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CONJUGACY CLASSES OF LEFT IDEALS

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ABSTRACT. Let A be a finite dimensional algebra over a field K, and let $C(A)$ be the set of conjugacy classes of left ideals in A. We prove that C(A) is finite if and only if the number of conjugacy classes of nilpotent left ideals in A is finite.

1 Introduction

Throughout the paper we assume A to be a finite dimensional algebra with unity over a field K. Let $U(A)$ be the unit group in A. By $L(A)$ we denote the set of left ideals of A. The equivalence relation $\sim$ can be introduced on $L(A)$ by identifying $L_1, L_2 \in L(A)$ such that $L_1 = L_2u$, for some $u \in U(A)$. The set of equivalence classes $L(A)/\sim$, denoted by $C(A)$, is clearly identical with the set of conjugacy classes of left ideals in A. Obviously for $u, v \in U(A)$ and $L_1, L_2 \in L(A)$ we have $L_1uL_2v = L_1L_2v$, so $C(A)$ admits a semigroup structure given by: $[L_1][L_2] := [L_1L_2]$.

The definition of $C(A)$ was introduced in [7]. The finiteness of $C(A)$ was proved to have some connection with the finite representation type property of finite dimensional algebras[1] (see Theorems 6 and 7 in [7]). A complete characterization of algebras for which the semigroup of conjugacy classes of left ideals is finite has been obtained for a certain class of basic[2] algebras with 2-nilpotent Jacobson radical (see [7], Theorem 12). This work aims at continuing the investigation of $C(A)$. In particular, we prove the following result.

Theorem 1.1 *Let A be a finite dimensional algebra over an arbitrary field K. The following conditions are equivalent:*

(1) C(A) is finite,

(2) the number of conjugacy classes of nilpotent left ideals in A is finite.

We note that the weaker form of Theorem 1.1, under the additional hypothesis that A is a basic algebra, appeared as an argument in the proof of Theorem 12 in [7].

The paper is divided into three sections. In Section 2 we give some preparatory facts on conjugacy classes of idempotents in a finite dimensional algebra. In Section 3 some semigroup-theoretical results are recalled for further usage. The proof of Theorem 1.1 is given in Section 4. We conclude this paper with a discussion of connections between the case of arbitrary algebras and the case of basic algebras.

[1] Recall that an algebra A is of finite representation type if and only if there are finitely many isomorphism classes of idemcomposable left A-modules of finite length.

[2] Recall that an algebra A with a complete set $\{e_1, e_2, \dots, e_n\}$ of primitive orthogonal idempotents is called basic is $e_iA \not\simeq e_jA$ (as left A-modules), for all $i \neq j$.

2 Conjugacy classes of idempotents

By $J(A)$ we denote the Jacobson radical of A. The image of $a \in A$ under the natural map $A \to A/J(A)$ shall be denoted by $\overline{a}$. We begin with some remarks concerning the conjugacy classes of idempotents in A and the principal left ideals generated by such elements.

Lemma 2.1 *The number of conjugacy classes of idempotents in A is finite.*

Proof. It is well known that every two decompositions of unity into a sum of primitive orthogonal idempotents are conjugate, see Theorem 3.4.1 in [4]. Hence the assertion follows easily. ■

The following result gathers certain well known equivalences concerning conjugacy classes of idempotents in A.

Theorem 2.2 *Let e, f be idempotents in A. The following conditions are equivalent:*

(1) $\overline{A}\overline{e} \simeq \overline{Af}$ as $\overline{A}$-modules,

(2) $Ae \simeq Af$ as A-modules,

(3) $\overline{e}$ and $\overline{f}$ are conjugate in $\overline{A}$,

(4) e and f are conjugate in A,

(5) $[\overline{A}\overline{e}] = [\overline{Af}]$ in $C(\overline{A})$,

(6) $[Ae] = [Af]$ in $C(A)$,

(7) e and f generate the same ideal in the multiplicative monoid $(A, \cdot)$.

Before we proceed with the proof let us remark that condition (7) does not usually appear in the ring-theoretic considerations. On the other hand, it plays an important role in the theory of connected algebraic monoids, see Corollary 6.8 in [9]. However, it proves to be very useful also in the setting of finite dimensional algebras (which are connected monoids if the field K is algebraically closed).

