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Indicated colorings of graphs

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INDICATED COLORINGS OF GRAPHS

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Abstract. Let us consider a game in which two players color a graph. The first player (Indicator) selects a vertex and the second player (Painter) colors it in a proper way in one of available colors from a fixed set. The goal of Indicator is to color the whole graph and the goal of Painter is to spoil it. The smallest number of colors which is necessary for Indicator to win the game (regardless of the strategy of Painter) will be called *indicated chromatic number* and denoted by $\chi_I(G)$. In this paper we will present some facts about indicated chromatic number, in particular we will present a class of graphs which shows that the difference between $\chi_I(G)$ and $\chi(G)$ can be arbitrarily big. We will also present some interesting conjectures and intuition for strategies of players and indicated chromatic number of random graphs.

1. Introduction

All graphs in this paper are assumed to be simple, i.e. finite, without loops and multiple edges. By $V(G)$ and $E(G)$ we denote the set of vertices and the set of edges of a graph G. Coloring of a graph is a function $c : V(G) \mapsto C = \{c_1, c_2, \ldots, c_k\}$. The elements of a set C are called *colors*. We say that coloring is *proper* if adjacent vertices v_i and v_j satisfy $c(v_i) \neq c(v_j)$, which means that adjacent vertices have different colors. The smallest number of colors needed to properly color a graph G is called *chromatic number* of a graph G and is denoted by $\chi(G)$.

Let us consider a game in which two players color a graph using colors from a fixed set. The first player (Indicator) selects a vertex and the second player (Painter) colors it in a proper way in one of available colors. The goal of Indicator is to color the whole graph and the goal of Painter is to achieve a bad partial coloring (that is, a coloring that cannot be extended to the proper coloring of the whole graph). The smallest number of colors which is necessary for Indicator to win the game (regardless of the strategy of Painter) will be called *indicated chromatic number* and denoted by $\chi_I(G)$.

One motivation to consider problems related to this chromatic number comes from a similar graph coloring game, in which players colors alternatively a graph but the first player tries to color it and second player wants to spoil it. This game firstly appears in Martin Gardner *Mathematical Games* in *Scientific American* ([8]) in 1981 and it was considered in many papers for same specific classes of graphs ([2], [6], [7], [10], [11], [13], [14]). A nice survey about this game can be found in [5].

2. Bounds for indicated chromatic number

We will begin with some trivial bounds.

Fact 1. *Every graph G satisfies*

$$\chi(G) \leq \chi_I(G) \leq \Delta(G) + 1.$$

Proof. If there is a winning strategy for Indicator the game will end with a proper coloring of a graph. So there should be at least as many colors as is required in any proper coloring, i.e. $\chi(G)$. On the other hand, if there are $\Delta(G) + 1$ colors then, no matter how the game will be played, there will be always an available color for each vertex, because it cannot have more than $\Delta(G)$ neighbors. □

As immediate consequence we have the following

Fact 2. *If K_n is clique on n vertices then*

$$\chi_I(K_n) = n,$$

moreover, if G contains K_n as a subgraph then

$$\chi_I(K_n) \geq n.$$

So, on cliques the indicated chromatic number is equal to the chromatic number. The same holds for bipartite graphs.

Fact 3. *If G is bipartite graph then*

$$\chi_I(G) = 2.$$

Proof. When there are only two colors and Indicator indicates a vertex which makes connected graph with already painted vertices, the Painter have no choice in putting colors – he has to put the same color as in a proper coloring with two colors. So, the strategy for Indicator is to present such vertices. It will effect in a proper coloring of connected component of graph. Each such component can be colored independently, so as a result, we will get a proper coloring of the whole graph. □

Better upper bound can be done using coloring number $\mathrm{col}(G)$. It is defined as the smallest k such that every subgraph of G has a vertex with degree lower than k.

Fact 4. *Every graph G satisfies*

$$\chi_I(G) \leq \mathrm{col}(G).$$

Proof. Firstly Indicator arranges vertices of a graph G in such order v_1, v_2, $\dots$, v_n that each vertex has less than $\mathrm{col}(G)$ neighbors with smaller index. He constructs such order from the end – taking from G, in each step, vertex with degree less then $\mathrm{col}(G)$. It is possible by definition of $\mathrm{col}(G)$. During the play, Indicator indicates vertices according to this order. In each part of the game Painter has more colors than it was used before, so it is always possible to put some color. □

This upper bound is quite good for some class of graphs.

