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Subspace semigroups of finite dimensional algebras

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SUBSPACE SEMIGROUPS OF FINITE DIMENSIONAL ALGEBRAS

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ABSTRACT. The aim of this term paper is to give an accessible exposition of selected results concerning the so-called subspace semigroups which arose in the context of developing geometric tools in the study of the representation theory of finite dimensional algebras. Main references on this subject are [7] and [8].

1 Introduction

Throughout the paper $\mathbb{K}$ is assumed to be an algebraically closed field. Assume that A is a finite dimensional $\mathbb{K}$-algebra. By $\mathcal{S}(A)$ we denote the set of $\mathbb{K}$-subspaces of A. A binary operation $* : \mathcal{S}(A) \times \mathcal{S}(A) \to \mathcal{S}(A)$ can be introduced on $\mathcal{S}(A)$ by assigning the following element of $\mathcal{S}(A)$ to an arbitrary pair (V, W) of subspaces of A:

$$V * W = \mathrm{lin}_{\mathbb{K}}\{vw \mid v \in V, w \in W\}. \tag{1.1}$$

It is obvious that $(\mathcal{S}, *)$ is a semigroup.

Our main goal is to consider the case when $A = \mathrm{M_n}(\mathbb{K})$ is the algebra of $n \times n$ matrices over $\mathbb{K}$. In Section 2 we recall the basics of the semigroup theory as well as the theory of algebraic monoids. We also give the structure of the multiplicative monoid of $\mathrm{M_n}(\mathbb{K})$, namely the description of the Green's relations on $\mathrm{M_n}(\mathbb{K})$ and the ideal structure of this semigroup. Next, following [8], we prove that the subspace semigroup $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ admits a finite chain of ideals, which are either regular or nilpotent. The size of the regular factors of the chain obtained in Section 3 appear to have a direct connection with the Morita equivalence classes of unitary subalgebras of $\mathrm{M_n}(\mathbb{K})$, which we demonstrate in the last Section.

2 Preliminaries

In order for this paper to remain as close to self-contained as possible, we introduce the basics of the semigroup theory and of the theory of algebraic monoids. We omit most proofs, referring to external sources, if necessary.

2.1 Semigroup theory

For basic results from the semigroup theory we refer to [3]. By S we denote an arbitrary semigroup, that is, a set S equipped with a binary operation $\cdot : S \times S \to S$ that is assumed to be associative. We write $a \cdot b$, or (if the context allows us) ab, to denote the product of two elements $a, b \in S$. If A, B are nonempty subsets of a semigroup S then by AB we denote the set $\{ab \mid a \in A, b \in B\}$.

Suppose that there exist elements $0, 1 \in S$ such that

$$0 \cdot s = 0 = s \cdot 0, \quad s \cdot 1 = s = 1 \cdot s, \quad \text{for all } s \in S.$$

An element 0 is called the **zero** of S, whereas an element 1 is called the **identity** of S. A semigroup with identity is called a **monoid**. If S is an arbitrary semigroup then by S^1 we denote the set S, if there is an identity in S, or $S \cup \{1\}$, if there is no identity in S. In the latter case we put $1 \cdot a = a = a \cdot 1$ for all $a \in S^1$. Note that $(S^1, \cdot)$ is a monoid.

Definition 2.1 *An element s of a semigroup S with 0 is called:*

- ***idempotent*** *if and only if $s^2 = s$,*
- ***nilpotent*** *if and only if $s^n = 0$, for some $n > 0$,*
- ***regular*** *if and only if $a = axa$, for some $x \in S$.*

A semigroup S is called nil (regular) if every element of S is nilpotent (regular). S is called nilpotent if $S^n = \{0\}$, for some natural number $n > 0$.

The following partial order may be introduced in the set $E(S)$ of all idempotents in S:

$$e \leq f \iff ef = fe = e, \quad e, f \in E(S).$$

A minimal non-zero idempotent in S is called a **primitive idempotent**.

Definition 2.2 *Let $\emptyset \neq I \subseteq S$. The set I is called:*

- ***a right ideal of S****, write $I <_r S$, if $IS \subseteq I$,*
- ***a left ideal of S****, write $I <_l S$, if $SI \subseteq I$,*
- ***an ideal of S****, write $I \lhd S$, if $SI \subseteq I \wedge IS \subseteq I$.*

Definition 2.3 (The Rees quotient) *Let $I \lhd S$. One can introduce the following binary operation $\circ$ in the set $(S \setminus I) \cup \{\theta\}$:*

$$a \circ b = \begin{cases} ab & , \text{ if } ab \notin I, \\ \theta & , \text{ if } ab \in I, \end{cases} \quad \text{for } a, b \in S \setminus I.$$

Moreover, for every $a \in (S \setminus I) \cup \{\theta\}$

$$a \circ \theta = \theta \circ a = \theta.$$

*The operation $\circ$ is clearly associative and the obtained structure is called the **Rees quotient** S/I of the semigroup S by the ideal $I \lhd S$. Note that θ is the zero element of S/I.*

The following equivalence relations are crucial in our investigations.

Definition 2.4 (The Green's relations) *Let S be a semigroup. The following relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}$ can be considered on S:*

$$\begin{aligned} a\mathcal{L}b &\iff S^1a = S^1b, \\ a\mathcal{R}b &\iff aS^1 = bS^1, \\ a\mathcal{H}b &\iff a\mathcal{L}b \wedge a\mathcal{R}b, \\ a\mathcal{J}b &\iff S^1aS^1 = S^1bS^1. \end{aligned}$$

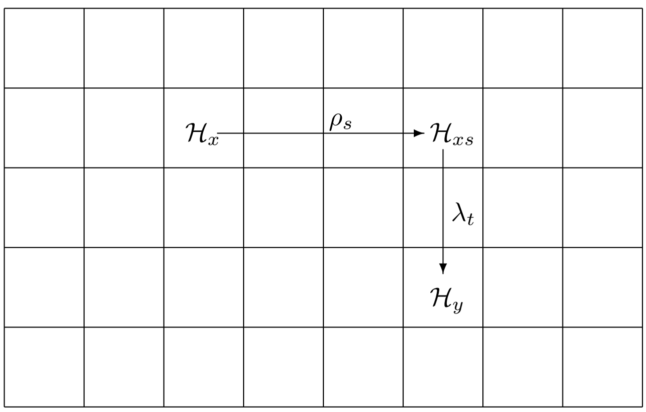
We also define $\mathcal{D}$ to be the smallest equivalence relation on S that contains $\mathcal{R}$ and $\mathcal{L}$. The $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{J}$-, $\mathcal{H}$-, $\mathcal{D}$- classes of an element a will be denoted by $\mathcal{L}_a, \mathcal{R}_a, \mathcal{J}_a, \mathcal{H}_a, \mathcal{D}_a$, respectively.

