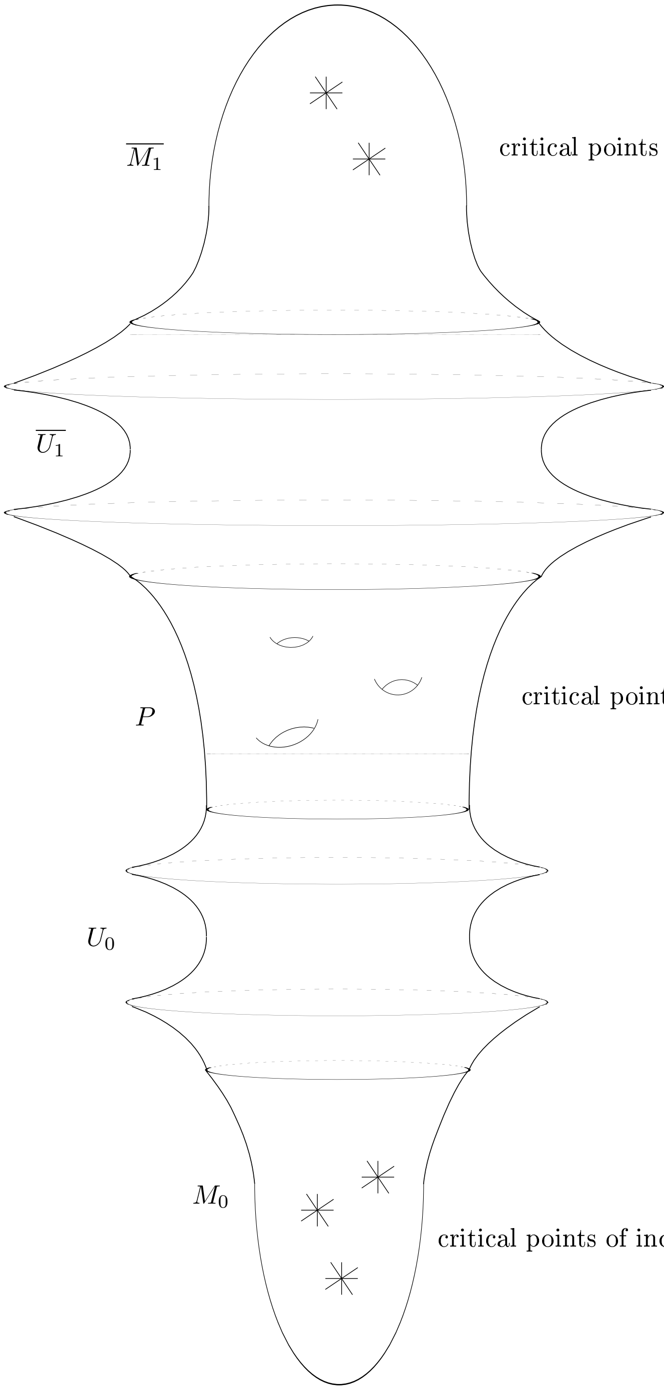


Figure 5.1: Generalized strong symplectic fold structure on M

structures η_1' and $\widetilde{\eta_1}$ are overtwisted, it is enough to make the two numbers $c_1(\xi)$ and $d_3(\xi)$ agree. This will be done by modifying concave SSF structures on S_0

and S_1.

The following technique of arranging $c_1(\xi)$ and $d_3(\xi)$ is well known and it was introduced in [13]. For the sake of completeness we will briefly describe the technique here. The presented argument is based on [15].

Let us start with arranging the invariant c_1. It follows from Equation (5.3) that $c_1(\ker(\widetilde{\eta_1})) = 0$. Note that at this point we still have $\widetilde{\eta_1} = e\eta_1$. We will now apply further modifications to the link $\mathbb{L}'$ and to the SSF structure on S_0 in order to make η_1' satisfy $c_1(\ker(\eta_1')) = 0$. We need to pay attention not to modify the already arranged contact framings of the knots in $\mathbb{L}'$. Note that by simultaneously applying a positive stabilization and a negative destabilization of a knot one can decrease its rotation by 2 without changing its framing. Similarly, we can increase the rotation by 2 by simultaneously applying a negative stabilization and a positive destabilization. As in the previous step of the proof, in order to be able to perform a knot's destabilization, we modify the SSF structure on S_0 by attaching an overtwisting SSF block to the free boundary of S_0. It is, therefore, possible to change the rotation number of a knot by an arbitrary even number. Since $P.D.c_1(\ker(\eta_1'))|_2 = 0$ (see Proposition 4.1. in [22]), it follows that we can modify η_1' so that $c_1(\ker(\eta_1')) = 0$.

Finally, let us deal with the d_3 invariant. Note that, according to Equation (5.4), a negative stabilization of a contact structure increases the d_3 invariant by 1, since it introduces a new surgery with respect to the contact framing $+1$. It follows that we can make the invariants agree by negatively stabilizing one of the contact structures η_1' or $\widetilde{\eta_1}$. This corresponds to attaching an overtwisting block to the free boundary of S_0 or $\overline{S_1}$ respectively.

The resulting contact structures $\widetilde{\eta_1}$ and η_1' are of the same homotopy class. To make them isotopic we need to ensure that they are both overtwisted. If necessary, this can be done by attaching a further overtwisting block to S_0 and S_1. □

Note that analogously we get the following result for manifolds with boundary.

