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DG-structure and derived categories

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Abstract

We describe an equivalence between a bounded derived category of coherent sheaves on a smooth projective variety Y and a derived category of modules over a DG category $\mathcal{C}$. The category $\mathcal{C}$ depends on a full exceptional collection ρ on Y. We propose two ways of calculating $\mathcal{C}$; one by means of A_∞ categories and the other by mutating ρ into a strong collection. As an example we present calculations for a collection on $\mathbb{P}^2$ blown up in two infinitely close points.

Introduction

In [HP] Hille and Perling described a way of constructing a toric surface $X(\sigma_Y)$ from a full exceptional collection σ_Y of line bundles on a rational surface Y. This construction identifies the Picard groups of X and Y and thus leads to an exceptional collection σ_X of line bundles on X. Bondal conjectured that X is in fact a degeneration of Y. However, if Y is a Hirzebruch surface $\mathbb{F}_a$ then for any $s \in \mathbb{Z}$ there exist an exceptional collection σ_Y which leads to $X(\sigma_Y)$ equal to $\mathbb{F}_{2s+a}$. This shows that the conjecture is not true. On the other hand in [HI] Hochenegger and Ilten proved that there always exists a sequence of deformations and degenerations from Y to X. The proof of this theorem is a consequence of another result from [HI], namely that all rational $\mathbb{C}^*$ surfaces of the given rank of Picard group are homogeneously deformation connected.

The work of Hochenegger and Ilten does not use the fact that σ_X and σ_Y are in some cases full exceptional collections. As a rational surface is determined by its derived category of coherent sheaves (see [Huy], §12) the deformation of the derived category might be considered as a deformation of the variety itself. Such deformations are called noncommutative. If both collections σ_X and σ_Y are strong and full then in order to find a noncommutative deformation one has to deform corresponding quivers. Examples of noncommutative deformations can be found in [HI] and [P1].

However, in some cases the noncommutative deformation can not be found just by looking at quivers. The easiest example is a strong exceptional collection σ on $\mathbb{P}^2$ blown up in two different point. It gives $X(\sigma)$ equal to $\mathbb{P}^2$ blown up in two infinitely close points with σ_X full but not strong. In order to find a noncommutative deformation in this case one has to understand how a full exceptional collection determines the derived category of coherent sheaves.

In this paper we cite main results by Bondal, Kapranov and Keller concerning this problem. We propose two ways of explicit calculations and pursue one of them in the mentioned case of a collection on $\mathbb{P}^2$ blown up in two infinitely close points.

The paper is organised as follows. In the first section we recall the definitions of additive, abelian and triangulated categories. Using DG categories and twisted complexes over them we define enhanced triangulated categories after [BK]. We use these notions to formulate in the second section the theorem of Bondal and Kapranov about the structure of an enhanced derived category. This theorem is compared with the result of Keller from [K3]. Both theorems concern in particular the structure of a bounded derived category $D^b(Y)$ of coherent sheaves on a smooth projective variety Y dependent on a full exceptional collection ρ. In order to recover $D^b(Y)$ one has to find a DG category $\mathcal{C}_\rho$. In the following sections we propose two ways of doing it. In the third section it is described how to determine $\mathcal{C}_\rho$ using A_∞ categories and the existence of minimal models. For this purpose we use Massey products as defined in [GM]. To illustrate the theory with an example we calculate Massey product for homomorphisms between line bundles appearing in the collection σ on $\mathbb{P}^2$ blown up in two infinitely close points. In the last section we show how to find $\mathcal{C}_\rho$ using mutations of exceptional collections. The calculations for the mentioned collection σ are presented and the category $\mathcal{C}_\sigma$ is given explicitly.

1 Enhanced triangulated categories

The following definitions are after [GM].

Definition 1.1. *An* additive *category is a category $\mathcal{A}$ such for any two objects A, $B \in \mathcal{A}$ the set $Hom(A, B)$ is an abelian group and the following axioms are satisfied.*

(A1) The composition of morphisms is bi-additive.

(A2) There exists a zero object $0 \in Ob(\mathcal{A})$ such that $Hom(0,0)$ is the zero group.

(A3) For any objects A, $B \in \mathcal{A}$ there exists an object $A \oplus B \in \mathcal{A}$ and morphisms

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \oplus B \underset{i_2}{\overset{p_2}{\rightleftarrows}} B$$

such that $p_1 i_1 = id_A$, $p_2 i_2 = id_B$, $i_1 p_1 + i_2 p_2 = id_{A \oplus B}$ and $p_2 i_1 = p_1 i_2 = 0$.

A category $\mathcal{A}$ is called *preadditive* if for any objects A and B in $\mathcal{A}$ the set $\mathrm{Hom}(A, B)$ is an abelian group and the axioms (A1) and (A2) are satisfied.

Definition 1.2. *An additive category $\mathcal{A}$ is* abelian *if for any morphism $\phi : A \to B$ there exists a sequence*

$$K \xrightarrow{k} A \xrightarrow{i} I \xrightarrow{j} B \xrightarrow{c} K'$$

such that $ji = \phi$, K is the kernel of ϕ, K' is the cokernel of ϕ and I is both the kernel of c and cokernel of k.

Definition 1.3. *A* triangulated *category is an additive category $\mathcal{A}$ together with an additive automorphism $T : \mathcal{A} \to \mathcal{A}$, called the translation functor, and a class of distinguished triangles. These triangles have to satisfy axioms (TR1) to (TR4).*

In order to simplify the notation we will write $A[n]$ instead of $T^n(A)$. A triangle is a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

and the morphism of triangles is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & A'[1]. \end{array}$$

The class of distinguished triangles needs to satisfy the following axioms:

(TR1) 1. For any object A in $\mathcal{A}$ there exists a distinguished triangle

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1] \ ,$$

2. Any triangle isomorphic to a distinguished triangle is itself distinguished,

3. Any morphism $f : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1].$$

(TR2) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

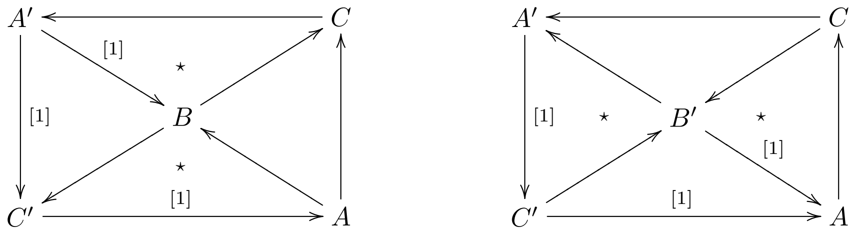
is distinguished.

(TR3) For given two distinguished triangles and two morphisms ϕ and ψ as in the diagram below

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \vdots{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & A'[1] \end{array}$$

there exists θ completing it to a diagram of a morphism of triangles.

To formulate the last axiom we need an octahedron diagram:



(upper cap) (lower cap)

where the triangles marked with a star are distinguished and other commute. Then the last axiom states that

(TR4) Any diagram of the type "upper cap" can be completed to an octahedron diagram.

The notion of a triangulated category demands a choice of the class of distinguished triangles. In order to find something close to triangulated categories but defined as a set with a given system of operations Bondal and Kapranov in [BK] introduced the notion of an enhanced triangulated category. To understand it we need some definitions from [BK].

Definition 1.4. *A* DG category *is a preadditive category $\mathcal{C}$ in which the abelian groups $Hom(A,B)$ are endowed with a $\mathbb{Z}$-grading and a differential d of degree one. The composition of morphisms*

$$Hom(A,B) \otimes Hom(B,C) \to Hom(A,C)$$

is a morphism of complexes and for any object $A \in \mathcal{C}$ the identity morphism id_A is a closed morphism of degree zero.

A *DG functor* between two DG categories $\mathcal{C}$ and $\mathcal{C}'$ is an additive functor $F : \mathcal{C} \to \mathcal{C}'$ that preserves the grading and differential on morphisms. DG functors between two DG categories form another DG category denoted DG-Fun$(\mathcal{C},\mathcal{C}')$ in the covariant case and DG-Fun$^o(\mathcal{C},\mathcal{C}')$ in the contravariant one.

From a DG category $\mathcal{C}$ one can construct its *cohomology category* $H(\mathcal{C})$ with the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}_{H(\mathcal{C})}(A,B)$ given by cohomology of $\mathrm{Hom}_{\mathcal{C}}(A,B)$. A further restriction to the zeroth cohomology gives a preadditive category $H^0(\mathcal{C})$.

Let $\mathcal{V}\mathrm{ect}_k$ be a DG category of complexes of vector spaces over the field k. A *right DG module* over C is a DG functor $M \in$ DG-Fun$^o(\mathcal{C},\mathcal{V}\mathrm{ect}_k)$. After [K3] we define a derived category $D(\mathcal{C})$ as a localization of H^0(DG-Fun$^o(\mathcal{C},\mathcal{V}\mathrm{ect}_k)$) with respect to the class of quasi-isomorphisms. By $D^b(\mathcal{C})$ we will denote the smallest strictly full triangulated subcategory of $D(\mathcal{C})$ containing modules M such that for every $C \in \mathcal{C}$ the graded vector space $H^*(M(C))$ is finitely dimensional. The Yoneda embedding gives a fuctor $h : \mathcal{C} \to$ DG-Fun$^o(\mathcal{C},\mathcal{V}\mathrm{ect}_k)$ which assigns to every $C \in \mathcal{C}$ a free module $h_C = \mathrm{Hom}_{\mathcal{C}}(-,C)$.

Definition 1.5. *A* twisted complex *over a DG category $\mathcal{C}$ is a set $\{(E_i)_{i\in\mathbb{Z}}, q_{ij} : E_i \to E_j\}$ where E_i's are objects in $\mathcal{C}$ equal to 0 for almost all i and q_{ij} are morphisms in $\mathcal{C}$ of degree $i-j+1$ such that $dq_{ij} + \sum_k q_{kj}q_{ik} = 0$.*

For a DG category $\mathcal{C}$ let $\mathcal{C}^{\oplus}$ denote the category obtained form $\mathcal{C}$ by adjoining formal finite direct sums. Then twisted complexes over $\mathcal{C}^{\oplus}$ form a DG category which is denoted by Pre-Tr$(\mathcal{C})$ and its zeroth cohomology category is denoted by Tr$(\mathcal{C})$.

A twisted complex $K = \{E_i, q_{ij}\}$ over $\mathcal{C}$ defines a contravariant functor $\alpha(K)$ from $\mathcal{C}$ into $C(\mathcal{A}b)$ – the category of complexes of abelian groups;

$$\alpha(K)(E) = \bigoplus \mathrm{Hom}_{\mathcal{C}}(E,E_i)[i]$$

with the differential $d+Q$ where $Q = \| q_{ij} \|$ and d is the differential in $\bigoplus \mathrm{Hom}_{\mathcal{C}}(E,E_i)[i]$.

Definition 1.6. *A DG-category $\mathcal{C}$ is* pretriangulated *if for every twisted complex $K \in$ Pre-Tr$(\mathcal{C})$ the corresponding contravariant functor $\alpha(K)$ is representable.*

With these definitions we can introduce the notion of an enhanced triangulated category.

Definition 1.7. *A triangulated category $\mathcal{A}$ is* enhanced *if there exists a pretriangulated category $\mathcal{C}$ together with an equivalence $H^0(\mathcal{C}) \to \mathcal{A}$.*

If $\mathcal{A}$ is an abelian category with enough injectives then the bounded derived category $D^b(\mathcal{A})$ is enhanced. Its enhancement is a DG subcategory of complexes over $\mathcal{A}$ consisting of complexes bounded from below, with all terms injective and almost all cohomology groups equal to zero.

2 Exceptional collections and generating sets

As described in [B] under some conditions a derived category of an abelian category can be presented as a derived category of modules over an algebra. In order to have such presentation a strong full exceptional collection is needed. Let $\mathcal{A}$ be an abelian category with enough injectives and let $\langle E_1, \dots, E_n\rangle$ be a strong exceptional collection. Then $D^b(\mathcal{A})$ is equivalent to the bounded derived category of modules over an algebra $\mathrm{End}(\bigoplus E_i)$.

The natural question to ask is what happens if an exceptional collection is full (generates the category $\mathcal{A}$ as the smallest triangulated category) but not strong (there exist non-zero higher Ext groups from E_i to E_j for $i < j$).

An answer to this question is given in [BK]. To state the result of this paper some further definitions are required.

Definition 2.1. *A twisted complex $C = \{E_{ij}, q_{ij}\}$ over a DG category $\mathcal{C}$ is called* one-sided *if $q_{ij} = 0$ for $i \geq j$.*

For a DG category $\mathcal{C}$ let $\widetilde{\mathcal{C}}$ denote the DG category obtained form $\mathcal{C}$ by adjoining formal translates of objects. The objects of $\widetilde{\mathcal{C}}$ are of the form $E[n]$ where $E \in \mathrm{ob}\mathcal{C}$ and $n \in \mathbb{Z}$. The morphisms in $\widetilde{\mathcal{C}}$ are given by

$$\mathrm{Hom}^i_{\widetilde{\mathcal{C}}}(E[n], F[m]) = \mathrm{Hom}^{i+m-n}_{\mathcal{C}}(E, F).$$

Denote by $\text{Pre-Tr}^+(\mathcal{C})$ the full DG subcategory of $\text{Pre-Tr}(\widetilde{\mathcal{C}})$ with one-sided twisted complexes as objects. Its zeroth cohomology category is denoted by $\mathrm{Tr}^+(\mathcal{C})$.

For objects $E_1, \dots, E_n$ of a triangulated category $\mathcal{C}$ let $\langle E_1, \dots, E_n\rangle_{\mathcal{C}}$ denote the smallest triangulated subcategory of $\mathcal{C}$ containing $E_1, \dots, E_n$.

Theorem 2.1 (Theorem 1 of §4 in [BK])**.** *Let $\mathcal{D}$ be a pretriangulated category, $E_1, \dots, E_n$ objects in $\mathcal{D}$ and $\mathcal{C} \subset \mathcal{D}$ the full DG subcategory on the objects E_i. Then $\langle E_1, \dots, E_n\rangle_{H^0(\mathcal{D})}$ is equivalent to $Tr^+(\mathcal{C})$ as a triangulated category.*

The category $\mathrm{Tr}^+(C)$ is a full triangulated subcategory of the category $\mathrm{Tr}(\widetilde{\mathcal{C}})$. Moreover, for any DG category $\mathcal{C}'$ the category $\mathrm{Tr}(C')$ is a full triangulated subcategory of $H^0(\text{DG-Fun}^{\circ}(\mathcal{C}', C(\mathcal{A}b)))$ – the zeroth cohomology category of DG modules over $\mathcal{C}'$. It follows that $\langle E_1, \dots, E_n\rangle_{H^0(\mathcal{D})}$ is a full triangulated subcategory of DG modules over some DG category.

Another theorem about an equivalence of a derived category with a category of modules over a DG category can be found in [K3]. In order to state it we need few more definitions. The following are after [K2].

Definition 2.2. *Let $\mathcal{E}$ be a full additive subcategory of an abelian category $\mathcal{F}$ closed under extensions and let $\mathcal{S}$ be a set of short exact sequences in $\mathcal{F}$ with terms in $\mathcal{E}$. Then the pair $(\mathcal{E}, \mathcal{S})$ is a* Quillen exact category.

Definition 2.3. *An object X in a Quillen exact category $(\mathcal{E}, \mathcal{S})$ is* injective *if every sequence in $\mathcal{S}$ of the form $0 \to X \to Y \to Z \to 0$ splits. Dually an object Z is* projective *if every sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{S}$ splits.*

Definition 2.4. *A* Frobenius category *is a Quillen exact category $(\mathcal{E}, \mathcal{S})$ with enough injectives, enough projectives and such that the class of projectives coincides with injectives.*

For every object E in a Frobenius category $(\mathcal{E}, \mathcal{S})$ there exists an injective object $I(E)$ and a sequence

$$E \xrightarrow{i_E} I(E) \xrightarrow{\pi_E} \Sigma(E)$$

in $\mathcal{S}$. The assignment $E \to \Sigma(E)$ can be extended to a functor $\Sigma : \mathcal{E} \to \mathcal{E}$.

From a Frobenius category $\mathcal{E}$ one constructs its stable category $\underline{\mathcal{E}}$.