Proof. We sketch the outline of the proof in the following steps:

$$\begin{array}{ccccccccccccc} & & & & & (6) & & & & & & & \\ & & & \swarrow & & & \nwarrow & & & & & & \\ (1) & \Rightarrow & (2) & & \Rightarrow & & & (4) & \Rightarrow & (3) & \Rightarrow & (5) & \Rightarrow & (1) \\ & & & \nwarrow & & & \swarrow & & & & & & \\ & & & & & (7) & & & & & & & \end{array}$$

To deduce (2) from (1) observe that $\overline{A}\overline{e} \simeq Ae/J(A)e$ and an isomorphism between $\overline{A}\overline{e}$ and $\overline{Af}$ can be lifted to an isomorphism between Ae and Af (see Proposition 17.18 in [1]).

Suppose that $Ae \simeq Af$ as A-modules. It is known that

$$Hom(Ae, Af) \simeq eAf, \tag{2.1}$$

and every homomorphism $\phi : Ae \to Af$ can be described by the formula $\phi(x) = x(eaf)$, for $x \in Ae$ where $a \in A$ (see Corollary 6.4 b in [8]). Take ϕ to be an isomorphism in $Hom(Ae, Af)$. There exists $fbe \in fAe$ such that $\phi^{-1}(y) = y(fbe)$. Hence we have:

$$e = eaf \cdot fbe, \quad f = fbe \cdot caf, \tag{2.2}$$

for some $a, b \in A$. Consider the following decompositions into direct sums of A-modules:

$$A = Ae \oplus A(1-e), \quad A = Af \oplus A(1-f).$$

From Krull-Schmidt theorem it follows that $A(1-e) \simeq A(1-f)$. Hence, using (2.1) again, we get

$$\begin{array}{c} 1-f = (1-f)c(1-e) \cdot (1-e)d(1-f) \\ 1-e = (1-e)d(1-f) \cdot (1-f)c(1-e) \end{array}, \tag{2.3}$$

for some $c, d \in A$. Put

$$u := eaf + (1-e)d(1-f), \quad v := fbe + (1-f)c(1-e).$$

According to (2.2) and (2.3) we have $uv = 1$ and $eu = uf$. Thus (2) $\Rightarrow$ (4).

The implications (4) $\Rightarrow$ (6) $\Rightarrow$ (2), as well as (4) $\Rightarrow$ (7) are obvious. Assume that (7) holds. From Theorem 2.20 in [3] it follows that there exist $a, b \in A$ such that eaf and fbe belong to the ideal $AeA = AfA$ of the semigroup $(A, \cdot)$ such that $(fbe)(eaf) = f, (eaf)(fbe) = e$. In view of 2.1 these two elements yield an isomorphism between the left A-modules Ae and Af, see (2.1). Hence (2) follows.

The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (1) are obvious.

■

3 Strongly π-regular semigroups

Although the statement of Theorem 1.1 concerns purely ring-theoretic properties of the algebra A, its proof depends heavily on certain semigroup-theoretic considerations as well as on some well known classical facts on semigroups. For the latter we refer the reader to [3]. In this section we introduce the class of strongly-π-regular semigroups and prove this class contains the multiplicative monoid of every finite dimensional algebra with unity. We follow Proposition 1.2 in [5].

Definition 3.1 *Recall that H is a subgroup of semigroup S if and only if H is a subgroup of the unit group $U(eSe)$ of the monoid eSe, for some idempotent $e \in S$.*

The semigroup S is called strongly π-regular if for every $s \in S$ there exists $n \geq 1$ such that s^n is contained in a subgroup of S.

Example 3.2 *The multiplicative monoid $M_n(D)$ of $n \times n$ matrices over a division ring D is strongly π-regular.*

Proof.[[6], Prop. 1.3] D^n is viewed as a right D-space and as a left $M_n(D)$-module. We interpret $M_n(D)$ as the endomorphism ring $\mathrm{End}_D(D^n)$. If $a \in M_n(D)$ then, by Fitting's lemma ([8], §5.3) there exist subspaces V, W of D^n such that $D^n = V \oplus W$ and for every $k \geq n$ we have $a^k V = 0$ and a^k maps isomorphically W onto W. Let $f \in M_n(D)$ be the map such that $f_{|W} = \mathrm{id}_W$ and $fV = 0$. Consider the following set:

$$M = \{x \in M_n(D) \,|\, xW = W,\, xV = 0\}.$$

It is obviously a monoid with unity equal to f. Therefore $M = fM_n(D)f$ and the element a^k is invertible in M. Hence a^k belongs to a subgroup of $M_n(D)$.■

Corollary 3.3 *The multiplicative monoid of a semisimple artinian ring R is strongly π-regular, with maximal subgroups isomorphic to finite direct products of certain full skew linear groups $GL_m(D)$ over division rings D.*