It can be shown that the relations $\mathcal{R}$ and $\mathcal{L}$ commute and $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ (see [3, 2.1]). Each $\mathcal{D}$-class of a semigroup S is a union of $\mathcal{R}$-classes and also a union of $\mathcal{L}$-classes. Moreover, $a\mathcal{D}b$ if and only if $\mathcal{R}_a \cap \mathcal{L}_b \neq \emptyset$ and, if and only if, $\mathcal{R}_b \cap \mathcal{L}_a \neq \emptyset$. If $a\mathcal{R}b$ in S, then

$$as = b, \quad bs' = a, \quad \text{for some } s, s' \in S.$$

In the above situation the right translation $\rho_s : S \to S$ defined by $\rho_s(x) = xs$ maps $\mathcal{L}_a$ to $\mathcal{L}_b$. The map $\rho_{s'} : S \to S$ maps $\mathcal{L}_b$ back to $\mathcal{L}_a$ and the composition of the maps $\rho_{s'}\rho_s : S \to S$ acts as the identity map on $\mathcal{L}_a$. Moreover, the map ρ_s preserves $\mathcal{R}$-classes in the sense that it maps each $\mathcal{H}$-class in $\mathcal{L}_a$ in a 1-1 manner onto the corresponding ($\mathcal{R}$-equivalent) $\mathcal{H}$-class of $\mathcal{L}_b$. There is a dual result for $\mathcal{L}$-classes. These results are known as Green's Lemma (see [3, 2.16]).

It is often useful to visualize a $\mathcal{D}$-class D of a semigroup S in the following way, called **the egg-box picture**. The elements of D may be arranged in a rectangular pattern, like an egg-box, the rows correspond to the $\mathcal{R}$-classes and the columns to the $\mathcal{L}$-classes contained in D. Each cell of the egg-box corresponds to an $\mathcal{H}$-class contained in D and one can easily show that no cell is empty.



There is more to be said about the rows and the columns of the egg-box in the case when we are dealing with the $\mathcal{D}$-class of an idempotent element. We recall certain facts regarding this matter.

Theorem 2.5 ([3, 2.14 - 2.17, 2.20]) *The following facts are satisfied in any semigroup S.*

(1) Any idempotent e of S is a right identity of $\mathcal{L}_e$, a left identity of $\mathcal{R}_e$ and a two-sided identity of $\mathcal{H}_e$. Thus no $\mathcal{H}$-class can contain more than one idempotent.

(2) Any $\mathcal{H}$-class containing an idempotent is a subgroup of S.

(3) If a and b are elements of S, then $ab \in \mathcal{R}_a \cap \mathcal{L}_b$ if and only if $\mathcal{R}_b \cap \mathcal{L}_a$ contains an idempotent. In that case:

$$a\mathcal{H}_b = \mathcal{H}_a b = \mathcal{H}_a\mathcal{H}_b = \mathcal{H}_{ab} = \mathcal{R}_a \cap \mathcal{L}_b.$$

(4) Let $e\mathcal{D}f$ be idempotents of S. Then there exist $a, a' \in S$ such that the mappings $x \mapsto a'xa$ and $y' \mapsto aya'$ are mutually inverse isomorphisms of $\mathcal{H}_e$ upon $\mathcal{H}_f$, and of $\mathcal{H}_f$ upon $\mathcal{H}_e$, respectively.

There is a wide class of semigroups, for which the Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide, which allows one to use the egg-box patterns on the $\mathcal{J}$-classes. While this class has not been completely described yet, certain important examples can be considered.

Definition 2.6 *A semigroup S is called π-**regular** if and only if a power of every element of S is regular. If a power of every element of S belongs to a subgroup of S, we say that S is **strongly** π-**regular** ($s\pi r$).*

Theorem 2.7 ([10, 1.4]) *Let S be a $s\pi r$ semigroup. Then $\mathcal{J} = \mathcal{D}$ on S. Moreover $a\,\mathcal{J}\,ab\,\mathcal{J}\,b$ if and only if $a\mathcal{L}e\mathcal{R}b$, for some idempotent e in S, and also $a\,\mathcal{J}\,ba\,\mathcal{J}\,b$ if and only if $a\mathcal{R}e\mathcal{L}b$, for some idempotent e in S.*

The following corollary follows from Theorems 2.5 and 2.7.

Collorary 2.8 *Assume that S is a $s\pi r$ semigroup. Let e, f be idempotents of S such that $e\,\mathcal{J}\,f$. If $e = ef = fe$, then $e = f$.*

Proof Since $e\,\mathcal{J}\,ef\,\mathcal{J}\,f$, Theorem 2.7 yields that $\mathcal{L}_e \cap \mathcal{R}_f$ contains an idempotent. It follows from the assertion (3) of Theorem 2.5 that $\mathcal{R}_e \cap \mathcal{L}_f$ contains $ef = e$. Thus $f\mathcal{L}e$. Moreover, $e\,\mathcal{J}\,fe\,\mathcal{J}\,f$. Thus $\mathcal{R}_e \cap \mathcal{L}_f$ contains an idempotent and, as before, $\mathcal{L}_e \cap \mathcal{R}_f$ contains $ef = e$. Thus $f\mathcal{R}e$. As a result, the idempotents e and f belong to the same $\mathcal{H}$-class of S. Since every $\mathcal{H}$-class may contain no more than one idempotent (the assertion (1) of Theorem 2.5), $e = f$. ■

Theorem 2.9 ([3, 2.11]) *If S is a $s\pi r$ semigroup then the following conditions are equivalent:*

- *an element $a \in S$ is regular,*
- *every element of $\mathcal{J}_a$ is regular.*

Collorary 2.10 *Let S be $s\pi r$ and take any $\mathcal{J}$-class J of S. If J contains an idempotent then it is regular.*

* * *

Definition 2.11 *Assume that S is a semigroup with zero. A two-sided ideal M of S is called* ***0-minimal*** *if $M \neq \{0\}$ and $\{0\}$ is the only two-sided ideal of S properly contained in M.*

Definition 2.12 *A semigroup S is called:*

- ***0-simple*** *if and only if $S^2 \neq \{0\}$ and every ideal in S is equal either to S or to $\{0\}$,*
- *a* ***null semigroup*** *if $S \neq \{0\}$ and $S^2 = \{0\}$.*