Theorem 5.2.4. *Let M be an oriented 4-dimensional manifold with boundary and let μ be a contact structure on $Q = \partial M$. Then M admits a generalized SSF structure with the contact boundary (Q, μ).*

Chapter 6

Properties of SSF-contact structures

In the following chapter, we study properties of SSF-contact structures. Let us recall that an SSF structure on a manifold M induces a contact form on $M \times S^1$ given by an explicit formula. Such a contact form is called an SSF-contact form. See Section 4.7 for details.

We determine some basic properties of SSF-contact structures. Moreover, we classify SSF-contact structures in dimension 3 up to homotopy of the corresponding 2-distributions. Finally, we discuss fillability of SSF-contact forms giving examples of classes of both fillable and non-fillable manifolds.

We organize the chapter in the following way.

In Section 6.1, we discuss general properties of SSF-contact manifolds. More specifically, we find a family of convex hypersurfaces, we give the concrete formula for the Reeb vector field and we discuss the transversality of the S^1-loops.

In Section 6.2, we classify SSF-contact structures in dimension 3 up to homotopy of the corresponding 2-distributions. In particular, we prove that for a given oriented surface Σ there exists an SSF-contact structure on $\Sigma \times S^1$ in every homotopy type of S^1-invariant cooriented 2-distributions.

In Section 6.3, we discuss the fillability of SSF-contact structures. We prove that SSF-contact structures on $(M \cup \overline{M}) \times S^1$ are fillable. Moreover, we discuss applications of the planar torsion to construct examples of non-fillable SSF-contact manifolds of dimension 3.

6.1 General properties of SSF-contact structures

Consider an SSF manifold

$$M = M_0 \cup_N \overline{M_1},$$

where on N there exists a contact structure η such that $(M_0, \mathrm{d}\lambda_0)$ and $(M_1, \mathrm{d}\lambda_1)$ are exact symplectic fillings of (N, η). Let α be the induced contact form on $M \times S^1$. Let us recall that the form α is defined by the following formula.

$$\alpha = \begin{cases} \lambda_1 + \mathrm{d}\theta & \text{on} \quad \mathring{M}_0 \times S^1, \\ f(t)\eta + g(t)\mathrm{d}\theta & \text{on} \quad \underline{(N \times [-1,1])} \times S^1, \\ \lambda_2 - \mathrm{d}\theta & \text{on} \quad \mathring{M}_1 \times S^1. \end{cases}$$

Here we use the notation from Section 4.7.

S^1-invariance

One can see from the formula that the form α is S^1-invariant. It follows that ∂_θ is a contact vector field of α and a hypersurface $M \times \{\theta\}$ is a convex hypersurface for each $\theta \in S^1$. Note, moreover, that $(N \times \{0\}) \times \{\theta\}$ is the dividing set on such a hypersurface with respect to the contact vector field ∂_θ. Indeed,

$$\alpha(\partial_\theta) = \begin{cases} 1, & \text{on} \quad \mathring{M}_0 \times S^1, \\ g(t), & \text{on} \quad (N \times [-1,1]) \times S^1, \\ -1, & \text{on} \quad \overline{\mathring{M}_1} \times S^1, \end{cases}$$

which is equal to 0 precisely on $(N \times \{0\}) \times S^1$ (see Figure 4.8).

Reeb vector field

Let us determine the formula for the Reeb vector field of the form α. The derivative of α is given by

$$\mathrm{d}\alpha = \begin{cases} \mathrm{d}\lambda_1, & \text{on} \quad \mathring{M}_0 \times S^1, \\ f(t)\mathrm{d}\eta + f'(t)\mathrm{d}t \wedge \eta + g'(t)\mathrm{d}t \wedge \mathrm{d}\theta, & \text{on} \quad (N \times [-1,1]) \times S^1, \\ \mathrm{d}\lambda_2, & \text{on} \quad \overline{\mathring{M}_1} \times S^1. \end{cases}$$

Observe that the vector field $R = \partial_\theta$ fulfills the Reeb vector field conditions on $\mathring{M}_0 \times S^1$. Indeed,

$$\mathrm{d}\alpha(R, \cdot) = 0, \quad \alpha(R) = 1.$$

Similarly, on $\overline{\mathring{M}_1} \times S^1$ the vector field $R = -\partial_\theta$ is the Reeb vector field.

Let us now focus on the set $(N \times [-1,1]) \times S^1$ and look for the Reeb vector field among the fields of the form

$$R = a(t)R_N + c(t)\partial_\theta,$$

where R_N is the Reeb vector field on (N, η). Let us calculate

$$\mathrm{d}\alpha(R, \cdot) = -a(t)f'(t)\mathrm{d}t - c(t)g'(t)\mathrm{d}t$$

and

$$\alpha(R) = a(t)f(t) + c(t)g(t).$$

It follows that in order to find R, we need to solve the following system of equations

$$\begin{aligned} -a(t)f'(t) - c(t)g'(t) &= 0 \\ a(t)f(t) + c(t)g(t) &= 1 \end{aligned}$$

After multiplying the first equation by $f(t)$ (which by assumption is nowhere zero, see condition (iii) in the definition of the contact SSF form) and pluging in $a(t)f(t)$ from the second equation we get

$$-\Big(1 - c(t)g(t)\Big)f'(t) - c(t)f(t)g'(t) = 0.$$

Thus we can get $c(t)$ as

$$c(t) = -\frac{f'(t)}{f(t)g'(t) - f'(t)g(t)}.$$

We can divide by $f(t)g'(t) - f'(t)g(t)$, since this expression is positive by assumption (see condition (iv) in the definition of the contact SSF form). Note that $c(t) = 0$ precisely for $t = 0$, $c(t)$ is positive for $t < 0$ and negative for $t > 0$. For the sake of completeness let us give the formula for $a(t)$.

$$a(t) = \frac{g'(t)}{f(t)g'(t) - f'(t)g(t)}$$

It follows that for $t = 0$ the Reeb vector field R is tangent to N and is equal to the scaled Reeb vector field of (N, η).