Definition 2.5. The stable category $\underline{\mathcal{E}}$ *of a Frobenius category $\mathcal{E}$ has the same objects as $\mathcal{E}$ and for $X, Y \in \underline{\mathcal{E}}$ one puts $Hom_{\underline{\mathcal{E}}}(X, Y) = Hom_{\mathcal{E}}(X, Y)/I(X, Y)$ where $I(X, Y) \subset Hom_{\mathcal{E}}(X, Y)$ is a set of morphisms which factor through some injective object in $\mathcal{E}$.*

The stable category $\underline{\mathcal{E}}$ is triangulated. The shift functor is given by Σ and distinguished triangles are obtained by means of the following commutative diagram for any morphism $f : X \to Y$

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & I(X) & \xrightarrow{\pi_X} & \Sigma(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow[g]{} & C(f) & \xrightarrow[h]{} & \Sigma(X) \end{array}$$

where the left square is cocartesian. Then the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} C(f) \xrightarrow{h} \Sigma(X)$$

is a distinguished triangle in $\underline{\mathcal{E}}$.

An important example of a Frobenius category is given in [K2]. For an abelian category $\mathcal{A}$ the category $\mathrm{Kom}^*(\mathcal{A})$ of complexes over $\mathcal{A}$ with pointwise split exact sequences is a Frobenius category. Moreover the stable category in this case is the homotopy category of $\mathrm{Kom}^*(\mathcal{A})$ with the standard triangulated structure. It follows that if an abelian category $\mathcal{A}$ has enough projectives or injectives then the derived category $D(\mathcal{A})$ is a stable category of a Frobenius category.

In particular the derived category of quasi-coherent sheaves on a projective variety X is a stable category of a Frobenius category $D(\mathrm{Qcoh}(X)) = \underline{\mathrm{Kom}(\mathrm{Qcoh}(X))}$.

If an additive category has infinite direct sums it is convenient to introduce the notion of compact (also called small) objects.

Definition 2.6. *An object X of an additive category $\mathcal{A}$ is* compact *if $Hom(X,-)$ commutes with infinite direct sums.*

Let X be a smooth projective variety. In [BvB] Bondal and Van den Bergh show that compact objects in the derived category of quasi-coherent sheaves $D(\mathrm{Qcoh}(X))$ are elements of bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(X))$ denoted by $D^b(X)$.

With these definitions and examples we are ready to state the

Theorem 2.2 (cf. §4.3 of [K3])**.** *Let $\mathcal{E}$ be a k-linear Frobenius category with infinite direct sums and let the stable category $\underline{\mathcal{E}}$ be generated by a set of compact objects $\mathcal{X} \subset \underline{\mathcal{E}}$. Then there exists a DG category $\mathcal{C}_{\mathcal{X}}$ and an equivalence of triangulated categories $s : \underline{\mathcal{E}} \to D(\mathcal{C}_{\mathcal{X}})$ giving rise to an equivalence between $\mathcal{X}$ and the full subcategory of $D(\mathcal{C}_{\mathcal{X}})$ formed by the free modules h_C for $C \in \mathcal{C}_{\mathcal{X}}$.*

The category of modules over a DG category $\mathcal{C}$ depends on the quasi-isomorphism class of $\mathcal{C}$. Hence, the category $\mathcal{C}_{\mathcal{X}}$ is determined up to quasi-isomorphisms.

The DG category $\mathcal{C}_{\mathcal{X}}$ has objects corresponding to elements of $\mathcal{X}$ and the morphism spaces are calculated by means of acyclic complexes with projective components

$$P_X^\bullet = \{\ldots \to P_X^n \to P_X^{n-1} \to \ldots\}, \quad P_X^i \in \mathcal{E}$$

such that $Z^0(P_X^\bullet) = X$ for every $X \in \mathcal{X}$.

Such complexes can be found for any set of objects in $\underline{\mathcal{E}}$. In particular a full exceptional collection $\sigma = \langle E_1, \ldots, E_n \rangle$ on a smooth projective variety X defines a DG category $\mathcal{C}_\sigma$ with $H(\mathcal{C}_\sigma)$ given by Ext groups between the E_i's. The collection σ defines a functor $s : D(\mathrm{Qcoh}(X)) \to D(\mathcal{C})$. It induces an equivalence between the category generated by E_i's and the category generated by the free modules h_{E_i}'s. The modules over $\mathcal{C}_\sigma$ corresponding to elements of $D^b(X)$ assign to every element of $\mathcal{C}_\sigma$ a complex in $D^b(\mathcal{C}_\sigma)$. It shows that $D^b(X)$ is equivalent to $D^b(\mathcal{C}_\sigma)$.

Remark: Bondal and Kapranov in [BK] consider modules over a DG category with infinitely many object – in order to get $\mathrm{Tr}^+(\mathcal{C})$ one has to adjoin formal translates of objects. However, Keller in [K3] uses DG categories with objects given by generators of $\underline{\mathcal{E}}$. In particular for an exceptional collection $\sigma = \langle E_1, \ldots, E_n \rangle$ the DG category C_σ has n objects.

The the proof of theorem 2.2 requires taking resolutions and so it does not give a method for calculating the DG category $\mathcal{C}_\sigma$. In the remaining sections we will propose two ways of finding explicitly the morphisms in $\mathcal{C}_\sigma$.

For calculations we will focus on X equal to $\mathbb{P}^2$ blown up in two infinitely closed points. Let H denote the divisor of a line on X - the pull-back of a class of a line on $\mathbb{P}^2$, let E_1 be the strict transform of the first exceptional divisor and E_2 the second exceptional divisor (X is obtained from $\mathbb{P}^2$ by blowing up one point and later blowing up a point on the first exceptional divisor). Then the intersection numbers are $H^2 = 1$, $E_1^2 = -2$, $E_2^2 = -1$, $H.E_1 = H.E_2 = 0$ and $E_1.E_2 = 1$. A direct calculation shows that $\sigma = \langle \mathcal{O}, \mathcal{O}(H - E_1 - E_2), \mathcal{O}(H - E_2), \mathcal{O}(H), \mathcal{O}(2H - E_1 - 2E_2) \rangle$ is an exceptional collection. It is not strong and the fact that it is full will be proved later.

For simplicity instead considering a DG category $\mathcal{C}_\sigma$ we work with a DG algebra A_X – the "path algebra" of $\mathcal{C}_\sigma$. The construction of A_X is analogous to the construction of a path algebra of a quiver. Elements of A_X correspond to morphism in $\mathcal{C}_\sigma$. The grading and differentials in A_X are the same as in $\mathcal{C}_\sigma$ and the composition is defined by

$$\alpha \circ_{A_X} \beta = \begin{cases} \gamma & \text{if} \quad \alpha \text{ and } \beta \text{ are composable in } \mathcal{C}_\sigma \text{ and } \alpha \circ_{\mathcal{C}_\sigma} \beta = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

3 A_∞ structure and minimal models

The first approach to this problem is in a language of A_∞ algebras and categories. Let us first give some definitions after [K1].

Definition 3.1. *An A_∞* algebra *is a $\mathbb{Z}$-graded vector space A together with graded operations*

$$m_n : A^{\otimes n} \to A, \quad n \geq 1$$

of degree $2-n$ *satisfying for any* n

$$\sum_{r+s+t=n}(-1)^{r+st}m_{r+1+t}(id^{\otimes r}\otimes m_s\otimes id^{\otimes t})=0.$$

Note that when these formulae are applied to elements, additional signs appear because of the Koszul sign rule:

$$(f\otimes g)(x\otimes y)=(-1)^{|x||g|}f(x)\otimes g(y),$$

where $|x|$, $|g|$ denote the degrees of x and g respectively.

The operation m_1 gives an A_∞ algebra A a structure of a complex. Its cohomology is denoted by $H(A)$. Any DG algebra can be regarded as an A_∞ algebra with m_1 given by the differential, m_2 given by composition and trivial m_i's for $i>2$.

Definition 3.2. *Let* K *be a field. An* A_∞ category $\mathcal{A}$ *over* K *consists of*

- *objects* $ob(\mathcal{A})$,
- *for any two* $A,B\in ob(\mathcal{A})$ *a* $\mathbb{Z}$ *- graded* K*-vector space* $Hom(A,B)$ *denoted also by* (A,B),
- *for any* $n\geq 1$ *and a sequence* $A_0,A_1,\ldots,A_n\in ob(\mathcal{A})$ *a graded map:*

$$m_n:(A_{n-1},A_n)\otimes\ldots\otimes(A_0,A_1)\to(A_0,A_n)$$

of degree $2-n$ *such that for any* n

$$\sum_{r+s+t=n}(-1)^{r+st}m_{r+1+t}(id^{\otimes r}\otimes m_s\otimes id^{\otimes t})=0.$$

Thus, an A_∞ algebra is an A_∞ category with one object.

As we work with projective varieties we will always assume that the base field is $\mathbb{C}$.

If $\mathcal{A}$ is an A_∞ category then for $A\in \mathrm{ob}(\mathcal{A})$ the morphism $e\in(A,A)$ is a *strict identity* if $m_2(\alpha,e)=\alpha$, $m_2(e,\beta)=\beta$ and $m_n(\ldots,e,\ldots)=0$ for $n\neq 2$ whenever it makes sense.

A morphism of A_∞ algebras $f:A\to B$ is a family of graded maps $f_n:A^{\otimes n}\to B$ of degree $1-n$ such that

$$\sum_{r+s+t=n}(-1)^{r+st}f_{r+1+t}(\mathrm{id}^{\otimes r}\otimes m_s\otimes \mathrm{id}^{\otimes t})=\sum_{i_1+\ldots+i_r=n}(-1)^p m_r(f_{i_1}\otimes\ldots\otimes f_{i_r})$$

where $p=(r-1)(i_1-1)+(r-2)(i_2-1)+\ldots+2(i_{r-2}-1)+(i_{r-1}-1)$.

There is a similar definition of an A_∞ functor $F:\mathcal{A}\to\mathcal{B}$.

A functor $F:\mathcal{A}\to\mathcal{B}$ is called a *quasi-isomorphism* if F_1 is a quasi-isomorphism. F is strict if $F_n(\ldots,e,\ldots)=0$ for $n\neq 2$ and for any strict identity e.

Notice, that if A is an A_∞ algebra and B is a graded vector space then a family $F=(F_i)$ of graded maps $F_i:A^{\otimes i}\to B$ of degree $1-i$ induce such an A_∞ structure on B that F becomes an A_∞ morphism. The analogous statement is true for A_∞ categories.

Any A_∞ algebra is quasi-isomorphic to its cohomologies as stated in [K1].

Theorem 3.1. *If* A *is an* A_∞*-algebra, then* $H(A)$ *admits an* A_∞*-algebra structure such that*

1. $m_1=0$ *and* m_2 *is induced from* m_2^A *and*
2. *there is an* A_∞*-quasi-isomorphism* $A\to H(A)$ *inducing the identity on homology.*

Moreover, this structure is unique up to a non unique A_∞*-isomorphism.*

The A_∞ algebra $H(A)$ is called the *minimal model* of A.

On the other hand for every A_∞ algebra A there exist an A_∞ quasi-isomorphism $\phi:A\to U(A)$ for some DG algebra $U(A)$ (Proposition 2.1. form [K1]). The DG algebra $U(A)$ is determined up to a DG quasi-isomorphism. Hence finding the A_∞ structure on $H(A_X)=\mathrm{Ext}^*(\bigoplus E_i,\bigoplus E_i)$ leads to a DG structure on A_X.

The theorem 2.2 can be generalized to a language of A_∞ algebras. It was done by Lefevre-Hasegawa in [L-H]. In order to formulate it we need the following definitions after [K1].

Definition 3.3. *A triangulated category* $\mathcal{C}$ *has* split idempotents *if every idempotent endomorphism admits a kernel.*

In [BS] Balmer and Schlichting prove that a bounded derived category of an abelian category has split idempotents.

Definition 3.4. *An object G of a triangulated category $\mathcal{C}$ is a* generator *if $\mathcal{C}$ is equal to the closure of G under shifts in both directions, extensions and passage to direct factors.*

Thus, if there is a full exceptional collection $\langle E_1, \ldots, E_n \rangle$ on a projective variety X then $\bigoplus_{i=1}^n E_i$ is a generator of $D^b(X)$.

Definition 3.5. *An A_∞* module *over an A_∞ category $\mathcal{A}$ is a contravariant functor from $\mathcal{A}$ to a DG $\mathcal{V}ect_k$ regarded as an A_∞ category. The category of modules over $\mathcal{A}$ is denoted by $Mod(\mathcal{A})$.*

For an A_∞ category A one can forget about the A_∞ structure and remember only the multiplication m_2. Then the category of modules over the algebra A will be denoted by $\mathrm{Mod}(A, m_2)$ and the category of graded right modules by $\mathrm{Gr\ mod}(A, m_2)$.

Definition 3.6. *For an A_∞ algebra A the* derived category $D_\infty(A)$ *is a localization of the category $Mod(\mathcal{A})$ (with degree zero morphisms) with respect to the class of quasi-isomorphism. The* perfect derived category $per(A)$ *is the closure of the free A-module of rank one under shifts in both directions, extensions and passage to direct factors .*

Now we can state the following.

Theorem 3.2 (Theorem 7.6.0.6 of [L-H]). *Let C be a K-linear Frobenius category and let its stable category $\underline{\mathcal{C}}$ has split idempotents and a generator G. Then there is a structure of A_∞ algebra on*

$$A = \bigoplus_{n \in \mathbb{Z}} Hom(G, G[n])$$

such that $m_1 = 0$, m_2 is given by composition and the functor:

$$\underline{\mathcal{C}} \to Gr\ mod(A, m_2), \quad U \to \bigoplus_{n \in \mathbb{Z}} Hom(G, U[n])$$

lifts to an equivalence of triangulated categories $\underline{\mathcal{C}} \to per(A)$.

Let us come back to the question about the DG algebra A_X determined by the full exceptional collection σ on X. By calculating Hom's and higher Ext groups between elements in the exceptional collection one can find $H(A_X)$. It is nontrivial in degree 0 and 1. In degree 0 it is given by a path algebra of the following quiver:

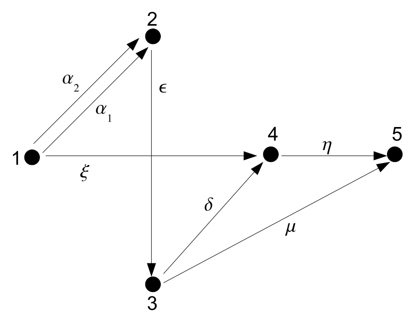


Figure 1: The quiver of the collection $\langle \mathcal{O}, \mathcal{O}(H - E_1 - E_2), \mathcal{O}(H - E_2), \mathcal{O}(H), \mathcal{O}(2H - E_1 - 2E_2) \rangle$ on X.

with one relation $\mu\epsilon\alpha_1 = \eta\delta\epsilon\alpha_2$. In degree 1 there is one nontrivial element τ between $\mathcal{O}(H - E_1 - E_2)$ and $\mathcal{O}(H - E_2)$, that is between the second and the third vertex.

The operation m_1 in $H(A_X)$ is always zero and m_2 is obtained from composition in $D^b(X)$. Due to degree restrictions the only nontrivial operation in $H(A_X)$ can be m_3 with one argument equal to τ.

We consider the A_∞ structure on $H(A_X)$ coming from the structure on $H(\mathcal{C}_\sigma)$ and thus m_3 can be non-zero only on composable arrows. Hence the only nontrivial m_3's can be: $m_3(\delta, \tau, \alpha_1)$, $m_3(\delta, \tau, \alpha_2)$, $m_3(\eta\delta, \tau, \alpha_1)$, $m_3(\eta\delta, \tau, \alpha_2)$, $m_3(\mu, \tau, \alpha_1)$, $m_3(\mu, \tau, \alpha_2)$ and $m_3(\eta, \delta, \tau)$. Moreover the relations among m_i's give

$$m_3(\eta\delta, \tau, \alpha_1) = m_2(\eta, m_3(\delta, \tau, \alpha_1)) + m_2(m_3(\eta, \delta, \tau), \alpha_1),$$

$$m_3(\eta\delta, \tau, \alpha_2) = m_2(\eta, m_3(\delta, \tau, \alpha_2)) + m_2(m_3(\eta, \delta, \tau), \alpha_2).$$

Let us write:

$$m_3(\delta, \tau, \alpha_1) = A\delta\epsilon\alpha_1 + B\delta\epsilon\alpha_2 + C\xi;$$

$$m_3(\delta, \tau, \alpha_2) = D\delta\epsilon\alpha_1 + E\delta\epsilon\alpha_2 + F\xi;$$

$$m_3(\mu,\tau,\alpha_1) = G\mu\epsilon\alpha_1 + H\mu\epsilon\alpha_2 + I\eta\delta\epsilon\alpha_1 + J\eta\xi;$$
$$m_3(\mu,\tau,\alpha_2) = K\mu\epsilon\alpha_1 + L\mu\epsilon\alpha_2 + M\eta\delta\epsilon\alpha_1 + N\eta\xi;$$
$$m_3(\eta,\delta,\tau) = P\eta\delta\epsilon + Q\mu\epsilon,$$

where $A,\dots,Q$ are any complex numbers.