Let a be a non-zero element of a semigroup S with 0. By $J(a)$ we denote the principal ideal S^1aS^1 of S generated by a, and by $\mathcal{J}_a$ the $\mathcal{J}$-class containing a, that is, the set of generators of $J(a)$. Let $I(a) = J(a) \setminus \mathcal{J}_a$. Then $I(a)$ is an ideal of S. The quotient $J(a)/I(a)$ is called a **principal factor** of S. Note that $J(a)/I(a) = \mathcal{J}_a \cup \{\theta\}$. Each principal factor of S is either 0-simple or null (see Lemma 2.39 in [3]). By a **principal series of a semigroup** S we mean a chain of ideals of S:

$$S = S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} = \{0\}, \tag{2.1}$$

such that there is no ideal of S strictly between S_i and S_{i+1}, for $i = 1, 2, \ldots, m$. By the **principal factors of the series** (2.1) we mean the Rees factors S_i/S_{i+1} $(i = 1, 2, \ldots, m)$. The factors of a principal series are isomorphic in some order to the principal factors of S (see, [3, 2.40]). Consequently, any two principal series of S have isomorphic factors. Of course, not all semigroups admit a principal series, since there exist semigroups with infinite chains of ideals.

2.2 The multiplicative monoid of $\mathrm{M_n}(\mathbb{K})$

In this paragraph we recall well known facts concerning the semigroup structure of the multiplicative monoid $\mathrm{M_n}(\mathbb{K})$. We start with a description of $\mathcal{J}$-, $\mathcal{L}$- and $\mathcal{R}$-classes of $\mathrm{M_n}(\mathbb{K})$. Next, we obtain the ideal structure of this monoid. As a corollary we prove that $\mathrm{M_n}(\mathbb{K})$ is regular and $s\pi$r, which implies that the Green's relations $\mathcal{J}$ and $\mathcal{D}$ coincide in this monoid.

By $\mathrm{Gl_n}(\mathbb{K})$ we denote the group of invertible matrices over the field $\mathbb{K}$. Then $\mathrm{Gl_n}(\mathbb{K})$ is obviously a subgroup of the monoid $\mathrm{M_n}(\mathbb{K})$. By $\mathrm{r}(a)$ we denote the rank of an element $a \in \mathrm{M_n}(\mathbb{K})$.

Proposition 2.13 *The following conditions are equivalent for* $a, b \in \mathrm{M_n}(\mathbb{K})$*:*

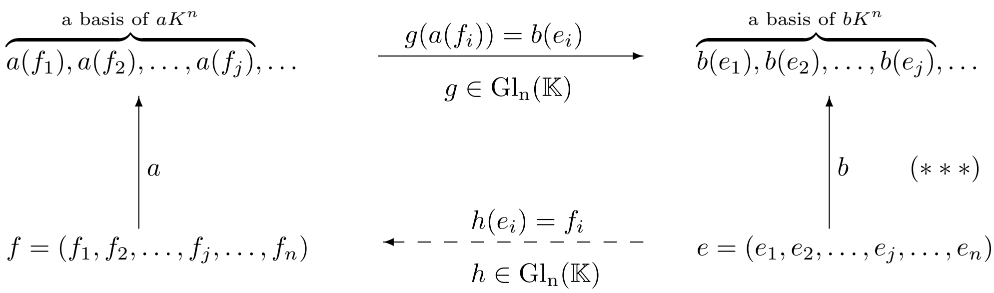
(1) $\mathrm{Gl_n}(\mathbb{K})\, a\, \mathrm{Gl_n}(\mathbb{K}) = \mathrm{Gl_n}(\mathbb{K})\, b\, \mathrm{Gl_n}(\mathbb{K})$.

(2) $\mathrm{M_n}(\mathbb{K})\, a\, \mathrm{M_n}(\mathbb{K}) = \mathrm{M_n}(\mathbb{K})\, b\, \mathrm{M_n}(\mathbb{K})$.

(3) $\mathrm{r}(a) = \mathrm{r}(b)$.

Proof It is clear that (1) $\Rightarrow$ (2). Assume that, according to (2), there exist matrices $p, q, r, s \in \mathrm{M_n}(\mathbb{K})$ such that $a = pbq$, $b = ras$. Observe that: $\mathrm{r}(a) \geq \mathrm{r}(ras) = \mathrm{r}(b)$. Analogously, one verifies that $\mathrm{r}(b) \geq \mathrm{r}(a)$. Hence $\mathrm{r}(a) = \mathrm{r}(b)$. Thus (2) $\Rightarrow$ (3). Now assume that $\mathrm{r}(a) = \mathrm{r}(b) = j$. The $\mathbb{K}$-spaces spanned by the columns of a and b are therefore isomorphic. We denote these spaces by $a\,\mathbb{K}^n$ and $b\,\mathbb{K}^n$. There exists $g \in \mathrm{Gl_n}(\mathbb{K})$ such that $ga\,\mathbb{K}^n = b\,\mathbb{K}^n$.

Let $E = \{e_1, \ldots, e_n\}$ be a basis of $\mathbb{K}^n$ such that $\{b(e_1), \ldots, b(e_j)\}$ is a basis of $b\,\mathbb{K}^n$. Having fixed E, we can choose a basis $F = \{f_1, \ldots, f_n\}$ of $\mathbb{K}^n$ such that $ga(f_i) = b(e_i)$, for $1 \leq i \leq j$. In short, we choose bases E, F of $\mathbb{K}^n$ such that the images, under a, b respectively, of the first j elements in E, F form the bases E', F' of $a\,\mathbb{K}^n$ and $b\,\mathbb{K}^n$. Note that g restricted to the first j rows and the first j columns is the transition matrix from E' to F':



It is therefore obvious that by taking the transition matrix $h \in \mathrm{Gl_n}(\mathbb{K})$ from E to F we receive the following equality: $b = gah$. Hence $b \in \mathrm{Gl_n}(\mathbb{K})\, a\, \mathrm{Gl_n}(\mathbb{K})$. Similarly, $a \in \mathrm{Gl_n}(\mathbb{K})\, b\, \mathrm{Gl_n}(\mathbb{K})$. Thus $\mathrm{Gl_n}(\mathbb{K})\, a\, \mathrm{Gl_n}(\mathbb{K}) = \mathrm{Gl_n}(\mathbb{K})\, b\, \mathrm{Gl_n}(\mathbb{K})$. ■

The last lemma yields the following description of the Green's relations on the monoid $\mathrm{M_n}(\mathbb{K})$:

Proposition 2.14 *The Green's relations on the monoid* $\mathrm{M_n}(\mathbb{K})$ *are of the following form:*

$$\begin{array}{lclcl} a\,\mathcal{J}\,b & \iff & \mathrm{Gl_n}(\mathbb{K})\, a\, \mathrm{Gl_n}(\mathbb{K}) = \mathrm{Gl_n}(\mathbb{K})\, b\, \mathrm{Gl_n}(\mathbb{K}) & \iff & \mathrm{r}(A) = \mathrm{r}(B), \\ a\mathcal{R}b & \iff & a\, \mathrm{Gl_n}(\mathbb{K}) = b\, \mathrm{Gl_n}(\mathbb{K}) & \iff & \mathrm{im}(a) = \mathrm{im}(b), \\ a\mathcal{L}b & \iff & \mathrm{Gl_n}(\mathbb{K})\, a = \mathrm{Gl_n}(\mathbb{K})\, b & \iff & \ker(a) = \ker(b). \end{array} \tag{2.2}$$

PROOF We prove the Proposition separately for each of the Green's relations:

- $a \mathcal{J} b \iff \mathrm{Gl_n}(\mathbb{K})\, a\, \mathrm{Gl_n}(\mathbb{K}) = \mathrm{Gl_n}(\mathbb{K})\, b\, \mathrm{Gl_n}(\mathbb{K}) \iff \mathrm{r}(a) = \mathrm{r}(b)$.

  This fact is an immediate consequence of Proposition 2.13.

- $a\mathcal{R}b \iff a\, \mathrm{Gl_n}(\mathbb{K}) = b\, \mathrm{Gl_n}(\mathbb{K}) \iff \mathrm{im}(a) = \mathrm{im}(b)$.

  The arguments are similar to the proof of implication (3) $\Rightarrow$ (1) in Proposition 2.13. Assume that $\mathrm{im}(a) = \mathrm{im}(b)$. The subspaces spanned by the columns of these matrices are therefore identical. Assuming the notation from $(***)$ we can take $g = \mathrm{id}$. Thus $ah = b$, $bh^{-1} = a$ for some $h \in \mathrm{Gl_n}(\mathbb{K})$. Therefore $a\, \mathrm{Gl_n}(\mathbb{K}) = b\, \mathrm{Gl_n}(\mathbb{K})$. It is clear that $a\, \mathrm{Gl_n}(\mathbb{K}) = b\, \mathrm{Gl_n}(\mathbb{K}) \Rightarrow a\, \mathrm{M_n}(\mathbb{K}) = b\, \mathrm{M_n}(\mathbb{K})$. It is left to prove that $a\mathcal{R}b \Rightarrow \mathrm{im}(a) = \mathrm{im}(b)$. However $\mathrm{im}(a) = (a\, \mathrm{M_n}(\mathbb{K}))\, \mathbb{K}^n = (b\, \mathrm{M_n}(\mathbb{K}))\, \mathbb{K}^n = \mathrm{im}(b)$.

- $a\mathcal{L}b \iff \mathrm{Gl_n}(\mathbb{K})\, a = \mathrm{Gl_n}(\mathbb{K})\, b \iff \ker(a) = \ker(b)$.

  Assume that $\ker(a)$ is of dimension $n - j$ and that it is equal to $\ker(b)$. Choose a basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathbb{K}^n$ such that the last $n-j$ vectors form a basis of $\ker(a)$. Since $\ker(a) = \ker(b)$, $b(u_i) = 0$, for $i > j$, and the images under a and b of the vectors u_i, for $i \leq j$, are linearly independent. Thus, there exists $g \in \mathrm{Gl_n}(\mathbb{K})$ such that $g(b(u_i)) = a(u_i)$, for $i \leq j$. Hence $gb = a$. Therefore we obtain $\mathrm{Gl_n}(\mathbb{K})\, a = \mathrm{Gl_n}(\mathbb{K})\, b$. The implication $\mathrm{Gl_n}(\mathbb{K})\, a = \mathrm{Gl_n}(\mathbb{K})\, b \Rightarrow \mathrm{M_n}(\mathbb{K})\, a = \mathrm{M_n}(\mathbb{K})\, b$ is clear. If $\mathrm{M_n}(\mathbb{K})\, a = \mathrm{M_n}(\mathbb{K})\, b$, then clearly $b(\ker(a)) = 0$ and $a(\ker(b)) = 0$. Hence $\ker(a) \subseteq \ker(b) \subseteq \ker(a)$. The result follows.

■

We are ready to give a description of the ideals of the monoid $\mathrm{M_n}(\mathbb{K})$.

Proposition 2.15 *Let $M_j = \{a \in \mathrm{M_n}(\mathbb{K}) \mid \mathrm{r}(a) \leq j\}, j = 0, 1, 2, \ldots, n$. Then M_j is a semigroup ideal of $\mathrm{M_n}(\mathbb{K})$ and*

$$\mathrm{M_n}(\mathbb{K}) \supset M_{n-1} \supset \ldots \supset M_1 \supset M_0 = \{0\} \tag{2.3}$$

is the lattice of semigroup ideals in $\mathrm{M_n}(\mathbb{K})$.

PROOF If a matrix a belongs to $I \lhd \mathrm{M_n}(\mathbb{K})$, then its $\mathcal{J}$-class $\mathcal{J}_a$ clearly belongs to I as well. Indeed, if $b \in \mathcal{J}_a$ then, according to Proposition 2.14,

$$b \in \mathrm{M_n}(\mathbb{K})\, a\, \mathrm{M_n}(\mathbb{K}) \subseteq \mathrm{M_n}(\mathbb{K})\, I\, \mathrm{M_n}(\mathbb{K}) \subseteq I.$$

Obviously, in $\mathrm{M_n}(\mathbb{K})\, a\, \mathrm{M_n}(\mathbb{K})$ we can find matrices of any rank less or equal to the rank of a. Therefore, I contains all matrices whose rank is less than or equal to the rank of a. Since $\mathrm{r}(ab) \leq \min\{\mathrm{r}(a), \mathrm{r}(b)\}$ it follows that M_i, for $0 \leq i \leq n$, are the only ideals in $\mathrm{M_n}(\mathbb{K})$. The inclusions of (2.3) are now clear. ■

The principal factors of $\mathrm{M_n}(\mathbb{K})$ are precisely the Rees quotients M_j/M_{j-1}, where $j = 1, 2, \ldots, n$, which are the sets of matrices of rank j with zero adjoined (note that this adjoined zero is not a zero matrix). It is clear that M_j/M_{j-1}

are not null semigroups and as such they must be 0-simple.