Transversality

From the form of the Reeb vector field indicated in the previous paragraph we see that the curves $\{z\} \times S^1$ are positively transverse to the contact structure for $z \in \mathring{M}_0 \cup \big(N \times [-1, 0)\big)$. Similarly, the curves $\{z\} \times S^1$ are negatively transverse to the contact structure for $z \in \mathring{M}_1 \cup \big(N \times (0, 1]\big)$. Moreover, since $\alpha(\partial_\theta) = 0$ on $\big(N \times \{0\}\big) \times S^1$, the curves are Legendrian for $z \in N \times \{0\}$.

6.2 Up to homotopy classification of SSF-contact structures in dimension 3

In this section, we study homotopy classes of hyperplane fields corresponding to SSF-contact structures. In particular, we show that for manifolds of dimension 3 there exists an SSF-contact structure in any homotopy class of S^1-invariant 2-distributions.

SSF-contact structures as maps to S^{2n}

Consider an SSF manifold M^{2n} such that $M \times S^1$ is parallelizable. Homotopy classes of $2n$-dimensional coorientable distributions on $M \times S^1$ are in one-to-one correspondence with homotopy classes of functions $M \times S^1 \to S^{2n}$. For a contact form α on $M \times S^1$ a corresponding function can be defined as the Reeb vector field of α projected to the fiber of a trivialization of $T(M \times S^1)$ and normalized.

If we restrict ourselves to S^1-invariant contact structures, it is enough to consider homotopy classes of functions $M \to S^{2n}$. By the Hopf theorem, such a homotopy class is uniquely determined by the degree of a function representing that class.

Let α be a contact form on $M \times S^1$ induced by an SSF structure on M. Let us here use the same notation as in Section 6.1.

As was noted in Section 6.1, the form α is S^1-invariant. Thus, from the discussion above it follows that the homotopy class of the induced $2n$-distribution corresponds to a homotopy class $[f_\alpha]$ of functions $M \to S^{2n}$ determined by the Reeb vector field and is uniquely determined by the degree of a function f_α.

In Section 6.1, we have found the following formula for the Reeb vector field of α

$$R = \begin{cases} \partial_\theta & \text{on} \quad \mathring{M}_0 \times S^1, \\ a(t)R_N + c(t)\partial_\theta & \text{on} \quad \underline{(N \times [-1,1])} \times S^1, \\ -\partial_\theta & \text{on} \quad \mathring{M}_1 \times S^1, \end{cases} \tag{6.1}$$

where the functions a and c are defined as

$$a(t) = \frac{g'(t)}{f(t)g'(t) - f'(t)g(t)}, \quad c(t) = -\frac{f'(t)}{f(t)g'(t) - f'(t)g(t)}$$

and R_N is the Reeb vector field on (N, η).

3-dimensional case

Let us now focus on the case of 3-dimensional SSF-contact manifolds, where the degrees of functions f_α can be analyzed explicitly. Let Σ be a closed, oriented surface and let

$$\Sigma = \Sigma_0 \cup_\Gamma \overline{\Sigma}_1$$

be an SSF decomposition of Σ determined by the union Γ of disjoint circles on Σ. The circles Γ form the separating hypersurface of the SSF structure. Denote by α the induced contact structure on $\Sigma \times S^1$. We are going to construct a function f_α, which corresponds to α as described above.

Let

$$h = (h_1, h_2, h_3) : \Sigma \times \{\theta\} \xrightarrow{\cong} \tilde{\Sigma} \subset \mathbb{R}^3$$

be an embedding of $\Sigma \times \{\theta\}$ into $\mathbb{R}^3$. We can choose h in such a way that $\tilde{\Gamma} = h(\Gamma)$ is contained in the XY plane and that a neighbourhood of each circle $\gamma \subset \tilde{\Gamma}$ in $\tilde{\Sigma}$ is of the form $\gamma \times [-\epsilon, \epsilon]$. Moreover, let the embedding h be such that the Reeb vector field R of α is equal to $\pm\partial_\theta$ outside the counter image of sets $\gamma \times [-\epsilon, \epsilon]$.

The diffeomorphism $h : \Sigma \times \{\theta\} \xrightarrow{\cong} \tilde{\Sigma}$ induces an isomorphism h_* of tangent bundles

$$h_* : \mathrm{T}(\Sigma \times \{\theta\}) \to \mathrm{T}\tilde{\Sigma}.$$

This allows us to construct an isomorphism of the following vector bundles

$$F : \mathrm{T}(\Sigma \times S^1) \mid_{\Sigma \times \{\theta\}} \to \mathrm{T}\mathbb{R}^3 \mid_{\tilde{\Sigma}}$$

by setting $F(v) = h_*(v)$ for $v \in T(\Sigma \times \{\theta\})$ and sending ∂_θ to the outward pointing vector field to $\tilde{\Sigma}$.

Then the value of a function $f_\alpha : \Sigma \to S^2$ for a point $p \in \Sigma$ can be defined by the following composition

$$p \mapsto R_{(p,\theta)} \overset{F}{\mapsto} F(R)_{h(p)} \overset{\|\pi\|}{\to} S^2,$$

where $R_{(p,\theta)}$ is the Reeb vector of the contact form α at the point (p, θ) and $\|\pi\|$ is the projection of $\mathrm{T}\mathbb{R}^3$ to a fiber via the standard trivialization of $\mathrm{T}\mathbb{R}^3$ followed by the normalization of a vector.