We want to consider the A_∞ structure up to an isomorphism. Let us assume that there is an isomorphism F of A_∞ algebras from $H(A_X)$ with a given A_∞ structure to $H(A_X)$ with a new A_∞ structure. We want to keep the composition in $H(A_X)$ obtained from $D^b(X)$ so F_1 has to be identity. F_i has degree $1-i$ so the only remaining non-trivial term in F can be F_2 and all the possibilities are:

$$F_2(\tau,\alpha_1) = a\epsilon\alpha_1 + b\epsilon\alpha_2,$$
$$F_2(\tau,\alpha_2) = c\epsilon\alpha_1 + d\epsilon\alpha_2,$$
$$F_2(\delta,\tau) = e\delta\epsilon,$$
$$F_2(\eta\delta,\tau) = f\mu\epsilon + g\eta\delta\epsilon,$$
$$F_2(\mu,\tau) = h\mu\epsilon + j\eta\delta\epsilon,$$

where $a,\dots,j$ are any complex numbers. The inverse morphism G is given by $G_1 = \mathrm{id}$, $G_2(x,y) = -F_2(x,y)$. The new isomorphic A_∞ structure on $H(A_X)$ is then given by

$$m_3'(\delta,\tau,\alpha_1) = (A+e-c)\delta\epsilon\alpha_1 + (B-d)\delta\epsilon\alpha_2 + C\xi;$$
$$m_3'(\delta,\tau,\alpha_2) = (D-c)\delta\epsilon\alpha_1 + (E+e-d)\delta\epsilon\alpha_2 + F\xi;$$
$$m_3'(\mu,\tau,\alpha_1) = (G+h-a)\mu\epsilon\alpha_1 + (H-b)\mu\epsilon\alpha_2 + (I+j)\eta\delta\epsilon\alpha_1 + J\eta\xi;$$
$$m_3'(\mu,\tau,\alpha_2) = (K+j-c)\mu\epsilon\alpha_1 + (L+h-d)\mu\epsilon\alpha_2 + M\eta\delta\epsilon\alpha_1 + N\eta\xi;$$
$$m_3'(\eta,\delta,\tau) = (P+g-e)\eta\delta\epsilon + (Q+f)\mu\epsilon,$$

Thus, up to an isomorphism $m_3(\eta,\delta,\tau) = 0$. Moreover an isomorphism F making $m_3(\eta,\delta,\tau) = 0$ does not change the values of remaining operations. They can be calculated by means of Massey products.

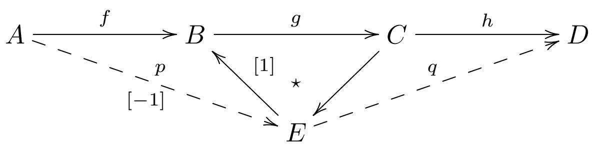
3.1 Massey products

The following definitions are again after [GM].

Let $\mathcal{A}$ be a triangulated category and let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

be a complex of objects in $\mathcal{A}$. Let E be a third vertex of a distinguished triangle containing g

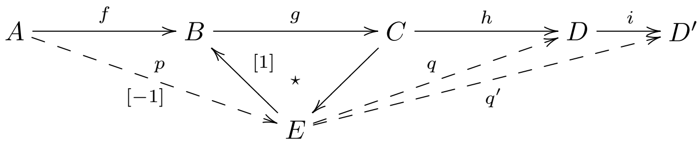


Since $gf = 0$ and $hg = 0$ there exist morphisms $p : A \to E[-1]$ and $q : E \to D$. Their composition is called the *triple Massey product* of h, g and f and is denoted by $\langle h,g,f\rangle$. It is an element in $\mathrm{Hom}(A, D[-1])$ unique up to $\mathrm{Hom}(B, D[-1]) \circ f + h[-1] \circ \mathrm{Hom}(A, C[-1]) \subset \mathrm{Hom}(A, D[-1])$.

Remark: For the following diagram

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} D'$$

if the Massey product $\langle h,g,f\rangle$ exists then $\langle ih,g,f\rangle = i\langle h,g,f\rangle$ because one can calculate it using the following diagram:



In algebraic topology one can find a similar operation namely the Toda bracket. The original definition can be found for example in [W]. [GM] translates it into a language of DG categories.

Definition 3.7. *Let $\mathcal{C}$ be a DG category and let*

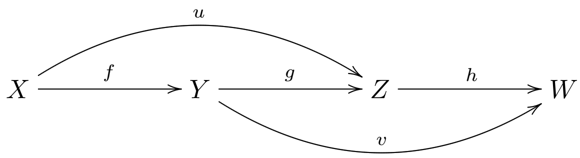
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

be a sequence of morphism in $\mathcal{C}$ which becomes a complex in $H^*\mathcal{C}$. As $gf = 0 = hg$ in $H^*\mathcal{C}$ there exist $u \in Hom_{\mathcal{C}}(X, Z)$ and $v \in Hom_{\mathcal{C}}(Y, W)$ such that $d(u) = gf$ and $d(v) = hg$. Then $\theta = hu - vf$ is the Toda bracket *of h, g and f.*

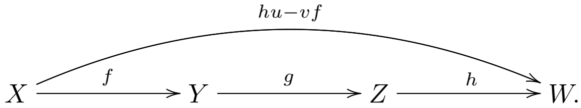
A derived category of an abelian category $\mathcal{A}$ is equivalent to the homotopy category of $K(\mathcal{A})$ if $\mathcal{A}$ has enough injectives. On the category $\mathrm{Kom}(\mathcal{A})$ of complexes over $\mathcal{A}$ one can introduce a differential δ such that δ-cocycles are morphisms of complexes and δ-coboundaries are morphism homotopic to zero. This gives $\mathrm{Kom}(\mathcal{A})$ a structure of DG category with the cohomology category $H(\mathrm{Kom}(\mathcal{A})) = K(\mathcal{A})$. Then using the description of exact triangles in $D(\mathcal{A})$ (by means of cylinders and cones) one can show that for a triple of morphisms in $D(\mathcal{A})$ the definitions of Toda bracket and Massey product coincide.

A derived category of an abelian category $\mathcal{A}$ with enough injectives has an enhancement given by the category $\mathrm{Kom}(\mathcal{A})$. Hence the Massey products provide information about the DG structure of the enhancement. In particular they might help to calculate the DG category $\mathcal{C}_\sigma$. Connection between triple Massey products and the operation m_3 on $H(\mathcal{C}_\sigma)$ is easier to see using Toda brackets.

Let $\mathcal{C}$ a DG category with four objects X, Y, Z and W and morphisms



such that $d(u) = gf$, $d(v) = hg$ and on remaining arrows the differential is zero. Then the cohomology category $H(\mathcal{C})$ has the form:



In order to find an A_∞ structure on $H(\mathcal{C})$ with trivial m_1 one has to consider an A_∞ quasi-isomorphism $F : H(\mathcal{C}) \to \mathcal{C}$ such that F_1 is the identity. The relations among F_i's and m_i's are the following:

$$F_1 \circ m_2 = m_2(F_1 \otimes F_1) + d \circ F_2,$$
$$F_1 \circ m_3 + F_2(m_2 \otimes \mathrm{id} - \mathrm{id} \otimes m_2) = m_2(F_1 \otimes F_2 - F_2 \otimes F_1) + d \circ m_3.$$

It follows that

$$\begin{aligned} F_2(g, f) &= -u, \\ F_2(h, g) &= -v, \\ F_3(h, g, f) &= (-1)^{|h|+1} hu + (-1)^{|h|+|g|+1} vf. \end{aligned}$$

Thus if $|h|$ and $|g|$ have the same parity then we have an equality $\langle h, g, f \rangle = \pm m_3(h, g, f)$.

In the case of the exceptional collection on X Massey products are well defined on three composable arrows one of which is τ. However, $\langle \eta, \delta, \tau \rangle$ is not defined because $\eta\delta$ is not zero.

Finally, the A_∞ structure on $H(A_X)$ is given by

$$\begin{aligned} m_3(\delta, \tau, \alpha_1) &= -\langle \delta, \tau, \alpha_1 \rangle, \\ m_3(\delta, \tau, \alpha_2) &= -\langle \delta, \tau, \alpha_2 \rangle, \\ m_3(\mu, \tau, \alpha_1) &= -\langle \mu, \tau, \alpha_1 \rangle, \\ m_3(\mu, \tau, \alpha_2) &= -\langle \mu, \tau, \alpha_2 \rangle, \\ m_3(\eta, \delta, \tau) &= 0. \end{aligned}$$

In order to calculate the Massey products on X one has to understand its geometry. As described before there are three linearly independent divisors on X. They are H - the pull-back of a line on $\mathbb{P}^2$ and two exceptional divisors E_1 and E_2. Let $p_0 \in \mathbb{P}^2$ be the first blown up point and $p_1 \in E_1$ – the second one. If p_0 lies on a line on $\mathbb{P}^2$ then the pull-back H' of this line intersects E_1 in one point. Moreover H' and E_1 can intersect in $p_1 \in E_1$. Such a line on X will be denoted by H''. The linear equivalences between these divisors are

$$H'' + E_2 \equiv H', \quad H' + E_1 + E_2 \equiv H.$$

As every arrow in the quiver corresponds to a morphism of line bundles there is a divisor associated to it - the divisor of zeros of the section. Let us choose the arrows in the quiver in such a way that:

$$Z(\alpha_1) = H'' + E_2, \quad Z(\alpha_2) = H', \quad Z(\epsilon) = E_1, \quad Z(\delta) = E_2,$$

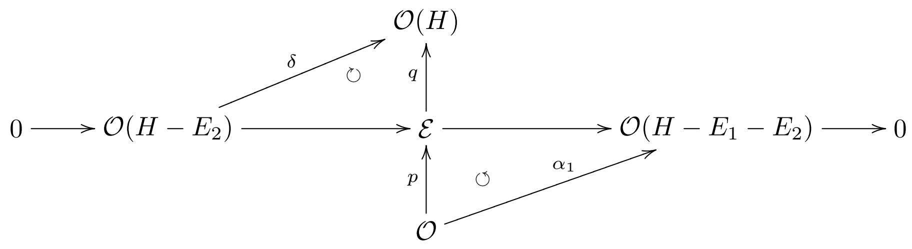
$$Z(\xi) = H, \quad Z(\eta) = H'', \quad Z(\mu) = H'$$

for fixed H and H'.

This point of view allows to calculate one Massey product. Let $\mathcal{E}[1]$ be defined by the distinguished triangle:

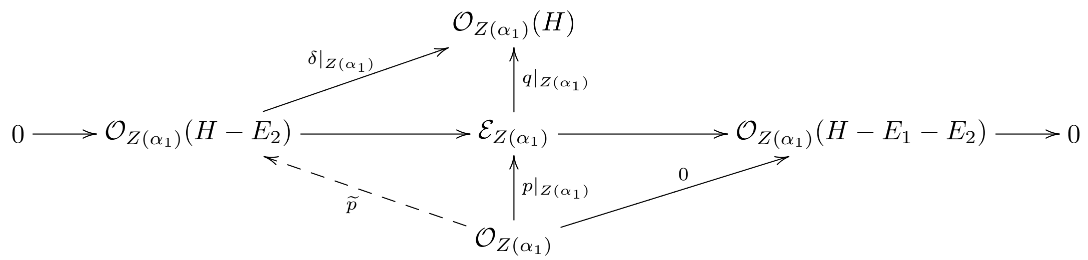
$$\mathcal{O}(H - E_1 - E_2) \xrightarrow{\tau} \mathcal{O}(H - E_2)[1] \longrightarrow \mathcal{E}[1] \longrightarrow \mathcal{O}(H - E_1 - E_2)[1].$$

Then the definition of Massey product given at the beginning of the section can be reformulated. For example $\langle \delta, \tau, \alpha_1 \rangle = qp$ for q and p as on the diagram:



In order to determine $\langle \delta, \tau, \alpha_1 \rangle \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}(H))/(\delta\epsilon\alpha_1 + \delta\epsilon\alpha_2)$ one has to understand sections $\delta\epsilon\alpha_1$ and $\delta\epsilon\alpha_2$ and the subspace of $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(H))$ generated by them. Notice that $Z(\delta\epsilon\alpha_1) = H'' + E_1 + 2E_2$ and $Z(\delta\epsilon\alpha_2) = H' + E_1 + E_2$ when $Z(\xi) = H$. It follows that every section of $\mathcal{O}(H)$ that is zero at $p_1 = E_1 \cap E_2$ belongs to the subspace generated by $\delta\epsilon\alpha_1$ and $\delta\epsilon\alpha_2$.

For any $x \in X$ such that $\alpha_1(x) = 0$ there exist $\widetilde{p} : \mathcal{O}_{Z(\alpha_1)} \to \mathcal{O}_{Z(\alpha_1)}(H - E_2)$ such that the following diagram commutes



Then $qp|_{Z(\alpha_1)} = \delta\widetilde{p}|_{Z(\alpha_1)}$. As $Z(\alpha_1) = H'' + E_2$ and $Z(\delta) = E_2$ it follows that $\delta|_{Z(\alpha_1)} \equiv 0$ and so for $p_1 = E_1 \cap E_2$ $qp(p_1) = q\widetilde{p}(p_1) = 0$. It shows that $\langle \delta, \tau, \alpha_1 \rangle = 0$ in $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(H))/(\delta\epsilon\alpha_1 + \delta\epsilon\alpha_2)$.

Unfortunately this method does not determine the values of the remaining Massey products.

Remark: There might be a way of finding the extension $\mathcal{E}$ by toric methods as described in [P2]. The combinatorial methods of toric varieties may also give an explicit algorithm for calculating Massey products of toric equivariant sheaves.

4 Mutations

Another way to find a DG structure on A_X is through mutations. It is based on the following lemma.

Lemma 4.1. *If a DG algebra A has only zeroth cohomology then it is quasi-isomorphic to $H^0(A)$.*

Proof. Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$. Then $I = (\bigoplus_{n<0} A_n) \oplus \ker(d_0)$ is an ideal in A and the inclusion $I \to A$ is a quasi-isomorphism. Moreover $J = \mathrm{Im}(d_{-1}) \oplus (\bigoplus_{n<0} A_n) \subset I$ is an ideal and $I \to I/J$ is a quasi-isomorphism. This proves that A is quasi-isomorphic to $I/J = H^0(A)$. □

There is an action of the braid group on the set of exceptional collections on a variety Y as described in [B]. If $\langle E, F \rangle$ is a pair of exceptional objects on Y then $\langle L_E F, E \rangle$ and $\langle F, R_F E \rangle$ are also exceptional pairs for $L_E F$ and $R_F E$ defined as follows by means of distinguished triangles in $D^b(Y)$.

$$L_E F \to \mathrm{Hom}(E, F) \otimes E \to F,$$

$$E \to \mathrm{Hom}(E, F)^* \otimes F \to R_F E.$$

If $\rho = \langle E_1, \ldots, E_n\rangle$ is an exceptional collection on Y then the left i-th mutation $L_i\rho$ and the right i-th mutation $R_i\rho$ are defined as follows

$$L_i\rho = \langle E_1, \ldots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots, E_n\rangle,$$

$$R_i\rho = \langle E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, E_{i+2}, \ldots, E_n\rangle.$$

A strong full exceptional collection ρ on a smooth projective variety Y leads to a DG category $\mathcal{C}_\rho$ concentrated in degree 0 as proved in the lemma 4.1. Determining how categories $\mathcal{C}_\rho$ and $\mathcal{C}_{L_i\rho}$ ($\mathcal{C}_{L_i\rho}$ respectively) are connected allows to find a category $\mathcal{C}_\sigma$ for any collection σ obtained by a sequence of mutations from a strong exceptional collection.

