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Arkadiusz Męcel

Uniwersytet Warszawski

Algebras with finitely many orbits with respect to the unit group

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University of Warsaw
Institute of Mathematics
Banacha 2, 02-097 Warsaw

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Algebras with finitely many orbits with respect to the unit group

Author: Arkadiusz Męcel
<am234204@students.mimuw.edu.pl>

Supervisor: prof. Jan Okniński
<okninski.mimuw.edu.pl>

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ALGEBRAS WITH FINITELY MANY ORBITS WITH RESPECT TO THE UNIT GROUP

ARKADIUSZ MĘCEL

ABSTRACT. We investigate unitary algebras over a field $\mathbb{K}$ with the following property: if B is the subgroup of the unit group $U(A)$ of the algebra A, then A has finitely many orbits with respect to the natural left group action of $B \times B$. Two important examples of B are considered: the whole unit group $U(A)$ and, in the finite-dimensional case, the Borel subgroup of $U(A)$. We construct certain orbit semigroup $\mathcal{O}_A$ connected with algebra A and discuss a number of similarieties between the contracted semigroup algebra $\mathbb{K}_0[\mathcal{O}_A]$ and A. We conclude with few questions open for furher studies.

Introduction

Let A be an unitary $\mathbb{K}$-algebra. Throughout this paper the field $\mathbb{K}$ is assumed to be infinite. Let $U(A)$ be the unit group of A. For any subgroup B of $U(A)$ one can consider the following group action $(B \times B) \times A \to A$ on A:

$$(b_1, b_2)a = b_1 a b_2^{-1}, \quad \text{for every } b_1, b_2 \in B,\ a \in A. \tag{0.1}$$

The orbits of this action will be called ***B*-orbits** of A. The B-orbit of an element $a \in A$ shall be denoted as BaB. Should it not lead to confusion, we will omit the name of the subgroup B and use the terminology of 'orbits in A'. The aim of this paper is to discuss two following questions:

Question 1 *Can we characterise algebras that have finitely many $U(A)$-orbits?*

Question 2 *If the set of orbits on A is finite, can we equip it with a semigroup action that would carry any important information on A?*

First of these two problems has been studied mostly in the finite-dimensional case. Partial answers were obtained, see [12]. The infinite dimensional case has not been studied much and no general characterisations are known, even in the class of simple algebras. Certain stronger (than (0.1)) $U(A)$-actions on A have been studied by Han in [5], [6] and by Hirano in [7]. However, all results that have been obtained are based on the classical theory of artinian algebras. As for present it is not yet known whether a non-artinian algebra can have finite number of $U(A)$-orbits with respect to the action defined in (0.1).

Question 2 is motivated by the role double-coset decompositions play in representation theory (see [3], chapter 8) as well as in algebraic geometry. Consider an algebraic group G. If B is the Borel subgroup of G, that is, a maximal closed solvable subgroup of G, then within certain important classes of algebraic groups we have the following decomposition, called the Bruhat decomposition of G:

$$G = \dot{\bigcup_{w \in W}} BwB,$$

where W is the so-called 'Weyl group' of G – a finite group generated by a set of involutions.[1] If G is, for instance, equal to the full linear group of invertible $n \times n$ matrices $Gl_n(\mathbb{F}_q)$ over a field $\mathbb{F}_q$ with q elements and if B is the subgroup of G consisting of all invertible upper-diagonal matrices, then W is isomorphic to the full permutation group S_n. Under certain conditions, it is possible to equip the set of B-orbits in G with the semigroup action that depends only on W (see [3], §67A). This construction leads to the notion of Hecke algebra and to classical results on complex representations of finite groups with BN-pair. An example of such result is the group algebra isomorphism $e\mathbb{C}[Gl_n(\mathcal{F}_q)]e \simeq \mathbb{C}[S_n]$, where $e = |B|^{-1} \sum_{b \in B} b \in \mathbb{C}[B]$ (see [3], 68.21). Similar results were obtained in the area of algebraic semigroups, see [15], [16].

This paper is divided into three sections. The first two are based on [12]. In the first part we consider unitary rings A with finitely many $U(A)$-orbits. We give examples and prove few facts about certain classes of such algebras. In the second part we consider the orbits on finite-dimensional algebras. Within this class the language of orbits can be translated into the language of one-sided ideals. It is proved that any algebra of finite representation type has finitely many orbits. In the third part we consider finite-dimensional algebras with finitely many two sided ideals. The B-orbits on A, where B is a Borel subgroup of the algebraic

[1] See [7], §1.7 or [8], chapter 28.

group $U(A)$, are considered. Prof. Okniński proposed a construction of a certain finite B-orbit semigroup $\mathcal{O}_A$ which contains important data on the algebra A. We present this construction along with some motivating results. In the end we state some open problems which we hope to work on in the future.

1 Algebras with finitely many $U(A)$-orbits

Let A be associative $\mathbb{K}$-algebra with 1. We will focus on such A that satisfy the following condition:

$$|\{U(A)aU(A)\,|\,a \in A\}| < \infty. \tag{1.1}$$

Obviously, (1.1) is satisfied by any division algebra D. Indeed, we have two $U(D)$-orbits on D: $\{0\}$ and $D \setminus \{0\} = U(D)$. We shall consider such case as a trivial one and restrict our attention to such algebras that have at least three orbits (or equivalently: a nonempty set of nonzero noninvertible elements).