From Formula (6.1) for the Reeb vector field it follows that for a point $p \in \Sigma$ away from the dividing set Γ the value of the function f_α is a normal vector to $\tilde{\Sigma}$ at the point $h(p)$ (pointing outwards or inwards $\tilde{\Sigma}$). As the point p approaches

Γ, the value of f_α leans to the surface $\tilde{\Sigma}$ to finally become tangent to $\tilde{\Gamma}$ for $p \in \Gamma$.

Our goal, now, is to describe a procedure of calculating the degree of the function f_α. For this reason let us consider the following operation on the surface Σ producing a new surface Σ' and let us call this operation the **reducing operation**. The operation will consist of the following two steps.

1. Cut the surface along a separating circle $\gamma \in \Gamma$.

2. Glue a disc to each of the two boundary circles produced in the step 1.

The reducing operation produces a new surface $\Sigma' = \Sigma \backslash \gamma \cup D^2 \cup D^2$ (see Figure 6.1). Let us consider the SSF structure on Σ' with $\Gamma \backslash \gamma$ as the dividing set. Denote by α' the induced contact structure on Σ' and by $f_{\alpha'}$ the corresponding function $\Sigma' \to S^2$.

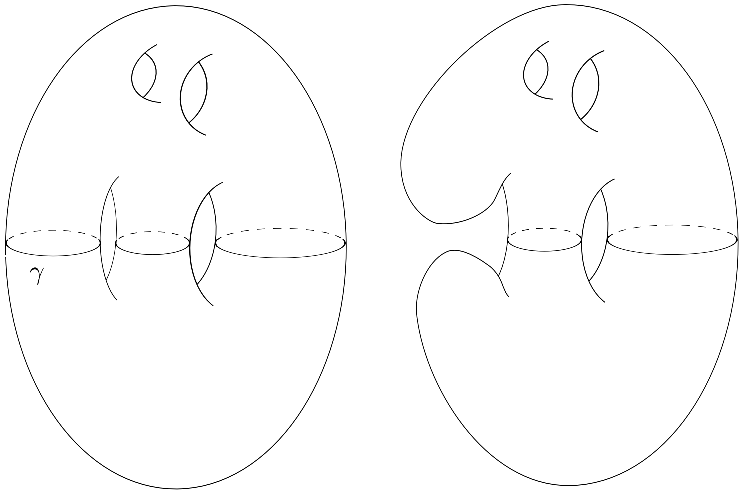


Figure 6.1: Reducing operation

The reducing operation does not change the degree of the function associated with the contact structure.

Lemma 6.2.1. $\deg(f_{\alpha'}) = \deg(f_\alpha)$.

Proof. Let us consider the point $(0, 0, 1) \in S^2$. By the form of the embedding h the counter image of $(0, 0, 1)$ by f_α is fully contained in $\mathring{\Sigma}_0 \sqcup \overline{\mathring{\Sigma}_1}$.

We can perform the reducing operation and attach the two discs in such a way that

$$f_{\alpha'}^{-1}(0, 0, 1) = f_\alpha^{-1}(0, 0, 1) \cup \{p_0, p_1\},$$

where p_0 and p_1 belong to the attached discs. Note that $f_{\alpha'}$ is orientation preserving at one of the points p_0 and p_1 and orientation reversing at the other. It follows that the degrees of f_α and $f_{\alpha'}$ are equal. $\square$

Note that the reducing operation allows us to calculate the degree of any function induced by an SSF structure. Indeed, by iterating the operation one can get rid of the separating set. Now, in order to calculate the degree of f_α, it is enough to know degrees of functions f_Σ. Here, a function $f_\Sigma : \Sigma \to S^2$ for a given oriented surface $\Sigma \subset \mathbb{R}^3$ is a normalized outward pointing normal vector field to Σ. One can easily prove the following lemma.

Lemma 6.2.2. *Let Σ be a closed oriented surface of genus g. Then*

$$\deg(f_\Sigma) = 1 - g.$$

Now, let us show that any number can be represented as a degree of a function corresponding to an SSF-contact structure.

Proposition 6.2.3. *For a given closed oriented surface Σ every integer can be represented as the degree of the function corresponding to an SSF-contact structure on $\Sigma \times S^1$.*

Proof. Let

$$\Sigma = \Sigma_0 \cup_\Gamma \overline{\Sigma}_1$$

be an SSF structure on Σ, let α be the implied contact form on $\Sigma \times S^1$ and let f_α be the function related to α as described above.

Consider the following modified SSF structure on Σ

$$\Sigma = (\Sigma_0 \cup D^2) \cup_{\Gamma \cup \gamma} \overline{(\Sigma_1 \backslash D^2)}$$

with a new contractible loop γ added to the dividing hypersurface. Denote by α' the contact structure associated with the new SSF structure. Then

$$\deg(f_{\alpha'}) = \deg(f_\alpha) + 1.$$

It can be seen by performing the reducing operation on the new structure along the separating circle γ.

Similarly, for the SSF structure

$$\Sigma = (\Sigma_0 \backslash D^2) \cup_{\Gamma \cup \gamma} \overline{(\Sigma_1 \cup D^2)}$$

the degree of the new function is given by

$$\deg(f_{\alpha'}) = \deg(f_\alpha) - 1.$$

By iterating one of these operations one can get a contact structure with an arbitrary degree of the induced function. □

Corollary 6.2.4. *For a given oriented surface Σ there exists an SSF-contact structure on $\Sigma \times S^1$ in every homotopy class of S^1-invariant cooriented 2-distributions.*

Remark 6.2.5. Let us formulate the above results in terms of the classical invariants d_2 and d_3 of the 2-distributions (see Section 4.2 in [18] for the definitions of d_2 and d_3). Let α be an SSF-contact structure. Then $d_2(\ker \alpha) = \deg(f_\alpha)$. Moreover, if the d_2-invariants of SSF-contact forms α_0 and α_1 are equal, then $d_3(\ker \alpha_0, \ker \alpha_1) = 0$.