Fact 5. *If G is a planar graph then*

$$\chi_I(G) \leq 6.$$

Fact 6. *If G is a outerplanar graph then*

$$\chi_I(G) \leq 3.$$

But in general, this upper bound is rather poor, for example bipartite graphs has $\chi_I(G) = 2$ but $\mathrm{col}(G)$ can be arbitrarily big.

What about distance from the lower bound χ? Firstly, we will show that there are examples of graphs G which satisfy strong inequality $\chi_I(G) > \chi(G)$.

Example 7. Graph G presented on Figure 1 has $\chi(G) = 3$ and $\chi_I(G) = 4$.

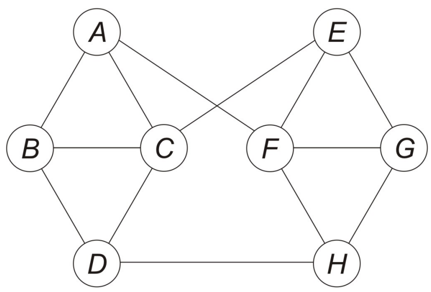

FIGURE 1. Example of a graph for which $\chi_I(G) > \chi(G)$.

Proof. Firstly, notice that $\chi(G) = 3$ and there is only one way of coloring this graph using 3 colors (modulo permutations of color names). This is so because any proper coloring of the subgraph consisting of vertices $ABCD$ requires that A and D has the same color. Similarly E and H must have the same color. Vertices D and H are adjacent, so their colors should be different. Now, it is easy to see, that there is only one way to put colors on the other vertices.

The maximum degree of the graph G equals to 3 so from Fact 1 we have $\chi_I(G) \leq 4$. The only thing which is left, is to prove that there is no winning strategy for Indicator when the game is played using three colors.

There is only one way of coloring this graph using 3 colors so during the play Indicator cannot leave to Painter any other decision besides "naming" a color (using them for the first time). In other case, Painter will choose a color which not corresponds to this unique coloring and the game will end with Painter's win. Such Indicator's moves will be called *forcing* moves. We will prove that such forcing play is impossible for Indicator. Let G_1 denotes the induced subgraph on vertices A, B, C, D and let G_2 be the induced subgraph on vertices E, F, G and H. If Indicator starts with two adjacent vertices from one of this subgraphs, he cannot do forcing move to the second of these subgraphs. If Indicator starts with two adjacent vertices, but one in G_1 and the other in G_2, he cannot do a forcing move, because there are no triangles containing an edge between G_1 and G_2. This completes the proof. □

Theorem 8. *There is no constant a such that $\chi_I(G) \leq \chi(G) + a$ for every graph G.*

Proof. Suppose that there is such a. Take G as a graph consisting of $a+1$ copies of graph presented in Figure 1 connected with all possible edges. Such a graph has $\chi(G) = 3(a+1)$ and $\chi_I(G) = 4(a+1)$, which gives a contradiction. □

The question about upper bound for χ_I as a function of χ is still open.

Conjecture 9. *There is a function f such that*

$$\chi_I(G) \leq f(\chi(G))$$

for every graph G.

3. Indicated chromatic number of subgraphs

Hardness in determining the indicated chromatic number is related with the fact that this number is not monotonic with respect to taking subgraphs.

Fact 10. *If $G \subset H$ than not always $\chi_I(G) \leq \chi_I(H)$.*

Proof. Let G be a graph presented on Figure 1. Let H be a full tripartite graph $K_{3,3,3}$. Since graph G can be colored using three colors and all colors are used at most three times, we have $G \subset H$. Notice that $\chi_I(H) = 3$, which is smaller than $\chi_I(G) = 4$. □

Even if we consider only induced subgraphs a similar result can be proved.

Fact 11. *If G is an induced subgraph of H than not always $\chi_I(G) \leq \chi_I(H)$.*

Proof. Let G be a graph presented in Figure 1. Let H be a graph consisting of graph G and one more vertex, denoted by I, connected with vertices A, B, D, E, G and H. We will show a strategy for Indicator to win the game on graph H using only three colors. Firstly, Indicator presents vertex I, which gets the first color. Then Indicator indicates vertex A, which has to get different color. In other moves, Indicator can always indicate a vertex which is connected with two vertices having different colors and not connected with vertex in the third color. It forces all the Painter's moves and end with good coloring of graph H with 3 colors. Since $\chi_I(G) = 4$ we get the assertion of this fact. □

Above results show that no local strategies can be applied global. In practice, it means that induction technics cannot be applied too.