Proposition 1.1 *Assume that A satisfies (1.1). Then the lattice $I(A)$ of two sided ideals of A is finite. If the base field $\mathbb{K}$ is infinite, then $I(A)$ consists only of principal ideals. In the class of finite-dimensional algebras over an infinite field, the finiteness of $I(A)$ is equivalent to the fact, that $I(A)$ is distributive.*

Proof. Observe that if $U \cap I \neq \varnothing$, for a nonzero $U(A)$-orbit U of A and for any ideal $I \triangleleft A$, then $I \supseteq U$. Every ideal of A is, therefore, a union of some $U(A)$-orbits. Since there are finitely many $U(A)$-orbits on A, there are also finitely many distinct unions of them, which implies the finiteness of $I(A)$.

Assume that the base field $\mathbb{K}$ is infinite. If, for some $a, b \in A$, the ideal $AaA + AbA$ was not principal, we would obtain an infinite family of two sided ideals of A: $\{A(a + \alpha b)A\,|\,\alpha \in \mathbb{K}\}$, which contradicts the fact that $I(A)$ is finite.

To prove the last assertion observe that any $\mathbb{K}$-algebra A can be treated as a right $A^0 \otimes_{\mathbb{K}} A$-module (where A^0 is the opposite algebra of A). Assume that A is finite-dimensional and the field $\mathbb{K}$ is infinite. From [13], §2.6, Ex.3, we can see that if $I(A)$ is distributive then it is also finite.[2] It is also known, that if $I(A)$ is not distributive, then one can give an infinite set of distinct two-sided ideals of A (see [13], §2.2, Ex. 4). The assertion follows. ■

Let $M_n(\mathbb{K})$ be the algebra of $n \times n$ matrices over field $\mathbb{K}$. For $m \in M_n(\mathbb{K})$ assume that $M_n(\mathbb{K})mM_n(\mathbb{K}) = \{xmy\,|\,x, y \in M_n(\mathbb{K})\}$ denotes the ideal of the multiplicative monoid $M_n(\mathbb{K})$, generated by m.

[2]The lattice of submodules of $A_{A^0 \otimes_{\mathbb{K}} A}$ is isomorphic to $I(A)$.

Proposition 1.2 *The following conditions are equivalent for* $a, b \in M_n(\mathbb{K})$*:*

(1) $Gl_n(\mathbb{K}) a Gl_n(\mathbb{K}) = Gl_n(\mathbb{K}) b Gl_n(\mathbb{K})$*,*

(2) $M_n(\mathbb{K}) a M_n(\mathbb{K}) = M_n(\mathbb{K}) b M_n(\mathbb{K})$*,*

(3) $rank(a) = rank(b)$*.*

PROOF. (1) ⇒ (2) is obvious. Suppose now that (2) is satisfied. Then there exist matrices $p, q, r, s \in M_n(\mathbb{K})$ such that $a = pbq, b = ras$. Therefore

$$rank(a) = rank(pbq) \leq rank(b).$$

The opposite inequality is proved in a similar way. Hence (3) follows. If $rank(a) = rank(b) = j$ then by means of Gauss elimination one can reduce matrices a,b to the same diagonal idempotent $e_j = \left(\begin{smallmatrix} Id_j & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_n(\mathbb{K})$. In other words, there exist invertible matrices u, v, x, y that $uav = xby = e_j$ which implies (1). ∎

Example 1.3 *Let* $\mathbb{K}$ *by any field. Then the matrix algebra* $M_n(\mathbb{K})$ *has exactly* $n+1$ *orbits.*

Example 1.3 shows that there is no bound on the number of $U(A)$-orbits on a simple algebra A. In a similar way we can easily prove, that for every division algebra D, the matrix ring $M_n(D)$ has $n+1$ $Gl_n(D)$-orbits. From Artin-Wedderburn theorem we get the following corollary.

Corollary 1.4 *Every semisimple left artinian* $\mathbb{K}$*-algebra* A *satisfies* (1.1).

Numerous well known algebras have infinitely many orbits, the polynomial ring $\mathbb{K}[x]$ being one of them. In fact, one can give the following, finite-dimensional, example.

Example 1.5 *Let* $\mathbb{K}$ *be any infinite field. The following matrix subalgebra* $A \subseteq M_4(\mathbb{K})$ *has infinitely many* $U(A)$*-orbits.*

$$A = \begin{pmatrix} \mathbb{K} & 0 & 0 & 0 \\ 0 & \mathbb{K} & 0 & 0 \\ \mathbb{K} & \mathbb{K} & \mathbb{K} & 0 \\ \mathbb{K} & \mathbb{K} & 0 & \mathbb{K} \end{pmatrix}$$

Proof. For any $\lambda \in \mathbb{K}$ consider the following matrices:

$$a_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & \lambda & 0 & 0 \\ \lambda^{-1} & \lambda & 0 & 0 \end{pmatrix}$$

One can easily see that the Jacobson radical $J(A)$ of A is equal to the ideal $\left(\begin{smallmatrix} 0 & 0 \\ M_2(\mathbb{K}) & 0 \end{smallmatrix}\right) \lhd A$. Let $D_4 \subseteq M_4(\mathbb{K})$ be the set of all 4×4 invertible diagonal matrices. Then $U(A/J(A)) \simeq D_4$, hence $U(A) = D_4 + J(A)$. Since $J(A)^2 = 0$, we have $U(A)a_\lambda U(A) = D_4 a_\lambda D_4$. This means that for any $\lambda, \mu \in \mathbb{K} \setminus \{0\}$ matrices a_λ and a_μ are in the same $U(A)$-orbit if and only if there exist $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{K} \setminus \{0\}$ such that

$$a_\mu = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \cdot a_\lambda \cdot \begin{pmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_3 b_1 \lambda & a_3 b_2 \lambda & 0 & 0 \\ a_4 b_1 \lambda^{-1} & a_4 b_2 \lambda & 0 & 0 \end{pmatrix}.$$