6.3 Fillability of SSF-contact structures

In this section, we are going to discuss fillability of SSF-contact structures.

In Subsection 6.3.1, we will construct an exactly fillable SSF-contact structure on $S^{2n} \times S^1$. In Subsection 6.3.2, we are going to generalize this construction to show that the natural SSF structure on doubles induces an exactly fillable SSF-contact structure. Finally, in Subsection 6.3.3, we will use the notion of planar torsion (see [41]) to give examples of non-fillable SSF-contact structures in dimension 3.

6.3.1 Fillable SSF-contact structures on $S^{2n} \times S^1$

In this section, we are going to consider standard contact structures on $S^{2n} \times S^1$. We will observe that the structures are fillable. Finally, we will point SSF-contact structures on $S^{2n} \times S^1$ isotopic to the standard structures.

Standard contact structures on $S^{2n} \times S^1$

Consider the following form α_{st} on $S^{2n} \times S^1$

$$\alpha_{\mathrm{st}} := z\mathrm{d}\theta + \sum_{i=1}^{n} (x_i \mathrm{d}y_i - y_i \mathrm{d}x_i),$$

where θ is the S^1-coordinate, $(z, x_1, y_1, \ldots, x_n, y_n)$ are the standard coordinates of $\mathbb{R}^{2n+1}$ and $S^{2n} \subset \mathbb{R}^{2n+1}$ is the inclusion of S^{2n} as the unit sphere.

Lemma 6.3.1. *α_{st} is an exactly fillable contact form on $S^{2n} \times S^1$.*

Proof. Consider the manifold $\mathbb{R}^{2n+1} \times S^1$ with the following symplectic form

$$\omega := \mathrm{d}z \wedge \mathrm{d}\theta + 2 \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

The vector field $Y = z\partial_z + \frac{1}{2} \sum_{i=1}^{n} (x_i \partial_{x_i} + y_i \partial_{y_i})$ is a Liouville vector field of ω. Indeed,

$$\mathscr{L}_Y \omega = \mathrm{d}\iota_Y \omega = \mathrm{d}\big(z\mathrm{d}\theta + \sum_{i=1}^{n} (x_i \mathrm{d}y_i - y_i \mathrm{d}x_i)\big) = \omega.$$

Moreover, the field Y is transverse to $S^{2n} \times S^1$. It follows that $\alpha_{\mathrm{st}} = \iota_Y \omega$ is a contact form on $S^{2n} \times S^1$ and $(D^{2n+1} \times S^1, \omega)$ is its exact symplectic filling. □

SSF-contact structures on $S^{2n} \times S^1$

Consider an SSF structure on S^{2n} associated with the decomposition

$$S^{2n} = D^{2n} \cup \overline{D^{2n}},$$

where D^{2n} is equipped with the symplectic form $\mathrm{d}\lambda$ for $\lambda = \sum_{i=1}^{n} (x_i \mathrm{d}y_i - y_i \mathrm{d}x_i)$. Denote by α the contact form on $S^{2n} \times S^1$ induced by the SSF structure.

Lemma 6.3.2. *The contact form α is isotopic to α_{st}.*

Proof. Consider the following projection

$$\pi : S^{2n} \to D^{2n},\ (x_i, y_i, z) \mapsto (x_i, y_i).$$

It induces a form $\hat{\lambda}$ on S^{2n} as

$$\hat{\lambda} = \pi^* \lambda = \sum_{i=1}^{n} (x_i \mathrm{d} y_i - y_i \mathrm{d} x_i).$$

Let $r = \sum_{i=1}^{n} (x_i^2 + y_i^2)$. For $r \neq 0$ we can calculate

$$\lambda = \sum_{i=1}^{n} (x_i \mathrm{d} y_i - y_i \mathrm{d} x_i) = r^2 \sum_{i=1}^{n} \left(\frac{x_i}{r} \mathrm{d}\left(\frac{y_i}{r}\right) - \frac{y_i}{r} \mathrm{d}\left(\frac{x_i}{r}\right) \right) = r^2 \eta,$$

where η is the standard contact form on the unit sphere S^{2n-1}. After lifting the form to S^{2n} we get

$$\hat{\lambda} = r^2 \eta = (1 - z^2)\eta.$$

Using this notation we get the following formulas for α_{st} and α

$$\alpha_{\mathrm{st}} = \begin{cases} \hat{\lambda} + z \mathrm{d}\theta, & \text{for} \quad z \in [\epsilon, 1], \\ (1 - z^2)\eta + z \mathrm{d}\theta, & \text{for} \quad z \in (-\epsilon, \epsilon), \\ \hat{\lambda} + z \mathrm{d}\theta, & \text{for} \quad z \in [-1, -\epsilon] \end{cases}$$

and

$$\alpha = \begin{cases} \hat{\lambda} + \mathrm{d}\theta, & \text{for} \quad z \in [\epsilon, 1], \\ f(z)\eta + g(z) \mathrm{d}\theta, & \text{for} \quad z \in (-\epsilon, \epsilon), \\ \hat{\lambda} - \mathrm{d}\theta, & \text{for} \quad z \in [-1, -\epsilon]. \end{cases}$$

Let us, first, focus on $z \in (-\epsilon, \epsilon)$. The condition for the form $f(z)\eta + g(z)\mathrm{d}\theta$ to be contact is equivalent to say that $f \neq 0$ and a vector from the origin to a point on the curve $(f(z), g(z))$ is never tangent to the curve at that point.