The object L_EF can be considered as a complex $\{0 \to \mathrm{Hom}(E, F) \otimes E \to F \to 0\}$ with non-zero terms in the 0th and 1st gradings. Analogously R_FE can be thought of as $\{0 \to E \to \mathrm{Hom}(E, F)^* \otimes F \to 0\}$ with non-zero terms in the (-1)st and 0th grading. For a complex $\mathbb{B} = \{\ldots B_{-1} \to B_0 \to B_1 \to \ldots\}$ with finitely many non-zero terms and any object A let $\mathrm{Hom}(A, \mathbb{B}) = \bigoplus \mathrm{Hom}(A, B_n)$ and $\mathrm{Hom}(\mathbb{B}, A) = \bigoplus \mathrm{Hom}(B_n, A)$ as proposed in [BK]. In the case of mutations we will only have to deal with complexes with two non-zero terms. If

$$\mathbb{B} \quad = \quad \{0 \longrightarrow B_0 \xrightarrow{\phi} B_1 \longrightarrow 0\}$$

is such a complex with ϕ of degree 0 then for any object A the grading in $\mathrm{Hom}(A, B)$ is given by

$$\mathrm{Hom}^i(A, \mathbb{B}) = \mathrm{Hom}^i(A, B_0) \oplus \mathrm{Hom}^{i-1}(A, B_1).$$

For $(f_0, f_1) \in \mathrm{Hom}^i(A, \mathbb{B})$ the differential d is given by $d(f_0, f_1) = (-df_0, \phi f_0 + df_1)$. Analogously

$$\mathrm{Hom}^i(\mathbb{B}, C) = \mathrm{Hom}^i(B_0, C) \oplus \mathrm{Hom}^{i+1}(B_1, C)$$

and the differential of $(g_0, g_1) \in \mathrm{Hom}^i(B, C)$ is $d(g_0, g_1) = (dg_0 - g_1\phi, -dg_1)$. Finally the composition is defined by $(g_0, g_1) \circ (f_0, f_1) = (-1)^{|f_0|}g_0f_0 + g_1f_1$. It is not a unique definition but with such a choice of signs the square of the differential is zero and the graded Leibniz rule is satisfied.

For a complex

$$\mathbb{D} \quad = \quad \{0 \longrightarrow D_{-1} \xrightarrow{\psi} D_0 \longrightarrow 0\}$$

again with ψ of degree 0 we have

$$\mathrm{Hom}^i(A, \mathbb{D}) = \mathrm{Hom}^{i+1}(A, D_{-1}) \oplus \mathrm{Hom}^i(A, D_0),$$

$$\mathrm{Hom}^i(\mathbb{D}, C) = \mathrm{Hom}^{i-1}(D_{i-1}, C) \oplus \mathrm{Hom}^i(D_0, C).$$

The differential of $(f_{-1}, f_0) \in \mathrm{Hom}^i(A, \mathbb{D})$ is given by $d(f_{-1}, f_0) = (df_{-1}, \psi f_{-1} - df_0)$ and of $(g_{-1}, g_0) \in \mathrm{Hom}(\mathbb{D}, C)$ by $d(g_{-1}, g_0) = (-dg_{-1} + g_0\psi, dg_0)$. The composition is $(g_{-1}, g_0) \circ (f_{-1}, f_0) = g_{-1}f_{-1} + (-1)^{|f_0|}g_0f_0$. Again, this choice of signs is not unique but it satisfies the graded Leibniz rule.

We showed how mutations act on the DG structure of exceptional collections and in order to calculate the DG structure of A_X one has to find a strong exceptional collection on X and mutate it to obtain σ.

Now we come back to calculations on X equal to $\mathbb{P}^2$ blown up in two infinitely close points with divisors H, E_1 and E_2. Let $\pi_1 : X \to \widetilde{X}$ be the blown down of E_2 ($\widetilde{X}$ is $\mathbb{P}^2$ blown up in one point) and let $\pi_2 : \widetilde{X} \to \mathbb{P}^2$ be the blown down of E_1. A full exceptional collection on $\mathbb{P}^2$ is $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$. A theorem of Orlov from [O] states that $\langle \pi_2^*(\mathcal{O}), \pi_2^*(\mathcal{O}(1)), \pi_2^*(\mathcal{O}(2)), \mathcal{O}_{E_1}\rangle$ is a full collection on $\widetilde{X}$. Furthermore $\langle \pi_1^*\pi_2^*(\mathcal{O}), \pi_1^*\pi_2^*(\mathcal{O}(1)), \pi_1^*\pi_2^*(\mathcal{O}(2)), \pi_i^*(\mathcal{O}_{E_1}), \mathcal{O}_{E_2}\rangle = \langle \mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H), \mathcal{O}_{E_1+E_2}, \mathcal{O}_{E_2}\rangle$ is a full exceptional collection on X. Here $\mathcal{O}_{E_1+E_2}$ denotes the structure sheaf of the reduced scheme $E_1 + E_2$. It fits to a short exact sequence

$$0 \to \mathcal{O}(-E_1 - E_2) \to \mathcal{O} \to \mathcal{O}_{E_1+E_2} \to 0.$$

All pullbacks are understood as derived functors. This collection is not strong but its mutation $\langle \mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H - E_1 - E_2), \mathcal{O}(2H), \mathcal{O}_{E_2}\rangle$ is. There is a sequence of mutations

- $\langle \mathcal{O},\ \mathcal{O}(H - E_1 - E_2),\ \mathcal{O}(H - E_2),\ \mathcal{O}(H),\ \mathcal{O}(2H - E_1 - 2E_2)\rangle$,
- $\langle \mathcal{O},\ \mathcal{O}(H - E_1 - E_2),\ \mathcal{O}(H),\ \mathcal{O}_{E_2},\ \mathcal{O}(2H - E_1 - 2E_2)\rangle$,
- $\langle \mathcal{O},\ \mathcal{O}(H),\ \mathcal{O}_{E_1+E_2},\ \mathcal{O}_{E_2},\ \mathcal{O}(2H - E_1 - 2E_2)\rangle$,
- $\langle \mathcal{O},\ \mathcal{O}(H),\ \mathcal{O}_{E_1+E_2},\ \mathcal{O}(2H - E_1 - E_2)[-1],\ \mathcal{O}_{E_2}\rangle$,
- $\langle \mathcal{O},\ \mathcal{O}(H),\ \mathcal{O}(2H - E_1 - E_2)[-1],\ \mathcal{O}(2H)[1],\ \mathcal{O}_{E_2}\rangle$

which we will use to determine the DG structure on A_X.

Remark: The DG structure of A_X is considered up to quasi-isomorphisms. Hence the calculated DG algebras $\widetilde{A}$ will be replaced by quasi-isomorphic ones with less elements. To do the calculations we will use the presentation of a finitely generated DG algebra as a set of all elements $\{a_i\}_{i\in I}$ and multiplication $\{a_ia_j = a_k\}_{i,j\in I}$ together with the datum about grading and differential. Then a set $\mathcal{J} \subset \widetilde{A}$ containing 0 is an ideal if for any $a_i \in \mathcal{J}$ and for any $j \in I$ the element $a_k = a_ia_j$ (or $a_k = a_ja_i$) also belongs to $\mathcal{J}$. If an ideal $\mathcal{J}$ consists only of elements cohomologicaly trivial then $\widetilde{A}/J$ is quasi-isomorphic to $\widetilde{A}$. On the other hand $A_0 \subset \widetilde{A}$ is a subalgebra if for $a_i, a_j \in A_0$ we have $a_ia_j \in A_0$. Thus $\widetilde{A} \setminus \{a_k | k \in K\}$ is a subalgebra if there is no equality of the form $a_ia_j = a_k$ for $i,j \notin K$ and $k \in K$. If again the set $\{a_k | k \in K\}$ consist of cohomologicaly trivial elements then $\widetilde{A} \setminus \{a_k | k \in K\} \subset \widetilde{A}$ is a quasi-isomorphic subalgebra.

Remark. For a mutation of an exceptional pair E, F the set $\mathrm{Hom}(E, L_EF)$ consist of elements $\{\alpha_i \otimes \mathrm{id}_E\}_{\alpha_i\in \mathrm{Hom}(E,F)}$ in degree 0 and $\{\alpha_i\}_{\alpha_i\in \mathrm{Hom}(E,F)}$ in degree 1. The differential is given by $d(\alpha_i \otimes \mathrm{id}_E) = \alpha_i$. If A is a DG algebra on an exceptional collection containing the pair E, F and $\widetilde{A}$ is the algebra of a collection containing L_EF, E, then $\widetilde{A} \setminus \{\alpha_i \otimes \mathrm{id}_E, \alpha_i\}_{\alpha_i\in \mathrm{Hom}(E,F)}$ is quasi-isomorphic to $\widetilde{A}$ (because the elements $\alpha_i \otimes \mathrm{id}_E$ and α_i can not be presented as compositions of elements of $\widetilde{A}$). The same is true for right mutations. Hence, in the following calculations the sets $\mathrm{Hom}(E, L_EF)$ and $\mathrm{Hom}(R_FE, F)$ will always be omitted.

For any two elements E and F of an exceptional collection $\mathrm{Hom}(E, F)$ is a finite dimensional $\mathbb{C}$-vector space. In analogy with arrows in a quiver we will denote by $\mathrm{Arr}(E, F)$ the basis of this space. Thus $\mathrm{Arr}(E, F) = \{\alpha, \beta, \gamma\}$ means that $\mathrm{Hom}(E, F) = \mathbb{C}^3$ with chosen basis α, β and γ.

- The exceptional collection $\langle \mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H - E_1 - E_2), \mathcal{O}(2H), \mathcal{O}_{E_2}\rangle$ is strong and the DG algebra of this collection is concentrated in degree 0. It can be presented as follows:

$$\begin{aligned}
\mathrm{Arr}(\mathcal{O}, \mathcal{O}(H)) &= \{\alpha_1, \alpha_2, \alpha_3\},\\
\mathrm{Arr}(\mathcal{O}(H), \mathcal{O}(2H - E_1 - E_2)) &= \{\beta_1, \beta_2\},\\
\mathrm{Arr}(\mathcal{O}(2H - E_1 - E_2), \mathcal{O}(2H)) &= \{\gamma\},\\
\mathrm{Arr}(\mathcal{O}(2H), \mathcal{O}_{E_2}) &= \{\delta\},\\
\mathrm{Arr}(\mathcal{O}, \mathcal{O}(2H - E_1 - E_2)) &= \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\},\\
\mathrm{Arr}(\mathcal{O}(H), \mathcal{O}(2H)) &= \{\zeta_1, \zeta_2, \zeta_3\},\\
\mathrm{Arr}(\mathcal{O}(2H - E_1 - E_2), \mathcal{O}_{E_2}) &= \{\eta\},\\
\mathrm{Arr}(\mathcal{O}, \mathcal{O}(2H)) &= \{\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6\}\\
\mathrm{Arr}(\mathcal{O}(H), \mathcal{O}_{E_2}) &= \{\kappa\},\\
\mathrm{Arr}(\mathcal{O}, \mathcal{O}_{E_2}) &= \{\lambda\}.
\end{aligned}$$

The composition is given by:

$$\begin{array}{lll}
\beta_1\alpha_1 = \epsilon_1, & \beta_1\alpha_2 = \epsilon_2, & \beta_1\alpha_3 = \epsilon_3,\\
\beta_2\alpha_1 = \epsilon_2, & \beta_2\alpha_2 = \epsilon_4, & \beta_2\alpha_3 = \epsilon_5,\\
\zeta_1\alpha_1 = \iota_3, & \zeta_1\alpha_2 = \iota_5, & \zeta_1\alpha_3 = \iota_6,\\
\zeta_2\alpha_1 = \iota_1, & \zeta_2\alpha_2 = \iota_2, & \zeta_2\alpha_3 = \iota_3,\\
\zeta_3\alpha_1 = \iota_2, & \zeta_3\alpha_2 = \iota_4, & \zeta_3\alpha_3 = \iota_5,\\
\kappa\alpha_1 = 0, & \kappa\alpha_2 = 0, & \kappa\alpha_3 = \lambda,\\
\gamma\beta_1 = \zeta_2, & \gamma\beta_2 = \zeta_3, & \eta\beta_1 = 0,\\
\eta\beta_2 = \kappa, & \delta\gamma = 0, & \gamma\epsilon_1 = \iota_1,\\
\gamma\epsilon_2 = \iota_2, & \gamma\epsilon_3 = \iota_3, & \gamma\epsilon_4 = \iota_4,\\
\gamma\epsilon_5 = \iota_5, & \eta\epsilon_1 = 0, & \eta\epsilon_2 = 0,\\
\eta\epsilon_3 = 0, & \eta\epsilon_4 = 0, & \eta\epsilon_5 = \lambda,\\
\delta\zeta_1 = \kappa, & \delta\zeta_2 = 0, & \delta\zeta_3 = 0,\\
\delta\iota_1 = 0, & \delta\iota_2 = 0, & \delta\iota_3 = 0,\\
\delta\iota_4 = 0, & \delta\iota_5 = 0, & \delta\iota_6 = \lambda.
\end{array}$$

Then the mutation $\mathcal{O}_{E_1+E_2}[-1] = \{\gamma \otimes \mathcal{O}(2H - E_1 - E_2) \to \mathcal{O}(2H)\}$ leads to

- the collection $\langle \mathcal{O}, \mathcal{O}(H), \mathcal{O}_{E_1+E_2}[-1], \mathcal{O}(2H-E_1-E_2), \mathcal{O}_{E_2}\rangle$ with

$$\begin{aligned}
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H)) &= \{\alpha_1,\alpha_2,\alpha_3\},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}_{E_1+E_2}[-1]) &= \left\{\begin{array}{ccc} & \gamma\otimes\beta_1, & \gamma\otimes\beta_2,\\ & \downarrow & \downarrow\\ \zeta_1, & \zeta_2, & \zeta_3\end{array}\right\}\begin{array}{c}0\\ \\ 1\end{array},\\
\mathrm{Arr}(\mathcal{O}_{E_1+E_2}[-1],\mathcal{O}(2H-E_1-E_2)) &= \{\gamma^*\otimes\mathrm{id}\},\\
\mathrm{Arr}(\mathcal{O}(2H-E_1-E_2),\mathcal{O}_{E_2}) &= \{\eta\},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}_{E_1+E_2}[-1]) &= \left\{\begin{array}{cccccc} \gamma\otimes\epsilon_1, & \gamma\otimes\epsilon_2, & \gamma\otimes\epsilon_3, & \gamma\otimes\epsilon_4, & \gamma\otimes\epsilon_5, & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \iota_1, & \iota_2, & \iota_3, & \iota_4, & \iota_5, & \iota_6\end{array}\right\}\begin{array}{c}0\\ \\ 1\end{array},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}(2H-E_1-E_2)) &= \{\beta_1,\beta_2\},\\
\mathrm{Arr}(\mathcal{O}_{E_1+E_2}[-1],\mathcal{O}_{E_2}) &= \left\{\begin{array}{c}\delta\\ \gamma^*\otimes\eta\end{array}\right\}\begin{array}{c}-1\\ 0\end{array},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}(2H-E_1-E_2)) &= \{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}_{E_2}) &= \{\kappa\},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}_{E_2}) &= \{\lambda\}.
\end{aligned}$$

The arrows inside the set represent the values of the differential while the numbers on the right denote gradings. For example there are five morphism from $\mathcal{O}(H)$ to $\mathcal{O}_{E_1+E_2}[-1]$. $\gamma\otimes\beta_1$ and $\gamma\otimes\beta_2$ are in degree zero when ζ_1, ζ_2 and ζ_3 are in degree one. Moreover $d(\gamma\otimes\beta_1)=\zeta_2$ and $d(\gamma\otimes\beta_2)=\zeta_3$.