This is equivalent to the existence of a solution to the following set of equations:

$$\begin{cases} a_3 b_1 \lambda = \mu \\ a_3 b_2 \lambda = \mu \\ a_4 b_1 \lambda^{-1} = \mu^{-1} \\ a_4 b_2 \lambda = \mu \end{cases}$$

We can easily see that $a_3 = a_4$ and $b_1 = b_2$. Hence $\lambda^2 = \mu^2$ which means that the number of $U(A)$-orbits is infinite. ■

It is to be noted that in Example 1.4 the set $\{a_\lambda \mid \lambda \in \mathbb{K}\}$ of representatives of different $U(A)$-orbits belongs to $J(A)$. In other words, an infinite set of $U(A)$-orbits could be found within the Jacobson radical of A. If $\mathbb{K}$ is an algebraically closed field then it is a special case of the following fact.

Theorem 1.6 ([12], Prop. 9) *The following conditions are equivalent for a finite-dimensional algebra A over an algebraically closed field $\mathbb{K}$.*

(1) there are finitely many $U(A)$-orbits on A,

(2) there are finitely many $U(A)$-orbits on $J(A)$.

Schetch of the proof. The whole proof is given in [12]. We would like to point out one interesting detail. Clearly (2) is a consequence of (1). To prove the reverse implication one can first consider the following additional statement:

(2') for any $e = e^2 \in A$ there are finitely many $U(eAe)$-orbits on $J(eAe)$.

We claim that $(2) \Rightarrow (2')$. Let $e = e^2 \in A$. Suppose that $x, y \in eAe$ are in the same $U(A)$-orbit, that is, there exist $u, v \in U(A)$ such that $y = uxv$. Then $y = (eue)x(eve)$. Similarly one shows that $x = rys$ for some $r, s \in eAe$. Therefore x and y generate the same ideal of the multiplicative monoid eAe. We claim that they have to belong to the same $U(eAe)$ orbit.

Lemma 1.7 *Let A be a finite-dimensional $\mathbb{K}$-algebra, where $\mathbb{K}$ is algebraically closed. Consider two elements x, y of A such that the ideals of the multiplicative monoid A, generated by these elements are equal. Then the $U(A)$-orbits of x, y are also equal.*

This statement is quite similar to the implication $(1) \Rightarrow (2)$ of Proposition 1.2. The field $\mathbb{K}$, however, is now assumed to be algebraically closed. Is it a necessary condition? We do not know. The only known proof of Lemma 1.7 comes from M. Putcha ([14], Prop. 6.1) and is entirely based on algebraic geometry and the fact that every $\mathbb{K}$-algebra can be viewed as a connected algebraic variety. No 'elementwise' proof is known.

We come back to the proof of implication $(2) \Rightarrow (2')$. Since $J(eAe) \subseteq J(A)$ we can see that Lemma 1.9 implies $(2) \Rightarrow (2')$. The rest of the proof, given in [12], is a standard induction on the cardinality n_A of a complete set of orthogonal primitive idempotents of A. ∎

It is natural to ask how complicated an algebra with finitely many $U(A)$-orbits can be? We do not know of any (right, left) nonartinian examples. The problem with considering the general case of Question 1 comes from lack of useful conditions equivalent to (1.1). There is, however, a large class of rings for which such a natural condition exists. The importance of this condition will be seen in the next section.

Definition 1.8 (Left (right) perfect ring) *We say that the ring R is left (right) perfect if every left (right) R-module has a projective cover.*

It is known that there exist right perfect rings which are not left perfect, so the notion of a perfect ring is not symmetric (see [2], Ex. (28.2)).

Lemma 1.9 ([12], Lemma 1) *Let R be a left perfect ring with identity. Let $x, y \in R$. Assume that $Rx = Ry$ is an equality of two left principal ideals of R. Then there exists $g \in U(R)$ such that $x = gy$.*

Proof. The proof of Lemma 1.9 can be found in [12]. It is based on the existence of the projecive covers for left (right) principal R-modules generated by x, y. ■

Corollary 1.10 *Let A be left (right) perfect algebra with unity. Let $x, y \in A$. The following conditions are equivalent.*

(1) $U(A)xU(A) = U(A)yU(A)$

(2) left ideals Ax, Ay (right ideals xA, yA) are conjugated.

There is one-to-one correspondence between $U(A)$-orbits and conjugation classes of left (right) principal ideals on A.

We can see that for a left perfect algebra A, the condition (1.1) is equivalent to the finiteness of the set of conjugacy classes of left principal ideals. It is noted in [12] that if a unitary algebra A has finitely many conjugacy classes of arbitrary left ideals, then it has to be artinian. It motivates the following result.

Theorem 1.11 *Suppose that the unitary $\mathbb{K}$-algebra A satisfies the following conditions:*

(1) A is right noetherian,

(2) A has finitely many $U(A)$-orbits.

Then A is right artinian.

Proof. It is sufficient to prove that the algebra A is left perfect. Then the assertion follows from [2], Ex. 28.9.

The following theorem of Bass gives us a very usesful characterisation of left perfect rings.

Theorem 1.12 (Bass, [2], 28.4) *The following conditions are equivalent for a ring R:*

(1) R is left (right) perfect,

(2) R has d.c.c for right (left) principal ideals.