The last part of this brief overview of basic properties of $\mathrm{M_n}(\mathbb{K})$ concerns the problems of regularity and strong π-regularity of this monoid. Proposition 2.13 yields that very $\mathcal{J}$-class of $\mathrm{M_n}(\mathbb{K})$ is of the form Ge_kG, where $G = \mathrm{Gl_n}(\mathbb{K})$ and e_k is the diagonal idempotent of rank k, where $0 \leq k \leq n$. Take a non-zero element $a = ge_kh$, where $g, h \in \mathrm{Gl_n}(\mathbb{K})$ and $k > 0$. Then a is a regular element in $\mathrm{M_n}(\mathbb{K})$ since $(ge_kh)(h^{-1}e_kg^{-1})(ge_kh) = ge_kh$. Thus $\mathrm{M_n}(\mathbb{K})$ is a regular semigroup. We will also need to be able to identify the $\mathcal{J}$- and $\mathcal{D}$-classes of $\mathrm{M_n}(\mathbb{K})$. According to Theorem 2.7 it is enough to prove that $\mathrm{M_n}(\mathbb{K})$ is sπr.

Proposition 2.16 *The multiplicative monoid of* $\mathrm{M_n}(\mathbb{K})$ *is strongly* π*-regular.*

Proof The space $\mathbb{K}^n$ may be viewed as as a left $\mathrm{M_n}(\mathbb{K})$-module. We interpret $\mathrm{M_n}(\mathbb{K})$ as the endomorphism ring $\mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$. If $a \in \mathrm{M_n}(\mathbb{K})$ then, by Fitting's lemma ([9, 5.3]), there exist subspaces V, W of $\mathbb{K}^n$ such that $\mathbb{K}^n = V \oplus W$ and for every $k \geq n$ we have $a^kV = 0$ and a^k maps isomorphically W onto W. Let $f \in \mathrm{M_n}(\mathbb{K})$ be the map such that $f|_W = \mathrm{id}_W$ and $fV = 0$. Consider the following set:

$$M = \{x \in \mathrm{M_n}(\mathbb{K}) \mid xW = W,\, xV = 0\}.$$

It is obviously a group (isomorphic to $\mathrm{Gl}(W)$) with identity equal to f. Hence a^k belongs to a subgroup of $\mathrm{M_n}(\mathbb{K})$. ■

2.3 Algebraic monoids

For an introduction to algebraic geometry and the theory of linear algebraic groups we refer to [5]. We also recall certain results from [10].

Definition 2.17 *A subset $X \subseteq \mathbb{K}^n$ is* ***closed*** *if it is the zero set of a collection of polynomials in $\mathbb{K}[x_1, x_2, \ldots, x_n]$. If $X \subseteq \mathbb{K}^n$ then the smallest closed set $Y \subseteq \mathbb{K}^n$ such that $X \subseteq Y$ is called the closure $\overline{X}$ of X. The topology defined with regard to these closed sets is called the* ***Zariski topology****.*

Definition 2.18 *Assume that $X \subseteq \mathbb{K}^n$ is closed. By the* ***coordinate ring*** *of X we mean:*

$$\mathbb{K}[X] = \mathbb{K}[x_1, x_2, \ldots, x_n]/I,$$

where $I = \{f \in \mathbb{K}[x_1, x_2, \ldots, x_n] \mid f(X) = 0\}$.

The coordinate ring of X should be thought of as the ring of polynomial functions on X. In fact, for any $P \in X$ an element $f \in \mathbb{K}[X]$ determines a polynomial map $X \to \mathbb{K}$ (usually also denoted by f) given by $f \mapsto f(P)$. This is well-defined, because all functions in I vanish on X by definition. If the function $f : X \to \mathbb{K}$ is identically zero, then $f \in I$, by definition, so $f = 0$ in $\mathbb{K}[X]$.

Definition 2.19 *Let $X \subseteq \mathbb{K}^n$ and $Y \subseteq \mathbb{K}^m$ be closed sets. Then a map*

$$\phi = (\phi_1, \dots, \phi_m) : X \to Y$$

*is a **morphism** if each $\phi_i \in \mathbb{K}[X]$. If this is the case then there is a natural homomorphism of $\mathbb{K}$-algebras $\phi^* : \mathbb{K}[Y] \to \mathbb{K}[X]$. We say that ϕ is **dominant** if $\overline{\phi(X)} = Y$.*

Definition 2.20 *Let X be a topological space. Then*

- *X is **irreducible** if X is non-empty and X is not a union of two proper closed subsets.*
- *X is **Noetherian** if it satisfies the descending chain condition on closed subsets.*
- *If X is expressible as a finite union of maximal irreducible (closed) subsets, we call these subsets the **irreducible components** of X. Every Noetherian space is a union of its irreducible components.*
- *Assume that X is irreducible. We say that the natural number $\dim(X)$ is the **dimension of** X if:*

  $$\dim(X) = \{\sup_{n \in \mathbb{N}} n : X \supset X_1 \supset X_2 \supset \ldots \supset X_n\},$$

  where X_i are closed and irreducible. The dimension of a Noetherian space is the supremum of the dimensions of the irreducible components of X.

Definition 2.21 *An **affine variety** is a closed subset of some $\mathbb{K}^n$. A **quasi-affine variety** is an open subset of an affine variety.*

Both classes of topological spaces that appear in Definition 2.21 belong to a more general class of the, so-called, algebraic varieties, which need not to be defined here (see [5, §2]). In this paper, we say that a topological space X is a **variety** only if X is either an affine, or a quasi-affine variety.

Example 2.22 *Let us give some important examples of varieties. Recall that for a matrix $a \in \mathrm{M_n}(\mathbb{K})$ by $\mathrm{r}(a)$ we denote the rank of a.*

(i) The following subset $M_r \subseteq \mathrm{M_n}(\mathbb{K})$ is an affine irreducible variety

$$M_r = \{a \in \mathrm{M_n}(\mathbb{K}) \mid \mathrm{r}(a) \leq r\}.$$

Note that M_r is the set of commons zero of all $(r+1) \times (r+1)$ minors.

(ii) The following subset J_r is a quasi-affine irreducible variety:

$$J_r = \{a \in \mathrm{M_n}(\mathbb{K}) \mid \mathrm{r}(a) = r\},$$

as it is equal to $M_{r+1} \setminus M_r$.