Theorem 3.4 *Let A be a finite dimensional algebra with unity. Then the multiplicative semigroup $(A, \cdot)$ of A is strongly π-regular.*

Proof.[See [5], Prop. 1.2] Let $a \in A$ be a non-nilpotent element. Since $A/J(A)$ is semisimple artinian, it follows from Corollary 3.3 that there exists $k \geq 1$ such that the image $\overline{b}$ of $b := a^k$ lies in a maximal subgroup G of the multiplicative semigroup $A/J(A)$. Hence, for each $i \geq 1$ there exists $x_i \in A$ such that $\overline{x_i} \in G$ and $\overline{b} = \overline{x_i b^{i+1}} \in G$. The nilpotency of $J(A)$ yields

$$(b - x_1 b^2)(b - x_2 b^3) \dots (b - x_n b^{n+1}) = 0, \text{ for some } n \geq 1.$$

After expanding the left side we get $b^n + wb^{n+1} = 0$, for some $w \in A$. Put $c := b^n$. It is clear that $c \neq 0$ and $c \in Ab^{n+1}$. Thus, for some $w_1, w_2, \dots, w_{n-1}, w_n \in A$ we get:

$$c = w_1 b^{n+1} = w_1 cb = w_1 w_2 b^{n+1} b = w_1 w_2 w_3 b^{n+1} b^2 = \dots = w_1 w_2 \dots w_n b^{2n}$$

and thus

$$c = zc^2, \text{ for some } z \in A. \tag{3.1}$$

Hence

$$c^2 = czc^2 = cz(zc^2)c = (cz^2c)c^2. \tag{3.2}$$

Put $g = cz^2c$. Since $\overline{b} \in G$ then $\overline{c} = \overline{b^n} \in G$ and the previous equation yields that also $\overline{g} \in G$ being, in fact, the identity element of this group. From the proof of Proposition 27.1 (on lifting idempotents modulo a nil-ideal) in [8] it follows that there exists e in the subring of A generated by the element g such that

$$e = e^2, \quad eg = ge, \quad g^r = eg^r, \quad \overline{e} = \overline{g}$$

for some $r \geq 1$. Thus, (3.2) yields

$$c^2 = gc^2 = g^r c^2 \in eA. \tag{3.3}$$

From (3.1) it follows that $Ac = Ac^2$, and therefore c and c^2 generate the same left ideal of the multiplicative semigroup A. But $c^2 \in eA$ (due to (3.3)) and $e \in cA$ (since e belongs to the subring of A generated by $g = cz^2c$), hence c, c^2, e generate the same two-sided ideal of the semigroup A. Since A is finite dimensional, the semigroup A has no infinite chains of idempotents. Hence the principal factor A_c of c in A is a completely 0-simple semigroup (see [3], Lemma 2.39 and Theorem 2.48) and c^2 is nonzero in A_c. Therefore c lies in a maximal subgroup of A_c, and thus also in a maximal subgroup of A. It follows that A is strongly π-regular, as claimed. ■

4 Proof of the main result

Recall that a partial order $\leq$ can be introduced on the set of all idempotents in A. If $e = e^2, f = f^2$ belong to A then we say that $e \leq f$ if and only if $ef = fe = e$.

Lemma 4.1 *Let $L \in L(A)$. The following conditions are equivalent for an idempotent $e \in L$:*

(1) e is a maximal idempotent in L,

(2) the left ideal $L(1-e)$ is contained in $J(A)$,

PROOF. First note that the ideal L can be decomposed into a direct sum of left ideals:

$$L = Ae \oplus L(1-e). \tag{4.1}$$

Assume that (1) holds. If $L \subseteq J(A)$ then (2) clearly holds. Thus, we may assume that $e \neq 0$. Suppose that $L(1-e) \not\subseteq J(A)$. Then there exists a nonzero $f = f^2 \in L(1-e)$ and (4.1) applied to $L(1-e)$ implies that

$$L = Ae \oplus Af \oplus L(1-e)(1-f) = Ae \oplus Af \oplus L(1-e-f+ef).$$

Observe that $f = f(1-e)$ so $fe = 0$. Thus:

$$\begin{aligned}(e+f-ef)^2 &= e^2 + f^2 + ef + fe - eef - efe - fef - eff + efef \\ &= e + f - ef.\end{aligned}$$

Obviously

$$e(e+f-ef) = e = (e+f-ef)e.$$

In other words, $e \leq e + f - ef$. Since e is maximal in L, we deduce that $e = e + f - ef$. Thus:

$$0 = f - ef \quad \Rightarrow \quad 0 = f(f - ef) = f^2 - fef = f^2 = f,$$

a contradiction. Therefore assertion (2) follows.