Suppose that A is not left perfect. In view of the theorem of Bass there exists an infinite strictly decreasing chain on right principal ideals in A:

$$a_1A \supsetneq a_2A \supsetneq a_3A \supsetneq \dots \tag{1.2}$$

Let $U = U(A)$. We may assume that $a_i \notin U$. Since A has finitely many U-orbits, there exist such $i, j \in \mathbb{N}$ that $Ua_iU = Ua_jU$, where $i > j$. Let $v, w \in U$ be such that $a_i = va_jw$. Then $a_iA = va_jA$. This implies that $va_jA \subsetneq a_jA \Rightarrow a_jA \subsetneq v^{-1}a_jA$. This contradicts our assumptions on A, since:

$$a_jA \subsetneq v^{-1}a_jA \subsetneq v^{-2}a_jA \subseteq \dots$$

is a strictly increasing chain of principal right ideals of A. Hence A is left perfect and the assertion follows. ∎

Theorem 1.11 yields an example of a simple $\mathbb{K}$-algebra that does not have finitely many $U(A)$-orbits. For example, the first Weyl algebra

$$A_1 = \mathbb{K}\langle x, y\rangle/(xy - yx - 1).$$

It is known that all Weyl algebras A_n are simple noetherian algebras which are not artinian (see [10], (3.17)).

In the beginning of this section it was noted that there is no general characterisation of unitary algebras satisfying (1.1), even within the class of simple algebras. If there is any example of a simple unitary algebra with finitely many $U(A)$-orbits, other than the matrix ring over division algebra, it must be a nonnoetherian one.

2 The conjugacy classes of left ideals

In this short section we assume A is a finite-dimensional $\mathbb{K}$-algebra with unity, where $\mathbb{K}$ is an infinite field. Such an algebra is obviosly left perfect and, as a consequence of Lemma 1.9, A has finitely many $U(A)$-orbits if and only if A has a finite number of conjugacy classes of left principal ideals. This criterion will prove very useful for the class of finite-dimensional algebras. Following [12] we will show that within this class, algebras satysfying (1.1) lie between two important classes: algebras of finite representation type, and algebras with distributive lattice of two sided ideals.

Let A be an algebra with identity over a field $\mathbb{K}$. By $S(A)$ we denote the set of all subspaces of A equipped with the operation $X \star Y = Span_{\mathbb{K}}(XY)$. It is called the subspace semigroup of A, [11]. Clearly, the set $L(A)$ of left ideals of A is a subsemigroup of $S(A)$. The set of equivalence classes of the relation defined on $S(A)$ by: $X \sim Y$ if $Xg = Y$, for some $g \in U(A)$ will be denoted by $S(A)/U(A)$. We write $[X]$ for the class of X. Recall that A is of finite representation type if there are finitely many isomorphism classes of finitely generated indecomposable left A-modules.

Theorem 2.1 ([12], Corollary 4) *Assume that A is a finite-dimensional algebra with identity over a field $\mathbb{K}$. If A is of finite representation type then $L(A)/U(A)$ equipped with the product $[I][J] = [IJ]$ is a finite semigroup.*

Proof. We begin with the following lemma.

Lemma 2.2 ([12], Corollary 3) *Let A be a finite-dimensional $\mathbb{K}$-algebra with identity. Assume that I, J are left ideals of A. The left A-modules $A/I, A/J$ are isomorphic if and only if $J = Ig$ for some $g \in U(A)$.*

Proof. Clearly, if $J = Ig$ for some $g \in U(A)$, then $a + J \to (a + J)g^{-1}$ is an isomorphism between left A-modules A/J and $A/(Jg^{-1}) = A/I$. Assume that $\psi : A/I \to A/J$ is an isomorphism of left A-modules. Let $\phi(1 + I) = x + J$. Then $A/J = \phi(A/I) = \{ax + J \,|\, a \in A\}$. Hence $Ax + J = A$. We claim that there exists $g \in U(A)$ such that $g + J = x + J$. In other words

$$(x + J) \cap U(A) \neq \varnothing. \tag{2.1}$$

If (2.1) holds, then:

$$I = ann_A(1 + I) = ann_A(x + J) = ann_A(g + J) = \{a \in A \,|\, ag \in J\} = Jg^{-1}$$

and the assertion follows.

Suppose that the intersection of (2.1) is empty. Then also $(x+J+J(A))\cap U(A)$ is empty and A can be assumed semisimple. If so, then $J = Ae$, for some idempotent $e \in A$. Therefore $Ax+J = Ax+Ae = A$ implies $Ax(1-e) = A(1-e)$. By Lemma 1.9 there exists $g \in U(A)$ such that $g(1-e) = x(1-e)$. Thus $g = x-xe+ge \in x+J$. The intersection (2.1) is indeed nonempty and the assertion follows. ■

Now we can prove Theorem 2.1. If I, J are left ideals of A then $IgJh = IJh$ for all $g, h \in U(A)$. Therefore the semigroup operation on $\mathcal{C}_A = L(A)/U(A)$ is well defined and associative. $\mathcal{C}_A$ is a finite semigroup. Indeed, every left A-module of the form A/I, where $I \in L(A)$, has dimension bounded by the dimension of A. By the hypothesis on A, there are finitely isomorphism classes of such modules. Hence, the assertion follows from Lemma 1.9. ■

A finite semigroup from Theorem 2.1 will be denoted as $\mathcal{C}_A$. We can see that $\mathcal{C}_A$ is the set of classes of left ideals of A under conjugation by elements of $U(A)$. We recall that the $U(A)$-orbits on A are the classes of left principal ideals of A under conjugation by elements of $U(A)$. Therefore we get the following corollary:

Corollary 2.3 ([12], Theorem 6) *Consider the following conditions for a finite-dimensional algebra A over an infinite field $\mathbb{K}$:*

(1) A is of finite representational type,

(2) $\mathcal{C}_A$ is finite,

(3) A has finitely many $U(A)$-orbits,

(4) the lattice of two sided ideals of A is distributive.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

It is to be noted that none of the implications can be reversed. In view of the description of algebras A of finite-representational type satisfying $J(A)^2 = 0$ ([13], Theorem 11.8), the appropriate examples are given by the Corollary 11 and Theorem 12 of [12].