One can choose a path of functions f_t and g_t from $f_0 = f$ and $g_0 = g$ to $f_1 = 1 - z^2$ and $g_1 = z$ so that this condition is fulfilled for every pair (f_t, g_t) and consequently the forms $f_t(z)\eta + g_t(z)\mathrm{d}\theta$ are contact (see Figure 6.2). The function g_t can be chosen in such a way that it is linear close to $z = \pm\epsilon$ with the slope increasing from 0 to 1 as t runs from 0 to 1.

Now let us consider $z \in [\epsilon, 1]$. Here the isotopy α_t between α and α_{st} can be constructed as

$$\alpha_t = \hat{\lambda} + l_t(z)\mathrm{d}\theta,$$

where l_t is the linear part of the function g_t close to ϵ. An easy calculation shows that α_t is a contact structure for every t.

The case of $z \in [-1, -\epsilon]$ is analogous.

The three described isotopies collectively form an isotopy from α_{st} to α. □

Corollary 6.3.3. *The contact structure α is strongly fillable.*

Remark 6.3.4. Note that in dimension 3 Lemma 6.3.2 follows from the Lutz's construction in [28], which implies that for a decomposition $\Sigma = \Sigma_0 \cup_\Gamma \overline{\Sigma}_1$ of a surface Σ the product $\Sigma \times S^1$ admits a unique (up to isotopy) S^1-invariant contact structure for which the loops $\{z\} \times S^1$ are positively transverse for z in the interior of Σ_0, negatively transverse for z in the interior of $\overline{\Sigma}_1$ and Legendrian for $z \in \Gamma$.

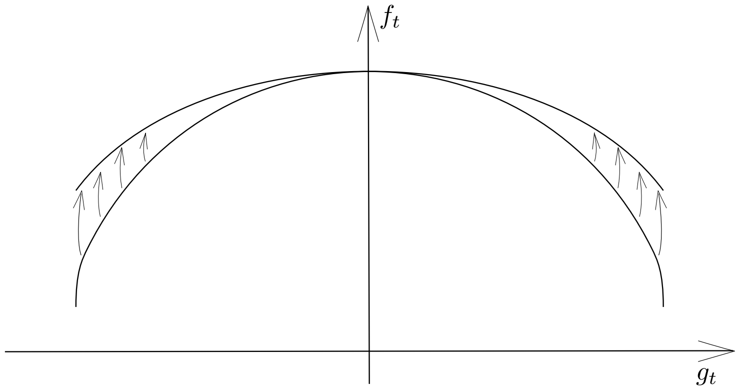


Figure 6.2: A path of functions (f_t, g_t)

6.3.2 Fillability of SSF-contact structures on $(M \cup \overline{M}) \times S^1$

Let $(M, \mathrm{d}\lambda)$ be an exact symplectic manifold with convex boundary.

The double of M admits a natural SSF structure (see Section 4.2). Denote by α the contact form on $(M \cup \overline{M}) \times S^1$ induced by the SSF structure.

Proposition 6.3.5. *The contact manifold* $((M \cup \overline{M}) \times S^1, \alpha)$ *admits an exact symplectic filling.*

Proof. Let us start with introducing a notation of the half disc D^2_+, which we will use throughout the proof.

$$D^2_+ := D^2 \cap \{x \leq 0\} = \{x^2 + y^2 \leq 1\} \cap \{x \leq 0\}$$

The boundary of D^2_+ consists of two parts $\partial D^2_+ = I_c \cup I_l$, where

$$I_c = \{x^2 + y^2 = 1, x \leq 0\}, \quad I_l = \{x = 0, y \in [-1, 1]\}.$$

Let us consider the manifold W defined as

$$W = (M \times [-1, 1]) \cup_f (\partial M \times D^2_+). \tag{6.2}$$

We have that

$$\partial(M \times [-1, 1]) = (\partial M \times [-1, 1]) \cup (M \times \{-1, 1\}),$$

and

$$\partial(\partial M \times D^2_+) = (\partial M \times I_c) \cup (\partial M \times I_l).$$

The gluing function f from the definition (6.2) of W relates the part $\partial M \times [-1, 1]$ of the boundary of $M \times [-1, 1]$ with the part $\partial M \times I_l$ of the boundary of $\partial M \times D^2_+$ and can be defined by

$$\begin{aligned} f : \partial M \times [-1, 1] &\to \partial M \times I_l, \\ (x, s) &\mapsto (x, (0, s)). \end{aligned}$$

See Figure 6.3 for a schematic picture of W.