Assume that $L(1-e) \subseteq J(A)$. Suppose that there exists an idempotent $f \in L$ such that $ef = fe = e$. Clearly $f - e \in L$ and $(f - e)e = 0$ so $f - e \in L(1 - e)$. Since $(f - e)^2 = f - e$, it follows that $f - e = 0$. Thus (2) $\Rightarrow$ (1). ■

Proof.[Theorem 1.1] We see at once that (1) implies (2). Assume that there are finitely many conjugacy classes of nilpotent left ideals in A. Consider $I, J \in L(A)$ that are not contained in $J(A)$. According to Lemma 4.1 there exist idempotents $e_I \in I$ and $e_J \in J$, maximal (respectively in I and J) such that the following equalities hold:

$$I = Ae_I \oplus I(1 - e_I), \quad J = Ae_J \oplus J(1 - e_J),$$

and $I(1 - e_I), J(1 - e_J) \subseteq J(A)$. By Lemma 2.1 we may assume that e_I and e_J are conjugate. Conjugating I or J we can assume that $e_I = e_J =: e$, so that

$$I = Ae \oplus I(1 - e), \quad J = Ae \oplus J(1 - e), \tag{4.2}$$

and $I(1 - e), J(1 - e) \subseteq J(A)$.

Since there are only finitely many conjugacy classes of left ideals in $J(A)$, we may also assume that $I(1 - e)$ and $J(1 - e)$ are conjugate. So there exists $g \in U(A)$ such that

$$I(1 - e) = J(1 - e)g. \tag{4.3}$$

The element g may be represented as a matrix of the form $g = \left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right)$, where

$$g_1 = ege, \quad g_2 = eg(1 - e), \quad g_3 = (1 - e)ge, \quad g_4 = (1 - e)g(1 - e).$$

Let

$$I_1 = eIe, \quad I_2 = eI(1 - e), \quad I_3 = (1 - e)Ie, \quad I_4 = (1 - e)I(1 - e)$$

$$J_1 = eJe, \quad J_2 = eJ(1 - e), \quad J_3 = (1 - e)Je, \quad J_4 = (1 - e)J(1 - e).$$

Substituting (4.3) we obtain

$$\left(\begin{smallmatrix} 0 & I_2 \\ 0 & I_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & J_2 \\ 0 & J_4 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} J_2 g_3 & J_2 g_4 \\ J_4 g_3 & J_4 g_4 \end{smallmatrix}\right). \tag{4.4}$$

It is also clear that there exists $h = \left(\begin{smallmatrix} h_1 & h_2 \\ h_3 & h_4 \end{smallmatrix}\right) \in U(A)$ such that

$$\left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} h_1 & h_2 \\ h_3 & h_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right) = 1.$$

Hence

$$g_3 h_2 + g_4 h_4 = 1 - e.$$

Therefore

$$J(1 - e) = J(1 - e)(g_3 h_2 + g_4 h_4)$$

From (4.4) we deduce that $J(1-e)g_3 = (J_2 \oplus J_4)g_3 = 0$ and $J(1-e)g_4 = I(1-e)$. Thus

$$J(1-e) = J(1-e)g_4h_4 = I(1-e)h_4. \tag{4.5}$$

One then easily checks that for every $n \in \mathbb{N}$ the following equalities hold:

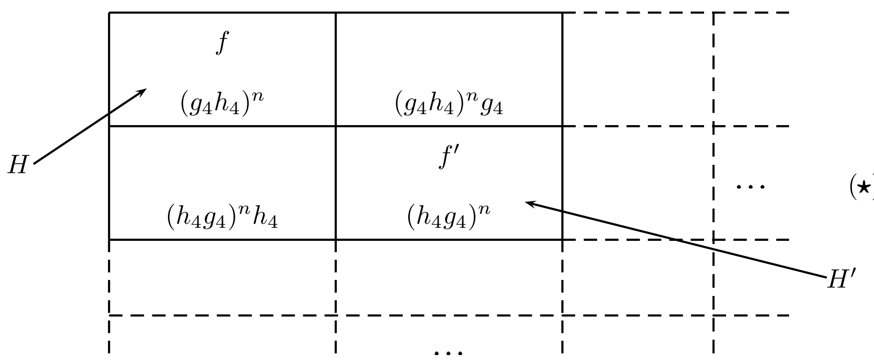
$$\begin{cases} J(1-e)(g_4h_4)^n = J(1-e) \\ I(1-e)(h_4g_4)^n = I(1-e). \end{cases} \tag{4.6}$$

