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DG categories of exceptional collections

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Abstract

Given a full exceptional collection we recall an equivalence between a bounded derived category of coherent sheaves on a smooth projective variety X and a derived category of modules over a DG category $\mathcal{C}$. We propose two explicit ways of calculating the category $\mathcal{C}$. The first one uses A_∞ categories and Massey products. The second is based on mutations of exceptional collections.

Introduction

The derived category of coherent sheaves $D^b(\mathrm{Coh}(X))$ on a smooth projective variety X has been for a long time an object of interest. Full and strong exceptional collections provide probably the easiest known description of it. The next step is understanding the structure of $D^b(\mathrm{Coh}(X))$ given a full exceptional collections with nontrivial higher Ext groups. A structure theorem was proved by Bondal and Kapranov in the beginning of 90's. It requires finding a DG category $\mathcal{D}$ with finitely many objects. However, we are not aware of any known algorithm allowing to actually calculate the category $\mathcal{D}$.

With this paper, we are trying to fill in the gap by proposing two ways of explicit calculations of the DG category $\mathcal{D}_\sigma$ associated to a full exceptional collection $\sigma = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle$ on X. One method is based on the existence of the minimal model for a DG category regarded as an A_∞ category. If due to degree reasons there are not many possibly nontrivial higher multiplications on $\mathrm{End}(\bigoplus \mathcal{E}_i)$ then the A_∞ structure on $H(\mathcal{D}_\sigma)$ can be calculated by means of the Massey products. It is also possible to determine the category $\mathcal{D}_\sigma$ via mutations from the collection σ to a collection τ for which the DG category $\mathcal{D}_\tau$ is known. In this approach the collection τ can either be a strong one or a collection in which the Massey products fully determine the A_∞ structure.

The paper is organised as follows. In the first section we recall the definitions of DG categories and the theory of enhanced triangulated categories. We state after [BK] the equivalence of a bounded derived category of coherent sheaves on a smooth projective variety and the category of twisted complexes over a DG category $\mathcal{C}$. Following [BLL] we also present the category of twisted complexes as a subcategory of the derived category of $\mathcal{C}$.

In the second section we recall the definition of an A_∞ category after [K1] and following [M] we give an algorithm for finding the A_∞ structure on a cohomology category of a DG category. After [L-H] we describe the bar and cobar construction to obtain for any A_∞ category a quasi-isomorphic DG category. We end this section with a description of modules over A_∞ categories.

We state after [L-H] the equivalence of derived categories of modules over A_∞ and DG categories. This allows to substitute the category $\mathcal{C}$ described by Bondal and Kapranov with a category $\mathcal{D}$ in which the morphisms spaces between objects are finite-dimensional.

In the third section we notice the similarity between the A_∞ operations on $H(\mathcal{D})$ and the Massey products in $\mathrm{End}(\bigoplus \mathcal{E}_i)$. We use it to determine the higher multiplications m_n's on $H(\mathcal{D})$ in some cases. Later, we describe an alternative approach. We recall after [B] mutations of exceptional collections and generalise this notion to mutations of associated DG categories. Finally, we use both methods to calculate the DG category associated to a full exceptional collection on $\mathbb{P}^2$ blown-up in three points.

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1 Equivalence of $D^b(\mathbf{Coh}(X))$ and $D^b(\mathcal{C}_\sigma)$

Let X be a smooth projective variety over $\mathbb{C}$. A coherent sheaf $\mathcal{E}$ on X is *exceptional* if $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}$ and $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $i > 0$. An ordered collection $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n\rangle$ of exceptional objects is an *exceptional collection* if $\mathrm{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i > j$ and $k \geq 0$. Moreover the collection σ is *strong* if also $\mathrm{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $k > 0$ and all i, j.

An exceptional collection σ is *full* if the smallest strictly full triangulated subcategory of $D^b(\mathrm{Coh}(X))$ containing σ is equal to $D^b(\mathrm{Coh}(X))$.

A strong full exceptional collection provides a description of the category $D^b(\mathrm{Coh}(X))$. In [B] Bondal has shown the following theorem

Theorem 1.1. *Let $\langle \mathcal{E}_1, \dots, \mathcal{E}_n\rangle$ be a full strong exceptional collection on a smooth projective variety X. Then $D^b(Coh(X))$, the bounded derived category of coherent sheaves on X is equivalent to $D^b(mod-A)$ a bounded derived category of finite dimensional right modules over an algebra $A = End(\bigoplus_{i=1}^n \mathcal{E}_i)$.*

Moreover the algebra A can be presented as a path algebra of a quiver Q with n vertices and no cycles.

In [BK] Bondal and Kapranov proved an analogous theorem for a collection σ which is full but not strong. Before formulating the result we will recall some definitions after [BK].

Definition 1.2. *A* DG category *is a preadditive category $\mathcal{C}$ in which the abelian groups $Hom_{\mathcal{C}}(A,B)$ are endowed with a $\mathbb{Z}$-grading and a differential ∂ of degree one. The composition of morphisms*

$$Hom_{\mathcal{C}}(A,B) \otimes Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,C)$$

is a morphism of complexes and for any object $C \in \mathcal{C}$ the identity morphism id_C is a closed morphism of degree zero.

For an element x in a graded vector space we will denote by $|x|$ the grading of x.

For simplicity we shall write $\mathcal{C}(A,B)$ instead of $\mathrm{Hom}_{\mathcal{C}}(A,B)$.

A DG category $\mathcal{C}$ is *ordered* if there exists a partial order $\preceq$ on the set of objects such that $\mathcal{C}(A,B) = 0$ for $B \preceq A$.

To a DG category $\mathcal{C}$ we associate two graded categories. The category $\mathcal{C}^{\text{gr}}$ obtained from $\mathcal{C}$ by forgetting the differentials on morphisms and the *homotopy category* $H(\mathcal{C})$ with the same objects as $\mathcal{C}$ and morphisms $H(\mathcal{C})(A, B)$ given by cohomology of $\mathcal{C}(A, B)$. A further restriction to the zeroth cohomology gives a preadditive category $H^0(\mathcal{C})$. Two object C, C' of a DG category $\mathcal{C}$ are *homotopy equivalent* if they become isomorphic in $H(\mathcal{C})$.

A *DG functor* between two DG categories $\mathcal{C}$ and $\mathcal{C}'$ is an additive functor $F : \mathcal{C} \to \mathcal{C}'$ that preserves the grading and differential on morphisms. DG functors between DG categories form a DG category. Let $F, G : \mathcal{C} \to \mathcal{C}'$ be DG functors. To construct a DG category DG-Fun$(\mathcal{C}, \mathcal{C}')$ we put $\text{Hom}^k(F, G)$ to be the set of natural transformations $t : F^{\text{gr}} \to G^{\text{gr}}[k]$. The differential ∂ is defined pointwise; for $t_C : F(C) \to G[k](C)$ we have $(\partial(t))_C = \partial(t_C) : F(C) \to G[k-1](C)$.

The category of contravariant DG functors is denoted by DG-Fun$^{\text{o}}(\mathcal{C}, \mathcal{C}')$. A functor $F : \mathcal{C} \to \mathcal{C}'$ is a quasi-isomorphism if $H(F) : H(\mathcal{C}) \to H(\mathcal{C}')$ is an isomorphism.

The category $\text{DGVect}_{\mathbb{C}}$ of complexes of vector spaces with homogenous morphisms $f : V^\bullet \to W^\bullet$, $f(V^i) \subset W^{i+k}$ and a differential

$$\partial(f)^i = f^{i+1}\partial_V - (-1)^{|f|}\partial_W f^{i+1}$$

is a DG category. Its homotopy category $H(\text{DGVect}_{\mathbb{C}})$ is the homotopy category of complexes.

A *right DG module* over a DG category C is a DG functor $M \in$ DG-Fun$^{\text{o}}(\mathcal{C}, \text{DGVect}_{\mathbb{C}})$. After [K2] we define a derived category $D(\mathcal{C})$ as a localization of $H^0(\text{DG-Fun}^{\text{o}}(\mathcal{C}, \text{DGVect}_{\mathbb{C}}))$ with respect to the class of quasi-isomorphisms. By $D^b(\mathcal{C})$ we will denote the subcategory of $D(\mathcal{C})$ formed by compact objects. The Yoneda embedding gives a functor $h : \mathcal{C} \to D^b(\mathcal{C})$ which assigns to every $C \in \mathcal{C}$ a free module $h_C = \text{Hom}_{\mathcal{C}}(-, C)$.

For a DG category $\mathcal{C}$ we define the category $\widehat{\mathcal{C}}$ of formal shifts. The objects of $\widehat{\mathcal{C}}$ are of the form $C[n]$ where $C \in \mathcal{C}$ and $n \in \mathbb{N}$. For elements $B[k]$ and $C[n]$ of $\widehat{\mathcal{C}}$ we put $\text{Hom}^l_{\widehat{\mathcal{C}}}(B[k], C[n]) = \text{Hom}^{l+n-k}_{\mathcal{C}}(B, C)$. For appropriate sign convention see [BLL].

Let $f \in \text{Hom}_{\mathcal{C}}(B, C)$ be a closed morphism of degree zero in a DG category $\mathcal{C}$ such that $B[1]$ is an object of $\mathcal{C}$. Then an object $D \in \mathcal{C}$ is the *cone* of f iff there exist morphisms

$$B[1] \xrightarrow{i} D \xrightarrow{p} B[1], \qquad\qquad C \xrightarrow{j} D \xrightarrow{s} C,$$

such that

$$pi = 1, \quad sj = 1, \quad si = 0, \quad pj = 0, \quad ip + js = 1.$$

It is proved in [BLL] that the cone of closed degree zero morphism is uniquely defined up to a DG isomorphism.

One can formally add cones of closed morphisms to a DG category $\mathcal{C}$ by considering the category $\mathcal{C}^{\text{pre-tr}}$ of one-sided twisted complexes over $\mathcal{C}$.

Definition 1.3. *A* one-sided twisted complex *over a DG category $\mathcal{C}$ is an expression $(\bigoplus_{i=1}^n C_i[r_i], q_{i,j})$ where C_i's are objects of $\mathcal{C}$, $r_i \in \mathbb{Z}$, $n \geq 0$ and $q_{i,j} \in Hom^1(C_i[r_i], C_j[r_j])$ such that $q_{i,j} = 0$ for $i \geq j$ and $\partial q + q^2 = 0$.*

One-sided twisted complexes over $\mathcal{C}$ form a DG category $\mathcal{C}^{pre\text{-}tr}$ with the morphism space between $C = (\bigoplus C_i[r_i], q)$ and $C' = (\bigoplus C'_j[r'_j], q')$ given by the set of matrices $f = (f_{i,j})$, $f_{i,j} : C_i[r_i] \to C'_j[r'_j]$. With a differential defined as

$$\partial(f) = (\partial f_{i,j}) + q'f - (-1)^{degf} fq$$

the category $\mathcal{C}^{pre\text{-}tr}$ is a DG category. We will denote its zeroth homotopy category $H^0(\mathcal{C}^{pre\text{-}tr})$ by $\mathcal{C}^{tr}$.