4. Monotonicity of indicated colorings

Let us introduce one more interesting problem related to this game. It is a similar question to the one for the game chromatic number stated by Zhu in [13]. Suppose that Indicator has a winning strategy in the game with k colors. Does he have a winning strategy when the game is played using $k + 1$ colors? It sounds strange. But, if we think for a while, we will get that intuitively it can be false. When there are more colors, Indicator has less possibilities to force Painter's moves. So, Painter has more possibilities to prevent from proper coloring. A weaker conjecture is also open.

Conjecture 12. *There is a function f such that for every graph G, if Indicator has a winning strategy using k colors, he has also a winning strategy using $f(k)$ colors and every greater number of colors.*

This statement is true for bipartite graphs and f equals identity.

Theorem 13. *In game on any bipartite graph B with $k \geq 2$ colors the Indicator has always a winning strategy.*

Proof. Let U_1 and U_2 be the set of vertices in the bipartition of graph B. We can assume that firstly the Indicator presents some vertex $v_1 \in U_1$ and the Painter puts 1. In next moves Indicator presents all the neighbors of v_1 in U_2 – they will get some colors from 2 to k. Afterwards, the Indicator presents all neighbors of those vertices which have color 2. They can get color 1 or at least 3. Indicator can repeat this strategy of showing neighbors of vertices colored 1 and neighbors of vertices colored 2. There cannot be any problem because no vertex in U_2 has color 1 and no vertex in U_1 has color 2. This will stop in the point, where all neighbors of all vertices colored 1 or 2 are colored. In this point, vertices colored 1 or 2 cannot be a part of bad coloring (they cannot have an uncolored neighbor which is adjacent to vertices in all the colors), because all of they neighbors are colored. So, we can forgot about them. The Indicator can now start his strategy from the beginning – present some vertex in U_1 and then neighbors of 1's and 2's. Similarly, new vertices colored 1 or 2 are in different parts of partitioning (not necessarily the same parts as the old vertices colored 1 and 2), so there cannot occur a bad partial coloring. □

5. Limitations on strategies

Another area of investigation is related to some specific strategies of players. For example we can limit Indicator to only *connected moves*, which means that colored subgraph has to be connected in each part of the game. Notice that when Indicator makes a connected move, he leaves less possibilities for Painter to choose color of a vertex, than indicating an isolated vertex. So, intuitively this limitation for Indicator's moves is natural and cannot change anything. It appears that this is false.

Example 14. In the game played on graph presented on Figure 2 using 4 colors Indicator wins the game only if he can make unconnected moves.

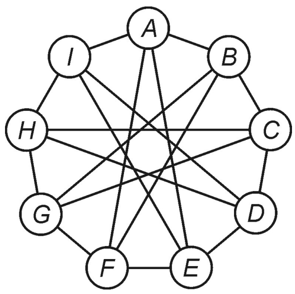

Figure 2. Example of graph on which connected moves are not good.

Proof. Notice that a vertex cannot be colored if it has all neighbors in different colors. So, if some vertex has two neighbors sharing the same color or it will be indicated by Indicator before at least one of his neighbors, it will be possible to be colored. Such vertex will be called *safe*. If at some moment of the game all vertices will be colored or safe, then Indicator wins the game.

In the first move Indicator presents vertex A. Assume that Painter puts color 1. In the second move, Indicator indicates vertex C. Painter can put color 1 or a new color, say color 2.

In the first case, vertex B will be safe. So, Indicator can present it at the end. Hence vertices F and G are safe. In the same way vertices E, H and vertices D, I are safe. In other words, if Indicator indicates vertices in order D, I, E, H, F, G, B, the game will end in good coloring.

In the second case, Indicator presents vertex B, which has to get new color – 3. Next, Indicator indicates vertices F and G. Whatever Painter does, vertices C and F will have the same color 2 or vertices A and G will have color 1. These are symmetric cases, so we can assume the first one holds. Notice that vertex I is safe because none of its neighbors can have color 2. So, its neighbors are also safe. All of the vertices are colored or safe, hence the game will end in good coloring.