Example 2.4 *Let $A = M_n(\mathbb{K})$. Then the semigroup $\mathcal{C}_A$ is isomorphic to the subsemigroup E of $M_n(\mathbb{K})$, consisting of the following idempotent matrices:*

$$0, \quad E_{11}, \quad E_{11} + E_{22}, \quad E_{11} + E_{22} + E_{33}, \quad \ldots \quad E_{11} + \ldots + E_{nn},$$

where E_{ij} are matrix units.

Proof. It is well known that every nonzero left ideal of $M_n(\mathbb{K})$ is conjugate to one of the ideals

$$M_n(\mathbb{K})(E_{11} + E_{22} + \ldots E_{ii}), \quad 1 \le i \le n.$$

One can easily check that the multiplication in E corresponds to the multiplication in $\mathcal{C}_A$. ∎

Most results concerning the semigroup $\mathcal{C}_A$ and its applications to the theory of finite-dimensional algebras are based only on the finiteness of $\mathcal{C}_A$. No applications involving the semigroup structure of $\mathcal{C}_A$ are known. The following theorem can be found in [12].

Theorem 2.5 ([12], Theorem 7) *A finite-dimensional algebra A oven an infinite field is of finite representation type if and only if for every $n \ge 1$ the semigroup $\mathcal{C}_{M_n(A)}$ is finite.*

3 The orbits with respect to the Borel subgroup

In this section we assume that A is a finite-dimensional unitary algebra over an algebraically closed field $\mathbb{K}$ with a distributive lattice of two sided ideas $I(A)$. By Proposition 1.1 algebra A has finitely many two-sided ideals.

By $J(A)$ we denote the Jacobson radical of A. We may assume that $A \subseteq M_n(\mathbb{K})$, for some $n \ge 1$ (for example using the regular representation). It is known, by the theorem of Malcev-Wedderburn, that A may be decomposed, as a linear space, to the direct sum of linear subspaces $\overline{A} \oplus J(A)$, where $\overline{A}$ is semisimple and isomorphic to

$$A/J(A) = \bigoplus_{i=1}^{s} M_{n_i}(\mathbb{K}).$$

Let $f_1, f_2, \ldots, f_s$ be minimal central idempotents of $\overline{A}$. Then f_i are orthogonal and $f_1 + ... + f_s = 1$. In other words, $\overline{A}$ can be viewed as the subalgebra of $M_n(\mathbb{K})$ of the form

$$\begin{pmatrix} M_{n_1}(\mathbb{K}) & 0 & 0 & \ldots & 0 \\ 0 & M_{n_2}(\mathbb{K}) & 0 & \ldots & 0 \\ 0 & 0 & M_{n_3}(\mathbb{K}) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & M_{n_s}(\mathbb{K}) \end{pmatrix}, \qquad (3.1)$$

where $n = n_1 + n_2 + \ldots + n_s$.

Moreover, the set $E = \{e_1, \ldots, e_n\}$ of rank one diagonal idempotents in $M_n(\mathbb{K})$ forms a complete set of primitive orthogonal idempotents of A.

Let $U(A)$ be the group of units of A. It is well known that $U(A) \simeq U(\overline{A}) + J(A)$. Every element of $U(\overline{A})$ can be viewed as an invertible matrix of the form (3.1). We will say that $B \subseteq U(A)$ is the **Borel subgroup** of $U(A)$ corresponding to the chosen set of primitive orthogonal idempotents $e_1, e_2, \ldots, e_n$ of A if $B = \overline{B} + J(A)$, where $\overline{B} \subseteq \overline{A}$ consists of all invertible matrices of form (3.1) that are upper triangular. One can easily see that B is indeed a maximal closed solvable subgroup of $U(A)$. We will consider B-orbits on A, that is, double cosets BxB, where $x \in A$.

Obviously, if A has finitely many B-orbits then it has also finitely many $U(A)$-orbits. Of course, if there is any example of an algebra A with finitely many $U(A)$-orbits and infinitely many $U(A)$-orbits it cannot be a basic one, since $U(A) = B$ iff A is a basic algebra[3].

We shall introduce certain action $\star$ on the set $\{BxB \,|\, x \in A\}$ of B-orbits on A. Let $X \subseteq A$. By X' we denote the linear subspace $span_{\mathbb{K}} X$ spanned by X. Let $x, y \in A$. We say that the B-orbit BaB, $a \in A$ is equal to the product $BxB \star ByB$

[3]By definition, an algebra A is basic iff $A/J(A)$ is isomorphic to the direct product of division algebras. Hence if A is finite-dimensional over $\mathbb{K}$ and basic, then $A/J(A)$ is isomorphic to the direct product of finitely many copies of $\mathbb{K}$.

if:

$$(BxByB)' = (BaB)'. \tag{3.2}$$

The action $\star$, provided it is well defined, is obviously associative since for any $x, y, z \in A$ we have

$$((BxByB)'(BzB))' = (BxByBzB)' = (BxB(ByBzB)')'.$$

Let $A_{ij} = e_i A e_j$, for some primitive idempotents $e_i, e_j \in E$. Put

$$A_E := \bigcup_{1 \le i,j \le n} A_{ij}.$$

Theorem 3.1 *Let A be a finite dimensional algebra with a distributive lattice of ideals over an algebraically closed field $\mathbb{K}$. The set $\mathcal{O}_A = \{BxB \,|\, x \in A_E\}$ equipped with the $\star$ action is a monoid with unity equal to B.*

PROOF. We begin with two lemmas.