Let $C = (\bigoplus C_i[r_i], q_{i,j})$ be an object of $\mathcal{C}^{\text{pre-tr}}$. The shift of C is defined as $C[1] = (\bigoplus C_i[r_i + 1], q_{i,j})$. The category $\mathcal{C}^{\text{pre-tr}}$ is closed under formal shifts.

Let $f : C \to D$ be a closed morphism of degree 0 in $\mathcal{C}^{\text{pre-tr}}$, where $C = (\bigoplus_{i=1}^n C_i[r_i], q_{i,j})$, $D = (\bigoplus_{i=1}^m D_i[s_i], p_{i,j})$. *The cone* of f is a twisted complex $C(f) = (\bigoplus_{i=1}^{n+m} E_i[t_i], u_{i,j})$, where

$$E_i = \begin{cases} C_i & \text{for } i \le n \\ D_{i-n} & \text{for } i > n \end{cases}$$

$$t_i = \begin{cases} r_i + 1 & \text{for } i \le n \\ s_i & \text{for } i > n \end{cases}$$

$$u_{i,j} = \begin{cases} q_{i,j} & \text{for } i, j \le n \\ f_{i,j-n} & \text{for } i \le n < j \\ p_{i-n,j-n} & \text{for } i, j > n \end{cases}$$

The functor $\mathrm{Tot} : (\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}} \to \mathcal{C}^{\text{pre-tr}}$ establishes a quasi-isomorphism between $(\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$ and $\mathcal{C}^{\text{pre-tr}}$. Let $C_i = (\bigoplus_{j=1}^{n_i} D_j^i[r_j^i], q_{jk}^i)$ be objects of $\mathcal{C}^{\text{pre-tr}}$ and let $C = (\bigoplus_{i=1}^n C_i[r_i], q_{ij})$ be a twisted complex in $(\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$. Then the *convolution* $\mathrm{Tot}(C)$ is equal to $(\bigoplus_{i=1}^n \bigoplus_{j=1}^{n_j^i} D_j^i[r_j^i + r_i], q_{jk}^i + q_{ij})$.

The DG category $\mathcal{C}$ is *pretriangulated* if the embedding $H^0(\mathcal{C}) \to \mathcal{C}^{tr}$ is an equivalence. The category $H^0(\mathcal{C})$ for a pretriangulated category is triangulated. If for a triangulated category $\mathcal{A}$ there exists a pretriangulated category $\mathcal{C}$ such that $\mathcal{A}$ is equivalent to $H^0(\mathcal{C})$ the category $\mathcal{A}$ is called *enhanced* with the *enhancement* $\mathcal{C}$.

Recall, that an additive category $\mathcal{A}$ is Karoubian if every projector splits. The category $\mathcal{C}^{\text{tr}}$ needs not to be Karoubian. As showed in [BLL] the category $D^b(\mathcal{C})$ is the Karoubization of $\mathcal{C}^{\text{tr}}$.

A standard example of an enhanced triangulated category is the bounded derived category $D^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ with enough injectives I. Its enhancement is $\underline{\mathrm{Kom}}^b(I)$, a DG subcategory of $\mathrm{Kom}(I)$ consisting of complexes bounded from below and with almost all cohomology groups equal to zero.

With these definitions we are ready to state the following theorem after [BK].

Theorem 1.4. *Let $\mathcal{D}$ be a pretriangulated category, $E_1, \ldots, E_n$ objects in $\mathcal{D}$ and $\mathcal{C} \subset \mathcal{D}$ the full DG subcategory on the objects E_i. Then the smallest triangulated subcategory of $H^0(\mathcal{D})$ containing $E_1, \ldots, E_n$ is equivalent to $\mathcal{C}^{tr}$ as a triangulated category.*

It follows that a full exceptional collection σ on a smooth projective variety X leads to an equivalence of $D^b(\mathrm{Coh}(X))$ and $\mathcal{C}_\sigma^{\text{tr}}$ for some DG category $\mathcal{C}_\sigma$. The category $\mathcal{C}_\sigma$ is a subcategory of the enhancement of $D^b(\mathrm{Coh}(X))$ and thus $H^i\mathrm{Hom}_{\mathcal{C}_\sigma}(E, F) = \mathrm{Ext}^i_{D^b(\mathrm{Coh}(X))}(E, F)$ for any elements E, F of σ.

Remark 1.5. The category $D^b(\mathrm{Coh}(X))$ for a smooth projective variety X is Karoubian. Hence, the category $D^b(\mathcal{C}_\sigma)$ is equivalent to $\mathcal{C}_\sigma^{\mathrm{tr}}$. We thus obtain an equivalence of $D^b(\mathrm{Coh}(X))$ and $D^b(\mathcal{C}_\sigma)$.

Remark 1.6. As shown in [K2] for two quasi-isomorphic DG categories $\mathcal{C}, \mathcal{C}'$ the derived categories $D(\mathcal{C})$ and $D(\mathcal{C}')$ are equivalent. Hence for a full exceptional collection σ on X the category $\mathcal{C}_\sigma$ is determined up to a quasi-isomorphism.

Remark 1.7. The group $\mathbb{Z}^n$ acts on the set of exceptional collections in $D^b(\mathcal{A})$ by translations. For $(a_1, \dots, a_n) \in \mathbb{Z}^n$ and an exceptional collection $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ the collection $\sigma[a_1, \dots, a_n] = \langle \mathcal{E}_1[a_1], \dots, \mathcal{E}_n[a_n] \rangle$ is also exceptional. The composition in the category $\mathcal{C}_{\sigma[a_1,\dots,a_n]}$ is the same as in the category $\mathcal{C}_\sigma$ and for $f \in \mathrm{Hom}(\mathcal{E}_i[a_i], \mathcal{E}_j[a_j])$ we put $\partial_{\sigma[a_1,\dots,a_n]}(f) = (-1)^{a_j - a_i} \partial_\sigma(f)$.

Remark 1.8. Let $\mathcal{C}_\sigma(\mathcal{E}_1, \mathcal{E}_n) = H(\mathcal{C}_\sigma(\mathcal{E}_1, \mathcal{E}_n)) \oplus P$ be any decomposition of the vector space $\mathcal{C}_\sigma(\mathcal{E}_1, \mathcal{E}_n)$. Morphisms in P form an ideal and $\mathcal{C}_\sigma$ is quasi-isomorphic to a DG category $\mathcal{C}'$ with $\mathcal{C}'(\mathcal{E}_1, \mathcal{E}_n) = H(\mathcal{C}_\sigma(\mathcal{E}_i, \mathcal{E}_{i+1}))$. Moreover, let $\mathcal{C}_\sigma(\mathcal{E}_i, \mathcal{E}_{i+1}) = H(\mathcal{C}_\sigma(\mathcal{E}_i, \mathcal{E}_{i+1})) \oplus P_i$ be such a decomposition for morphisms between adjacent objects. Then the category $\mathcal{C}''$ with $\mathcal{C}''(\mathcal{E}_i, \mathcal{E}_{i+1}) = H(\mathcal{C}_\sigma(\mathcal{E}_i, \mathcal{E}_{i+1}))$ is a subcategory of $\mathcal{C}_\sigma$ quasi-isomorphic to it. It proves that the category $\mathcal{C}_\sigma$ is always quasi-isomorphic to a DG category $\widetilde{\mathcal{C}}_\sigma$ with $\widetilde{\mathcal{C}}_\sigma(\mathcal{E}_1, \mathcal{E}_n) = H(\widetilde{\mathcal{C}_\sigma}(\mathcal{E}_1, \mathcal{E}_n))$, $\widetilde{\mathcal{C}}_\sigma(\mathcal{E}_i, \mathcal{E}_{i+1}) = H(\widetilde{\mathcal{C}_\sigma}(\mathcal{E}_i, \mathcal{E}_{i+1}))$.

Remark 1.9. If a DG category $\mathcal{C}$ has only zeroth cohomology then it is quasi-isomorphic to $H(\mathcal{C}) = H^0(\mathcal{C})$. Indeed, for $C_1, C_2 \in \mathcal{C}$ let $\mathcal{C}(C_1, C_2) = \bigoplus_{n \in \mathbb{Z}} \mathcal{C}^n(C_1, C_2)$ with differential

$$\partial^n_{C_1,C_2} : \mathcal{C}^n(C_1, C_2) \to \mathcal{C}^{n+1}(C_1, C_2)$$

and let $\mathcal{C}_I$ be a DG category such that $\mathrm{ob}\, \mathcal{C}_I = \mathrm{ob}\, \mathcal{C}$ and

$$\mathcal{C}_I(C_1, C_2) = \bigoplus_{n \le -1} \mathcal{C}^n(C_1, C_2) \oplus \ker \partial^0_{C_1,C_2}.$$

Then the natural inclusion functor $\mathcal{C}_I \to \mathcal{C}$ is a quasi-isomorphism. Let also

$$J_{C_1,C_2} = \mathrm{Im}(\partial^{-1}_{C_1,C_2}) \oplus (\bigoplus_{n<0} \mathcal{C}^n(C_1, C_2))$$

for any C_1 and C_2 in $\mathcal{C}$ and consider a category $\mathcal{C}_{I/J}$ with $\mathrm{ob}\, \mathcal{C}_{I/J} = \mathrm{ob}\, \mathcal{C}$ and

$$\mathcal{C}_{I/J}(C_1, C_2) = \mathcal{C}_I(C_1, C_2)/J_{C_1,C_2}.$$

Then $\mathcal{C}_{I/J}$ is isomorphic to $H(\mathcal{C})$ and the natural functor $\mathcal{C}_I \to \mathcal{C}_{I/J}$ is a quasi-isomorphism. Hence, Theorem 1.4 is a generalisation of the Theorem 1.1.

The definition of $\mathcal{C}_\sigma$ from the theorem 1.4 requires taking injective resoultions and thus is not convenient for calculations. In the remaining part of the paper we will present two ways of finding $\mathcal{C}_\sigma$. We will always assume that X is a smooth projective variety and $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ is a full exceptional collection in $D^b(\mathrm{Coh}(X))$.

2 A_∞ categories

We will use A_∞ categories. We start by recalling definitions and facts we shall use after [K1] and [L-H].