According to Theorem 3.4 the multiplicative semigroup of A is strongly π-regular. It follows that $(g_4h_4)^n \in H, (h_4g_4)^n \in H'$, for some $n \in \mathbb{N}$, where H, H' are maximal subgroups in the monoid $((1-e)A(1-e), \cdot)$. Put

$$s := (g_4h_4)^n g_4 \in (1-e)A(1-e).$$

Observe that $sh_4 \in H$ hence there exists $h_4' \in H$ such that $sh_4h_4' = (g_4h_4)^n$. Thus $(g_4h_4)^n$ and s are in the same $\mathcal{R}$-class[3] of the monoid $(1-e)A(1-e)$. In the same manner we can see that s and $(h_4g_4)^n$ are in the same $\mathcal{L}$-class of $(1-e)A(1-e)$. It follows that $(g_4h_4)^n, (h_4g_4)^n$ are in the same $\mathcal{J}$-class of the monoid $(1-e)A(1-e)$.

Let f, f' be the unities of H and H', respectively. Since the maximal subgroups of every semigroup S are precisely the $\mathcal{H}$-classes of S containing idempotents (see [3], Ex. 1 in §2.3), then $(g_4h_4)^n \mathcal{H} f$ and $(h_4g_4)^n \mathcal{H} f'$ and it follows that f, f' are in the same $\mathcal{J}$-class of the monoid $((1-e)A(1-e), \cdot)$. The egg-box pattern on this class (see [3], page 48) is of the following form[4]:



(⋆)

According to (4.6) (and knowing that $f, f' \in (1-e)A(1-e)$) we deduce that:

$$\begin{cases} J(1-e) = J(1-e)f &= Jf \\ I(1-e) = I(1-e)f' &= If' \end{cases}. \tag{4.7}$$

[3]For the definition and basic facts concerning Green relations see Chapter 2 of [3].

[4]Note that if f, f' not only lie in the same $\mathcal{J}$-class of $((1-e)A(1-e), \cdot)$, but also in the same $\mathcal{R}$– or $\mathcal{L}$– class of this monoid, then the egg-box pattern on their $\mathcal{J}$-class becomes simpler. We deal only with the general case, leaving the remaining details for the reader.

Implication (7) $\Rightarrow$ (4) of Theorem 2.2 allows us to find $u \in U((1-e)A(1-e))$ such that in $(1-e)A(1-e)$ we have $u^{-1}fu = f'$. Consider $\widehat{u} =: e + u$. Clearly $\widehat{u}^{-1} f \widehat{u} = f'$. Hence

$$\begin{aligned} J\widehat{u} &= (Ae \oplus Jf)\widehat{u} \\ &= Ae\widehat{u} \oplus Jf\widehat{u} \\ &= Ae \oplus J\widehat{u}\widehat{u}^{-1}f\widehat{u}. \end{aligned}$$

Obviously $\widehat{u} \in U(A)$. Thus, replacing J by its conjugate $J\widehat{u}$, we may assume that

$$\begin{cases} J = Ae \oplus Jf' \\ I = Ae \oplus If' \end{cases}.$$

We are, therefore, allowed to assume that $f = f'$ and

$$(g_4h_4)^n, (h_4g_4)^n \in H = H' = U(fAf).$$

According to ($\star$) also $(g_4h_4)^n g_4, (h_4g_4)^n h_4 \in H$. Put

$$z = 1 - f + (g_4h_4)^n g_4.$$

Observe that since $(g_4h_4)^n g_4 \in U(fAf)$ then $z \in U(A)$. Clearly $ef = 0$ and it follows that:

$$\begin{aligned} Jz &= (Ae \oplus Jf)z \\ &= Ae \oplus Jfz \\ &\overset{(4.7)}{=} Ae \oplus J(1-e)(g_4h_4)^n g_4 \\ &\overset{(4.6)}{=} Ae \oplus J(1-e)g_4 \\ &\overset{(4.5)}{=} Ae \oplus I(1-e)h_4g_4 \\ &\overset{(4.6)}{=} Ae \oplus I(1-e) = \\ &\overset{(4.2)}{=} I. \end{aligned}$$

Thus $[I] = [J]$ in $C(A)$. This proves that $C(A)$ is finite. ■

Corollary 4.2 *Assume that A is a finite dimensional algebra with unity. Assume that A has n conjugacy classes of idempotents (see Lemma 2.1) and that the number of conjugacy classes of nilpotent left ideals in A is finite and equal to m. Then the semigroup $C(A)$ is finite and $|C(A)| \le nm$.*