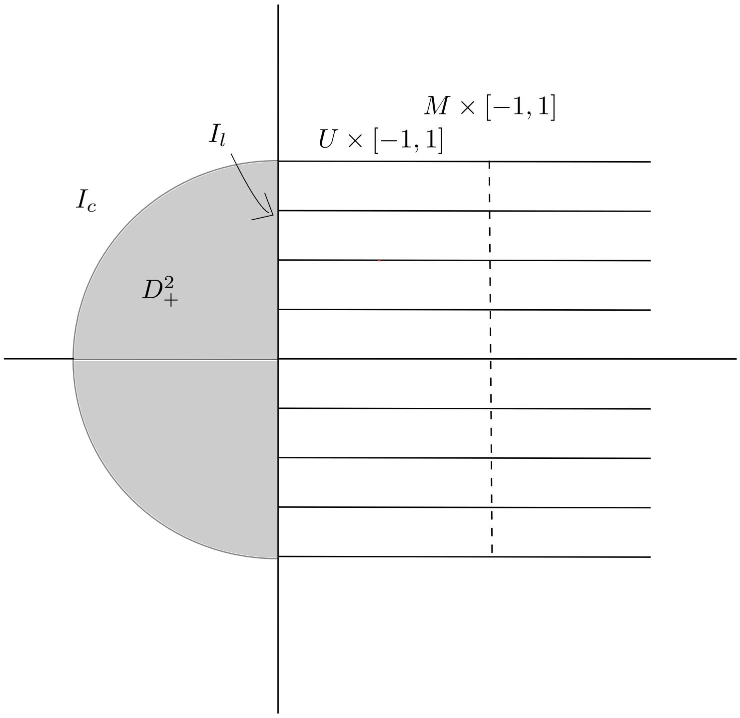


Figure 6.3: Filling W of an SSF-contact structure on $(M \cup \overline{M}) \times S^1$

The boundary of W is given by the formula

$$\partial W = M \times \{-1, 1\} \cup \partial M \times I_c.$$

Note that ∂W is diffeomorphic to the double of M. Therefore, $W \times S^1$ is a topological filling of $(M \cup \overline{M}) \times S^1$. Our goal is to construct an exact symplectic structure $\mathrm{d}\mu$ on $W \times S^1$ convex on the boundary.

Since $(M, \mathrm{d}\lambda)$ has a convex boundary, we can choose a collar neighbourhood $U \cong \partial M \times [\frac{1}{2}, 1]$ of the boundary $M \times \{1\}$ of M so that

$$\lambda = \frac{1}{2} r^2 \eta,$$

where η is a contact form on ∂M and $r \in [\frac{1}{2}, 1]$.

Now, on $(U \times [-1,1]) \cup_f (\partial M \times D^2_+) \subset W$ we can define a map π to $\mathbb{R}^2$ as

$$\begin{aligned} \pi(p, r, t) &= (1 - r, t) && \text{on } U \times [-1, 1], \\ \pi(p, x, y) &= (x, y) && \text{on } \partial M \times D^2_+, \end{aligned}$$

where $p \in \partial M$, $r \in [\frac{1}{2}, 1]$, $t \in [-1, 1]$ and $(x, y) \in D^2_+$. The map π together with the smooth coordinates on ∂M produces smooth coordinates on W around the points of the submanifold along which we glue the two pieces of W together. We will use these coordinates to prove the smoothness of the constructed symplectic form $\mathrm{d}\mu$.

Let us define the form μ as

$$\mu = \begin{cases} \lambda + t\mathrm{d}\theta & \text{on} \quad M \times [-1,1], \\ \frac{1}{2}(1-x)^2\eta + y\mathrm{d}\theta & \text{on} \quad \partial M \times D^2_+. \end{cases}$$

We see from the definition of the function π and from the form of λ close to the boundary that the form μ is smooth. Let us calculate the derivative of μ.

$$\mathrm{d}\mu = \begin{cases} \mathrm{d}\lambda + \mathrm{d}t\mathrm{d}\theta & \text{on} \quad M \times [-1,1] \\ \frac{1}{2}(1-x)^2\mathrm{d}\eta - (1-x)\mathrm{d}x\eta + \mathrm{d}y\mathrm{d}\theta & \text{on} \quad \partial M \times D^2_+ \end{cases}$$

The form $(\mathrm{d}\mu)^n$ is then given by

$$(\mathrm{d}\mu)^{n+1} = \begin{cases} (n+1)(\mathrm{d}\lambda)^n \mathrm{d}t\,\mathrm{d}\theta & \text{on} \quad M \times [-1,1], \\ \frac{n(n+1)}{2^{n-1}}(1-x)^{2n-1}\eta(\mathrm{d}\eta)^{n-1}\mathrm{d}x\mathrm{d}y\mathrm{d}\theta & \text{on} \quad \partial M \times D^2_+. \end{cases}$$

We see that $(\mathrm{d}\mu)^{n+1} \neq 0$, what proves that $\mathrm{d}\mu$ is symplectic.

Consider the following vector field Y on W

$$Y = \begin{cases} Y_M + t\partial_t & \text{on} \quad M \times [-1,1], \\ -\frac{1}{2}(1-x)\partial_x + y\partial_y & \text{on} \quad \partial M \times D^2_+, \end{cases}$$

where Y_M is a Liouville vector field on $(M, \mathrm{d}\lambda)$. Then Y is a Liouville vector field on $(W, \mathrm{d}\mu)$. Indeed,

$$\iota_Y \mathrm{d}\mu = \begin{cases} \iota_{Y_M}\mathrm{d}\lambda + t\mathrm{d}\theta & \text{on} \quad M \times [-1,1], \\ \frac{1}{2}(1-x)^2\eta + y\mathrm{d}\theta & \text{on} \quad \partial M \times D^2_+. \end{cases}$$

Since Y_M is Liouville for $\mathrm{d}\lambda$, we have $\iota_{Y_M}\mathrm{d}\lambda = \lambda$ and, consequently, $\iota_Y \mathrm{d}\mu = \mu$.

Note that Y points outwards along the boundary of W. It follows that $(W, \mathrm{d}\mu)$ is an exact symplectic filling of $(\partial W, \mu)$. Similarly as in the proof of Lemma 6.3.2 one can see that the manifold $(\partial W, \mu)$ is contactomorphic to $((M \cup \overline{M}) \times S^1, \alpha)$. □

6.3.3 Planar torsion and fillability of SSF-contact 3-manifolds

In Section 3.3, we have mentioned the notion of planar torsion. Here, we are going to use planar torsion to give examples of non-fillable SSF-contact structures.