Remark 2.1. The graded Leibniz rule usually has the form $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$. However, in the case of morphisms in the category $\mathrm{DGVect}_{\mathbb{C}}$ we have $\partial(fg) = f\partial(g) + (-1)^{\deg g}\partial(f)g$. Thus, all the definitions below differ from the ones given in [K1] by an appropriate sign change.

Definition 2.2. *An* A_∞ category $\mathcal{A}$ *over* $\mathbb{C}$ *consists of*

- *objects* $ob(\mathcal{A})$,
- *for any two* $A, B \in ob(\mathcal{A})$ *a* $\mathbb{Z}$ *- graded* $\mathbb{C}$*-vector space* $\mathcal{A}(A, B)$,
- *for any* $n \geq 1$ *and a sequence* $A_0, A_1, \ldots, A_n \in ob(\mathcal{A})$ *a graded map:*

$$m_n : \mathcal{A}(A_{n-1}, A_n) \otimes \ldots \otimes \mathcal{A}(A_0, A_1) \to \mathcal{A}(A_0, A_n)$$

of degree $2 - n$ *such that for any* n

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0.$$

Note that when these formulae are applied to elements additional signs appear because of the Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{|y||f|} f(x) \otimes g(y).$$

An A_∞ *algebra* is an A_∞ category with one object.

An A_∞ category $\mathcal{A}$ is *ordered* if there exists a partial order $\preceq$ on the set $\mathrm{ob}(\mathcal{A})$ such that $\mathcal{A}(A, A') = 0$ for $A' \preceq A$.

The operation m_1 gives for any $A, B \in \mathrm{ob}(\mathcal{A})$ a structure of a complex on $\mathcal{A}(A, B)$. The homotopy category with respect to m_1 is denoted by $H(\mathcal{A})$.

The A_∞ category is *minimal* if the operation m_1 is trivial.

Any DG category can be regarded as an A_∞ category with m_1 given by the differential, m_2 given by composition and trivial m_i's for $i > 2$.

For any set S there exists an A_∞ category $\mathbb{C}\{S\}$. The objects of $\mathbb{C}\{S\}$ are elements of S and

$$\mathbb{C}\{S\}(s_1, s_2) = \begin{cases} \mathbb{C} & \text{if } s_1 = s_2, \\ 0 & \text{otherwise} \end{cases}$$

All operations m_n in $\mathbb{C}\{S\}$ are trivial.

Definition 2.3. *A functor of* A_∞ *categories* $F : \mathcal{A} \to \mathcal{B}$ *is a map* $F_0 : ob(\mathcal{A}) \to ob(\mathcal{B})$ *and a family of graded maps*

$$F_n : \mathcal{A}(A_{n-1}, A_n) \otimes \ldots \otimes \mathcal{A}(A_0, A_1) \to \mathcal{B}(F_0(A_0), F_0(A_n))$$

of degree $1 - n$ *such that*

$$\sum_{r+s+t=n} (-1)^{rs+t} F_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum_{i_1+\ldots+i_r=n} (-1)^p m_r(F_{i_1} \otimes \ldots \otimes F_{i_r}),$$

where $p = (r-1)(i_r - 1) + (r-2)(i_{r-1} - 1) + \ldots + 2(i_3 - 1) + (i_2 - 1)$.

Composition of functors is given by

$$(F \circ G)_n = \sum_{i_1 + \ldots + i_s = n} F_s \circ (G_{i_1} \otimes \ldots \otimes G_{i_s}).$$

A functor $F : \mathcal{A} \to \mathcal{B}$ is called a *quasi-isomorphism* if F_1 is a quasi-isomorphism.

An A_∞ category $\mathcal{A}$ is *strictly unital* if for any object $A \in \mathrm{ob}(\mathcal{A})$ there exists $1_A \in \mathcal{A}(A, A)$ of degree 0 such that for any objects B, C in $\mathcal{A}$ and any morphisms $\phi \in \mathcal{A}(A, B)$, $\psi \in \mathcal{A}(C, A)$ we have $m_2(\phi, 1_A) = \phi$ and $m_2(1_A, \psi) = \psi$. Moreover for $n \neq 2$ the operation m_n equals 0 if any of its argument is equal to 1_A.

In particular, for any set S the category $\mathbb{C}\{S\}$ is strictly unital.

The category $\mathcal{A}$ is *homologically unital* if there exist units for the homotopy category $H(\mathcal{A})$.

Here, we cite the theorem proved by Lefevre-Hasegawa in [L-H] about a connection between strictly and homologicaly unital categories.

Theorem 2.4. *Minimal homolgically unital A_∞ category is quasi-isomorphic to a minimal strictly unital A_∞ category.*

Remark 2.5. A minimal A_∞ category is equal to its homotopy category. Hence, a quasi-isomorphism F given by the above theorem satisfies $F_1 = \mathrm{id}$.

An A_∞ category $\mathcal{A}$ is *augmented* if there exists a strict unit preserving functor $\epsilon : \mathbb{C}\{\mathrm{ob}(\mathcal{A})\} \to \mathcal{A}$. Then $\mathcal{A}$ decomposes as $\mathcal{A} = \mathbb{C}\{\mathrm{ob}(\mathcal{A})\} \oplus \bar{\mathcal{A}}$.

2.1 Minimal model

Any A_∞ category $\mathcal{A}$ is quasi-isomorphic to its homotopy category $H(\mathcal{A})$ as stated in [K1].

Theorem 2.6. *If $\mathcal{A}$ is an A_∞ category, then $H(\mathcal{A})$ admits an A_∞ category structure such that*

1. $m_1 = 0$ *and* m_2 *is induced from* $m_2^{\mathcal{A}}$ *and*
2. *there is an A_∞-quasi-isomorphism $\mathcal{A} \to H(\mathcal{A})$ inducing the identity on cohomology.*

Moreover, this structure is unique up to a non unique A_∞-isomorphism.

The A_∞ category $H(\mathcal{A})$ is called the *minimal model* of $\mathcal{A}$.

If $\mathcal{C}$ is a DG category then its minimal model $H(\mathcal{C})$ is quasi-isomorphic to a strictly unital A_∞ category. Indeed, the category $\mathcal{C}$ is strictly unital and hence homologically unital. It follows that its homotopy category is also homologically unital and the Theorem 2.4 guarantees that there exists a stricly unital minimal category quasi-isomorphic to $H(\mathcal{C})$.

Following [M] we describe an explicit construction of a minimal model for a DG category.

Let $\mathcal{C}$ be a DG category with differential ∂ and composition μ. In order to calculate its minimal model, the A_∞ structure on $H(\mathcal{C})$ one has to define maps $p : \mathcal{C} \to H(\mathcal{C})$ and $i : H(\mathcal{C}) \to \mathcal{C}$ such that both p and i are identities on objects of $\mathcal{C}$ and for any $C, C' \in \mathcal{C}$ maps $p : \mathcal{C}(C, C') \to H(\mathcal{C})(C, C')$ and $i : H(\mathcal{C})(C, C') \to \mathcal{C}(C, C')$ are morphisms of complexes of degree 0. Also a homogeneous map $h : \mathcal{C}(C, C') \to \mathcal{C}(C, C')$ of degree -1 is needed. These have to satisfy the following conditions

$$pi = \mathrm{id}, \quad ip = \mathrm{id} - \partial(h), \quad h^2 = 0.$$

The choice of i, p and h satisfying the above conditions is equivalent to the choice of splittings of the following exact sequences:

$$0 \longrightarrow \mathrm{Ker}\partial^n \longrightarrow \mathcal{C}^n(C_i, C_j) \underset{{}_{ij}s_1^n}{\rightleftarrows} \overline{\mathcal{C}^n(C_i, C_j)} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{C}^{n-1}(C_i, C_j) \xrightarrow{\partial^{n-1}} \mathrm{Ker}\partial^n \underset{{}_{ij}s_2^n}{\rightleftarrows} H^n\mathcal{C}(C_i, C_j) \longrightarrow 0$$

for all $C_i, C_j \in \mathrm{ob}(\mathcal{C})$ and all n.

Definition 2.7. *The triple $i : H(\mathcal{C}) \to \mathcal{C}$, $p : \mathcal{C} \to H(\mathcal{C})$, $h : \mathcal{C} \to \mathcal{C}$ is* compatible *with a finite family $\{{}_{ij}s_1^{n_k}, {}_{ij}s_2^{m_l}\}$ of splittings if $i|_{H^{m_l}(C_i,C_j)} = {}_{ij}s_2^{m_l}$ for all m_l, $h|_{Ims_2} = 0 = h|_{Ims_1}$ and $Im(h) = Im(s_1)$ for all s_1 in the family.*

We define the operations λ_n of degree $2 - n$ on the category $\mathcal{C}$ as follows. We put $\lambda_1 = -h^{-1}$, $\lambda_2 = \mu$ and

$$\lambda_n = \sum_{k=1}^{n-1} (-1)^{k+1} \mu(h(\lambda_{n-k}), h(\lambda_k)).$$

For $n > 1$ the higher multiplications m_n on $H(\mathcal{C})$ are defined as

$$m_n = p\lambda_n i^{\otimes n}.$$

2.2 The universal DG category

For any A_∞ category $\mathcal{A}$ there exists a DG category $U(\mathcal{A})$ and an A_∞ quasi-isomorphism $\mathcal{A} \to U(\mathcal{A})$. To define the category $U(\mathcal{A})$ we need the following definitions (see further [L-H]).

Definition 2.8. *A* DG cocategory *$\mathcal{B}$ consists of*

- *the set of objects $B_i \in ob(\mathcal{B})$,*
- *for any pair of objects $B_i, B_j \in ob(\mathcal{B})$ a complex of $\mathbb{C}$-vector spaces $\mathcal{B}(B_i, B_j)$ with a differential d^{ij} of degree 1 and*
- *a coassociative cocomposition – a family of linear maps*

$$\Delta : \mathcal{B}(B_i, B_j) \to \sum_{B_k \in ob(\mathcal{B})} \mathcal{B}(B_k, B_j) \otimes \mathcal{B}(B_i, B_j).$$

These data have to satisfy the condition

$$\Delta \circ d = (d \otimes id + id \otimes d) \circ \Delta.$$

For any set S the A_∞ category $\mathbb{C}\{S\}$ is also a DG cocategory.