PROOF. In the previous proof we take two left ideals I, J of A such that e, f are their respective maximal idempotents. From Lemma 4.1 it follows that:

$$\begin{cases} I = Ae \oplus I(1-e) \\ J = Af \oplus J(1-f) \end{cases},$$

where $I(1-e), J(1-f)$ are left ideals of A and belong to the Jacobson radical $J(A)$ of A. Assuming that e, f are conjugate and that $[I(1-e)] = [J(1-e)]$, we prove that $[I] = [J]$. The assertion follows. ■

5 Basic versus non-basic algebras

One of the motivations for the study of the semigroup $C(A)$ is to look for new (semigroup-theoretic) invariants of finite dimensional algebras. Notice that for every simple algebra $A = M_n(D)$ the semigroup $C(A)$ can be identified with $S = \{f_1, \dots, f_n, 0\}$, where $f_i f_j = f_j$ and $f_i 0 = 0 f_i = 0$ for all i, j. Therefore, having also in mind several other aspects of representation theory of finite dimensional algebras [2], in this section we assume the field K to be algebraically closed.

Definition 5.1 *Let A be a finite dimensional algebra and let $\{e_1, \dots, e_n\}$ be a complete set of orthogonal primitive idempotents of A. Recall that the directed graph $\Gamma = (V, E)$ is called the quiver of A, which we denote by $\Gamma(A)$, if the vertex set of Γ is of the form $V = \{1, 2, \dots, n\}$ and the edge set E is equal to $\{(i,j) \,|\, e_i J(A) e_j \neq 0\}$ (see, [8], §6.4). Associated with any quiver $\Gamma = (V, E)$ is the separated quiver $\Gamma^s = (V^s, E^s)$ where $V^s = V \times \{0, 1\}$ and $E^s = \{((i,0),(j,1)) \,|\, (i,j) \in E\}$. If A is a finite dimensional algebra with quiver $\Gamma = \Gamma(A)$, then the separated quiver Γ^s, corresponding to Γ, is called the separated quiver of A. Denote this separated quiver by $\Gamma^s(A)$.*

At a first glance it seems that having obtained the proof of the main result of this paper we should be able to generalize Theorem 12 of [7], which depended so much on the "basic" version of Theorem 1.1.

Theorem 5.2 *[Theorem 12, [7]] Let A be a finite dimensional basic algebra with a distributive lattice of ideals over an algebraically closed field K and such that $J(A)^2 = 0$. Then the following conditions are equivalent*

1. *C(A) is finite,*
2. *the separated graph $\Gamma^s(A)$ of A has no cycles (orientation ignored) and $\dim(eJ(A)) \leq 3$ for every primitive idempotent $e \in A$.*

One can easily give some examples of non-basic algebras (even in the class of algebras with 2-nilpotent Jacobson radical) which violate the criterion given above, in particular the bound on the dimension of $eJ(A)$.

Example 5.3 *Consider $A = M_n(K[x]/(x^2))$.*

It is obvious that $A = \overline{A} \oplus J(A)$ as vector spaces, where $J(A)^2 = 0$ and:

$$\overline{A} = \begin{pmatrix} K & K & \dots & K \\ K & K & \dots & K \\ \vdots & \vdots & \ddots & \vdots \\ K & K & \dots & K \end{pmatrix}, \quad J(A) = \begin{pmatrix} Kx & Kx & \dots & Kx \\ Kx & Kx & \dots & Kx \\ \vdots & \vdots & \ddots & \vdots \\ Kx & Kx & \dots & Kx \end{pmatrix}.$$

By $e_{ij}, i,j = 1, \dots, n$, we denote the matrix units that belong to $\overline{A}$. Let $L \in L(A)$. By L_i we denote the i-th row of L, defined as the linear space $e_{ii}L$. Clearly $L = L_1 + \dots + L_n$.

Observe that since $e_{ij}L \subseteq L$ for every i, j, all rows of L are equal, as K-subspaces of K^n. Thus the ideal L is of the form

$$L = \begin{pmatrix} L_1 \\ L_1 \\ \dots \\ L_1 \end{pmatrix}$$

Let $d(L)$ stand for the linear dimension of L_1. Assume $L \subseteq J(A)$. Every invertible element of A is of the form $u = u' + j$, where $u' \in U(\overline{A}) = GL_n(K)$ and $j \in J(A)$. It follows that $Lu = Lu'$. If M, N are subspaces of the space of row vectors K^n of equal dimension, then there exists an element $u \in GL_n(K)$ such that $Mu = N$. As a result: for every L, L' contained in $J(A)$ the following equivalence occurs:

$$[L] = [L'] \in C(A) \Leftrightarrow d(L) = d(L').$$

It follows that $C(M_n(K[x]/(x^2)))$ contains exactly $n+1$ conjugacy classes of left nilpotent ideals. According to Theorem 1.1 the semigroup $C(M_n(K[x]/(x^2)))$ is finite[5].

