ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Andrzej Grzesik

Uniwersytet Jagielloński

On the maximum number of C_5's in a triangle-free graph

Praca semestralna nr 2

(semestr zimowy 2010/11)

Opiekun pracy: Tomasz Łuczak

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

ON THE MAXIMUM NUMBER OF C_5'S IN A TRIANGLE-FREE GRAPH

ANDRZEJ GRZESIK

ABSTRACT. We show that a triangle-free graph on n vertices contains at most $\left(\frac{n}{5}\right)^5$ cycles of length five. It settles in the affirmative a conjecture of Erdős.

In [2] Erdős conjectured that the number of cycles of length 5 in a triangle-free graph of order n is at most $\left(\frac{n}{5}\right)^5$ and the maximum is achieved by a graph which is a blow-up of C_5. Győri [4] showed that a triangle-free graph of order n contains no more than $c\left(\frac{n+1}{5}\right)^5$ copies of C_5, where $c = \frac{16875}{16384} < 1.03$. Recently this bound has been improved further by Füredi. In this note we settle Erdős conjecture in the affirmative using flag algebras.

Let us first introduce basic notation, and recall some facts and results we shall use later on. An r-graph G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is a family of r-element subsets of $V(G)$ called edges. For a (large) r-graph G on n vertices and a (small) r-graph A on k vertices let $C_A(G)$ be the set of all k-element subsets of $V(G)$ which span r-graph A as an induced subgraph of G. The A-*density* of a r-graph G is defined as

$$d_A(G) = \frac{|C_A(G)|}{\binom{n}{k}}.$$

Thus, if A is just a single edge, $d_A(G)$ becomes the standard edge density.

For a family $\mathcal{F}$ of forbidden r-graphs we define the *Turán number* of $\mathcal{F}$ as

$$\mathrm{ex}_A(n, \mathcal{F}) = \max\{|C_A(G)| : G \text{ is } \mathcal{F}\text{-free}, |V(G)| = n\}$$

and by the *Turán density* $\pi_A(\mathcal{F})$ of $\mathcal{F}$ we mean

$$\pi_A(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{k}}.$$

It is easy to show, using a 'blow-up' argument similar to the one we use in the proof of Theorem 3 below, that the limit above exists.

Let us now sketch the main idea behind the flag algebras approach introduced by Razborov [6] (see also Baber and Talbot [1]).

Let us fix some r-graph A on k vertices and let $\mathcal{F}$ be a family of r-graphs whose Turán density we wish to compute or bound from above.

To this end we consider the family $\mathcal{H}$ of all $\mathcal{F}$-free r-graphs on l vertices, up to isomorphism. Clearly, if l is small we can list all elements $\mathcal{H}$ either by hand or by computer search.

For $H \in \mathcal{H}$ and a large $\mathcal{F}$-free r-graph G, let $p(H;G)$ denote the probability that a random l-element set from $V(G)$ induces a subgraph isomorphic to H. Thus, $d_A(G) = p(A;G)$. By averaging over all l-element subsets of $V(G)$, we can express the A-density of G as

$$d_A(G) = \sum_{H \in \mathcal{H}} d_A(H)p(H;G), \tag{1}$$

and hence

$$d_A(G) \leq \max_{H \in \mathcal{H}} d_A(H). \tag{2}$$

The above bound on $d_A(G)$ is, in general, rather poor. Using the flag-algebras approach one can sometimes improve it significantly.

Thus, define a *type* $\sigma = (G_\sigma, \theta)$ as an r-graph G_σ together with a bijective map $\theta : [|\sigma|] \longrightarrow V(G_\sigma)$. By the order $|\sigma|$ of σ we mean $|V(G_\sigma)|$. Given a type σ we define a σ-*flag* $F = (G_F, \theta)$ as a r-graph with an injective map θ that induces type σ. A flag $F = (G_F, \theta)$ is called *admissible* if G_F is $\mathcal{F}$-free.

In other words, for a given family $\mathcal{F}$ and a type σ (i.e. an r-graph with all vertices labelled by $[|\sigma|] = \{1, \ldots, |\sigma|\}$) an admissible σ-flag of order m is a $\mathcal{F}$-free r-graph on m vertices, which has $|\sigma|$ labelled vertices which induces σ.

Let us fix a type σ and an integer $m \leq (l + |\sigma|)/2$. This bound on m ensures that an r-graph on l vertices can contain two subgraphs on m vertices overlapping in exactly $|\sigma|$ vertices. Let $\mathcal{F}_m^\sigma$ be the set of all admissible σ-flags of order m, up to isomorphism. Furthermore, by Θ_H we denote the set of all injections from $[|\sigma|]$ to $V(H)$. Finally, for a given $F_a, F_b \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta_H$ let $p(F_a, F_b, \theta_H; H)$ be the probability that if we choose a random m-elements set $V_a \subseteq V(H)$ with $\mathrm{im}(\theta) \subset V_a$ and then select a random m-element set $V_b \subseteq V(H)$ such that $\mathrm{im}(\theta) = V_a \cap V_b$, then induced σ-flags are isomorphic to F_a and F_b respectively.