A functor Φ between DG cocategories $\mathcal{B}$ and $\mathcal{B}'$ preserves the grading and differentials on morphisms and satisfies the condition

$$\Delta \circ \Phi = (\Phi \otimes \Phi) \circ \Delta.$$

A DG cocategory $\mathcal{B}$ is *counital* if it admits a counit, a functor $\eta : \mathcal{B} \to \mathbb{C}\{\text{ob}(\mathcal{B})\}$. The category $\mathcal{B}$ is *coaugmented* if it is counital and admits a coaugmentation functor $\varepsilon : \mathbb{C}\{\text{ob}(\mathcal{B})\} \to \mathcal{B}$ such that the composition $\eta\varepsilon$ is the identity on $\mathbb{C}\{\text{ob}(\mathcal{B})\}$.

Let $\mathcal{B}$ be a coaugmented DG cocategory. The category $\bar{\mathcal{B}}$ has the same set of objects as $\mathcal{B}$ and $\bar{\mathcal{B}}(B_i, B_j) = \ker \varepsilon$.

For an augemnted A_∞ category $\mathcal{A}$ one can define its *bar* DG cocategory $B_\infty(\mathcal{A})$. Recall that $\mathcal{A} = \bar{\mathcal{A}} \oplus \mathbb{C}\{\text{ob}(\mathcal{A})\}$. Then $B_\infty(\mathcal{A}) = T^c(S\bar{\mathcal{A}})$ is a tensor cocategory of the suspension of $\bar{\mathcal{A}}$. Here $S\bar{\mathcal{A}}$ denotes the category $\bar{\mathcal{A}}$ with a shift in a morphisms spaces $(S\bar{\mathcal{A}})^n(A, A') = \bar{\mathcal{A}}^{n+1}(A, A')$. $S\bar{\mathcal{A}}$ is not an A_∞ category, however the operations m_n is $\bar{\mathcal{A}}$ define graded maps of degree 1 in $S\bar{\mathcal{A}}$.

$$b_n : S\bar{\mathcal{A}}(A_{n-1}, A_n) \otimes \ldots \otimes S\bar{\mathcal{A}}(A_0, A_1) \to S\bar{\mathcal{A}}(A_0, A_n),$$
$$b_n = -s \circ m_n \circ \omega^{\otimes n},$$

where $s : V \to SV$ is a suspension of a graded vector space V and $\omega = s^{-1}$.

The cocategory $B_\infty(\mathcal{A})$ has the same objects as $\mathcal{A}$ and

$$B_\infty(\mathcal{A})(A, A') = S\bar{\mathcal{A}}(A, A') \oplus \bigoplus_{A_1 \in \text{ob}(\mathcal{A})} S\bar{\mathcal{A}}(A_1, A') \otimes S\bar{\mathcal{A}}(A, A_1)$$
$$\oplus \bigoplus_{A_1, A_2 \in \text{ob}(\mathcal{A})} S\bar{\mathcal{A}}(A_2, A') \otimes S\bar{\mathcal{A}}(A_1, A_2) \otimes S\bar{\mathcal{A}}(A, A_1) \oplus \ldots$$

for $A \neq A'$. In the case $A = A'$ we have

$$B_\infty(\mathcal{A})(A, A) = 1_A \oplus S\bar{\mathcal{A}}(A, A) \oplus \bigoplus_{A_1 \in \text{ob}(\mathcal{A})} S\bar{\mathcal{A}}(A_1, A) \otimes S\bar{\mathcal{A}}(A, A_1)$$
$$\oplus \bigoplus_{A_1, A_2 \in \text{ob}(\mathcal{A})} S\bar{\mathcal{A}}(A_2, A) \otimes S\bar{\mathcal{A}}(A_1, A_2) \otimes S\bar{\mathcal{A}}(A, A_1) \oplus \ldots$$

where the degree of 1_A is zero. To simplify the notation we shall write $(\alpha_n, \ldots, \alpha_1)$ for $\alpha_n \otimes \ldots \otimes \alpha_1$. The differential in $B_\infty(\mathcal{A})$ is given by

$$d(\alpha_n, \ldots, \alpha_1) =$$
$$\sum_{k=1}^{n} \sum_{l=1}^{n-k+1} (-1)^{|\alpha_{l-1}| + \ldots + |\alpha_1|} (\alpha_n, \ldots \alpha_{l+k}, b_k(\alpha_{l+k-1}, \ldots, \alpha_l), \alpha_{l-1}, \ldots \alpha_1),$$

for $(\alpha_n, \ldots, \alpha_1) \in B_\infty(\mathcal{A})(A, A')$. The cocomposition is given by

$$\Delta(\alpha_n, \ldots, \alpha_1) =$$
$$1_{A'} \otimes (\alpha_n, \ldots, \alpha_1) + (\alpha_n, \ldots, \alpha_1) \otimes 1_A + \sum_{l=1}^{n-1} (\alpha_n, \ldots, \alpha_{l+1}) \otimes (\alpha_l, \ldots, \alpha_1).$$

With these definitions $B_\infty(\mathcal{A})$ is an augmented DG cocategory.

Remark 2.9. Let $\mathcal{A}$ be an ordered A_∞ category with finitely many objects, finitely dimensional morphisms spaces and such that $\bar{\mathcal{A}}(A,A) = 0$ for any object $A \in \mathrm{ob}\mathcal{A}$. Then the DG cocategory $B_\infty(\mathcal{A})$ also satisfies these conditions; i.e. is ordered, has finitely many objects, finitely dimensional morphisms spaces and $\overline{B_\infty(\mathcal{A})}(A,A) = 0$ for any object A.

Analogously, to an augmented DG cocategory $\mathcal{B}$ via a *cobar* construction one can assign a DG category $\Omega(\mathcal{B})$. Let $\mathcal{B}$ be a DG cocategory with a differential d and cocomposition Δ. It's cobar DG category is equal to $T(S^{-1}\bar{\mathcal{B}})$. Here $S^{-1}\bar{\mathcal{B}}$ denotes the shift of the cocategory $\mathcal{B}$ and $T(S^{-1}\bar{\mathcal{B}})$ is the tensor DG category of it. As before the morphisms spaces in $T(S^{-1}\bar{\mathcal{B}})$ are given by

$$\Omega(\mathcal{B})(B,B') = S^{-1}\bar{\mathcal{B}}(B,B') \oplus \bigoplus_{B_1 \in \mathrm{ob}(\mathcal{B})} S^{-1}\bar{\mathcal{B}}(B_1,B') \otimes S^{-1}\bar{\mathcal{B}}(B,B_1)$$
$$\oplus \bigoplus_{B_1,B_2 \in \mathrm{ob}(\mathcal{B})} S^{-1}\bar{\mathcal{B}}(B_2,B') \otimes S^{-1}\bar{\mathcal{B}}(B_1,B_2) \otimes S^{-1}\bar{\mathcal{B}}(B,B_1) \oplus \dots$$

for $B \neq B'$ and by

$$\Omega(\mathcal{B})(B,B) = 1_B \oplus S^{-1}\bar{\mathcal{B}}(B,B) \oplus \bigoplus_{B_1 \in \mathrm{ob}(\mathcal{B})} S^{-1}\bar{\mathcal{B}}(B_1,B) \otimes S^{-1}\bar{\mathcal{B}}(B,B_1)$$
$$\oplus \bigoplus_{B_1,B_2 \in \mathrm{ob}(\mathcal{B})} S^{-1}\bar{\mathcal{B}}(B_2,B) \otimes S^{-1}\bar{\mathcal{B}}(B_1,B_2) \otimes S^{-1}\bar{\mathcal{B}}(B,B_1) \oplus \dots$$

for 1_B – a morphism in degree zero. The composition in $\Omega(\mathcal{B})$ is defined by concatenation and the differential ∂ on the morphisms spaces is

$$\partial = \sum 1 \otimes \ldots \otimes 1 \otimes (d + \Delta) \otimes 1 \otimes \ldots \otimes 1.$$

Remark 2.10. If an ordered DG cocategory $\mathcal{B}$ has finitely many objects, finitely dimensional morphisms spaces between the objects and satisfies the condition $\bar{\mathcal{B}}(B,B) = 0$ then the same is true about $\Omega(\mathcal{B})$.

For an augmented A_∞ category $\mathcal{A}$ its universal DG category $U(A)$ is defined as $\Omega(B_\infty(\mathcal{A}))$. There is a natural map $\mathcal{A} \to U(\mathcal{A})$. It is proved in [L-H] that this map extends to a functor and is an A_∞ quasi-isomorphism (Lemme 2.3.4.3 of [L-H]). Moreover, for an A_∞ quasi-isomorphisms ϕ the functor $U(\phi)$ is a quasi-isomorphism of DG categories.

2.3 A_∞ modules

Definition 2.11. *An A_∞ module over an A_∞ category $\mathcal{A}$ is an A_∞ functor $M : \mathcal{A} \to DGVect_{\mathbb{C}}$. A morphism of modules $G : M \to N$ is given by a family $\{G_A : M_0(A) \to N_0(A)\}_{A \in ob\mathcal{A}}$ of morphisms in $DGVect_{\mathbb{C}}$. Moreover, the diagram*

$$\begin{array}{ccc} M_0(A_0) & \xrightarrow{M_n(\alpha_{n-1}\otimes\ldots\otimes\alpha_0)} & M_0(A_n) \\ \downarrow{\scriptstyle G_{A_0}} & & \downarrow{\scriptstyle G_{A_n}} \\ N_0(A_0) & \xrightarrow{N_n(\alpha_{n-1}\otimes\ldots\otimes\alpha_0)} & N_0(A_n) \end{array}$$

has to commute for any $n \in \mathbb{N}$, $A_i \in ob\mathcal{A}$ and $\alpha_i \in \mathcal{A}(A_i, A_{i+1})$.

The category of A_∞ modules over an A_∞ category $\mathcal{A}$ will be denoted as $\mathrm{Mod}_\infty^{\mathrm{strict}}\mathcal{A}$. This notation agrees with the notation in [L-H].

The morphism G of modules is a *quasi-isomorphism* of A_∞ modules if G_A is a quasi-isomorphism of complexes for any $A \in \mathrm{ob}\mathcal{A}$.

The *derived category* $D_\infty(\mathcal{A})$ of an A_∞ category $\mathcal{A}$ is defined as a localization of the category $\mathrm{Mod}_\infty^{\mathrm{strict}}\mathcal{A}$ with respect to the class of quasi-isomorphisms.

Remark 2.12 (Lemme 2.4.3.2 of [L-H])**.** For a DG category $\mathcal{C}$ the derived categories $D(\mathcal{C})$ and $D_\infty(\mathcal{C})$ are equivalent.

Remark 2.13. In the case of A_∞ algebra there can be more morphism between modules, but there is no difference on the level of derived categories; see [L-H].

For an A_∞ category $\mathcal{A}$ the derived category $D_\infty(\mathcal{A})$ is equivalent to the derived category of the universal algebra $D(U(\mathcal{A}))$ (see[L-H]).