Planar torsion was introduced in [40] by the means of blown up summed open books (a generalization of open book decompositions also defined in [40]).

Definition 6.3.6. Let N be a compact 3-manifold. A **blown up summed open book** on N is a fibration

$$\pi : N \setminus (B \cup \mathscr{I}) \to S^1,$$

where $B \subset N \setminus \partial N$ (the **binding**) is an oriented link and $\mathscr{I} \subset N \setminus \partial N$ (the **interface**) is a set of disjoint 2-tori.

The connected components of the fibres are called **pages**.

A contact structure ξ on N is said to be **supported by** π, if there exists a Reeb vector field of ξ positively transverse to the pages and positively tangent to their oriented boundaries and, moreover, the characteristic foliation of ξ on each boundary or interface torus has closed leaves in the homology class of the meridian.

A blown up summed open book is called **irreducible**, if the fibres are connected. A general blown up summed open book π on a manifold N can be decomposed to a union of **irreducible subdomains**

$$N = N_1 \cup N_2 \cup \dots N_n,$$

where N_i are manifolds with irreducible blown up summed open book structures π_i such that pages of π_i are pages of π and they are glued along tori from the interface of π. A blown up summed open book is called **symmetric**, if it has no boundary and it contains exactly two irreducible subdomains each of pages of the same topological type and empty binding and interface.

Note that 3-dimensional SSF-contact manifolds admit a natural blown up summed open book. Indeed, as discussed in Section 4.2, any decomposition of a surface Σ into a sum of two surfaces Σ_0 and $\overline{\Sigma}_1$ along their common boundary Γ is an SSF-decomposition. Then, one can consider a blown up summed open book on $\Sigma \times S^1$ with an empty binding, interface equal to $\Gamma \times S^1$ and a fibration π given by the projection to S^1 preserving the orientation of S^1 on $\Sigma_0 \times S^1$ and reversing the orientation on $\Sigma_1 \times S^1$. The SSF-contact structure induced by the decomposition $\Sigma = \Sigma_0 \cup_\Gamma \overline{\Sigma}_1$ is supported by π. Indeed, the Reeb vector field, as calculated in Section 6.1, is positively transverse to the pages. Moreover, since $\partial_\theta \in \xi$ and the Reeb vector field is tangent to Γ we see that the characteristic foliation of ξ on each interface torus has closed leaves in the homology class of the meridian.

Definition 6.3.7. Consider a contact 3-manifold and an integer $k \geq 0$. We say that the manifold has **planar k-torsion**, if it admits a contact embedding of a connected contact manifold (N, ξ) with a blown up summed open book π supporting ξ and fulfilling the following conditions.

- There exists a planar irreducible subdomain $N^P \subset N$ with pages of $k+1$ boundary components.
- $N \setminus N^P$ is not empty.
- π is not symmetric.

In [41], it is shown that planar torsion is an obstruction to fillability.

Planar torsions can be naturally used to define a family of non-fillable SSF-contact structures. Indeed, consider an SSF structure induced by a decomposition $\Sigma = \Sigma_0 \cup_\Gamma \overline{\Sigma}_1$, where Σ_0 and Σ_1 are not diffeomorphic and one of Σ_0 and Σ_1 contains a planar connected component. Such an SSF structure determines the SSF-contact structure on $\Sigma \times S^1$ which admits a blown up summed open book as described above. This open book determines a planar torsion of the contact structure, what implies that the contact structure is non-fillable. See Figure 6.4 for an example of an SSF structure inducing a non-fillable SSF-contact manifold with a planar torsion.

The proposition below follows from the discussion in this and the previous section.

Proposition 6.3.8. *For any oriented surface Σ the manifold $\Sigma \times S^1$ admits both fillable and non-fillable SSF-contact structures.*

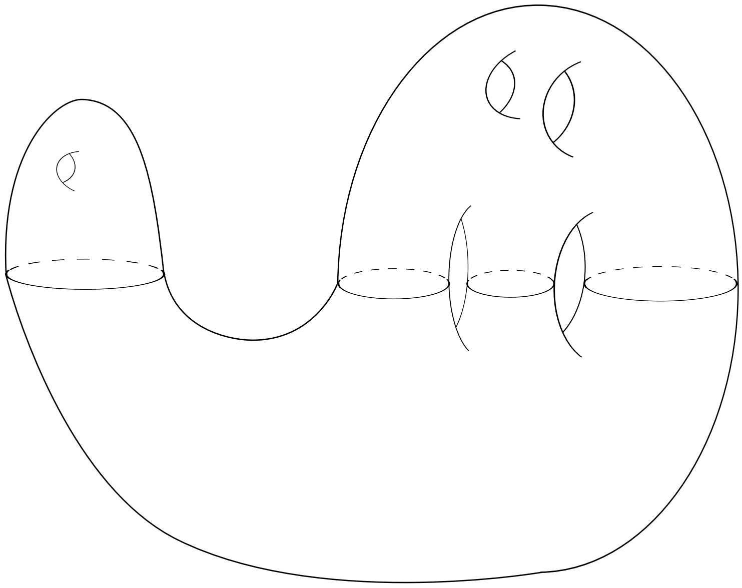


Figure 6.4: An SSF structure inducing a non-fillable SSF-contact manifold with planar torsion 3

Proof. A fillable structure can be got by decomposing Σ into a sum of two diffeomorphic surfaces with boundary. To get a non-fillable structure we can decompose Σ in such a way that one of Σ_i has a planar connected component. $\square$

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