Proposition 2.23 ([10, 2.14]) *The dimension of* $\mathbb{K}^n$ *is equal to* n*. Let* X *be an irreducible variety. Then:*

- *If* $X' \neq X$ *is a non-empty closed subset of* X*, then* $\dim(X') < \dim(X)$*.*
- *If* U *is a non-empty open subset of* X*, then* $\overline{U} = X$ *and* $\dim(U) = \dim(X)$*.*

Theorem 2.24 ([10, 2.19, 2.21]; [5, 4.4]) *Let* $\phi : X \to Y$ *be a dominant morphism of irreducible varieties and* $\dim(X) = n, \dim(Y) = m$*. Then:*

(i) $\phi(X)$ *contains a non-empty open subset of* $\overline{\phi(X)}$*.*

(ii) $m \leq n$*.*

(iii) Let W *be a closed irreducible subset of* Y*. If* Z *is an irreducible component of* $\phi^{-1}(W)$ *such that the restriction of* $\phi|_Z : Z \to W$ *is dominant, then* $\dim(Z) \geq \dim(W)+n-m$*. In particular if* $y \in \phi(X)$*, then each component of* $\phi^{-1}(y)$ *has dimension at least* $n - m$*.*

(iv) Let $\epsilon_\phi(x)$ *be the maximum dimension of any component of* $\phi^{-1}(\phi(x))$ *containing* $x \in X$*. Then for all positive integers* n *the following subsets of* X *are closed in* X*:*

$$\{x \in X \,|\, \epsilon_{\phi(x)} \geq n\}.$$

Definition 2.25 *A (linear) algebraic semigroup* $S = (S, \circ)$ *is an (affine) variety* S *along with an associative product map* $\circ : S \times S \to S$ *which is also a morphism of varieties. A homomorphism between algebraic semigroups* S, S' *is a semigroup homomorphism* $\Psi : S \to S'$ *which is also a morphism of varieties.*

Definition 2.26 *A connected semigroup* S *is a linear algebraic semigroup whose underlying variety is irreducible.*[1]

Any finite dimensional algebra over $\mathbb{K}$ is a connected algebraic semigroup. In particular, the matrix algebra $\mathrm{M_n}(\mathbb{K})$ belongs to this class. By G we denote the group of invertible elements in $\mathrm{M_n}(\mathbb{K})$, namely $\mathrm{Gl_n}(\mathbb{K})$. Let $X \subseteq \mathrm{M_n}(\mathbb{K})$ and

$$E(X) = \{x \in X \,|\, x^2 = x\}.$$

The following lemma will be needed in the next Section.

Lemma 2.27 *Assume that* $J = Ge_kG$ *is the* $\mathcal{J}$*-class of the diagonal idempotent of rank* k *in* $\mathrm{M_n}(\mathbb{K})$*. Let*

$$U(J) = \{a \in J \,|\, a^2 \in J\}.$$

Then $U(J)$ *is a non-empty open subset of* J*, hence an irreducible variety, and we have an onto (hence dominant) morphism*

$$\theta_J : U(J) \to E(J).$$

In particular, $E(J)$ *is irreducible.*

[1] For algebraic groups, the topological terms 'irreducible' and 'connected' coincide. It is not the case in the class of algebraic semigroups.

Proof Let $a \in \mathrm{M_n}(\mathbb{K})$. If the characteristic polynomial of a is

$$(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n),$$

let:

$$\rho_m(a) = \sum_{i_1<i_2<\dots<i_m} \alpha_{i_1}\dots\alpha_{i_m},$$

for $m = 0, 1, \dots, n$. Since $(-1)^m \rho_m(a)$ is a coefficient of the characteristic polynomial, $\rho_m : \mathrm{M_n}(\mathbb{K}) \to \mathbb{K}$ is a morphism. Let

$$E(J) = \{e \in M_k \mid e^2 = e, \rho_k(e) = 1\}.$$

Then E_r is a closed subset of $\mathrm{M_n}(\mathbb{K})$. Let:

$$U(J) = \{a \in J \mid a^2 \in J\}.$$

Then $U(J)$ is a non-empty open subset of J and hence an irreducible variety. Let $a \in U$. Then the characteristic polynomial of a is:

$$x^n + \beta_1 x^{n-1} + \dots + \beta_k x^{n-k}$$

with:

$$\beta_i = (-1)^i \rho_i(a), \quad i = 1, 2, \dots, k, \text{ and } \beta_k \neq 0.$$

Observe that if $a \in U(J)$, then $\mathcal{H}_a \subseteq U(J)$. Then in the $\mathcal{H}$-class (which is a group) of a in $\mathrm{M_n}(\mathbb{K})$ we have the following element:

$$a^{-1} = \frac{-1}{\beta_k}(a^{k-1} + \beta_1 a^{k-2} + \dots + \beta_{k-1}).$$

The map $a \mapsto a^{-1}$ is a morphism of the variety $U(J)$. Note that $aa^{-1} \in E(J)$. We therefore have a surjective morphism $\theta_J : U(J) \to E(J)$ given by:

$$\theta_J(a) = aa^{-1}.$$

As $U(J)$ is irreducible, also $E(J)$ is irreducible. ■

3 Closure semigroups

In this section we consider the ideal structure of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. The structure of this monoid is more complicated than the one of $\mathrm{M_n}(\mathbb{K})$. Yet, as we will see, certain similarities occur. When dealing with semigroups with infinite chains of ideals, the notion of 0-simple semigroups is not satisfactory, thus we introduce a more appropriate class.

Definition 3.1 *A **completely** 0**-simple semigroup** S is a* 0*-simple semigroup containing a primitive idempotent e. Alternatively, a semigroup is called completely* 0*-simple if it is* 0*-simple and $s\pi r$, see [10, 1.7].*

Assume that $\mathcal{S} = \{S_t\}_{t\in T}$ is a family of subsemigroups of a semigroup S with zero such that for any two members S', S'' of $\mathcal{S}$, $S' \cap S'' = \{0\}$. The union of all members of $\mathcal{S}$ is called a **0-disjoint union of semigroups** $\{S_t\}_{t\in T}$.

The aim of this section if to prove the following theorem.

Theorem 3.2 *The semigroup $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ has a finite ideal chain with each factor either a nilpotent semigroup or a 0-disjoint union of completely 0-simple semigroups.*

It is important to note, that the 0-disjoint unions that appear in the finite ideal chain of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ may consist of infinitely many summands, which makes this particular semigroup different from the multiplicative semigroup $(A, \cdot)$ of any finite dimensional unitary algebra A over an algebraically closed field $\mathbb{K}$. It is known that $(A, \cdot)$ is a connected, sπr (see [10, 3.18]) algebraic monoid, since $(A, \cdot)$ is isomorphic to a closed submonoid of $\mathrm{M_n}(\mathbb{K})$, for some n. It follows from [6, 2.14, 3.5] that $(A, \cdot)$ admits a finite chain of ideals with each factor either a nilpotent semigroup or a completely 0-simple semigroup.