The compositions are given as follows:

$$\begin{array}{rrr}
\gamma\otimes\beta_1\circ\alpha_1=\gamma\otimes\epsilon_1, & \gamma\otimes\beta_1\circ\alpha_2=\gamma\otimes\epsilon_2, & \gamma\otimes\beta_1\circ\alpha_3=\gamma\otimes\epsilon_3,\\
\gamma\otimes\beta_2\circ\alpha_1=\gamma\otimes\epsilon_2 & \gamma\otimes\beta_2\circ\alpha_2=\gamma\otimes\epsilon_4, & \gamma\otimes\beta_2\circ\alpha_3=\gamma\otimes\epsilon_5,\\
\zeta_1\alpha_1=\iota_3, & \zeta_1\alpha_2=\iota_5, & \zeta_1\alpha_3=\iota_6,\\
\zeta_2\alpha_1=\iota_1, & \zeta_2\alpha_2=\iota_2, & \zeta_2\alpha_3=\iota_3,\\
\zeta_2\alpha_1=\iota_2, & \zeta_3\alpha_2=\iota_4, & \zeta_3\alpha_3=\iota_5,\\
\beta_1\alpha_1=\epsilon_1, & \beta_1\alpha_2=\epsilon_2, & \beta_1\alpha_3=\epsilon_3,\\
\beta_2\alpha_1=\epsilon_2, & \beta_2\alpha_2=\epsilon_4, & \beta_2\alpha_3=\epsilon_5,\\
\kappa\alpha_1=0, & \kappa\alpha_2=0, & \kappa\alpha_3=\lambda,\\
\gamma^*\otimes\mathrm{id}\circ\gamma\otimes\beta_1=\beta_1, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\beta_2=\beta_2, & \gamma^*\otimes\mathrm{id}\circ\zeta_1=0,\\
\gamma^*\otimes\mathrm{id}\circ\zeta_2=0, & \gamma^*\otimes\mathrm{id}\circ\zeta_3=0, & \delta\circ\gamma\otimes\beta_1=0,\\
\delta\circ\gamma\otimes\beta_2=0, & \delta\zeta_1=\kappa, & \delta\zeta_2=0,\\
\delta\zeta_3=0, & \gamma^*\otimes\eta\circ\gamma\otimes\beta_1=0, & \gamma^*\otimes\eta\circ\gamma\otimes\beta_2=\kappa,\\
\gamma^*\otimes\eta\circ\zeta_1=0, & \gamma^*\otimes\eta\circ\zeta_2=0, & \gamma^*\otimes\eta\circ\zeta_3=0,\\
\eta\circ\gamma^*\otimes\mathrm{id}=\gamma^*\otimes\eta, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_1=\epsilon_1, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_2=\epsilon_2,\\
\gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_3=\epsilon_3, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_4=\epsilon_4, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_5=\epsilon_5,\\
\gamma^*\otimes\mathrm{id}\circ\iota_1=0, & \gamma^*\otimes\mathrm{id}\circ\iota_2=0, & \gamma^*\otimes\mathrm{id}\circ\iota_3=0,\\
\gamma^*\otimes\mathrm{id}\circ\iota_4=0, & \gamma^*\otimes\mathrm{id}\circ\iota_5=0, & \gamma^*\otimes\mathrm{id}\circ\iota_6=0,\\
\delta\circ\gamma\otimes\epsilon_1=0, & \delta\circ\gamma\otimes\epsilon_2=0, & \delta\circ\gamma\otimes\epsilon_3=0,\\
\delta\circ\gamma\otimes\epsilon_4=0, & \delta\circ\gamma\otimes\epsilon_5=0, & \delta\iota_1=0,\\
\delta\iota_2=0, & \delta\iota_3=0, & \delta\iota_4=0,\\
\delta\iota_5=0, & \delta\iota_6=\lambda & \gamma^*\otimes\eta\circ\gamma\otimes\epsilon_1=0,\\
\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_2=0, & \gamma^*\otimes\eta\circ\gamma\otimes\epsilon_3=0, & \gamma^*\otimes\eta\circ\gamma\otimes\epsilon_4=0,\\
\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_5=\lambda, & \gamma^*\otimes\eta\circ\iota_1=0, & \gamma^*\otimes\eta\circ\iota_2=0,\\
\gamma^*\otimes\eta\circ\iota_3=0, & \gamma^*\otimes\eta\circ\iota_4=0, & \gamma^*\otimes\eta\circ\iota_5=0,\\
\gamma^*\otimes\eta\circ\iota_6=0, & \eta\beta_1=0, & \eta\beta_2=\kappa,\\
\eta\epsilon_1=0, & \eta\epsilon_2=0, & \eta\epsilon_3=0,\\
\eta\epsilon_4=0, & \eta\epsilon_5=\lambda. &
\end{array}$$

This DG algebra is quasi-isomorphic to the following one

$$
\begin{aligned}
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H)) &= \{\alpha_1,\alpha_2,\alpha_3\},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}_{E_1+E_2}[-1]) &= \{\zeta_1(\mathrm{deg}1)\},\\
\mathrm{Arr}(\mathcal{O}_{E_1+E_2}[-1],\mathcal{O}(2H-E_1-E_2)) &= \{\gamma^*\otimes \mathrm{id}\},\\
\mathrm{Arr}(\mathcal{O}(2H-E_1-E_2),\mathcal{O}_{E_2}) &= \{\eta\},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}_{E_1+E_2}[-1]) &= \left\{\begin{array}{ccc} \gamma\otimes\epsilon_3, & \gamma\otimes\epsilon_5, & \\ \downarrow & \downarrow & \\ \iota_3, & \iota_5, & \iota_6 \end{array}\right\}\begin{array}{c} 0 \\ \\ 1 \end{array},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}(2H-E_1-E_2)) &= \{\beta_1,\beta_2\},\\
\mathrm{Arr}(\mathcal{O}_{E_1+E_2}[-1],\mathcal{O}_{E_2}) &= \left\{\begin{array}{c} \delta \\ \gamma^*\otimes\eta \end{array}\right\}\begin{array}{c} -1 \\ 0 \end{array},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}(2H-E_1-E_2)) &= \{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}_{E_2}) &= \{\kappa\},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}_{E_2}) &= \{\lambda\}.
\end{aligned}
$$

with composition:

$$
\begin{aligned}
\zeta_1\alpha_1 &= \iota_3, & \zeta_1\alpha_2 &= \iota_5, & \zeta_1\alpha_3 &= \iota_6,\\
\beta_1\alpha_1 &= \epsilon_1, & \beta_1\alpha_2 &= \epsilon_2, & \beta_1\alpha_3 &= \epsilon_3,\\
\beta_2\alpha_1 &= \epsilon_2, & \beta_2\alpha_2 &= \epsilon_4, & \beta_2\alpha_3 &= \epsilon_5,\\
\kappa\alpha_1 &= 0, & \kappa\alpha_2 &= 0, & \kappa\alpha_3 &= \lambda,\\
\gamma^*\otimes\mathrm{id}\circ\zeta_1 &= 0, & \delta\zeta_1 &= \kappa, & \gamma^*\otimes\eta\circ\zeta_1 &= 0,\\
\eta\circ\gamma^*\otimes\mathrm{id} &= \gamma^*\otimes\eta, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_3 &= \epsilon_3, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_5 &= \epsilon_5,\\
\gamma^*\otimes\mathrm{id}\circ\iota_3 &= 0, & \gamma^*\otimes\mathrm{id}\circ\iota_5 &= 0, & \gamma^*\otimes\mathrm{id}\circ\iota_6 &= 0,\\
\delta\circ\gamma\otimes\epsilon_3 &= 0, & \delta\circ\gamma\otimes\epsilon_5 &= 0, & \delta\iota_3 &= 0,\\
\delta\iota_5 &= 0, & \delta\iota_6 &= \lambda & \gamma^*\otimes\eta\circ\gamma\otimes\epsilon_3 &= 0,\\
\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_5 &= \lambda, & \gamma^*\otimes\eta\circ\iota_3 &= 0, & \gamma^*\otimes\eta\circ\iota_5 &= 0,\\
\gamma^*\otimes\eta\circ\iota_6 &= 0, & \eta\beta_1 &= 0, & \eta\beta_2 &= \kappa,\\
\eta\epsilon_1 &= 0, & \eta\epsilon_2 &= 0, & \eta\epsilon_3 &= 0,\\
\eta\epsilon_4 &= 0, & \eta\epsilon_5 &= \lambda.
\end{aligned}
$$

The next mutation on the list is $\{\mathcal{O}(2H-E_1-E_2)\to\eta^*\otimes\mathcal{O}_{E_2}\}=\mathcal{O}(2H-E_1-2E_2)[1]$. It gives the following collection

- $\langle\mathcal{O},\mathcal{O}(H),\mathcal{O}_{E_1+E_2}[-1],\mathcal{O}_{E_2},\mathcal{O}(2H-E_1-2E_2)[1]\rangle$. The morphism between the elements of this collection are

$$
\begin{aligned}
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H)) &= \{\alpha_1,\alpha_2,\alpha_3\},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}_{E_1+E_2}[-1]) &= \{\zeta_1(\mathrm{deg}1)\},\\
\mathrm{Arr}(\mathcal{O}_{E_1+E_2}[-1],\mathcal{O}_{E_2}) &= \left\{\begin{array}{c} \delta \\ \gamma^*\otimes\eta \end{array}\right\}\begin{array}{c} -1 \\ 0 \end{array},\\
\mathrm{Arr}(\mathcal{O}_{E_2},\mathcal{O}(2H-E_1-2E_2)[1]) &= \{\eta^*\otimes\mathrm{id}\},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}_{E_1+E_2}[-1]) &= \left\{\begin{array}{ccc} \gamma\otimes\epsilon_3, & \gamma\otimes\epsilon_5, & \\ \downarrow & \downarrow & \\ \iota_3, & \iota_5, & \iota_6 \end{array}\right\}\begin{array}{c} 0 \\ \\ 1 \end{array},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}_{E_2}) &= \{\kappa\},\\
\mathrm{Arr}(\mathcal{O}_{E_1+E_2}[-1],\mathcal{O}(2H-E_1-2E_2)[1]) &= \left\{\begin{array}{cc} \gamma^*\otimes\mathrm{id}, & \eta^*\otimes\delta, \\ \downarrow & \\ \eta^*\otimes\gamma^*\otimes\eta & \end{array}\right\}\begin{array}{c} -1 \\ \\ 0 \end{array},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}_{E_2}) &= \{\lambda\},
\end{aligned}
$$

$$\operatorname{Arr}(\mathcal{O}(H), \mathcal{O}(2H-E_1-2E_2)[1]) = \left\{ \begin{array}{cc} \beta_1, & \beta_2 \\ & \downarrow \\ & \eta^*\otimes\kappa \end{array} \right\} \begin{array}{c} -1 \\ \\ 0 \end{array},$$

$$\operatorname{Arr}(\mathcal{O}, \mathcal{O}(2H-E_1-2E_2)[1]) = \left\{ \begin{array}{ccccc} \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & \epsilon_5, \\ & & & & \downarrow \\ & & & & \eta^*\otimes\lambda \end{array} \right\} \begin{array}{c} -1 \\ \\ 0 \end{array}.$$

The composition is given by

$$\begin{array}{rrr}
\zeta_1\alpha_1 = \iota_3, & \zeta_1\alpha_2 = \iota_5, & \zeta_1\alpha_3 = \iota_6, \\
\kappa\alpha_1 = 0, & \kappa\alpha_2 = 0, & \kappa\alpha_3 = \lambda, \\
\beta_1\alpha_1 = \epsilon_1, & \beta_1\alpha_2 = \epsilon_2, & \beta_1\alpha_3 = \epsilon_3, \\
\beta_2\alpha_1 = \epsilon_2, & \beta_2\alpha_2 = \epsilon_4, & \beta_2\alpha_3 = \epsilon_5, \\
\eta^*\otimes\kappa\circ\alpha_1 = 0, & \eta^*\otimes\kappa\circ\alpha_2 = 0, & \eta^*\otimes\kappa\circ\alpha_3 = \eta^*\otimes\lambda, \\
\delta\zeta_1 = \kappa, & \gamma^*\otimes\eta\circ\zeta_1 = 0, & \gamma^*\otimes\mathrm{id}\circ\zeta_1 = 0, \\
\eta^*\otimes\delta\circ\zeta_1 = \eta^*\otimes\kappa, & \eta^*\otimes\gamma^*\otimes\eta\circ\zeta_1 = 0, & \eta^*\otimes\mathrm{id}\circ\delta = \eta^*\otimes\delta, \\
\eta^*\otimes\mathrm{id}\circ\gamma^*\otimes\eta = \eta^*\otimes\gamma^*\otimes\eta, & \delta\circ\gamma\otimes\epsilon_3 = 0, & \delta\circ\gamma\otimes\epsilon_5 = 0, \\
\delta\iota_3 = 0, & \delta\iota_5 = 0, & \delta\iota_6 = \lambda \\
\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_3 = 0, & \gamma^*\otimes\eta\circ\gamma\otimes\epsilon_5 = \lambda, & \gamma^*\otimes\eta\circ\iota_3 = 0, \\
\gamma^*\otimes\eta\circ\iota_5 = 0, & \gamma^*\otimes\eta\circ\iota_6 = 0, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_3 = \epsilon_3, \\
\gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_5 = \epsilon_5, & \gamma^*\otimes\mathrm{id}\circ\iota_3 = 0, & \gamma^*\otimes\mathrm{id}\circ\iota_5 = 0, \\
\gamma^*\otimes\mathrm{id}\circ\iota_6 = 0, & \eta^*\otimes\delta\circ\gamma\otimes\epsilon_3 = 0, & \eta^*\otimes\delta\circ\gamma\otimes\epsilon_5 = 0, \\
\eta^*\otimes\delta\circ\iota_3 = 0, & \eta^*\otimes\delta\circ\iota_5 = 0, & \eta^*\otimes\delta\circ\iota_6 = \eta^*\otimes\lambda, \\
\eta^*\otimes\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_3 = 0, & \eta^*\otimes\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_5 = \eta^*\otimes\lambda, & \eta^*\otimes\gamma^*\otimes\eta\circ\iota_3 = 0, \\
\eta^*\otimes\gamma^*\otimes\eta\circ\iota_5 = 0, & \eta^*\otimes\gamma^*\otimes\eta\circ\iota_6 = 0, & \eta^*\otimes\mathrm{id}\circ\kappa = \eta^*\otimes\kappa, \\
\eta^*\otimes\mathrm{id}\circ\lambda = \eta^*\otimes\lambda. & &
\end{array}$$