Let $\mathcal{A}$ and $\mathcal{A}'$ be A_∞ categories and $F : \mathcal{A} \to \mathcal{A}'$ an A_∞ quasi-isomorphism. Then $U(F) : \mathcal{A} \to \mathcal{A}'$ is a quasi-isomorphism of DG categories. It follows that $D(U(\mathcal{A}))$ is equivalent to $D(U(\mathcal{A}'))$ (see [K2]) and hence $D_\infty(\mathcal{A})$ is equivalent to $D_\infty(\mathcal{A}')$. Thus, we have proved, after [L-H] the

Proposition 2.14. *Let $\mathcal{A}$, $\mathcal{A}'$ be quasi-isomorphic A_∞ categories. Then categories $D_\infty(\mathcal{A})$ and $D_\infty(\mathcal{A}')$ are equivalent.*

Let us come back to the case of exceptional collections. Recall, that X is a smooth projective variety and σ – a full exceptional collection on X. We know already that there exist a DG category $\mathcal{C}_\sigma$ such that the derived category of coherent sheaves on X, $D^b(\mathrm{Coh}(X))$ is equivalent to $D^b(\mathcal{C}_\sigma)$, where $D^b(\mathcal{C}_\sigma) \subset D(\mathcal{C}_\sigma)$ is a subcategory spanned by compact objects. Moreover, the morphisms in the homotopy category $H(\mathcal{C}_\sigma)$ can be calculated via Ext groups in the category $D^b(\mathrm{Coh}(X))$. In particular, the minimal model of $\mathcal{C}_\sigma$, $H(\mathcal{C}_\sigma)$ is an ordered A_∞ category with finitely many objects and finitely dimensional morphisms space. According to the theorem 2.4 and remark 2.5 there exists a quasi-isomorphic strictly unital A_∞ category $\widetilde{\mathcal{C}_\sigma}$ with the same objects and morphism spaces. By remarks 2.9 and 2.10 the DG category $U(\widetilde{\mathcal{C}_\sigma})$ is an ordered DG category with finitely many objects and finitely dimensional morphisms spaces. It is A_∞ quasi-isomorphic to the original category $\mathcal{C}_\sigma$. From the proposition 2.14 and the remark 2.12 we deduce the following

Theorem 2.15. *Let X be a smooth projective variety and let $\sigma = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle$ be a full exceptional collection on X. There exists a DG category $\mathcal{D}_\sigma$ with objects $E_1, \ldots, E_n$ and finitely dimensional morphisms spaces such that*

(i) the bounded derived category of coherent sheaves on X, $D^b(Coh(X))$ is equivalent to the bounded derived category of $\mathcal{D}_\sigma$ and

(ii) for any i, j we have $H^\bullet(\mathcal{D}_\sigma(E_i, E_j)) = Ext^\bullet_X(\mathcal{E}_i, \mathcal{E}_j)$.

Remark 2.16. The category $\mathcal{D}_\sigma^{\text{pre-tr}}$ is pretriangulated and gives an enhancement of $D^b(\mathrm{Coh}(X))$.

In the next sections we will propose two methods of calculating the category $\mathcal{D}_\sigma$.

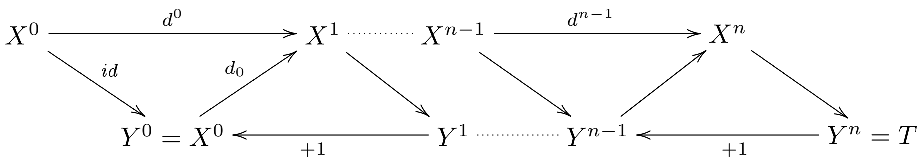
3 Calculating the category $\mathcal{D}_\sigma$ by Massey products

The triangulated category $D^b(\mathrm{Coh}(X))$ might be enough to determine the A_∞ category $H(\mathcal{C}_\sigma)$ and hence the DG category $\mathcal{D}_\sigma$. To see it we will use Massey products. We start by recalling some definitions after [GM].

Definition 3.1. *Let*

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n$$

be a complex in a triangulated category $\mathcal{D}$. A convolution *of this complex is any object T such that there exists the following diagram*



where all triangles $Y^i \to X^{i+1} \to Y^{i+1} \to Y^i[1]$ are distinguished and the composition of $X^i \to Y^i \to X^{i+1}$ is equal to $d^i : X^i \to X^{i+1}$.

Note, that for a convolution T we have a map of degree 0 from X^n to T and a map of degree n from T to X^0.

Knowing convolutions one can define Massey products μ_n.

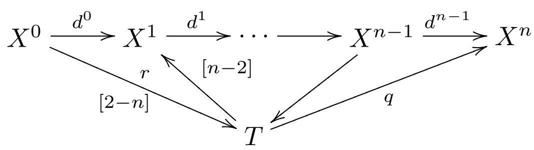
Definition 3.2. *For a pair of morphisms*

$$X^0 \xrightarrow{f} X^1 \xrightarrow{g} X^2$$

the double Massey product μ_2 is defined as $\mu_2(g, f) = gf$. Let

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n$$

be a complex in a triangulated category $\mathcal{D}$ and let T be a convolution of $X^1 \to \ldots \to X^{n-1}$. The Massey product *$\mu_n(d^{n-1}, \ldots, d^0)$ is defined as a set $\{qr\}$ where q and r are morphisms*



such that the composition $X^0 \to T \to X^1$ is d^0 and $X^{n-1} \to T \to X^n$ is d^{n-1}. The morphisms $X^{n-1} \to T$ and $T \to X^1[n-2]$ arise from the definition of the convolution.

Proposition 3.3 (see [GM])**.** *If the set $\mu_n(d^{n-1}, \ldots, d^0)$ is not empty then $0 \in \mu_{n-1}(d^{n-2}, \ldots, d^0)$ and $0 \in \mu_{n-1}(d^{n-1}, \ldots, d^1)$. Moreover, the complex $X^0 \to \ldots \to X^n$ has at least one convolution if and only if $0 \in \mu_n(d_{n-1}, \ldots, d_0)$.*

3.1 Massey products and minimal model

Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian category with enough injectives I. The bounded derived category of $\mathcal{A}$, $\mathcal{D} = D^b(\mathcal{A})$ has an enhancement given by the DG category $\mathcal{C} = \underline{\mathrm{Kom}}^b(I)$.

Lemma 3.4. *Let $f_1, \ldots f_n$, for $f_i : X_{i-1} \to X_i$, be a sequence of morphisms in $\mathcal{D}$ and let $v \in \mu_n(f_n, \ldots, f_1)$. Then there exists a choice of ${}_{i-1,i+k-1}s_2^{|\lambda_k(f_j,\ldots,f_{j-k+1})|}$ and ${}_{i-1,i+k-1}s_1^{|\lambda_k(f_j,\ldots,f_{j-k+1})|-1}$ for $1 < k < n$ and $k \leq j \leq n$ such that for any triple i, p, h compatible with these splittings we have $v = p\lambda_n(i(f_n), \ldots, i(f_1))$ and $0 = p\lambda_k(i(f_j), \ldots, i(f_{j-k+1}))$.*

Proof. We will show that for a complex

$$X_0 \xrightarrow{\bar{f}_0} X_1[n_1] \xrightarrow{\bar{f}_1} \cdots \longrightarrow X_k[n_k] \xrightarrow{\bar{f}_k} X_{k+1}[n_{k+1}]$$

of closed morphisms of degree 0 in $\mathcal{C}$ the convolution T of it is given by a complex

$$T = X_1[n_1 + k - 1] \oplus X_2[n_2 + k - 2] \oplus \ldots \oplus X_k[n_k]$$

with a differential $\partial = (\partial_1, \ldots, \partial_k)$, $\partial_l : T \to X_l[n_l + k - l]$ equal to

$$\partial_l(x_1, \ldots, x_l) = \partial_{X_l[n_l+k-l]}(x_l) + (-1)^{k-l} \sum_{i=1}^{l-1} (-1)^{(|\lambda_i|-1)n_{l-i}} h\lambda_i(\bar{f}_{l-1}, \ldots, \bar{f}_{l-i})(x_{l-i}).$$

A closed morphism of degree 0 $\bar{r} : X_0 \to T[1-k]$, $\bar{r} = (\bar{r}_1, \ldots, \bar{r}_k)$ is given by

$$\bar{r}_l(x_0) = (-1)^{l+1} h\lambda_l(\bar{f}_{l-1}, \ldots, \bar{f}_0)(x_0)$$

and a morphism $\bar{q} : T \to X_{k+1}[n_{k+1}]$, $\bar{q} = \sum_{l=0}^{k-1} \bar{q}_l$, $\bar{q}_{k-l} : X_l[n_l + k - l] \to X_{k+1}[n_{k+1}]$ is equal to

$$\bar{q}_{k-l}(x_l) = (-1)^{(|\lambda_{k-l+1}|-1)n_l} h\lambda_{k-l+1}(\bar{f}_k, \ldots, \bar{f}_l).$$

Hence,

$$\begin{aligned}
\bar{q}\bar{r} &= (-1)^{k+1} \bar{f}_{k+1} h\lambda_k(\bar{f}_k, \ldots, \bar{f}_1) - (-1)^{(|\lambda_k|-1)|\bar{f}_0|} h\lambda_k(\bar{f}_{k+1}, \ldots, \bar{f}_2)\bar{f}_1 \\
&+ \sum_{i=2}^{k-1} (-1)^{i+1+(|\lambda_{k-i+1}|-1)(|\bar{f}_1|+\ldots+|\bar{f}_i|)} h\lambda_{k-i+1}(\bar{f}_{k+1}, \ldots, \bar{f}_{i+1}) h\lambda_i(\bar{f}_i, \ldots, \bar{f}_1) \\
&= \lambda_{k+1}(\bar{f}_{k+1}, \ldots, \bar{f}_1).
\end{aligned}$$

$\bar{r}$ and $\bar{q}$ are closed morphisms of degree 0 in $\mathcal{C}$ and can be treated as morphisms in $\mathcal{D}$. Together with T they define the Massey product in the category $\mathcal{D}$. Maps $\bar{r}$ and $\bar{q}$ determine also the value of $\lambda_l(\bar{f}_{i+l-1}, \ldots, \bar{f}_i)$ for any $l \leq k$. Moreover, $p\lambda_l(\bar{f}_{i+l-1}, \ldots, \bar{f}_i) = 0 \in \mu_l(f_{l+i-1}, \ldots, f_i)$ as only with such a choice the Massey product μ_{k+1} is defined.