Definition 3.3 *Let M be a connected linear algebraic monoid defined over an algebraically closed field. By the* ***closure monoid of*** *M we mean the set $\mathcal{C}(M)$ of all closed irreducible non-empty subsets of M with multiplication:*

$$X \cdot Y = \overline{XY},$$

where $\overline{Z}$ stands for the closure of $Z \subseteq M$. Note that M is the submonoid of $\mathcal{C}(M)$.

Observe that a map $\mu : \mathcal{C}(\mathrm{M_n}(\mathbb{K})) \to \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ defined by

$$\mu(X) = \mathrm{lin}_{\mathbb{K}}(X) \tag{3.1}$$

is an onto homomorphism because $\overline{X} \subseteq \mathrm{lin}_{\mathbb{K}}(X)$, for every $X \subseteq \mathrm{M_n}(\mathbb{K})$. Thus the proof of the main result reduces to the following fact:

Theorem 3.4 *The semigroup $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ has a finite ideal chain with each factor either a nilpotent semigroup or a 0-disjoint union of completely 0-simple semigroups.*

Before we proceed to the proof let us note that Theorem 3.4 holds for any finite dimensional algebra A, instead of $\mathrm{M_n}(\mathbb{K})$. We give a general idea behind the proof. According to Theorem 2.9 we can consider the regular $\mathcal{J}$-classes of A. If we take a regular $\mathcal{J}$-class J of $(A, \cdot)$ then it can be easily proved that J is an irreducible variety and that the set $\mathcal{C}(J)$ of all closed irreducible or empty subsets of J with operation $X \cdot Y = \overline{XY \cap J}$ is a semigroup. It follows from [10, 3.28] that $(A, \cdot)$ has finitely many regular $\mathcal{J}$-classes $J_1, J_2, \ldots, J_k$ and that there exist a finite chain of ideals

$$(A, \cdot) = I_1 \supset I_2 \supset \ldots \supset I_m \supset I_{m+1} = \{0\} \tag{3.2}$$

such that every factor of (3.2) is either nilpotent of isomorphic to $\mathcal{C}(J_i)$, for some $i \in \{1, 2, \dots, k\}$. So the (generalized) assertion of Theorem 3.4 reduces to the case of the semigroup $\mathcal{C}(J_i)$. However, technical details occur if we try to prove these results in full generality, so we restrict our interest to $\mathrm{M_n}(\mathbb{K})$ only.

* * *

We shall prove Theorem 3.4 in the following steps.

- Step 1. We prove that:

$$\mathcal{C}(\mathrm{M_n}(\mathbb{K})) = I_n \supset I_{n-1} \supset \dots \supset I_1 \supset I_0 = \{0\},$$

  where I_i/I_{i-1} are isomorphic to $\mathcal{C}(J_i)$ and J_i are equal to the sets of matrices of rank i.

- Step 2. Let J be a $\mathcal{J}$-class of $\mathrm{M_n}(\mathbb{K})$. We construct a finite set of ideals $\{\mathcal{C}_i \mid 1 \leq i \leq q\}$ of $\mathcal{C}(J)$ such that

$$\mathcal{C}(J) = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \dots \supset \mathcal{C}_q \supset \mathcal{C}_{q+1} = \{0\}.$$

- Step 3. Let $\widetilde{\mathcal{C}_i} = \mathcal{C}_i/\mathcal{C}_{i+1}$, for $i = 0, 1, \dots, q$. Then $\widetilde{\mathcal{C}_i}$ has a finite ideal chain with each factor either a nilpotent semigroup or a 0-disjoint union of completely 0-simple semigroups.

We begin with Step 1. From Proposition 2.13 we know that there are exactly n non-zero $\mathcal{J}$-classes in $\mathrm{M_n}(\mathbb{K})$, and each $\mathcal{J}$-class J_d consists of all matrices of a fixed rank $0 < d \leq n$, that is $J_d = Ge_dG$, where e_d is the diagonal matrix of rank d, for $d = \{1, 2, \dots, n\}$. Let $J_0 = \{0\}$. Recall that the principal series of $\mathrm{M_n}(\mathbb{K})$ is of the following form:

$$\mathrm{M_n}(\mathbb{K}) = M_n \supset M_{n-1} \supset \dots \supset M_1 \supset \{0\},$$

where

$$M_k = \bigcup_{i=0}^{k} Ge_iG,\ k = 0, 1, \dots, n.$$

Consider the following subsets of $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$:

$$I_j := \{X \in \mathcal{C}(\mathrm{M_n}(\mathbb{K})) \mid X \subseteq M_j\},\ \text{where}\ j = 0, 1, \dots, n.$$

Observe that I_j are ideals in $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$. Moreover $I_{j+1}/I_j \simeq \mathcal{C}(Ge_{j+1}G)$, for $j = 0, 1, \dots, n-1$. We obtain the following series of inclusions:

$$\mathcal{C}(\mathrm{M_n}(\mathbb{K})) = I_n \supset I_{n-1} \supset \dots \supset I_1 \supset I_0 = \{0\}. \tag{3.3}$$

That concludes the considerations of Step 1. In order to prove Theorem 3.4 it is, therefore, enough to show that $\mathcal{C}(Ge_kG)$ has a finite ideal chain with each factor either a nilpotent semigroup or a 0-disjoint union of completely 0-simple semigroups.

Step 2. Assume that e is any non-zero idempotent of $\mathrm{M_n}(\mathbb{K})$. According to Proposition 2.13, the $\mathcal{J}$-class of e is equal to $J := GeG$ and from Proposition 2.15 we deduce that $\mathcal{C}(GeG)$ is equal to one of the factors in (3.3). By $G(e)$ we denote the unit group of the monoid $e\,\mathrm{M_n}(\mathbb{K})\,e$. It is precisely the $\mathcal{H}$-class of e of J. According to [10, 3.26], $G(e)$ is an algebraic subgroup of J. Since $\mathcal{J} = \mathcal{D}$ in $\mathrm{M_n}(\mathbb{K})$, it follows from Green's Lemma that all $\mathcal{H}$-classes of J are closed in J. Let X be a non-zero element of $\mathcal{C}(J)$. By an $\mathcal{H}$-class of X we mean a non-empty intersection of X with an $\mathcal{H}$-class of J. Let $\underline{d}(X)$ denote the minimum of the dimensions of the $\mathcal{H}$-classes of X. Let $\underline{d}(0) = \infty$. Then clearly:

$$\underline{d}(X \cdot Y) \geq \max\{\underline{d}(X), \underline{d}(Y)\}, \qquad \text{for } X, Y \in \mathcal{C}(J). \tag{3.4}$$