This collection again can be simplified. Moreover the elements $\mathcal{O}_{E_1+E_2}[-1]$ and $\mathcal{O}(2H-E_1-2E_2)[1]$ can be shifted to degree 0. After this operations the collection is $\langle \mathcal{O}, \mathcal{O}(H), \mathcal{O}_{E_1+E_2}, \mathcal{O}_{E_2}, \mathcal{O}(2H-E_1-2E_2)\rangle$ and the morphism are

$$\begin{aligned}
\operatorname{Arr}(\mathcal{O}, \mathcal{O}(H)) &= \{\alpha_1, \alpha_2, \alpha_3\}, \\
\operatorname{Arr}(\mathcal{O}(H), \mathcal{O}_{E_1+E_2}) &= \{\zeta_1\}, \\
\operatorname{Arr}(\mathcal{O}_{E_1+E_2}, \mathcal{O}_{E_2}) &= \left\{ \begin{array}{c} \delta \\ \gamma^*\otimes\eta \end{array} \right\} \begin{array}{c} 0 \\ 1 \end{array}, \\
\operatorname{Arr}(\mathcal{O}_{E_2}, \mathcal{O}(2H-E_1-2E_2)) &= \{\eta^*\otimes\mathrm{id}\,(\mathrm{deg}1)\}, \\
\operatorname{Arr}(\mathcal{O}, \mathcal{O}_{E_1+E_2}) &= \left\{ \begin{array}{ccc} \gamma\otimes\epsilon_3, & \gamma\otimes\epsilon_5, & \\ \downarrow & \downarrow & \\ \iota_3, & \iota_5, & \iota_6 \end{array} \right\} \begin{array}{c} -1 \\ \\ 0 \end{array},
\end{aligned}$$

$$\begin{aligned}
\operatorname{Arr}(\mathcal{O}(H), \mathcal{O}_{E_2}) &= \{\kappa\}, \\
\operatorname{Arr}(\mathcal{O}_{E_1+E_2}, \mathcal{O}(2H-E_1-2E_2)) &= \left\{ \begin{array}{cc} \gamma^*\otimes\mathrm{id}, & \eta^*\otimes\delta, \\ \downarrow & \\ \eta^*\otimes\gamma^*\otimes\eta & \end{array} \right\} \begin{array}{c} 1 \\ \\ 2 \end{array}, \\
\operatorname{Arr}(\mathcal{O}, \mathcal{O}_{E_2}) &= \{\lambda\}, \\
\operatorname{Arr}(\mathcal{O}(H), \mathcal{O}(2H-E_1-2E_2)) &= \left\{ \begin{array}{cc} \beta_1, & \beta_2 \\ & \downarrow \\ & \eta^*\otimes\kappa \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array}, \\
\operatorname{Arr}(\mathcal{O}, \mathcal{O}(2H-E_1-2E_2)) &= \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}.
\end{aligned}$$

with compositions

$$
\begin{array}{rrr}
\zeta_1\alpha_1 = \iota_3, & \zeta_1\alpha_2 = \iota_5, & \zeta_1\alpha_3 = \iota_6, \\
\kappa\alpha_1 = 0, & \kappa\alpha_2 = 0, & \kappa\alpha_3 = \lambda, \\
\beta_1\alpha_1 = \epsilon_1, & \beta_1\alpha_2 = \epsilon_2, & \beta_1\alpha_3 = \epsilon_3, \\
\beta_2\alpha_1 = \epsilon_2, & \beta_2\alpha_2 = \epsilon_4, & \beta_2\alpha_3 = 0, \\
\eta^*\otimes\kappa\circ\alpha_1 = 0, & \eta^*\otimes\kappa\circ\alpha_2 = 0, & \eta^*\otimes\kappa\circ\alpha_3 = 0, \\
\delta\zeta_1 = \kappa, & \gamma^*\otimes\eta\circ\zeta_1 = 0, & \gamma^*\otimes\mathrm{id}\circ\zeta_1 = 0, \\
\eta^*\otimes\delta\circ\zeta_1 = \eta^*\otimes\kappa, & \eta^*\otimes\gamma^*\otimes\eta\circ\zeta_1 = 0, & \eta^*\otimes\mathrm{id}\circ\delta = \eta^*\otimes\delta, \\
\eta^*\otimes\mathrm{id}\circ\gamma^*\otimes\eta = \eta^*\otimes\gamma^*\otimes\eta, & \delta\circ\gamma\otimes\epsilon_3 = 0, & \delta\circ\gamma\otimes\epsilon_5 = 0, \\
\delta\iota_3 = 0, & \delta\iota_5 = 0, & \delta\iota_6 = \lambda \\
\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_3 = 0, & \gamma^*\otimes\eta\circ\gamma\otimes\epsilon_5 = \lambda, & \gamma^*\otimes\eta\circ\iota_3 = 0, \\
\gamma^*\otimes\eta\circ\iota_5 = 0, & \gamma^*\otimes\eta\circ\iota_6 = 0, & \gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_3 = \epsilon_3, \\
\gamma^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_5 = 0, & \gamma^*\otimes\mathrm{id}\circ\iota_3 = 0, & \gamma^*\otimes\mathrm{id}\circ\iota_5 = 0, \\
\gamma^*\otimes\mathrm{id}\circ\iota_6 = 0, & \eta^*\otimes\delta\circ\gamma\otimes\epsilon_3 = 0, & \eta^*\otimes\delta\circ\gamma\otimes\epsilon_5 = 0, \\
\eta^*\otimes\delta\circ\iota_3 = 0, & \eta^*\otimes\delta\circ\iota_5 = 0, & \eta^*\otimes\delta\circ\iota_6 = 0, \\
\eta^*\otimes\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_3 = 0, & \eta^*\otimes\gamma^*\otimes\eta\circ\gamma\otimes\epsilon_5 = 0, & \eta^*\otimes\gamma^*\otimes\eta\circ\iota_3 = 0, \\
\eta^*\otimes\gamma^*\otimes\eta\circ\iota_5 = 0, & \eta^*\otimes\gamma^*\otimes\eta\circ\iota_6 = 0, & \eta^*\otimes\mathrm{id}\circ\kappa = \eta^*\otimes\kappa, \\
\eta^*\otimes\mathrm{id}\circ\lambda = 0. & &
\end{array}
$$

The next mutation is obtained by $\mathcal{O}(H-E_1-E_2) = \{\zeta_1\otimes\mathcal{O}(H)\to\mathcal{O}_{E_1+E_2}\}$. It gives the collection

- $\langle\mathcal{O},\mathcal{O}(H-E_1-E_2),\mathcal{O}(H),\mathcal{O}_{E_2},\mathcal{O}(2H-E_1-2E_2)\rangle$ with morphism

$$
\begin{aligned}
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H-E_1-E_2)) &= \left\{\begin{array}{ccccc} \zeta_1\otimes\alpha_1, & \gamma\otimes\epsilon_3, & \zeta_1\otimes\alpha_2, & \gamma\otimes\epsilon_5, & \zeta_1\otimes\alpha_3 \\ \iota_3, & & \iota_5, & & \iota_6 \end{array}\right\}\begin{array}{c} 0 \\ 1 \end{array}, \\
\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}(H)) &= \{\zeta_1\otimes\mathrm{id}\}, \\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}_{E_2}) &= \{\kappa\}, \\
\mathrm{Arr}(\mathcal{O}_{E_2},\mathcal{O}(2H-E_1-2E_2)) &= \{\eta^*\otimes\mathrm{id}\,(\mathrm{deg}1)\}, \\
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H)) &= \{\alpha_1,\alpha_2,\alpha_3\}, \\
\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}_{E_2}) &= \left\{\begin{array}{cc} \delta, & \\ \downarrow & \\ \zeta_1\otimes\kappa, & \gamma^*\otimes\eta \end{array}\right\}\begin{array}{c} -1 \\ \\ 0 \end{array}, \\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}(2H-E_1-2E_2)) &= \left\{\begin{array}{cc} \beta_1, & \beta_2 \\ & \downarrow \\ & \eta^*\otimes\kappa \end{array}\right\}\begin{array}{c} 0 \\ \\ 1 \end{array}, \\
\mathrm{Arr}(\mathcal{O},\mathcal{O}_{E_2}) &= \{\lambda\}, \\
\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}(2H-E_1-2E_2)) &= \left\{\begin{array}{cccc} \zeta_1^*\otimes\beta_1, & \zeta_1^*\otimes\beta_2, & \eta^*\otimes\delta, & \gamma^*\otimes\mathrm{id}, \\ & \zeta_1^*\otimes\eta^*\otimes\kappa & & \eta^*\otimes\gamma^*\otimes\eta \end{array}\right\}\begin{array}{c} 0 \\ 1 \end{array}, \\
\mathrm{Arr}(\mathcal{O},\mathcal{O}(2H-E_1-2E_2)) &= \{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\}.
\end{aligned}
$$

where

$$
\begin{array}{ll}
d(\zeta_1\otimes\alpha_1) = \iota_3, & d(\gamma\otimes\epsilon_3) = \iota_3, \\
d(\zeta_1\otimes\alpha_2) = \iota_5, & d(\gamma\otimes\epsilon_5) = \iota_5, \\
d(\zeta_1\otimes\alpha_3) = \iota_6, & d(\zeta_1^*\otimes\beta_2) = \zeta_1^*\otimes\eta_1^*\otimes\kappa, \\
d(\eta^*\otimes\delta) = -\zeta_1^*\otimes\eta_1^*\otimes\kappa, & d(\gamma^*\otimes\mathrm{id}) = \eta^*\otimes\gamma^*\otimes\eta.
\end{array}
$$

The composition of morphisms is

$$
\begin{array}{rr}
\zeta_1^*\otimes\mathrm{id}\circ\zeta_1\otimes\alpha_1 = \alpha_1, & \zeta_1^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_3 = 0, \\
\zeta_1^*\otimes\mathrm{id}\circ\zeta_1\otimes\alpha_2 = \alpha_2, & \zeta_1^*\otimes\mathrm{id}\circ\gamma\otimes\epsilon_5 = 0 \\
\zeta_1^*\otimes\mathrm{id}\circ\zeta_1\otimes\alpha_3 = \alpha_3, & \zeta_1^*\otimes\mathrm{id}\circ\iota_3 = 0, \\
\zeta_1^*\otimes\mathrm{id}\circ\iota_5 = 0, & \zeta_1^*\otimes\mathrm{id}\circ\iota_6 = 0,
\end{array}
$$

$$\delta \circ \zeta_1 \otimes \alpha_1 = 0, \qquad \delta \circ \gamma \otimes \epsilon_3 = 0,$$
$$\delta \circ \zeta_1 \otimes \alpha_2 = 0, \qquad \delta \circ \gamma \otimes \epsilon_5 = 0,$$
$$\delta \circ \zeta_1 \otimes \alpha_3 = 0, \qquad \delta\iota_3 = 0,$$
$$\delta\iota_5 = 0, \qquad \delta\iota_6 = \lambda$$
$$\zeta_1^* \otimes \kappa \circ \zeta_1 \otimes \alpha_1 = 0, \qquad \zeta_1^* \otimes \kappa \circ \gamma \otimes \epsilon_3 = 0,$$
$$\zeta_1^* \otimes \kappa \circ \zeta_1 \otimes \alpha_2 = 0, \qquad \zeta_1^* \otimes \kappa \circ \gamma \otimes \epsilon_5 = 0,$$
$$\zeta_1^* \otimes \kappa \circ \zeta_1 \otimes \alpha_3 = \lambda, \qquad \gamma^* \otimes \eta \circ \zeta_1 \otimes \alpha_1 = 0,$$
$$\gamma^* \otimes \eta \circ \gamma \otimes \epsilon_3 = 0, \qquad \gamma^* \otimes \eta \circ \zeta_1 \otimes \alpha_2 = 0,$$
$$\gamma^* \otimes \eta \circ \gamma \otimes \epsilon_5 = \lambda, \qquad \gamma^* \otimes \eta \circ \zeta_1 \otimes \alpha_3 = 0,$$
$$\gamma^* \otimes \eta \circ \iota_3 = 0, \qquad \gamma^* \otimes \eta \circ \iota_5 = 0,$$
$$\gamma^* \otimes \eta \circ \iota_6 = 0, \qquad \zeta_1^* \otimes \beta_1 \circ \zeta_1 \otimes \alpha_1 = \epsilon_1,$$
$$\zeta_1^* \otimes \beta_1 \circ \gamma \otimes \epsilon_3 = 0, \qquad \zeta_1^* \otimes \beta_1 \circ \zeta_1 \otimes \alpha_2 = \epsilon_2,$$
$$\zeta_1^* \otimes \beta_1 \circ \gamma \otimes \epsilon_5 = 0, \qquad \zeta_1^* \otimes \beta_1 \circ \zeta_1 \otimes \alpha_3 = \epsilon_3,$$
$$\zeta_1^* \otimes \beta_1 \circ \iota_3 = 0, \qquad \zeta_1^* \otimes \beta_1 \circ \iota_5 = 0,$$
$$\zeta_1^* \otimes \beta_1 \circ \iota_6 = 0, \qquad \zeta_1^* \otimes \beta_2 \circ \zeta_1 \otimes \alpha_1 = \epsilon_2,$$
$$\zeta_1^* \otimes \beta_2 \circ \gamma \otimes \epsilon_3 = 0, \qquad \zeta_1^* \otimes \beta_2 \circ \zeta_1 \otimes \alpha_2 = \epsilon_4,$$
$$\zeta_1^* \otimes \beta_2 \circ \gamma \otimes \epsilon_5 = 0, \qquad \zeta_1^* \otimes \beta_2 \circ \zeta_1 \otimes \alpha_3 = 0,$$
$$\zeta_1^* \otimes \beta_2 \circ \iota_3 = 0, \qquad \zeta_1^* \otimes \beta_2 \circ \iota_5 = 0,$$
$$\zeta_1^* \otimes \beta_2 \circ \iota_6 = 0, \qquad \eta^* \otimes \delta \circ \zeta_1 \otimes \alpha_1 = 0,$$
$$\eta^* \otimes \delta \circ \gamma \otimes \epsilon_3 = 0, \qquad \eta^* \otimes \delta \circ \zeta_1 \otimes \alpha_2 = 0,$$
$$\eta^* \otimes \delta \circ \gamma \otimes \epsilon_5 = 0, \qquad \eta^* \otimes \delta \circ \zeta_1 \otimes \alpha_3 = 0,$$
$$\eta^* \otimes \delta \circ \iota_3 = 0, \qquad \eta^* \otimes \delta \circ \iota_5 = 0,$$
$$\eta^* \otimes \delta \circ \iota_6 = 0, \qquad \gamma^* \otimes \mathrm{id} \circ \zeta_1 \otimes \alpha_1 = 0,$$
$$\gamma^* \otimes \mathrm{id} \circ \gamma \otimes \epsilon_3 = \epsilon_3, \qquad \gamma^* \otimes \mathrm{id} \circ \zeta_1 \otimes \alpha_2 = 0,$$
$$\gamma^* \otimes \mathrm{id} \circ \gamma \otimes \epsilon_5 = 0, \qquad \gamma^* \otimes \mathrm{id} \circ \zeta_1 \otimes \alpha_3 = 0,$$
$$\gamma^* \otimes \mathrm{id} \circ \iota_3 = 0, \qquad \gamma^* \otimes \mathrm{id} \circ \iota_5 = 0,$$
$$\gamma^* \otimes \mathrm{id} \circ \iota_6 = 0, \qquad \zeta_1^* \otimes \eta^* \otimes \kappa \circ \zeta_1 \otimes \alpha_1 = 0,$$
$$\zeta_1^* \otimes \eta^* \otimes \kappa \circ \gamma \otimes \epsilon_3 = 0, \qquad \zeta_1^* \otimes \eta^* \otimes \kappa \circ \zeta_1 \otimes \alpha_2 = 0,$$
$$\zeta_1^* \otimes \eta^* \otimes \kappa \circ \gamma \otimes \epsilon_5 = 0, \qquad \zeta_1^* \otimes \eta^* \otimes \kappa \circ \zeta_1 \otimes \alpha_3 = 0$$
$$\zeta_1^* \otimes \eta^* \otimes \kappa \circ \iota_3 = 0, \qquad \zeta_1^* \otimes \eta^* \otimes \kappa \circ \iota_5 = 0,$$
$$\zeta_1^* \otimes \eta^* \otimes \kappa \circ \iota_6 = 0, \qquad \eta^* \otimes \gamma^* \otimes \eta \circ \zeta_1 \otimes \alpha_1 = 0,$$
$$\eta^* \otimes \gamma^* \otimes \eta \circ \gamma \otimes \epsilon_3 = 0, \qquad \eta^* \otimes \gamma^* \otimes \eta \circ \zeta_1 \otimes \alpha_2 = 0,$$
$$\eta^* \otimes \gamma^* \otimes \eta \circ \gamma \otimes \epsilon_5 = 0, \qquad \eta^* \otimes \gamma^* \otimes \eta \circ \zeta_1 \otimes \alpha_3 = 0,$$
$$\eta^* \otimes \gamma^* \otimes \eta \circ \iota_3 = 0, \qquad \eta^* \otimes \gamma^* \otimes \eta \circ \iota_5 = 0,$$
$$\eta^* \otimes \gamma^* \otimes \eta \circ \iota_6 = 0, \qquad \kappa \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \kappa,$$
$$\beta_1 \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \beta_1, \qquad \beta_2 \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \beta_2,$$
$$\eta^* \otimes \kappa \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \eta^* \otimes \kappa, \qquad \eta^* \otimes \mathrm{id} \circ \kappa = \eta^* \otimes \kappa,$$
$$\kappa\alpha_1 = 0, \qquad \kappa\alpha_2 = 0,$$
$$\kappa\alpha_3 = \lambda, \qquad \beta_1\alpha_1 = \epsilon_1,$$
$$\beta_1\alpha_2 = \epsilon_2, \qquad \beta_1\alpha_3 = \epsilon_3,$$
$$\beta_2\alpha_1 = \epsilon_2, \qquad \beta_2\alpha_2 = \epsilon_4,$$
$$\beta_2\alpha_3 = 0, \qquad \eta^* \otimes \kappa \circ \alpha_1 = 0,$$
$$\eta^* \otimes \kappa \circ \alpha_2 = 0, \qquad \eta^* \otimes \kappa \circ \alpha_3 = 0,$$
$$\eta^* \otimes \mathrm{id} \circ \delta = \eta^* \otimes \delta, \qquad \eta^* \otimes \mathrm{id} \circ \zeta_1^* \otimes \kappa = \zeta_1^* \otimes \eta^* \otimes \kappa,$$
$$\eta^* \otimes \mathrm{id} \circ \gamma^* \otimes \eta = \eta^* \otimes \gamma^* \otimes \eta, \qquad \eta^* \otimes \mathrm{id} \circ \lambda = 0.$$