The dimension of $eJ(A)$ could be arbitrarily large in the previous example. Clearly, the lattice of ideals of A is distributive. This shows that a direct generalization of Theorem 5.1 to the non-basic case is not possible.

Definition 5.4 *Assume that A is a K-algebra with a complete set $\{e_1, e_2, \dots, e_n\}$ of primitive orthogonal idempotents. A basic algebra associated to A is the algebra*

$$A^b = e_A A e_A$$

where $e_A = e_{j_1} + \dots + e_{j_t}$, and $e_{j_1}, \dots, e_{j_t}$ are chosen such that $e_{j_i}A \not\simeq e_{j_s}A$ for $i \neq s$ and each $A-$module e_jA is isomorphic to one of the modules $e_{j_1}A, \dots, e_{j_t}A$.

As an example take $A = M_n(K)$. Then as e_A we can take any of the diagonal idempotents of rank one in A and $A^b \simeq K$.

One checks that the algebra A^b does not depend on the choice of the sets $e_1, e_2, \dots, e_n$ and $e_{j_1}, \dots, e_{j_t}$, up to a K-algebra isomorphism (see [2], 6.5). The algebra A^b is basic and there is an equivalence of categories (of left modules) $modA \simeq modA^b$.

We conclude with two basic observations concerning the relation between the finiteness of $C(A)$ and the finiteness of $C(A^b)$.

Proposition 5.5 *Assume that $e = e^2$ is an idempotent of a finite dimensional algebra A. Let $\phi : C(eAe) \to C(A)$ be defined as follows: $\phi([L]) = [AL]$. Then ϕ is well defined and it is a semigroup monomorphism.*

[5] In fact, Theorem 1.1 is not necessary to deduce finiteness of $C(M_n(K[x]/(x^2)))$. It is obvious that algebra $A = K[x]/(x^2)$ is of finite representation type. Therefore, the matrix algebra $M_n(A)$ is of finite representation type as well. From Theorem 6 in [7] we deduce that $C(M_n(A))$ is finite.

Proof. Assume that $[L_1] = [L_2]$ in $C(eAe)$. Then there exists $g \in U(eAe)$ such that

$$L_1 = L_2 g. \tag{5.1}$$

Observe that $g + (1 - e) \in U(A)$. Take $h \in U(eAe)$ such that $gh = e$. Then

$$(g + (1 - e))(h + (1 - e)) = gh + (1 - e)h + g(1 - e) + (1 - e) = e + (1 - e) = 1.$$

Since

$$AL_2 g = AL_2 eg = AL_2 e(g + (1 - e)),$$

(5.1) yields

$$AL_1 = AL_2 g = AL_2(g + 1 - e) \Rightarrow [AL_1] = [AL_2].$$

The map ϕ is therefore well defined. The multiplicativity of ϕ is clear. Indeed, for any $[L_1], [L_2] \in C(eAe)$ we have:

$$\begin{aligned} \phi([L_1][L_2]) &= \phi([L_1 L_2]) \\ &= [AL_1 L_2] \\ &= [AL_1 eAeL_2] \\ &= [AL_1 AL_2] \\ &= [AL_1][AL_2] = \phi([L_1])\phi([L_2]). \end{aligned}$$

If $[AL_1] = [AL_2]$ then it follows that $AL_1 = AL_2 g$ for some $g \in U(A)$. Hence $L_1 = eAL_1 = eAL_2 g = L_2 g$, so that $L_1 = L_2(ege)$. A similar argument shows that $L_2 = L_1(ehe)$ for some $h \in U(A)$. Let $g' = ege, h' = ehe$. Proceeding as in the proof of Theorem 1.1 we find maximal subgroups H, H' of the multiplicative monoid eAe such that $(h'g')^n \in H, (g'h')^n \in H'$ for some $n \geq 1$. As before, it follows that $(h'g')^n, (g'h')^n$ generate the same ideal of this monoid and:

$$L_1 = L_1 h'g' = L_1(h'g')^n, \quad L_2 = L_2 g'h' = L_2(g'h')^n \tag{5.2}$$

Let f, f' stand for the unit elements of H, H' respectively. Then, similarly to (4.7) we have:

$$L_1 = L_1 f, \quad L_2 = L_2 f' \tag{5.3}$$

and by Theorem 2.2 the idempotents $f \in H, f' \in H'$ are conjugate in eAe. Replacing L_2 by its conjugate we may assume that $f = f'$ and thus $H = H'$. Therefore $(h'g')^n h', (g'h')^n g' \in H \subseteq U(fAf)$. Consider $z := e - f + (h'g')^n h'$. Then $z \in U(eAe)$, because $(h'g')^n h' \in U(fAf)$. Clearly $f(e - f) = 0$. From (5.2) and (5.3) we deduce that:

$$L_1 z = L_1 fz = L_1(h'g')^n h' = L_1 h' = L_2$$

Therefore the ideal classes $[L_1]$ and $[L_2]$ are equal in $C(eAe)$. The injectivity of ϕ follows. ∎

Corollary 5.6 *If $C(A)$ is finite then $C(A^b)$ is finite.*

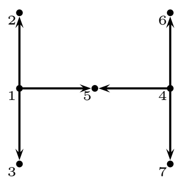
Note that the finiteness of $C(A^b)$ does not imply the finiteness of C(A). Indeed, suppose that the finiteness of $C(A)$ implies that $C(M_n(A))$ is finite, for all $n \geq 1$. It follows from Theorem 7 in [7] that A is of finite representation type. However, one can easily give an example of such algebra that is of infinite representation type and that possess finitely many conjugacy classes of left ideals.

Example 5.7 *Consider the following subalgebra A of the matrix algebra $M_7(K)$:*

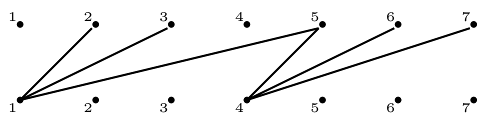
$$\begin{pmatrix} K & K & K & 0 & K & 0 & 0 \\ 0 & K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K & K & K & K \\ 0 & 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K \end{pmatrix}.$$

Then algebra A is of infinite representation type. However, $C(A)$ is finite.

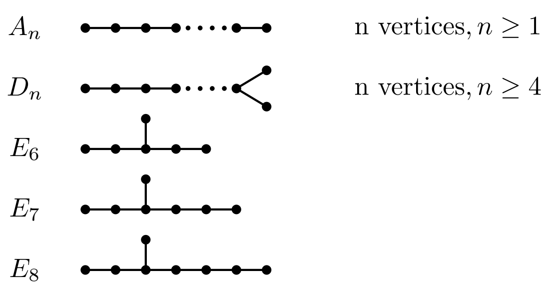
Proof. Since the set of diagonal idempotents of rank one in A forms the complete set of orthogonal primitive idempotents of A, one easily verifies that the quiver $\Gamma(A)$ is of the following form:



The separated graph $\Gamma^s(A)$ is, therefore, of the following form:



We can see that $\Gamma^s(A)$ has no cycles. Observe that A is a basic algebra and that $J(A)^2 = 0$. It is also clear that $\dim(eJ(A)) \leq 3$, for every primitive idempotent e of A. It follows from Theorem 5.2 that the semigroup $C(A)$ is finite. However, the separated graph $\Gamma^s(A)$ is not a disjoint union of the following graphs (the, so-called, Dynkin diagrams):



Thus, according to Theorem 11.8 of [8], the algebra A is of infinite representation type. ■

References

[1] ANDERSON F., FULLER K.: *Rings and Categories of Modules*, 2nd. ed., GTM 13, Springer, 1992.

[2] ASSEM I., SIMSON D., SKOWROŃSKI A.: *Elements of the Representation Theory of Associative Algebras. 1: Techniques of Representation Theory*, London Mathematical Society, Student Texts 65, Cambridge 2006.

[3] CLIFFORD A.H., PRESTON G.B.: *The algebraic theory of semigroup*, Vol I, Mathematical Surveys 7, Amer. Math. Soc., 1967.

[4] DROZD Y., KIRICHENKO V.: *Finite Dimensional Dlgebras*, Springer, 1994.

[5] JESPERS E., OKNIŃSKI J.: *Descending chain conditions and graded rings*, J. Algebra 178 (1995), 458-479.

[6] OKNIŃSKI J.: *Semigroups of Matrices*, Series in Algebra, 6. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[7] OKNIŃSKI J., RENNER L.: *Algebras with finitely many orbits,* J. Algebra 264 (2003), 479-495.

[8] PIERCE R.: *Associative Algebras*, GTM 88, Springer, 1982.

[9] PUTCHA M.: *Linear Algebraic Monoids*, London Mathematical Society Lecture Notes Studies 133, Cambridge University Press, 1988.