Lemma 3.2 *Let A be a finite dimensional algebra with a distributive lattice $I(A)$ of ideals over an algebraically closed field $\mathbb{K}$. Then:*

(1) A_{ii} is of the form $\mathbb{K}[x]/(x^{t_i})$, for some $t_i \ge 1$,

(2) A_{ij} is a uniserial $A_{ii} - A_{ij}$ bimodule, that is, the lattice of subbimodules of A_{ii} forms a single chain.

PROOF. We begin with the proof of (1). A_{ii} is a local algebra, since it is isomorphic to the endomorphism algebra of the idecomposable module Ae of A. One can also observe that A_{ii} has finitely many two sided ideals. Indeed, if $I \lhd A_{ii}$ then $AIA \lhd A$. Moreover, $AIA \cap A_{ii} = I$. Therefore we get an embedding of the lattice $I(A_{ii})$ of ideals of A_{ii} to the lattice $I(A)$ of ideals of A which is, by our assumption, finite. Thus, by Proposition 1.1, every ideal of A is principal. Hence the ideals of A_{ii} are also principal.

Let the Jacobson radical $J(A_{ii})$ of A_{ii} be generated by $x \in A_{ii}$. Consider the quotient algebra $A_{ii}^{(2)} = A_{ii}/J(A_{ii})^2$. Let $\overline{x}$ be the image of x in $A_{ii}^{(2)}$. The elements of $A_{ii}^{(2)}\overline{x}A_{ii}^{(2)} \lhd A_{ii}^{(2)}$ are annihilated by $J(A_{ii})$. A_{ii} is a local algebra, hence $A_{ii}\overline{x}A_{ii} = \mathbb{K}\overline{x}$. In a similar way one can prove that in the quotient algebra $A_{ii}^{(k)} = A_{ii}/J(A_{ii})^k$ the ideal $J(A_{ii})^{k-1}/J(A_{ii})^k$ is generated by the image $\overline{x}$ of

x mod $J(A_{ii})^k$. Thus it is of the form $\mathbb{K}\overline{x}^{k-1}$. Since A_{ii} is finitely-dimensional, the radical $J(A_{ii})$ is nilpotent. Hence (1) follows.

In a similar way one shows that every A_{ij} is a uniserial $A_{ii} - A_{jj}$ bimodule. Namely, the lattice of subbimodules of A_{ij} embedds into $I(A)$ via the map $M \to AMA$, so it is distributive. In view of (1) this easily implies that $J(A)^k \cap A_{ij}$ has dimension ≤ 1 modulo $J(A)^{k+1}$, for every $k \geq 1$. This can be used to show that for every $a, b \in A_{ij}$ the bimodules $A_{ii}aA_{jj}$ and $A_{ii}bA_{jj}$ are comparable under inclusion. ∎

Lemma 3.3 *Let A be a finite dimensional algebra with a distributive lattice of ideals over an algebraically closed field $\mathbb{K}$. Then:*

(1) every $BxB, 0 \neq x \in A_E$, intersects nontrivially only one e_iAe_j, namely the one with $x \in e_ixe_j$.

(2) if $x, y \in A_E$, then $BxB = ByB$ if and only if $A_{ii}xA_{jj} = A_{ii}yA_{jj}$, where $x = e_ixe_j$.

Proof. We begin with the proof of (1). Let $\overline{x}$ be the image of $x \in A$ modulo $J(A)$. Suppose that for some $y, z \in A$ we have:

$$Be_ize_jB = Be_kye_lB, \qquad \text{where } i \neq k. \tag{3.3}$$

Observe that $e_iBe_k \subseteq \overline{e_i}\overline{B}\overline{e_k} + J(A)$ and $e_kBe_i \subseteq \overline{e_k}\overline{B}\overline{e_i} + J(A)$. The elements of form $\overline{e_i}\overline{B}\overline{e_k}$ are either all zero, or they are of matrix form (3.1), where the only nonzero enties are at the intersection of the i-th row and k-th column. Same for $\overline{e_k}\overline{B}\overline{e_i}$ – they are all 0, or they form the subset of matrices of form (3.1) with nonzero entries at the (k,i) coordinate. Therefore, from the definition of B, either $e_iBe_k \subseteq J(A)$, or $e_kBe_i \subseteq J(A)$. Assume that $e_iBe_k \subseteq J(A)$. Then:

$$e_ize_j \in e_iBe_ize_jB \overset{(3.3)}{=} e_iBe_kye_lB \subseteq J(A)e_kye_lB. \tag{3.4}$$

Observe that the elements: e_ize_j and e_kye_l must, as they belong to the same B-orbit, belong to the same power of $J(A)$. However, (3.4) implies that if $e_kye_l \subseteq J(A)^k$ for some $k \geq 0$, then $e_ize_j \in J(A)^{k+1}$. This would lead to contradiction, hence $i = k$. A similar argument shows that we also have $j = l$ in (3.3).