Putting

$$\mu_1 = \zeta_1 \otimes \alpha_1 + \gamma \otimes \epsilon_3, \qquad \mu_2 = \zeta_1 \otimes \alpha_1 - \gamma \otimes \epsilon_3,$$

$$\mu_3 = \zeta_1 \otimes \alpha_2 + \gamma \otimes \epsilon_5, \qquad \mu_4 = \zeta_1 \otimes \alpha_2 - \gamma \otimes \epsilon_5,$$
$$\nu_1 = \zeta_1^* \otimes \beta_2 + \eta^* \otimes \delta, \qquad \nu_2 = \zeta_1^* \otimes \beta_2 - \eta^* \otimes \delta$$

and simplifying leads to the DG structure given by:

$$\begin{aligned}
\mathrm{Arr}(\mathcal{O}, \mathcal{O}(H - E_1 - E_2)) &= \{\mu_2, \mu_4\},\\
\mathrm{Arr}(\mathcal{O}(H - E_1 - E_2), \mathcal{O}(H)) &= \{\zeta_1 \otimes \mathrm{id}\},\\
\mathrm{Arr}(\mathcal{O}(H), \mathcal{O}_{E_2}) &= \{\kappa\},\\
\mathrm{Arr}(\mathcal{O}_{E_2}, \mathcal{O}(2H - E_1 - 2E_2)) &= \{\eta^* \otimes \mathrm{id}\,(\mathrm{deg}1)\},\\
\mathrm{Arr}(\mathcal{O}, \mathcal{O}(H)) &= \{\alpha_1, \alpha_2, \alpha_3\},\\
\mathrm{Arr}(\mathcal{O}(H - E_1 - E_2), \mathcal{O}_{E_2}) &= \left\{ \begin{array}{cc} \delta, & \\ \downarrow & \\ \zeta_1 \otimes \kappa, & \gamma^* \otimes \eta \end{array} \right\} \begin{array}{c} -1 \\ \\ 0 \end{array},\\
\mathrm{Arr}(\mathcal{O}(H), \mathcal{O}(2H - E_1 - 2E_2)) &= \left\{ \begin{array}{cc} \beta_1, & \beta_2 \\ & \downarrow \\ & \eta^* \otimes \kappa \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array},\\
\mathrm{Arr}(\mathcal{O}, \mathcal{O}_{E_2}) &= \{\lambda\},\\
\mathrm{Arr}(\mathcal{O}(H - E_1 - E_2), \mathcal{O}(2H - E_1 - 2E_2)) &= \left\{ \begin{array}{cccc} \zeta_1^* \otimes \beta_1, & \nu_1, & \nu_2 & \gamma^* \otimes \mathrm{id}, \\ & & \downarrow & \downarrow - \\ & & \zeta_1^* \otimes \eta^* \otimes \kappa & \eta^* \otimes \gamma^* \otimes \eta \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array},\\
\mathrm{Arr}(\mathcal{O}, \mathcal{O}(2H - E_1 - 2E_2)) &= \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}.
\end{aligned}$$

The compositions are:

$$\begin{array}{rrr}
\zeta_1^* \otimes \mathrm{id} \circ \mu_2 = \alpha_1, & \zeta_1^* \otimes \mathrm{id} \circ \mu_4 = \alpha_2, & \delta\mu_2 = 0,\\
\delta\mu_4 = 0, & \zeta_1^* \otimes \kappa \circ \mu_2 = 0, & \zeta_1^* \otimes \kappa \circ \mu_4 = 0,\\
\gamma^* \otimes \eta \circ \mu_2 = 0, & \gamma^* \otimes \eta \circ \mu_4 = -\lambda, & \zeta_1^* \otimes \beta_1 \circ \mu_2 = \epsilon_1,\\
\zeta_1^* \otimes \beta_1 \circ \mu_4 = \epsilon_2, & \nu_1\mu_2 = \epsilon_2, & \nu_1\mu_4 = \epsilon_4,\\
\nu_2\mu_2 = \epsilon_2, & \nu_2\mu_4 = \epsilon_4, & \gamma^* \otimes \mathrm{id} \circ \mu_2 = -\epsilon_3,\\
\gamma^* \otimes \mathrm{id} \circ \mu_4 = 0, & \zeta_1^* \otimes \eta^* \otimes \kappa \circ \mu_2 = 0, & \zeta_1^* \otimes \eta^* \otimes \kappa \circ \mu_4 = 0,\\
\eta^* \otimes \gamma^* \otimes \eta \circ \mu_2 = 0, & \eta^* \otimes \gamma^* \otimes \eta \circ \mu_4 = 0, & \kappa \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \kappa,\\
\beta_1 \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \beta_1, & \beta_2 \circ \zeta_1^* \otimes \mathrm{id} = \frac{1}{2}(\nu_1 + \nu_2), & \eta^* \otimes \kappa \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \eta^* \otimes \kappa,\\
\eta^* \otimes \mathrm{id} \circ \kappa = \eta^* \otimes \kappa, & \kappa\alpha_1 = 0, & \kappa\alpha_2 = 0,\\
\kappa\alpha_3 = \lambda, & \beta_1\alpha_1 = \epsilon_1, & \beta_1\alpha_2 = \epsilon_2,\\
\beta_1\alpha_3 = \epsilon_3, & \beta_2\alpha_1 = \epsilon_2, & \beta_2\alpha_2 = \epsilon_4,\\
\beta_2\alpha_3 = 0, & \eta^* \otimes \kappa \circ \alpha_1 = 0, & \eta^* \otimes \kappa \circ \alpha_2 = 0,\\
\eta^* \otimes \kappa \circ \alpha_3 = 0, & \eta^* \otimes \mathrm{id} \circ \delta = \frac{1}{2}(\nu_1 - \nu_2), & \eta^* \otimes \mathrm{id} \circ \zeta_1^* \otimes \kappa = \zeta_1^* \otimes \eta^* \otimes \kappa,\\
\eta^* \otimes \mathrm{id} \circ \gamma^* \otimes \eta = \eta^* \otimes \gamma^* \otimes \eta, & \eta^* \otimes \mathrm{id} \circ \lambda = 0. &
\end{array}$$

The final mutation is obtained by $\mathcal{O}(H - E_2) = \{\kappa \otimes \mathcal{O}(H) \to \mathcal{O}_{E_2}\}$. It leads to

- the collection $\langle \mathcal{O}, \mathcal{O}(H - E_1 - E_2), \mathcal{O}(H - E_2), \mathcal{O}(H), \mathcal{O}(2H - E_1 - 2E_2) \rangle$.
  Calculations give the following morphism between elements of this collection:

$$\begin{aligned}
\mathrm{Arr}(\mathcal{O}, \mathcal{O}(H - E_1 - E_2)) &= \{\mu_2, \mu_4\},\\
\mathrm{Arr}(\mathcal{O}(H - E_1 - E_2), \mathcal{O}(H - E_2)) &= \left\{ \begin{array}{cc} \kappa \otimes \zeta_1^* \otimes \mathrm{id}, & \delta \\ \gamma^* \otimes \eta, & \zeta_1^* \otimes \kappa \end{array} \right\} \begin{array}{c} 0 \\ 1 \end{array},\\
\mathrm{Arr}(\mathcal{O}(H - E_2), \mathcal{O}(H)) &= \{\kappa^* \otimes \mathrm{id}\},\\
\mathrm{Arr}(\mathcal{O}(H), \mathcal{O}(2H - E_1 - 2E_2)) &= \left\{ \begin{array}{cc} \beta_1, & \beta_2 \\ & \downarrow \\ & \eta^* \otimes \kappa \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array},
\end{aligned}$$

$$\mathrm{Arr}(\mathcal{O},\mathcal{O}(H-E_2)) = \left\{ \begin{array}{ccc} \kappa\otimes\alpha_1, & \kappa\otimes\alpha_2, & \kappa\otimes\alpha_3 \\ & & \downarrow \\ & & \lambda \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array},$$

$$\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}(H)) = \{\zeta_1\otimes\mathrm{id}\},$$

$$\mathrm{Arr}(\mathcal{O}(H-E_2),\mathcal{O}(2H-E_1-2E_2)) = \left\{ \begin{array}{ccc} \kappa^*\otimes\beta_1, & \kappa^*\otimes\beta_2, & \eta^*\otimes\mathrm{id}, \\ & & \kappa^*\otimes\eta^*\otimes\kappa \end{array} \right\} \; 0//1 \;,$$

$$\mathrm{Arr}(\mathcal{O},\mathcal{O}(H)) = \{\alpha_1,\alpha_2,\alpha_3\},$$

$$\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}(2H-E_1-2E_2)) = \left\{ \begin{array}{cccc} \zeta_1^*\otimes\beta_1, & \nu_1, & \nu_2 & \gamma^*\otimes\mathrm{id}, \\ & & \downarrow & \downarrow - \\ & & \zeta_1^*\otimes\eta^*\otimes\kappa & \eta^*\otimes\gamma^*\otimes\eta \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array},$$

$$\mathrm{Arr}(\mathcal{O},\mathcal{O}(2H-E_1-2E_2)) = \{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\},$$

with

$$d(\kappa\otimes\zeta_1^*\otimes\mathrm{id}) = \zeta_1^*\otimes\kappa, \qquad d(\delta) = \zeta_1^*\otimes\kappa,$$
$$d(\kappa^*\otimes\beta_2) = \kappa^*\otimes\eta^*\otimes\kappa, \qquad d(\eta^*\otimes\mathrm{id} = \kappa^*\otimes\eta^*\otimes\kappa.$$

The composition of morphisms is the following:

$$\kappa\otimes\zeta_1^*\otimes\mathrm{id}\circ\mu_2 = \kappa\otimes\alpha_1, \qquad \kappa\otimes\zeta_1^*\otimes\mathrm{id}\circ\mu_4 = \kappa\otimes\alpha_2,$$
$$\delta\mu_2 = 0, \qquad \delta\mu_4 = 0,$$
$$\gamma^*\otimes\eta\circ\mu_2 = 0, \qquad \gamma^*\otimes\eta\circ\mu_4 = -\lambda,$$
$$\zeta_1^*\otimes\kappa\circ\mu_2 = 0, \qquad \zeta_1^*\otimes\kappa\circ\mu_4 = 0,$$
$$\zeta_1^*\otimes\mathrm{id}\circ\mu_2 = \alpha_1, \qquad \zeta_1^*\otimes\mathrm{id}\circ\mu_4 = \alpha_2,$$
$$\zeta_1^*\otimes\beta_1\circ\mu_2 = \epsilon_1, \qquad \zeta_1^*\otimes\beta_1\circ\mu_4 = \epsilon_2,$$
$$\nu_1\mu_2 = \epsilon_2, \qquad \nu_1\mu_4 = \epsilon_4,$$
$$\nu_2\mu_2 = \epsilon_2, \qquad \nu_2\mu_4 = \epsilon_4,$$
$$\gamma^*\otimes\mathrm{id}\circ\mu_2 = -\epsilon_3, \qquad \gamma^*\otimes\mathrm{id}\circ\mu_4 = 0,$$
$$\zeta_1^*\otimes\eta^*\otimes\kappa\circ\mu_2 = 0, \qquad \zeta_1^*\otimes\eta^*\otimes\kappa\circ\mu_4 = 0,$$
$$\eta^*\otimes\gamma^*\otimes\eta\circ\mu_2 = 0, \qquad \eta^*\otimes\gamma^*\otimes\eta\circ\mu_4 = 0,$$
$$\kappa^*\otimes\mathrm{id}\circ\kappa\otimes\zeta_1^*\otimes\mathrm{id} = \zeta_1^*\otimes\mathrm{id}, \qquad \kappa^*\otimes\mathrm{id}\circ\delta = 0,$$
$$\kappa^*\otimes\mathrm{id}\circ\gamma^*\otimes\eta = 0, \qquad \kappa^*\otimes\mathrm{id}\circ\zeta_1^*\otimes\kappa = 0,$$
$$\kappa^*\otimes\beta_1\circ\kappa\otimes\zeta_1^*\otimes\mathrm{id} = \zeta_1^*\otimes\beta_1, \qquad \kappa^*\otimes\beta_1\circ\delta = 0,$$
$$\kappa^*\otimes\beta_1\circ\gamma^*\otimes\eta = 0, \qquad \kappa^*\otimes\beta_1\circ\zeta_1^*\otimes\kappa = 0,$$
$$\kappa^*\otimes\beta_2\circ\kappa\otimes\zeta_1^*\otimes\mathrm{id} = \frac{1}{2}(\nu_1+\nu_2), \qquad \kappa^*\otimes\beta_2\circ\delta = 0,$$
$$\kappa^*\otimes\beta_2\circ\gamma^*\otimes\eta = 0, \qquad \kappa^*\otimes\beta_2\circ\zeta_1^*\otimes\kappa = 0,$$
$$\eta^*\otimes\mathrm{id}\circ\kappa\otimes\zeta_1^*\otimes\mathrm{id} = 0, \qquad \eta^*\otimes\mathrm{id}\circ\delta = \frac{1}{2}(\nu_1-\nu_2),$$
$$\eta^*\otimes\mathrm{id}\circ\gamma^*\otimes\eta = \eta^*\otimes\gamma^*\otimes\eta, \qquad \eta^*\otimes\mathrm{id}\circ\zeta_1^*\otimes\kappa = \zeta_1^*\otimes\eta^*\otimes\kappa,$$
$$\kappa^*\otimes\eta^*\otimes\kappa\circ\kappa\otimes\zeta_1^*\otimes\mathrm{id} = \zeta_1^*\otimes\eta^*\otimes\kappa, \qquad \kappa^*\otimes\eta^*\otimes\kappa\circ\delta = 0,$$
$$\kappa^*\otimes\eta^*\otimes\kappa\circ\gamma^*\otimes\eta = 0, \qquad \kappa^*\otimes\eta^*\otimes\kappa\circ\zeta_1^*\otimes\kappa = 0,$$
$$\beta_1\circ\kappa^*\otimes\mathrm{id} = \kappa^*\otimes\beta_1, \qquad \beta_2\circ\kappa^*\otimes\mathrm{id} = \kappa^*\otimes\beta_2,$$
$$\eta^*\otimes\kappa\circ\kappa^*\otimes\mathrm{id} = \kappa^*\otimes\eta^*\otimes\kappa, \qquad \kappa^*\otimes\mathrm{id}\circ\kappa\otimes\alpha_1 = \alpha_1,$$
$$\kappa^*\otimes\mathrm{id}\circ\kappa\otimes\alpha_2 = \alpha_2, \qquad \kappa^*\otimes\mathrm{id}\circ\kappa\otimes\alpha_3 = \alpha_3,$$
$$\kappa^*\otimes\mathrm{id}\circ\lambda = 0, \qquad \kappa^*\otimes\beta_1\circ\kappa\otimes\alpha_1 = \epsilon_1,$$
$$\kappa^*\otimes\beta_1\circ\kappa\otimes\alpha_2 = \epsilon_2, \qquad \kappa^*\otimes\beta_1\circ\kappa\otimes\alpha_3 = \epsilon_3,$$
$$\kappa^*\otimes\beta_1\circ\lambda = 0, \qquad \kappa^*\otimes\beta_2\circ\kappa\otimes\alpha_1 = \epsilon_2,$$
$$\kappa^*\otimes\beta_2\circ\kappa\otimes\alpha_2 = \epsilon_4, \qquad \kappa^*\otimes\beta_2\circ\kappa\otimes\alpha_3 = 0,$$
$$\kappa^*\otimes\beta_2\circ\lambda = 0, \qquad \eta^*\otimes\mathrm{id}\circ\kappa\otimes\alpha_1 = 0,$$

$$\eta^* \otimes \mathrm{id} \circ \kappa \otimes \alpha_2 = 0, \qquad \eta^* \otimes \mathrm{id} \circ \kappa \otimes \alpha_3 = 0,$$
$$\eta^* \otimes \mathrm{id} \circ \lambda = 0, \qquad \kappa^* \otimes \eta^* \otimes \kappa \circ \kappa \otimes \alpha_1 = 0,$$
$$\kappa^* \otimes \eta^* \otimes \kappa \circ \kappa \otimes \alpha_2 = 0, \qquad \kappa^* \otimes \eta^* \otimes \kappa \circ \kappa \otimes \alpha_3 = 0,$$
$$\kappa^* \otimes \eta^* \otimes \kappa \circ \lambda = 0, \qquad \beta_1 \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \beta_1,$$
$$\beta_2 \circ \zeta_1^* \otimes \mathrm{id} = \frac{1}{2}(\nu_1 + \nu_2), \qquad \eta^* \otimes \kappa \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \eta^* \otimes \kappa,$$
$$\beta_1\alpha_1 = \epsilon_1, \qquad \beta_1\alpha_2 = \epsilon_2,$$
$$\beta_1\alpha_3 = \epsilon_3, \qquad \beta_2\alpha_1 = \epsilon_2,$$
$$\beta_2\alpha_2 = \epsilon_4, \qquad \beta_2\alpha_3 = 0,$$
$$\eta^* \otimes \kappa \circ \alpha_1 = 0, \qquad \eta^* \otimes \kappa \circ \alpha_2 = 0,$$
$$\eta^* \otimes \kappa \circ \alpha_3 = 0,$$