Consider a positive semidefinite square matrix $Q = (q_{ab})$ of dimension $|\mathcal{F}_m^\sigma|$ and put

$$c_H(\sigma, m, Q) = \sum_{F_a, F_b \in \mathcal{F}_m^\sigma} q_{ab} \mathbb{E}_{\theta \in \Theta_H}[p(F_a, F_b, \theta; H)].$$

The following fact (see [1] or [6]) is crucial for the flag-algebras approach.

Lemma 1. *For t types σ_i , $m_i \leq (l + |\sigma_i|)/2$, positive semidefinite matrices Q_i of dimension $|\mathcal{F}^{\sigma_i}_{m_i}|$, and $H \in \mathcal{H}$ let*

$$c_H = \sum_{i=1}^{t} c_H(\sigma_i, m_i, Q_i).$$

Then

$$\sum_{H \in \mathcal{H}} c_H p(H; G) + o(1) \geq 0.$$

If we combine the above lemma with (1) we get

$$d_A(G) \leq \sum_{H \in \mathcal{H}} (d_A(H) + c_H) p(H; G) + o(1).$$

Hence

$$d_A(G) \leq \max_{H \in \mathcal{H}} (d_A(H) + c_H) + o(1)$$

and consequently

$$\pi_A(\mathcal{F}) \leq \max_{H \in \mathcal{H}} (d_A(H) + c_H). \tag{3}$$

Since c_H may be negative, for an appropriate choices of the σ_i, m_i, and Q_i, this bound may be significantly better than the bound given by (2).

Note that now one can bound the Turán density by solving the following semidefinite programming problem: given σ_i and m_i we are to find positive semidefinite matrices Q_i which minimizes the bound for $\pi_A(\mathcal{F})$ given by (3).

The main result of this note is given by the following theorem.

Theorem 2. $\pi_{C_5}(K_3) \leq \frac{24}{625}$.

Proof. Consider $l = 5$, $m = 4$ and three types on 3 vertices – σ_0 stands for a graph with no edges, the type σ_1 has one edge and σ_2 has two. In each case we consider $m = 4$. It can be easily checked that there are 14 triangle-free graphs on 5 vertices and 8 admissible σ_0-flags (which corresponding variables form matrix P), 6 admissible σ_1-flags (we denote the corresponding matrix by Q) and 5 admissible σ_2-flags (matrix R). Computing all appearances of each pairs of flags in each graph, we infer that

$$
\begin{aligned}
\pi_{C_5} \quad \leq \quad & \frac{1}{120}\max\{120p_{11},\ 12p_{11}+24p_{12}+24p_{13}+24p_{15}+12q_{11},\\
& 8p_{12}+8p_{13}+8p_{14}+8p_{15}+8p_{16}+8p_{17}+4p_{22}+4p_{23}+4p_{55}+\\
& \quad +8q_{12}+8q_{13}+4r_{11},\\
& 12p_{14}+12p_{16}+12p_{17}+12p_{18}+6q_{22}+6q_{33}+12r_{13},\\
& 48p_{18}+24r_{33},\\
& 16p_{23}+16p_{25}+16p_{35}+8q_{11}+16q_{14},\\
& 8p_{27}+8p_{36}+8p_{45}+8q14+8q_{24}+8q_{34}+4q_{44}+4r_{11},\\
& 4p_{23}+4p_{24}+4p_{25}+4p_{26}+4p_{34}+4p_{35}+4p_{37}+4p_{56}+4p_{57}+\\
& \quad +4q_{12}+4q_{13}+4q_{15}+4q_{16}+4q_{23}+4r_{12}+4r_{14},\\
& 4p_{27}+4p_{28}+4p_{36}+4p_{38}+4p_{45}+4p_{58}+4q_{15}+4q_{16}+4q_{25}+\\
& \quad +4q_{36}+4r_{13}+2r_{22}+4r_{23}+4r_{34}+2r_{44},\\
& 8p_{44}+8p_{66}+8p_{77}+16q_{23}+16r_{15},\\
& 4p_{48}+4p_{68}+4p_{78}+4q_{26}+4q_{35}+2q_{55}+2q_{66}+4r_{15}+4r_{23}+\\
& \quad +4r_{25}+4r_{34}+4r_{35}+4r_{45},\\
& 12p_{88}+24r_{35}+12r_{55},\\
& 4p_{46}+4p_{47}+4p_{67}+4q_{24}+4q_{26}+4q_{34}+4q_{35}+4q_{45}+4q_{46}+\\
& \quad +4r_{12}+4r_{14}+4r_{24},\\
& 20q_{56}+20r_{24}+120\},
\end{aligned}
$$

where the maximum is taken over all possible coefficients which p_{ij}, q_{ij}, r_{ij} such that all the respective matrices P, Q, and R are positive semidefinite.