We have proved (1).

Let $BxB = ByB$. (1) implies that $x, y \in A_{ij}$, for some i, j. Thus $y = e_i y e_j \in e_i B e_i x e_j B e_j \subseteq e_i BxB e_j \subseteq A_{ii} x A_{jj}$. Similarly we get that $x \in A_{ii} y A_{jj}$. One implication of (2) follows. Assume now, that $A_{ii} x A_{jj} = A_{ii} y A_{jj}$. It suffices to show that $x \in ByB$. From Lemma 3.2 we may assume that A_{ii} is of the form $\mathbb{K}[z]/(z^k)$ for some $k \geq 1$ and A_{ij} is a cyclic A_{ii}-module. Let $A_{ij} = A_{ii} w$, where $w \in A_{ij}$. Every A_{ii}-submodule of A_{ij} is of the form $z^r A_{ii} w$, where $r = 0, 1, \dots$ (since A_{ij} is uniserial). Then $A_{ii} x = z^s A_{ii} w = z^s A_{ij}$ for some s. Hence $A_{ii} x = A_{ii} x A_{jj}$. Similarly we get that $A_{ii} y = A_{ii} y A_{jj}$. From the assumption we get that $A_{ii} x = A_{ii} y$. Hence there exists $u \in A_{ii}$ that $x = uy$. Also $y = vx$ for some $v \in A_{jj}$. Hence $x = uvx \Rightarrow uv \in A_{ii}$. Suppose that uv is noninvertible in A_{ii}. Then since A_{ii} is local, uv must be nilpotent, so $x = uvx = (uv)^2 x = \dots$ and $x = 0$, a contradiction. Thus $x = uy$, for some $u \in U(A_{ii})$ and $t = u + \sum_{k \neq i} e_k \in B$. It implies that $x = uy = ty \in ByB$. Hence $A_{ii} x A_{jj} = A_{ii} y A_{jj}$ and (2) follows. ■

We shall move to the proof of Theorem 3.1. Let $x \in A_{ij}, y \in A_{kl}$. Let $a \in e_i A e_l$ be an element such that $A_{ii} a A_{ll}$ is the unique maximal submodule of the $A_{ii} - A_{ll}$ bimodule A_{il} among those generated by elements of xBy. Such a exists since A_{il} is uniserial. We will prove that $BaB = BxB \star ByB$.

The set $BxBByB$ is the sum of some B-orbits, namely $B = \bigcup_m Ba_m B$, for some $a_m \in xBy \subset A_{ij}$. Of course $(BaB)' \subseteq (BxByB)'$. Observe that

$$(BaB)' = (B' a B')' \supseteq A_{ii} a A_{jj}.$$

Since a generates the maximal submodule of the uniserial module A_{ij}, we have $A_{ii} a A_{ii} \supseteq A_{ii} a_m A_{jj}$ for all a_m. Thus $(BaB)' \supset (BxBByB)'$. Hence BaB is indeed the $\star$ product of BxB and ByB in view of (3.2). What is left to prove is that if there is any $b \in xBy$ such that $(BbB)' = (BaB)'$ then $A_{ii} a A_{jj} = A_{ii} b A_{jj}$. Indeed, $e_i (BaB)' e_i = (e_i B e_i)' a (e_j B e_j)' = A_{ii} a A_{jj}$. The same for b, which ends the proof. ■

Example 3.4 *Let $A = M_n(\mathbb{K})$ for some $n \geq 1$.*

Observe that if $\{E_{ij} \,|\, 1 \leq i, j \leq n\}$ is the set of matrix units in $M_n(\mathbb{K})$, then $E = \{E_{ii} \,|\, 1 \leq i \leq n\}$ can be chosen as the full set of primitive orthogonal idempotents in A. We construct the semigroup $\mathcal{O}_A$ with respect to E. We can see that $A_{ij} = \mathbb{K}E_{ij}$ Thus $\mathcal{O}_A$ consists of the orbits of form $BE_{ij}B$ along with the zero element. The formula (3.2) translates into the following one.

$$(BE_{ij}B) \circ (BE_{kl}B) = \begin{cases} BE_{il}B, & \text{for } j \leq k \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, one can observe that $\mathbb{K}_0[\mathcal{O}_A] \simeq M_n(\mathbb{K})$. Indeed, consider the set $T \subset M_n(\mathbb{K})$ of the following form:

$$T = \{f_{ij} = E_{i,j} + E_{i,j+1} + \ldots + E_{i,n} \,|\, i,j = 1,\ldots,n\} \cup \{0\}.$$

One can see that T forms a multiplicative subsemigroup of $M_n(\mathbb{K})$. Its nonzero elements are linearly independent. Since $|T| = n^2 + 1$, then $\mathbb{K}_0[T] \simeq M_n(\mathbb{K})$. One can check that

$$f_{ij}f_{kl} = \begin{cases} f_{il}, & j \leq k \\ 0, & j > k. \end{cases}$$

Proposition 3.5 *Let A be a finite dimensional algebra with a distributive lattice of ideals over an algebraically closed field $\mathbb{K}$. Then $|\mathcal{O}_A| = \dim A$. Moreover $\mathbb{K}_0[\mathcal{O}_A]$ and A are isomorphic modulo their radicals.*