If we write

$$o_1 = \kappa \otimes \zeta_1^* \otimes \mathrm{id} + \delta, \qquad o_2 = \kappa \otimes \zeta_1^* \otimes \mathrm{id} - \delta,$$
$$\rho_1 = \kappa^* \otimes \beta_2 + \eta^* \otimes \mathrm{id}, \qquad \rho_2 = \kappa^* \otimes \beta_2 1\eta^* \otimes \mathrm{id},$$

and simplify then the final DG structure of A_X is given by:

$$\mathrm{Arr}(\mathcal{O}, \mathcal{O}(H - E_1 - E_2)) = \{\mu_2, \mu_4\},$$
$$\mathrm{Arr}(\mathcal{O}(H - E_1 - E_2), \mathcal{O}(H - E_2)) = \left\{ \begin{array}{c} o_2, \\ \gamma^* \otimes \eta \end{array} \right\} \begin{array}{c} 0 \\ 1 \end{array},$$
$$\mathrm{Arr}(\mathcal{O}(H - E_2), \mathcal{O}(H)) = \{\kappa^* \otimes \mathrm{id}\},$$
$$\mathrm{Arr}(\mathcal{O}(H), \mathcal{O}(2H - E_1 - 2E_2)) = \{\beta_1\},$$
$$\mathrm{Arr}(\mathcal{O}, \mathcal{O}(H - E_2)) = \left\{ \begin{array}{ccc} \kappa \otimes \alpha_1, & \kappa \otimes \alpha_2, & \kappa \otimes \alpha_3 \\ & & \downarrow \\ & & \lambda \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array},$$
$$\mathrm{Arr}(\mathcal{O}(H - E_1 - E_2), \mathcal{O}(H)) = \{\zeta_1 \otimes \mathrm{id}\},$$
$$\mathrm{Arr}(\mathcal{O}(H - E_2), \mathcal{O}(2H - E_1 - 2E_2)) = \{\kappa^* \otimes \beta_1, \rho_1\},$$
$$\mathrm{Arr}(\mathcal{O}, \mathcal{O}(H)) = \{\alpha_1, \alpha_2, \alpha_3\},$$
$$\mathrm{Arr}(\mathcal{O}(H - E_1 - E_2), \mathcal{O}(2H - E_1 - 2E_2)) = \left\{ \begin{array}{ccc} \zeta_1^* \otimes \beta_1, & \nu_1, & \gamma^* \otimes \mathrm{id}, \\ & & \downarrow \\ & & \eta^* \otimes \gamma^* \otimes \eta \end{array} \right\} \begin{array}{c} 0 \\ \\ 1 \end{array},$$
$$\mathrm{Arr}(\mathcal{O}, \mathcal{O}(2H - E_1 - 2E_2)) = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\},$$

with composition:

$$o_2\mu_2 = \kappa \otimes \alpha_1, \qquad o_2\mu_4 = \kappa \otimes \alpha_2, \qquad \gamma^* \otimes \eta \circ \mu_2 = 0,$$
$$\gamma^* \otimes \eta \circ \mu_4 = \lambda, \qquad \zeta_1^* \otimes \mathrm{id} \circ \mu_2 = \alpha_1, \qquad \zeta_1^* \otimes \mathrm{id} \circ \mu_4 = \alpha_2,$$
$$\zeta_1^* \otimes \beta_1 \circ \mu_2 = \epsilon_1, \qquad \zeta_1^* \otimes \beta_1 \circ \mu_4 = \epsilon_2, \qquad \nu_1\mu_2 = \epsilon_2,$$
$$\nu_1\mu_4 = \epsilon_4, \qquad \gamma^* \otimes \mathrm{id} \circ \mu_2 = \epsilon_3, \qquad \gamma^* \otimes \mathrm{id} \circ \mu_4 = 0,$$
$$\eta^* \otimes \gamma^* \otimes \eta \circ \mu_2 = 0, \qquad \eta^* \otimes \gamma^* \otimes \eta \circ \mu_4 = 0, \qquad \kappa^* \otimes \mathrm{id} \circ o_2 = \zeta_1^* \otimes \mathrm{id},$$
$$\kappa^* \otimes \mathrm{id} \circ \gamma^* \otimes \eta = 0, \qquad \kappa^* \otimes \beta_1 \circ o_2 = \zeta_1^* \otimes \beta_1, \qquad \kappa^* \otimes \beta_1 \circ \gamma^* \otimes \eta = 0,$$
$$\rho_1 o_2 = \nu_1, \qquad \rho_1 \circ \gamma^* \otimes \eta = \eta^* \otimes \gamma^* \otimes \eta, \qquad \beta_1 \circ \kappa^* \otimes \mathrm{id} = \kappa^* \otimes \beta_1,$$
$$\kappa^* \otimes \mathrm{id} \circ \kappa \otimes \alpha_1 = \alpha_1, \qquad \kappa^* \otimes \mathrm{id} \circ \kappa \otimes \alpha_2 = \alpha_2, \qquad \kappa^* \otimes \mathrm{id} \circ \kappa \otimes \alpha_3 = \alpha_3,$$
$$\kappa^* \otimes \mathrm{id} \circ \lambda = 0, \qquad \kappa^* \otimes \beta_1 \circ \kappa \otimes \alpha_1 = \epsilon_1, \qquad \kappa^* \otimes \beta_1 \circ \kappa \otimes \alpha_2 = \epsilon_2,$$
$$\kappa^* \otimes \beta_1 \circ \kappa \otimes \alpha_3 = \epsilon_3, \qquad \kappa^* \otimes \beta_1 \circ \lambda = 0, \qquad \rho_1 \circ \kappa \otimes \alpha_1 = \epsilon_2,$$
$$\rho_1 \circ \kappa \otimes \alpha_2 = \epsilon_4, \qquad \rho_1 \circ \kappa \otimes \alpha_3 = 0, \qquad \rho_1\lambda = 0,$$
$$\beta_1 \circ \zeta_1^* \otimes \mathrm{id} = \zeta_1^* \otimes \beta_1, \qquad \beta_1\alpha_1 = \epsilon_1, \qquad \beta_1\alpha_2 = \epsilon_2,$$
$$\beta_1\alpha_3 = \epsilon_3.$$

This calculations lead to the same structure on cohomologies as described before.

Returning to the notation from section 3 we can write that

$$
\begin{aligned}
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H-E_1-E_2)) &= \{\alpha_1,\alpha_2\},\\
\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}(H-E_2)) &= \left\{\begin{array}{c}\epsilon,\\ \tau\end{array}\right\}\begin{array}{c}0\\1\end{array},\\
\mathrm{Arr}(\mathcal{O}(H-E_2),\mathcal{O}(H)) &= \{\delta\},\\
\mathrm{Arr}(\mathcal{O}(H),\mathcal{O}(2H-E_1-2E_2)) &= \{\eta\},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H-E_2)) &= \left\{\begin{array}{ccc}\epsilon\alpha_1, & \epsilon\alpha_2, & \chi\\ & & \downarrow\\ & & \tau\alpha_2\end{array}\right\}\begin{array}{c}0\\ \\1\end{array},\\
\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}(H)) &= \{\delta\epsilon\},\\
\mathrm{Arr}(\mathcal{O}(H-E_2),\mathcal{O}(2H-E_1-2E_2)) &= \{\eta\delta,\mu\},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}(H)) &= \{\delta\epsilon\alpha_1,\delta\epsilon\alpha_2,\xi\},\\
\mathrm{Arr}(\mathcal{O}(H-E_1-E_2),\mathcal{O}(2H-E_1-2E_2)) &= \left\{\begin{array}{ccc}\eta\delta\epsilon, & \mu\epsilon, & \omega,\\ & & \downarrow\\ & & \mu\tau\end{array}\right\}\begin{array}{c}0\\ \\1\end{array},\\
\mathrm{Arr}(\mathcal{O},\mathcal{O}(2H-E_1-2E_2)) &= \{\eta\delta\epsilon\alpha_1,\mu\epsilon\alpha_1,\eta\xi,\mu\epsilon\alpha_2\},
\end{aligned}
$$

with relations:

$$
\begin{aligned}
\tau\alpha_1 &= 0, & \eta\delta\epsilon\alpha_2 &= \mu\epsilon\alpha_1,\\
\omega\alpha_1 &= \eta\xi, & \omega\alpha_2 &= 0,\\
\mu\tau\alpha_2 &= 0, & \delta\tau &= 0,\\
\delta\chi &= \xi, & \mu\chi &= 0.
\end{aligned}
$$

This algebra can be presented as a path algebra of a "DG quiver", that is a quiver in which some arrows are in degree zero and some are in degree one. This quiver is shown on the Figure 2.

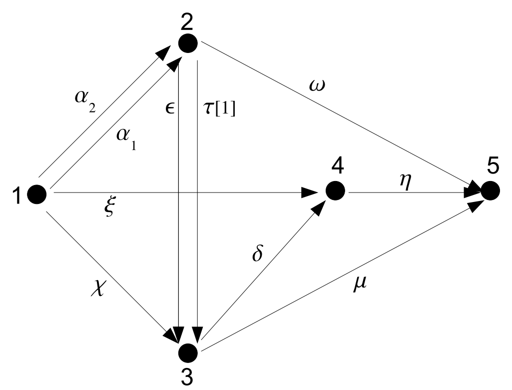


Figure 2: The "DG quiver" of the collection $\langle\mathcal{O},\mathcal{O}(H-E_1-E_2),\mathcal{O}(H-E_2),\mathcal{O}(H),\mathcal{O}(2H-E_1-2E_2)\rangle$ on X.

The relations and non-zero differentials are given by:

$$
\begin{aligned}
\tau\alpha_1 &= 0, & \eta\delta\epsilon\alpha_2 &= \mu\epsilon\alpha_1,\\
\omega\alpha_1 &= \eta\xi, & \omega\alpha_2 &= 0,\\
\mu\tau\alpha_2 &= 0, & \delta\tau &= 0,\\
\delta\chi &= \xi, & \mu\chi &= 0,\\
d(\chi) &= \tau\alpha_2, & d(\omega) &= \mu\tau.
\end{aligned}
$$

In order to calculate the operation m_3 on $H(A_X)$ we again use the fact that a morphism of graded vector spaces and an A_∞ structure on one of them induces the A_∞ structure on the other one.

Let us consider an A_∞ morphism $f : H(A_X) \to A_X$. The $f = (f_i)_{i=1,2,\dots}$ for $f_i : H(A_X)^{\otimes n} \to A_X$ of degree $1-i$. The algebras A_X and $H(A_X)$ have non-zero terms only in degrees zero and one so the only nontrivial f_i's can be f_1 and f_2. Moreover f_2 can be nontrivial only if one of its arguments is of degree one. The composition in $H(A_X)$ is induced from A_X so f_1 is an inclusion. Let us denote by $\widetilde{m}_i$ the A_∞ operations on $H(A_X)$. Then $\widetilde{m}_1 = 0$ and $\widetilde{m}_2$ is the composition. We are looking for $f_2 : H(A_X) \otimes H(A_X) \to A_X$ and $\widetilde{m}_3$ such that

$$f_1 \circ \widetilde{m}_2 = m_2(f_1 \otimes f_1) + d \circ f_2,$$

$$f_1 \circ \widetilde{m}_3 + f_2(\widetilde{m}_2 \otimes \mathrm{id}) - f_2(\mathrm{id} \otimes \widetilde{m}_2) = m_2(f_1 \otimes f_2) - m_2(f_2 \otimes f_1).$$

When applying these formulae to the elements we have to remember about the Koszul sign rule.

The final result is:

$$\begin{aligned} f_2(\tau, \alpha_1) &= 0, & \widetilde{m}_3(\delta, \tau, \alpha_1) &= 0,\\ f_2(\delta, \tau,) &= 0, & \widetilde{m}_3(\delta, \tau, \alpha_2) &= \xi,\\ f_2(\tau, \alpha_1) &= \chi, & \widetilde{m}_3(\mu, \tau, \alpha_1) &= \eta\xi,\\ f_2(\eta\delta, \tau,) &= 0, & \widetilde{m}_3(\mu, \tau, \alpha_2) &= 0,\\ & & \widetilde{m}_3(\eta, \delta, \tau) &= 0. \end{aligned}$$

This calculations coincide with the result obtained by calculating the triple Massey product $\langle \delta, \tau, \alpha_1 \rangle$.

References

[BS] P. Balmer, M. Schlichting, *Idempotent Completion of Triangulated Categories*, J. Algebra 236, 819–834, 2001.

[B] A. Bondal, *Representations of associative algebras and coherent sheaves*, Math. USSR Izviestiya, 34(1), 23-42, 1990.

[BK] A. Bondal, M. Kapranov, *Enhanced triangulated categories*, Mat. Sb. 181 (1990), no. 5, 669–683; translation in Math. USSR-Sb. 70, no. 1, 93–107, 1991.

[BvB] A. Bondal, M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. 3 (1), 1–36, 258, 2003.

[GM] S. Gelfand, Y. Manin, *Methods of Homological Algebra*, Springer, Berlin, 1996.

[Huy] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, 2006.

[HI] A. Hochenegger, N.O. Ilten, *Families of Divisors on T-Varieties and Exceptional Sequences on $\mathbb{C}^*$-Surfaces*, *arXiv:0906.4292*, 2009.

[HP] L. Hille, M. Perling, *Exceptional sequences of invertible sheaves on rational surfaces*, *arXiv:0810.1936*, 2008.

[K1] B. Keller, *A-infinity algebras, modules and functor categories*, Trends in representation theory of algebras and related topics, 67–93, Contemp. Math., 406, Amer. Math. Soc., Providence, RI, 2006.

[K2] B. Keller, *Derived categories and their uses*, Handbook of algebra, Vol. 1, 671–701, North-Holland, Amsterdam, 1996.

[K3] B. Keller, *Deriving DG categories*, Ann. Scient. Ec. Norm. Sup. 27, 63 - 102, 1994.

[L-H] Kenji Lefevre-Hasegawa, *Sur les A_∞-categories*, These de doctorat, Universite Denis Diderot Paris 7, available at B. Keller's homepage, 2003.

[O] D. O. Orlov, *Projective bundles, monoidal transformations and derived categories of coherent sheaves*, Izv. Akad. Nauk SSSR Ser.Mat. 56 (1992), pp.852-862; English transl. in Math. USSR Izv. 38, 133-141, 1993.

[P1] M. Perling, *Examples for exceptional sequences of invertible sheaves on rational surfaces*, *arxiv:0904.0529*, 2009.

[P2] M. Perling, *Graded Rings and Equivariant Sheaves on Toric Varieties*, Math. Nachr. 263/264, 181–197, 2004.

[W] G. Whitehead, *Recent advances in homotopy theory*, , CBMS Reg. Conf. Ser. Math., vol. 5, Amer. Math. Soc., Providence, RI, 1970.