We will proceed by induction. Assume that $v \in \mu_n(f_2, f_1, f_0)$. Then, according to the definition $v = q \circ r$, for a pair of morphisms in $\mathcal{D}$. Let us write $\bar{f}_j = i(f_j)$ for any inclusion $i : \mathcal{D} \to \mathcal{C}$. In the category $\mathcal{C}$ we have

$$X_0 \xrightarrow{\bar{f}_0} X_1[n_1] \xrightarrow{\bar{f}_1} X_2[n_2] \xrightarrow{\bar{f}_2} X_3[n_3].$$

The convolution T of $X_1[n_1] \to X_2[n_2]$ is the cone of $\bar{f}_1$,

$$T = X_1[n_1+1] \oplus X_2[n_2]$$

with a differential $\partial(x_1, x_2) = (\partial_{X_1[n_1+1]}(x_1), \bar{f}_1(x_1) + \partial_{X_2[n_2]}(x_2))$.

Let $\bar{r} : X_0 \to T[-1]$ be a morphism in $\mathcal{D}$ lifting r. It is a closed morphism of degree 0 and commutes with $\bar{f}_0 : X_0 \to X_1[n_1]$ hence is given by

$$\bar{r}(x_0) = (\bar{f}_0(x_0), \bar{r}_2(x_0)),$$

with

$$\begin{aligned} \bar{r}_2 \partial_{X_0} &= -\bar{f}_1 \bar{f}_0 + \partial_{X_2[n_2-1]}, \\ \partial(\bar{r}_2) &= -\lambda_2(\bar{f}_1, \bar{f}_0). \end{aligned}$$

Thus, one can choose ${}_{0,2}s_2^{n_2}$ and ${}_{0,2}s_1^{n_2-1}$ in such a way that for any compatible h the equality $h\lambda_2(\bar{f}_1, \bar{f}_0) = \bar{r}_2$ will be satisfied.

Analogously, a morphism of degree 0, $\bar{q} : T \to X_3[n_3]$ is given by $\bar{q}(x_1, x_2) = \bar{q}_1(x_1) + \bar{f}_2(x_2)$. It is closed if $\bar{q}\partial_T = \partial_{X_2[n_2]}\bar{q}$ which leads to

$$\begin{aligned} &\bar{q}_1 \partial_{X_1[n_1+1]}(x_1) + \bar{f}_2 \bar{f}_1(x_1) + \bar{f}_2 \partial X_2[n_2](x_2) = \partial_{X_3[n_3]} q_1(x_1) + \partial_{X_3[n_3]} \bar{f}_2(x_2), \\ &\bar{q}_1 \partial_{X_1[n_1+1]} + \bar{f}_2 \bar{f}_1 = \partial_{X_3[n_3]} \bar{q}_1, \\ &\partial(\bar{q}_1) = (-1)^{n_1} \lambda_2(\bar{f}_2, \bar{f}_1). \end{aligned}$$

Again, there exists such a choice of ${}_{1,3}s_2^{n_3-n_1}$ and ${}_{1,3}s_1^{n_3-n_1-1}$ that for any compatible h we will have $h(\lambda_2(\bar{f}_2, \bar{f}_1)) = \bar{q}_1$.

Finally

$$\begin{aligned} \mu_3(f_2, f_1, f_0) = &p(\bar{q}\bar{r}) = p(\bar{q}_1 \bar{r}_1 + \bar{q}_2 \bar{r}_2) \\ &= p((-1)^{n_1} h\lambda_2(\bar{f}_2, \bar{f}_1)\bar{f}_0 - \bar{f}_2 h\lambda_2(\bar{f}_1, \bar{f}_0)) = p(\lambda_3(\bar{f}_2, \bar{f}_1, \bar{f}_0)). \end{aligned}$$

Now, let us consider

$$X_0 \xrightarrow{\bar{f}_0} X_1[n_1] \longrightarrow \cdots \longrightarrow X_k[n_k] \xrightarrow{\bar{f}_k} X_{k+1}[n_{k+1}] \xrightarrow{\bar{f}_{k+1}} X_{k+2}[n_{k+2}]$$

in $\mathcal{C}$ lifting a complex in the category $\mathcal{D}$.

We have

$$\begin{aligned} T(X_1[n_1] \to \ldots \to X_{k+1}[n_{k+1}]) &= \mathrm{Cone}(T(X_1[n_1] \to \ldots \to X_k[n_k]) \to X_{k+1}[n_{k+1}]) \\ &= X_1[n_1+k] \oplus X_2[n_2+k-1] \oplus \ldots \oplus X_{k+1}[n_{k+1}]. \end{aligned}$$

For $l = 1, \ldots, k$ the differential is

$$\partial_l = \partial_{X_l[n+l+k+1-l]} + (-1)^{k+1+l}(\sum_{i=1}^{l-1}(-1)^{(|\lambda|_i-1)n_{l-i}} h\lambda_i(\bar{f}_{l-1}, \ldots, \bar{f}_{l-i}).$$

and

$$\begin{aligned} \partial_{k+1} &= \partial_{X_{k+1}[n_{k+1}]} + \sum_{l=1}^{k}(-1)^{(|\lambda_{k-l+1}|-1)n_l} h\lambda_{k-l+1}(\bar{f}_k, \ldots, \bar{f}_l) \\ &= \partial_{X_{k+1}[n_{k+1}]} + \sum_{i=1}^{k}(-1)^{(|\lambda_i|-1)n_{k+1-i}} h\lambda_i(\bar{f}_k, \ldots, \bar{f}_{k+1-i}). \end{aligned}$$

A morphism $\bar{r} : X_0 \to X_1[n_1] \oplus \ldots \oplus X_{k+1}[n_{k+1} - k]$ is given by $(\bar{r}_1, \ldots, \bar{r}_{k+1})$, $\bar{r}_l : X_0 \to X_l[n_l - l + 1]$. For $l = 1, \ldots, k$

$$\bar{r}_k = (-1)^{l+1} h\lambda_l(\bar{f}_{l-1}, \ldots, \bar{f}_0).$$

$\bar{r}$ is closed so $\bar{r}\partial_{X_0} = \partial_{T[-k]}\bar{r}$, in particular $\bar{r}_{k+1}\partial_{X_0} = \partial_{k+1}\bar{r}$. This gives

$$\bar{r}_{k+1}\partial_{X_0} = \partial_{X_{k+1}[n_{k+1}-k]}\bar{r}_{k+1} + (-1)^k \sum_{l=1}^{k} (-1)^{(|\lambda_{k-l+1}|-1)n_l} h\lambda_{k-l+1}(\bar{f}_k, \ldots, \bar{f}_l)\bar{r}_l$$
$$= \partial_{X_{k+1}[n_{k+1}-k]}\bar{r}_{k+1}$$
$$+ (-1)^k \sum_{l=1}^{k} (-1)^{(|\lambda_{k-l+1}|-1)n_l + l - 1} h\lambda_{k-l+1}(\bar{f}_k, \ldots, \bar{f}_l)\lambda_l(\bar{f}_{l-1}, \ldots, \bar{f}_0),$$
$$\bar{r}_{k+1}\partial_{X_0} - \partial_{X_{k+1}[n_{k+1}-k]}\bar{r}_{k+1} = (-1)^k \lambda_{k+1}(\bar{f}_k, \ldots, \bar{f}_0),$$
$$\partial\bar{r}_{k+1} = (-1)^k \lambda_{k+1}(\bar{f}_k, \ldots, \bar{f}_0).$$

Hence, as before one can choose such splittings s_1 and s_2 that

$$\bar{r}_{k+1} = (-1)^k h\lambda_{k+1}(\bar{f}_k, \ldots, \bar{f}_0)$$

for any compatible h.

A morphism $\bar{q} : X_1[n_1 + k] \oplus \ldots \oplus X_{k+1}[n_{k+1}] \to X_{k+2}[n_{k+2}]$ is a sum of morphisms

$$\bar{q}_{k-l} : X_{l+1}[n_{l+1} + k - l] \to X_{k+2}[n_{k+2}].$$

For $l = 1, \ldots, k$

$$\bar{q}_{k-l} = (-1)^{(|\lambda_{k-l+1}|-1)n_{l+1}} h\lambda_{k-l+1}(\bar{f}_{k+1}, \ldots, \bar{f}_{l+1}).$$

To find $\bar{q}_k$ we again use the fact that $\bar{q}$ is a morphisms of complexes, that is $\bar{q}\partial_T = \partial_{X_{k+2}[n_{k+2}]}\bar{q}$. Writing down this equality for elements of the form $(x_1, 0, \ldots, 0) \in T$ we obtain

$$\partial_{X_{k+2}[n_{k+2}]}\bar{q}_k = \bar{q}_k\partial_{X_1[n_1+k]}$$
$$+ \sum_{l=1}^{k} (-1)^{(|\lambda_{k-l+1}|-1)n_{l+1} + k - l + (|\lambda_l|-1)n_1} h\lambda_{k-l+1}(\bar{f}_{k+1}, \ldots, \bar{f}_{l+1}) h\lambda_l(\bar{f}_l, \ldots, \bar{f}_1),$$
$$\bar{q}_k\partial_{X_1} - \partial_{X_{k+2}[n_{k+2}-k]}\bar{q}_k$$
$$= (-1)^{k+n_1} \sum_{l=1}^{k} (-1)^{(|\lambda_{k-l+1}|-1)n_{l+1} + k - l + (|\lambda_l|-1)n_1}$$
$$h\lambda_{k-l+1}(\bar{f}_{k+1}, \ldots, \bar{f}_{l+1}) h\lambda_l(\bar{f}_l, \ldots, \bar{f}_1)$$
$$= (-1)^{k + |\lambda_{k-l+1}|n_1 + |\lambda_l|n_1 + n_1}$$
$$\sum_{l=1}^{k} (-1)^{(|\lambda_{k-l+1}|-1)(|f_1| + \ldots + |f_l|) + k - l} h\lambda_{k-l+1}(\bar{f}_{k+1}, \ldots, \bar{f}_{l+1}) h\lambda_l(\bar{f}_l, \ldots, \bar{f}_1)$$
$$= (-1)^{kn_1} \mu_{k+1}(\bar{f}_{k+1}, \ldots, \bar{f}_1),$$
$$\partial\bar{q}_k = (-1)^{(|\lambda_{k+1}|-1)n_1} \lambda_{k+1}(\bar{f}_{k+1}, \ldots, \bar{f}_1)$$

and there exists splittings s_1 and s_2 such that

$$\bar{q}_k = (-1)^{(|\lambda_{k+1}|-1)n_1} h\lambda_{k+1}(\bar{f}_{k+1}, \ldots, \bar{f}_1)$$

for any compatible h.