PROOF. The lenght of the uniserial $A_{ii} - A_{jj}$ bimodule A_{ij} is equal to the dimension of A_{ij} (as its subbimodules are the same as submodules of A_{ij} as a cyclic left or right module over the ring $\mathbb{K}[t]/(t^i)$, for some i, see Lemma 3.2). Therefore by Lemma 3.2 we have:

$$\dim A = \sum_{i,j} \dim A_{ij} = \sum_{i,j} |\{BxB \,|x \in A_{ij}\}| = |\mathcal{O}_A|$$

Let $N = \{BxB \,|\, x \in J(A) \cap A_E\}$. It is clear that N is a nil ideal of $\mathcal{O}_A$. Therefore it is nilpotent and we get $\mathbb{K}_0[N] \subseteq J(\mathbb{K}_0[\mathcal{O}_A])$. Consider the homomorphism $\phi : \mathbb{K}_0[\mathcal{O}_A] \to \mathbb{K}_0[\mathcal{O}_{\overline{A}}]$ determined by the natural map $A \to \overline{A}$. Clearly ϕ is surjective. Let $A_1, \ldots, A_r$ be the simple blocks of $\overline{A}$. It is easy to see that $\mathcal{O}_{\overline{A}} \simeq \mathcal{O}_{\overline{A_1}} \times \mathcal{O}_{\overline{A_2}} \times \ldots \times \mathcal{O}_{\overline{A_r}}$. By Example 3.4 we have $\mathbb{K}_0[\mathcal{O}_{\overline{A_i}}] \simeq A_i$. It follows easily that $\mathbb{K}_0[\mathcal{O}_{\overline{A}}] \simeq \overline{A}$. Since $\mathbb{K}_0[N] \simeq \ker(\phi)$ and $\dim \mathbb{K}_0[N] = |N| - 1 \geq \dim J(A)$, comparing the dimensions we get $J(\mathbb{K}_0[\mathcal{O}_A]) = \ker(\phi) = \mathbb{K}_0[N]$ and $\mathbb{K}_0[\mathcal{O}_A]/\mathbb{K}_0[N] \simeq \overline{A}$. ■

We can see that the structure of $\mathbb{K}_0[\mathcal{O}_A]$ resembles the structure of A. In fact, one can prove that if A is finite-dimensional and the lattice of ideals of A is distributive, then the algebras A and $\mathbb{K}_0[\mathcal{O}_A]$ are isomorphic modulo the squares of the radical. The quivers of A and $\mathbb{K}_0[\mathcal{O}_A]$ are, therefore, isomorphic. It follows that essential information on representation theory of the algebra A can be recovered from $\mathcal{O}_A$ (or even from the orbit semigroup of the basic algebra B_A of A, see [13], §6.6, Prop. b.). Nevertheless, as for present we do not have any example of algebra A where $A \ncong \mathbb{K}_0[\mathcal{O}_A]$.

* * *

In the end we would like to point out some problems we hope to investigate in the future. Let A be a finite-dimensional $\mathbb{K}$-algebra, where $\mathbb{K}$ is algebraically closed and the lattice $I(A)$ of two sided ideals of A is distributive. Two objects, connected with algebra A, have been introduced in this paper:

- the semigroup $\mathcal{C}_A$ of conjugacy classes of left principal ideals of A,
- the orbit semigroup $\mathcal{O}_A$.

Very little is known about the semigroup structure of $\mathcal{C}_A$ and $\mathcal{O}_A$. It can, most likely, be connected with the representation theory of A, as Theorem 2.5 and Proposition 3.5 indicate. It is known that the representation theory of finite-dimensional algebras can be reduced to considering the class of basic algebras (see [1], Collorary 6.10). Therefore one would like to investigate connections between $\mathcal{C}_{A_0}$ and $\mathcal{C}_A$ as well as between $\mathcal{O}_{A_0}$ and $\mathcal{O}_A$, where $A_0 \subseteq A$ is a basic subalgebra of A (see [1], 6.3).

The semigroups $\mathcal{C}_A$ and $\mathcal{O}_A$ are well defined due to some specific algebraic properties of A. One can also give an example of the orbit semigroup that is based on certan geometric properties of A. Assume that A has finitely many U-orbits, where $U = U(A)$ is the group of invertible elements in A. Then for all $x, y \in A$ we have:

$$UxU \cdot UyU = \dot{\bigcup_{i=1}^{n}} Ua_iU, \quad \text{where } a_i \in A. \tag{3.5}$$

The Zariski closure of $UxUyU$ is the sum of finitely many Zariski-closed subsets $\overline{Ua_iU}$, where $1 \le i \le n$. Consider the map $\phi : U \times U \times U \to \overline{UxUyU}$ defined by:

$\phi(u_1, u_2, u_3) = u_1 x u_2 y u_3$. It is clearly a dominant morphism of algebraic sets (see [14], p. 13). From Theorem 2.13 of [14] it follows that $\phi(U \times U \times U)$ contains a nonempty open subset of $\overline{UxUyU}$. The latter set is in fact an algebraic variety and thus, by (3.5), there exists an element $a \in A$ such that $\overline{UxUyU} = \overline{UaU}$. We may assume that $a = a_i$ for some $i \in \{1, \ldots, n\}$. Moreover, if $\overline{Ua_iU} = \overline{Ua_jU}$ for some j then the dense subsets Ua_iU, Ua_jU of $\overline{Ua_iU}$ must have a nonempty intersection. Since these are U-orbits, they must be equal. It is easy to see that we obtain a semigroup operation on the set $\mathcal{U}_A$ of all $U(A)$-orbits of A. Its structure and its relation to the algebra A are, as well as in the cases of $\mathcal{C}_A$ and $\mathcal{O}_A$, yet to be investigated.

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