As P, Q and R we take the following positive semidefinite matrices

$$
\frac{1}{625}\begin{pmatrix}
24 & -36 & -36 & 24 & -36 & 24 & 24 & -36\\
-36 & 277 & 97 & -79 & 97 & -79 & -259 & 54\\
-36 & 97 & 277 & -79 & 97 & -259 & -79 & 54\\
24 & -79 & -79 & 247 & -259 & 67 & 67 & -36\\
-36 & 97 & 97 & -259 & 277 & -79 & -79 & 54\\
24 & -79 & -259 & 67 & -79 & 247 & 67 & -36\\
24 & -259 & -79 & 67 & -79 & 67 & 247 & -36\\
-36 & 54 & 54 & -36 & 54 & -36 & -36 & 54
\end{pmatrix},
$$

$$
\frac{1}{2500}\begin{pmatrix}
1728 & -1551 & -1551 & -1308 & 687 & 687\\
-1551 & 2336 & 742 & 908 & 2557 & -4084\\
-1551 & 742 & 2336 & 908 & -4084 & 2557\\
-1308 & 908 & 908 & 1728 & -254 & -254\\
687 & 2557 & -4084 & -254 & 15264 & -14424\\
687 & -4084 & 2557 & -254 & -14424 & 15264
\end{pmatrix},
$$

$$\frac{1}{625}\begin{pmatrix} 1512 & 568 & -380 & 568 & -376 \\ 568 & 475 & -191 & 0 & -93 \\ -380 & -191 & 192 & -191 & -2 \\ 568 & 0 & -191 & 475 & -93 \\ -376 & -93 & -2 & -93 & 190 \end{pmatrix}.$$

Then, for an upper bound for $\pi_{C_5}(K_3)$ we get

$$\begin{aligned} \pi_{C_5}(K_3) \quad \leq \quad & \max\left\{\frac{24}{625}, \frac{24}{625}, \frac{24}{625}, \frac{24}{625}, \frac{24}{625}, \frac{24}{625}, \frac{322}{9375}, \frac{2355}{62500}, \frac{24}{625}, \frac{24}{625},\right. \\ & \left.\frac{24}{625}, \frac{24}{625}, -\frac{126}{6250}, \frac{24}{625}\right\} = \frac{24}{625}. \end{aligned}$$

□

Erdős' conjecture is a straightforward consequence of the above result.

Theorem 3. *The number of cycles of length 5 in a triangle-free graph of order n is at most $\left(\frac{n}{5}\right)^5$.*

Proof. Suppose that there is a triangle-free graph G on n vertices which has at least $\left(\frac{n}{5}\right)^5 + \epsilon$ cycles C_5, where $\epsilon > 0$. Then, we can construct triangle-free graphs G_{nN} consisting of n sets of N independent vertices and all edges between vertices in different sets according to edges in graph G.

Such graph G_{nN} has nN vertices and at least $\left(\left(\frac{n}{5}\right)^5 + \epsilon\right) N^5$ cycles C_5. Thus, the Turán density is at least

$$\pi_{C_5}(K_3) \geq \lim_{N\to\infty} \frac{\left(\frac{nN}{5}\right)^5 + \epsilon N^5}{\binom{nN}{5}} = \frac{24}{625} + \frac{120\epsilon}{n^5} > \frac{24}{625},$$

which contradicts Theorem 2. □

Acknowledgement

After finishing preparation of this manuscript I have learned that a similar result has been independently obtained by Hatami, Hladký, Král, Norine and Razborov ([5]). They have shown even stronger result – that the set of extremal graphs for this problem consists precisely of almost balanced blow-ups of a single C_5.

I would like also to thank Tomasz Łuczak for introducing me to the topic of flag algebras, suggesting using it to solve Erdős' pentagon conjecture and valuable comments and remarks.

References

[1] R. Baber, J. Talbot, *Hypergraphs do jump*, Combinatorics, Probability and Computing, Available on CJO 13 Jul 2010 doi:10.1017/S0963548310000222.

[2] P. Erdős, *On some problems in graph theory, combinatorial analysis and combinatorial number theory*, Graph Theory and Combinatorics, Proc. Conf. Hon. P. Erdös, Cambridge 1983, 1-17 (1984).

[3] Z. Füredi, *private communication.*

[4] E. Győri, *On the number of C_5's in a triangle-free graph*, Combinatorica **9**(1) (1989) 101-102.

[5] H. Hatami, J. Hladký, D. Král, S. Norine, A. Razborov, *On the Number of Pentagons in Triangle-Free Graphs*, arXiv:1102.1634v1, 2011.

[6] A. Razborov, *Flag Algebras*, Journal of Symbolic Logic, Vol. 72, No 4, 2007, pages 1239-1282.

Andrzej Grzesik, Theoretical Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków

E-mail address: Andrzej.Grzesik@uj.edu.pl