Finally

$$\begin{aligned}
\mu_{k+2}(f_{k+1}, \ldots, f_0) &= p(\bar{q}\bar{r}) = p(\sum_{l=0}^{k} \bar{q}_{k-l}\bar{r}_{l+1}) \\
&= p(\sum_{l=0}^{k}(-1)^{l+(|\lambda_{k-l+1}|-1)n_{l+1}} h\lambda_{k-l+1}(\bar{f}_{k+1}, \ldots, \bar{f}_{l+1})h\lambda_{l+1}(\bar{f}_l, \ldots, \bar{f}_0)) \\
&= p(\sum_{l=0}^{k}(-1)^{l+(|\lambda_{k-l+1}|-1)(|\bar{f}_0|+\ldots+|\bar{f}_l|)} h\lambda_{k-l+1}(\bar{f}_{k+1}, \ldots, \bar{f}_{l+1})h\lambda_{l+1}(\bar{f}_l, \ldots, \bar{f}_0)) \\
&= p(\lambda_{k+2}(\bar{f}_{k+1}, \ldots, \bar{f}_0)).
\end{aligned}$$

□

With this lemma we are ready to formulate the

Theorem 3.5. *Let $\mathcal{D}$ be an enhanced triangulated category, $\mathcal{E}_1, \ldots, \mathcal{E}_n$ objects in $\mathcal{D}$ and $\widetilde{\mathcal{C}}$ the full DG subcategory of the enhancement of $\mathcal{D}$ with objects $\mathcal{E}_i$. Let also $(f_1^1, \ldots, f_{n_1}^1), \ldots (f_1^j, \ldots, f_{n_j}^j)$ be a family of morphisms in $H(\widetilde{\mathcal{C}})$ such that for all i and $1 < l < n_i$ any two of $\lambda_l(f_{k+l-1}^i, \ldots, f_k^i)$ do not lie in the same space $\widetilde{\mathcal{C}}^n(\mathcal{E}_{i_1}, \mathcal{E}_{i_2})$ and finally let $v^i \in \mu_{n_i}(f_{n_i}^i, \ldots, f_1^i)$. Then, up to an A_∞ isomorphism, in the minimal model for $\widetilde{\mathcal{C}}$ one has $v^i = m_{n_i}(f_{n_i}^i, \ldots, f_1^i)$ and $0 = m_l(f_{k+l-1}^i, \ldots, f_k^i)$ for $l < n_i$.*

Proof. The lemma 3.4 tells us that for morphisms $f_1^k, \ldots, f_{n_k}^k$ and v^k there exists a choice of splittings s_1 and s_2 such that $v^k = p\lambda_{n_k}(i(f_{n_k}^k), \ldots, i(f_1^k))$ for any compatible triple i, p and h. The conditions of the theorem imply that for all $k = 1, \ldots, j$ the choice of these splittings is independent. Combining it with the result of Merkulov recalled in 2.1 we obtain the proof of the theorem.

□

Remark 3.6. In the case when a triple Massey product is not defined, one can still deduce something about the operation m_3. Let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$$

be morphisms in $H(\widetilde{(\mathcal{C}})$ such that $\beta\alpha \neq 0$ and $\gamma\beta = 0$. Then

$$\begin{aligned}
m_3(\gamma, \beta, \alpha) &= p\lambda_3(i(\gamma), i(\beta), i(\alpha)) \\
&= p((-1)^{|\alpha|}h(i(\gamma)i(\beta))i(\alpha) - i(\gamma)h(i(\beta)i(\alpha))).
\end{aligned}$$

Fixing ${}_{A,C}s_2^{|\beta|+|\alpha|}$ one can assume that $i(\beta)i(\alpha) = i(\beta\alpha)$ and hence $h(i(\beta)i(\alpha)) = 0$. Then where $\partial\lambda_3(i(\gamma), i(\beta), i(\alpha)) = (-1)^{|\alpha|}i(\gamma)i(\beta\alpha)$.

If the category $\mathcal{C}$ has infinite dimensional morphisms spaces and finite dimensional cohomology groups then there exists a choice of ${}_{A,C}s_2^{|\alpha|+|\beta|}$ and ${}_{C,D}s_2^{|\gamma|}$ such that $\partial\lambda_3(i(\gamma), i(\beta), i(\alpha)) \neq 0$. Finally, there exists a choice of ${}_{A,D}s_1^{|\alpha|+|\beta|+|\gamma|-1}$ such that $\lambda_3(i(\gamma), i(\beta), i(\alpha)) \in \mathrm{Ker}p$ which gives $m_3(\gamma, \beta, \alpha) = 0$.

4 Calculating the category $\mathcal{D}_\sigma$ via mutations

In the previous section we have proved that in some cases the DG category $\mathcal{D}_\sigma$ can be determined by means of Massey products. However, the theorem 3.5 does not determine the A_∞ category $H(\mathcal{C}_\sigma)$ for any collection σ and moreover calculating Massey products of arbitrary morphisms in a triangulated category is not always an easy task. Therefore, in this section we will present another algorithm allowing to determine the DG category $\mathcal{D}_\sigma$. It uses a mutation of the collection σ to a collection ρ for which the DG category $\mathcal{D}_\rho$ is known. In particular, it determines the DG categories of all exceptional collections obtained via mutations from a strong one.

We begin with recalling the definitions from [B]. Let $\langle \mathcal{E}, \mathcal{F} \rangle$ be an exceptional pair in $D^b(\mathrm{Coh}(X))$ for a smooth projective variety X. Then $\langle L_\mathcal{E}\mathcal{F}, \mathcal{E} \rangle$ and $\langle \mathcal{F}, R_\mathcal{F}\mathcal{E} \rangle$ are also exceptional pairs for $L_\mathcal{E}\mathcal{F}$ and $R_\mathcal{F}\mathcal{E}$ defined by means of distinguished triangles in $D^b(\mathrm{Coh}(X))$.

$$L_\mathcal{E}\mathcal{F} \to \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F} \to L_\mathcal{E}\mathcal{F}[1],$$
$$\mathcal{E} \to \mathrm{Hom}(\mathcal{E}, \mathcal{F})^* \otimes \mathcal{F} \to R_\mathcal{F}\mathcal{E} \to \mathcal{E}[1].$$

Here $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is treated as a complex of $\mathbb{C}$-vector spaces with trivial differential, $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \bigoplus \mathrm{Hom}^k(\mathcal{E}, \mathcal{F})$ for $\mathrm{Hom}^k(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{D^b(\mathrm{Coh}(X))}(\mathcal{E}, \mathcal{F}[k])$. For an element E of a $D^b(\mathrm{Coh}(X))$ and a complex $V^\bullet$ the tensor product is defined by $E \otimes V^\bullet = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{i=0}^{\dim V^k} E[-k]$.

If $\rho = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle$ is an exceptional collection in $D^b(\mathrm{Coh}(X))$ then the left i-th mutation $L_i\rho$ and the right i-th mutation $R_i\rho$ are defined as follows

$$L_i\rho = \langle \mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, L_{\mathcal{E}_i}\mathcal{E}_{i+1}, \mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_n \rangle,$$
$$R_i\rho = \langle \mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, \mathcal{E}_{i+1}, R_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_n \rangle.$$

Twisted complexes provide a description of the categories $\mathcal{D}_{L_i\rho}$ and $\mathcal{D}_{R_i\rho}$ by means of $\mathcal{D}_\rho$. To see it we need to define a tensor product of a twisted complex with a complex of vector spaces.

Let $\mathcal{C}$ be a DG category with finitely dimensional vector spaces, $C \in \mathcal{C}^{\text{pre-tr}}$ be a twisted complex and let $V^\bullet$ be a finite dimensional complex of vector spaces with the differential $\partial^i : V^i \to V^{i+1}$. $C \otimes V$ is defined as $(\bigoplus_{\{i|V^i \neq 0\}} C[-i]^{\oplus \dim V^i}, q_{i,j}) \in (\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$. The morphisms $q_{i,i+1}$ are induced by the differential ∂^i tensored with the identity on C and $q_{i,j} = 0$ for $j \neq i+1$.

Now, let $C, D \in \mathcal{C}^{\text{pre-tr}}$ be twisted complexes. There exist closed morphisms of degree 0

$$\phi : C \otimes \mathcal{C}^{\text{pre-tr}}(C, D) \to D,$$
$$\psi : C \to \mathcal{C}^{\text{pre-tr}}(C, D)^* \otimes D.$$

The morphism $\phi_{i,0} : C[-i]^{\oplus \dim \mathrm{Hom}^i(C,D)} \to D$ is given by morphisms of degree i between C and D and ψ is defined analogously. Now we define new twisted complexes over $\mathcal{C}$

$$L_C D = \mathrm{Tot}(C(\phi)[-1]),$$
$$R_D C = \mathrm{Tot}(C(\psi)).$$

Let $\rho = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on a smooth projective variety X and let $\mathcal{D}_\rho$ with objects $E_1, \dots, E_n$ be the category described in the theorem 2.15. We define two full subcategories of $\mathcal{D}_\rho^{\text{pre-tr}}$; $\mathcal{D}_\rho^{L_i}$ with objects $E_1, \dots, E_i, L_{E_i}E_{i+1}, E_{i+2}, \dots E_n$ and $\mathcal{D}_\rho^{R_i}$ with $E_1, \dots, E_{i-1}, R_{E_{i+1}}E_i, E_{i+1}, \dots, E_n$.

Proposition 4.1. *The category $\mathcal{D}_\rho^{L_i}$ satisfies the condition of the theorem 2.15 for the exceptional collection $L_i\rho$. Analogous statement is true for $\mathcal{D}_\rho^{R_i}$.*

Proof. The theorem 2.15 guarantees that the category $\mathcal{D}_\rho$ has finite dimensional vector spaces. Hence, mutations of twisted complexes over $\mathcal{D}_\rho$ are well defined. Furthermore, according to the remark 2.16 the category $D^b(\mathrm{Coh}(X))$ is equivalent to $(\mathcal{D}_\rho)^{\mathrm{tr}}$. Under this equivalence the object $L_{\mathcal{E}_i}\mathcal{E}_{i+1}$ corresponds to $L_{E_i}E_{i+1}$. This proves that condition (ii) of the theorem 2.15 is satisfied. Finally, the condition (ii), the theorem 1.4 and the fact that the category $D^b(\mathrm{Coh}(X))$ is Karoubian show that the condition (i) of the theorem 2.15 is also satisfied. □

5 Example

Let X be a toric surface with a fan presented in the Figure 1.

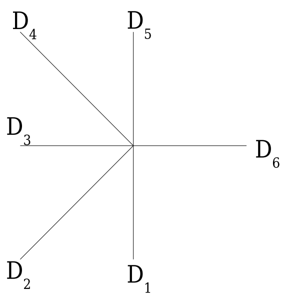


Figure 1: The fan of X.