It can be quite easily checked that no strategy using only connected moves is good. The Painter should put the same color only on vertices in distance 2 on the outer cycle and only when the vertex between them is colored. No more than 8 moves can be done. □

Notice that when the game is played on this graph with 3 colors then Indicator wins. He should just indicate vertices which are connected with two vertices having different colors. So, this example shows that when we limit Indicator to connected moves, then Conjecture 12 is false for the identity function.

6. Indicated chromatic number of random graphs

Let $G(n,p)$ stands for the probability space of all labeled graphs on n vertices, where every edge appears independently with probability p. Define $b = \frac{1}{1-p}$. We will determine the order of magnitude of the indicated chromatic number of $G(n,p)$.

Theorem 15. *If $\varepsilon > 0$ is a constant then* ***whp***[1]

$$\left(\frac{1}{2} + o(1)\right)\frac{n}{\log_b np} \le \chi_I(G(n,p)) \le (2+\varepsilon)\frac{n}{\log_b np}.$$

Proof. It is well known by the results of Bollobás [4] and Łuczak [12] that **whp** $\chi(G(n,p))$ is equal the above lower bound. Thus, the content of the theorem is the upper bound. This method where used by Bohman, Frieze and Sudakov in [3] to prove similar result for game chromatic number.

Let the number of colors be $k = (2+\varepsilon)\frac{n}{\log_b np}$ and let $\mathcal{C} = (C_1, C_2, \ldots, C_k)$ be a collection of pair-wise disjoint sets. For a vertex v let

$$A(v,\mathcal{C}) = \{i \in [k] : v \text{ is not adjacent to any vertex of } C_i\}$$

and set

$$a(v,\mathcal{C}) = |A(v,\mathcal{C})|.$$

Note that $A(v,\mathcal{C})$ is the set of available colors for uncolored vertex v when the partial coloring is given by the sets in $\mathcal{C}$.

Let

$$\alpha = 2+\varepsilon, \quad \beta = k(1-p)^{n/k} = \frac{\alpha n}{(np)^{1/\alpha}\log_b np}, \quad \gamma = \frac{9n\ln n}{\beta}$$

and

$$B(\mathcal{C}) = \{v : a(v,\mathcal{C}) < \beta/2\}.$$

[1] A sequence of events X_n occurs with high probability (**whp**) if $\lim_{n\to\infty}\mathbb{P}(X_n) = 1$

We will show that with high probability every partial coloring of the vertex set has the property that there are at most γ vertices with less than $\beta/2$ available colors.

Lemma 16. ***Whp**, for all partial collections $\mathcal{C}$*

$$|B(\mathcal{C})| \leq \gamma.$$

Proof. For some fixed $\mathcal{C}$ and every uncolored vertex v, the number of available colors is the sum of independent variables X_i, where $X_i = 1$ if v has no neighbors in C_i. Then $\mathbb{P}(X_i = 1) = (1-p)^{|C_i|}$ and since $(1-p)^x$ is a convex function we have

$$\begin{aligned} \mathbb{E}(a(v,\mathcal{C})) &= \sum_{i=1}^{k} (1-p)^{|C_i|} \leq k(1-p)^{(|C_1|+|C_2|+\ldots+|C_k|)/k} \leq \\ &\leq k(1-p)^{n/k} = \beta. \end{aligned}$$

I follows from the Chernoff bound (see [1] or [9]) that

$$\mathbb{P}(a(v,\mathcal{C}) \leq \beta/2) \leq e^{-\beta/8}.$$

Thus,

$$\mathbb{P}(\exists \mathcal{C} \text{ with } |B(\mathcal{C})| > \gamma) \leq k^n \binom{n}{\gamma} e^{-\beta\gamma/8} = o(1).$$

□

Set t_0 to be the last time for which Indicator presents a vertex with at least $\beta/2$ available colors, i.e.

$$t_0 = \min\{t : a(v, \mathcal{C}_t) \geq \beta/2 \text{ for all } v\},$$

where $\mathcal{C}_t$ denotes the collection of color classes when t vertices remain uncolored. From the above lemma we get $t_0 \leq \gamma$. It means that at some point where the number of uncolored vertices is less than γ, every vertex still has at least $\beta/2$ available colors. In particular, if $\beta/2 > \gamma$ the Indicator will win the game since no vertex will ever run out of colors. It can be easily calculate that condition $\beta/2 > \gamma$ holds for n big enough. □

Hence, our result shows that the indicated chromatic number is (up to a multiplicative constant) of the same asymptotic order as their chromatic number.

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