X is obtained from $\mathbb{P}^2$ by blowing-up two points and then blowing up one point on an exceptional divisor. The Picard group of X has basis H, E_1, E_2 and E_3 with

$$H^2 = 1, \qquad E_1^2 = -1, \qquad E_2^2 = -2, \qquad E_3^2 = -1,$$
$$HE_i = 0, \qquad E_1E_i = 0, \qquad E_2E_3 = 1.$$

In this basis the toric divisors are:

$$D_1 = E_1, \qquad D_2 = H - E_1 - E_2 - E_3, \qquad D_3 = E_2,$$
$$D_4 = E_3, \qquad D_5 = H - E_2 - 2E_3, \qquad D_6 = H - E_1.$$

In [HP] Hille and Perling prove that the collection $\langle \mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(H - E_2 - E_3), \mathcal{O}(H - E_3), \mathcal{O}(H), \mathcal{O}(2H - E_2 - 2E_3) \rangle$ is full. It is not strong, there is one nontrivial Ext^1 group between $\mathcal{O}(H - E_2 - E_3)$ and $\mathcal{O}(H - E_3)$. The quiver of this collection has the form presented in the Figure 2. There is one relation

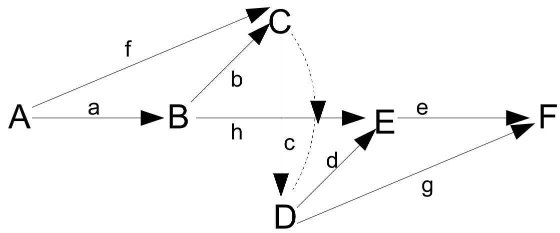


Figure 2: The quiver of the collection $\langle \mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(H - E_2 - E_3), \mathcal{O}(H - E_3), \mathcal{O}(H), \mathcal{O}(2H - E_2 - 2E_3)\rangle$ on X.

$edcba = icf$. The dotted arrow represents the nontrivial Ext group.

Calculating Massey product for morphisms in this collection requires finding a nontrivial extension

$$0 \to \mathcal{O}(H - E_3) \to \mathcal{E} \to \mathcal{O}(H - E_2 - E_3) \to 0.$$

After one mutation we obtain a collection in which calculating Massey products is significantly easier. We have a short exact sequence

$$0 \to \mathcal{O}(E_1) \to \mathcal{O}(H - E_2 - E_3) \to \mathcal{O}_{H-E_1-E_2-E_3} \to 0$$

so up to a translation, the 2-nd left mutation of the considered collection is $\langle \mathcal{O}, \mathcal{O}_{H-E_1-E_2-E_3}, \mathcal{O}(E_1), \mathcal{O}(H-E_3), \mathcal{O}(H), \mathcal{O}(2H-E_2-2E_3)\rangle$. This collection has nontrivial first Ext groups from $\mathcal{O}_{H-E_1-E_2-E_3}$ to $\mathcal{O}(E_1)$, $\mathcal{O}(H - E_3)$, $\mathcal{O}(H)$ and $\mathcal{O}(2H - E_2 - 2E_3)$. The quiver of the this collection is presented in the Figure 3.

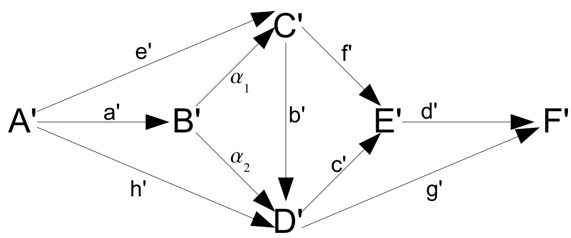


Figure 3: The quiver of the collection $\langle \mathcal{O}, \mathcal{O}_{H-E_1-E_2-E_3}, \mathcal{O}(E_1), \mathcal{O}(H - E_3), \mathcal{O}(H), \mathcal{O}(2H - E_2 - 2E_3)\rangle$ on X.

The arrows α_1 and α_2 are of degree 1. The relations in the quiver are

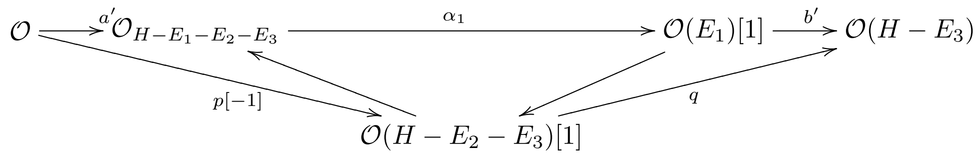
$$d'c'b'e' = g'h', \qquad \alpha_1 a' = 0, \qquad \alpha_2 a' = 0,$$
$$b'\alpha_1 = 0, \qquad g'\alpha_2 = 0, \qquad f'\alpha_1 = c'\alpha_2.$$

The values of $m_3(b', \alpha_1, a')$, $m_3(d', f'\alpha_1, a')$, $m_3(f', \alpha_1, a')$, $m_3(g', \alpha_2, a')$ and $m_3(c', \alpha_2, a')$ determine the A_∞ structure of the quiver presented in the Figure

3. Using the theorem 3.5 and the remark 3.6 we get that

$$
\begin{aligned}
&m_3(b', \alpha_1, a') \in \mu_3(b', \alpha_1, a'), && m_3(d', f'\alpha_1, a') = 0, \\
&m_3(g', \alpha_2, a') \in \mu_3(g', \alpha_2, a'), && m_3(f', \alpha_1, a') = 0, \\
&m_3(c', \alpha_2, a') = 0.
\end{aligned}
$$

Calculating μ_3 for morphisms in this collection is not difficult. For example $h' \in \mu_3(b', \alpha_1, a')$.



In the language of toric divisor the zeros of b' are $Z(b') = D_2 + D_3$. For the map p we have $Z(p) = D_4 + D_5$. Finally, for the zero locus of q is $Z(q) = D_3$. It follows that $Z(qp) = D_3 + D_4 + D_5$ and hence $qp = h'$.

Similarly one can show that $\mu_3(g', \alpha_2, a') = d'f'e'$.

Finally we obtain the following A_∞ structure

$$
\begin{aligned}
&m_3(b', \alpha_1, a') = h', && m_3(c'b', \alpha_1, a') = c'h', \\
&m_3(d'c'b', \alpha_1, a') = d'c'h', && m_3(d'f', \alpha_1, a') = 0, \\
&m_3(d', f'\alpha_1, a') = 0, && m_3(f', \alpha_1, a') = 0, \\
&m_3(d'c', \alpha_2, a') = 0, && m_3(g', \alpha_2, a') = d'f'e', \\
&m_3(c', \alpha_2, a') = 0, && m_3(d', f', \alpha_1) = 0, \\
&m_3(c', b', \alpha_1) = 0, && m_3(d'c', b', \alpha_1) = 0, \\
&m_3(d', c'b', \alpha_1) = 0, && m_3(d', c', \alpha_2) = 0.
\end{aligned}
$$

The operation having as the last argument α_1 or α_2 are trivial because there are no morphism of degree 0 from B' to neither E nor F.

With these information we can calculate the DG category of the collection $\langle \mathcal{O}, \mathcal{O}_{H-E_1-E_2-E_3}, \mathcal{O}(E_1), \mathcal{O}(H-E_3), \mathcal{O}(H), \mathcal{O}(2H-E_2-2E_3)\rangle$. It is presented in the Figure 4.

Again, arrows α_1 and α_2 are in degree 1. The relations and differentials in the quiver are

$$
\begin{aligned}
d'c'b'e' &= g'b'u_1 - g'u_3a' (= g'h'), & f'\alpha_1 &= c'\alpha_2, \\
d'f'e' &= g'u_2 - u_4a', & f'u_1 &= c'u_2, \\
\partial(u_1) &= \alpha_1 a', & \partial(u_2) &= \alpha_2 a', \\
\partial(u_3) &= b'\alpha_1, & \partial(u_4) &= g'\alpha_2.
\end{aligned}
$$

Now we can mutate the above DG category to obtain the DG category of the collection $\langle \mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(H-E_2-E_3), \mathcal{O}(H-E_3), \mathcal{O}(H), \mathcal{O}(2H-E_2-2E_3)\rangle$. Here, $\mathcal{O}(H-E_2-E_3)$ is a right mutation of $\mathcal{O}_{H-E_1-E_2-E_3}$ over $\mathcal{O}(E_1)$ and can be written as a twisted complex, where the morphism is α_1.

$$\mathcal{O}(H-E_2-E_3) \simeq \mathcal{O}_{H-E_1-E_2-E_3} \to \mathcal{O}(E_1),$$

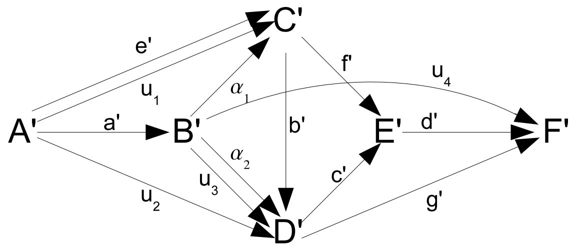


Figure 4: The DG category of the collection $\langle \mathcal{O}, \mathcal{O}_{H-E_1-E_2-E_3}, \mathcal{O}(E_1), \mathcal{O}(H-E_3), \mathcal{O}(H), \mathcal{O}(2H-E_2-2E_3)\rangle$ on X.

Up to a quasi-isomorphism we obtain the DG category in the Figure 5.

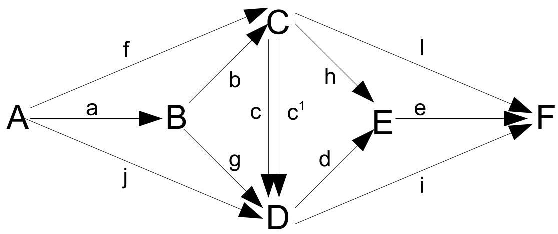


Figure 5: The DG category of the collection $\langle \mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(H-E_2-E_3), \mathcal{O}(H-E_3), \mathcal{O}(H), \mathcal{O}(2H-E_2-2E_3)\rangle$ on X.

The arrow c^1 is in degree 1. We have following relations and differentials

$$\begin{aligned} ehba &= ij, & edcba &= icf, & edga &= lf,\\ ig &= lb, & dj &= hf, & edc^1ba &= ic^1f,\\ \partial(h) &= dc^1, & \partial(j) &= c^1f, & & \\ \partial(l) &= ic^1, & \partial(g) &= c^1b. & & \end{aligned}$$

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