Let $q := \underline{d}(J)$. For $0 \leq d \leq q$, let:

$$\mathcal{C}_d = \mathcal{C}_d(J) = \{X \in \mathcal{C}(J) \,|\, \underline{d}(X) \geq d\}.$$

Let $\mathcal{C}_{q+1} = \{0\}$. Then we clearly obtain the following series of ideals of $\mathcal{C}(J)$:

$$\mathcal{C}(J) = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \ldots \supset \mathcal{C}_q \supset \mathcal{C}_{q+1} = \{0\}. \tag{3.5}$$

Step 3. Note that the following partial order may be introduced in the set of all non-zero $\mathcal{J}$-classes of any semigroup S:

$$\mathcal{J}_1 \geq \mathcal{J}_2 \iff S^1 a S^1 \supseteq S^1 b S^1, \text{ for any } a \in \mathcal{J}_1, b \in \mathcal{J}_2\,.$$

It is clear that a regular $\mathcal{J}$-class is 0-simple (i.e. the ideal generated by this class is a 0-simple semigroup) only if it is a minimal element with respect to this ordering. The $\mathcal{J}$-classes $\mathcal{J}_1$ and $\mathcal{J}_2$ and said to **comparable** if and only if $\mathcal{J}_1 \leq \mathcal{J}_2$ or $\mathcal{J}_1 \geq \mathcal{J}_2$. We will prove the following fact.

Proposition 3.5 *Assume that J is a $\mathcal{J}$-class of $\mathrm{M_n}(\mathbb{K})$ and that $\underline{d}(J) = q$. Let $\widetilde{\mathcal{C}_i} = \mathcal{C}_i/\mathcal{C}_{i+1}$, for $i = 0, 1, \ldots, q$, be the factors of (3.5). Then the non-zero regular $\mathcal{J}$-classes of $\widetilde{\mathcal{C}_i}$ are not comparable and there exist ideals $\widetilde{I}$, $\widetilde{I}'$ of $\widetilde{\mathcal{C}_i}$ such that:*

$$\widetilde{\mathcal{C}_i} \supseteq \widetilde{I}' \supseteq \widetilde{I} \supseteq \{0\}$$

and

(1) $\widetilde{I}^5 = 0$.

(2) $\widetilde{I}'/\widetilde{I}$ *is a* 0*-disjoint union of its completely* 0*-simple ideals.*

(3) $\widetilde{\mathcal{C}_i}^5 \subseteq \widetilde{I}'$.

Proof We begin with the following lemma.

Lemma 3.6 *If J is a regular $\mathcal{J}$-class of $\mathcal{C}(J)$ then $J \cup \{0\}$ is completely* 0*-simple.*

Proof Let $\underline{J}$ be a non-zero regular $\mathcal{J}$-class of $\mathcal{C}(J)$. We have to prove that $\underline{J}$ is 0-simple and that it contains a primitive idempotent. Every non-zero regular $\mathcal{J}$-class $\underline{J}$ of $\mathcal{C}(J)$ contains an idempotent H. It is therefore enough to prove that:

(i) H may be chosen as a closed connected subgroup of the $\mathcal{H}$-class $\mathcal{H}_e$ of e.

(ii) The non-zero regular $\mathcal{J}$-classes of $\mathcal{C}(J)$ are in one-to-one correspondence with the conjugacy classes of closed connected subgroups of $\mathcal{H}_e$.

(iii) If H_1 and H_2 are closed connected subgroups of $\mathcal{H}_e$ such that $\underline{J}_1, \underline{J}_2$ are the $\mathcal{J}$-classes of H_1, H_2 respectively, then the following equivalence holds:

$$\underline{J}_1 \geq \underline{J}_2 \iff xH_1x^{-1} \subseteq H_2, \text{ for some } x \in \mathcal{H}_e.$$

(iv) $\underline{J}$ contains a primitive idempotent.

Let A be a non-zero idempotent in $\mathcal{C}(J)$. Then in $\mathrm{M_n}(\mathbb{K})$, $S = \overline{A}$ is a connected semigroup with $\overline{S^2} = S$. Let f be a maximal idempotent in S. By [10, 5.10], the Rees factor $S/\overline{SfS}$ is a nil semigroup and so it is nilpotent, see [6, 2.14]. Hence $S = \overline{SfS}$. In $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$, $S = S \cdot fSf \cdot S$. Thus $S\,\mathcal{J}\,fSf$ in $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$. So $A\,\mathcal{J}\,H$ in $\mathcal{C}(J)$, where the unit group H of fSf is an idempotent. Since $\mathrm{M_n}(\mathbb{K})$ is connected, all idempotents in J are conjugate, see [10, 6.1]. We may therefore assume that $f = e$, i.e. $H = A$ is the unit group of $f\overline{A}f$. Assertion (i) follows.

Let H_1 and H_2 be closed connected subgroups of $\mathcal{H}_e$. Suppose there exist $X, Y \in \mathcal{C}(J)$ such that $XH_1Y = H_2$. This implies that there exist x, y in $\mathcal{H}_e$ such that $xH_1y \subseteq H_2$. Then $xy \in H_2$ and $xH_1x^{-1} \subseteq H_2$. It follows that $H_1\,\mathcal{J}\,H_2$ if and only if H_1 and H_2 are conjugate. Thus (ii) and (iii) follow.

Suppose now that $E \in \mathcal{C}(J)$ is an idempotent such that $E \leq H$. According to Definition 3.1 we have to show that $H = E$. If $E \cdot H = H \cdot E = E$, then $E \subseteq eSe \cap J = H$. Therefore E is a closed connected subgroup of $\mathcal{H}_e$ contained in H. By the above, H must be a minimal idempotent in its $\mathcal{J}$-class and (iv) follows. ∎

Let $\widetilde{I}$ consist of all $X \in \widetilde{\mathcal{C}}_i$ with $\widetilde{\mathcal{C}}_i X \widetilde{\mathcal{C}}_i$ not containing a non-zero regular element. Clearly $\widetilde{I}$ is an ideal of $\widetilde{\mathcal{C}}_i$. Let $\widetilde{I}'$ denote the union of $\widetilde{I}$ and the regular elements of $\widetilde{\mathcal{C}}_i$. Incomparability of the regular $\mathcal{J}$-classes of $\mathcal{C}(J)$ implies that $\widetilde{\mathcal{C}}_i L \widetilde{\mathcal{C}}_i \subseteq L \cup \widetilde{I}$, for any regular $\mathcal{J}$-class L of $\widetilde{\mathcal{C}}_i$. It follows that $\widetilde{I}'$ is an ideal. Moreover $\widetilde{I}'/\widetilde{I}$ is regular. Hence (2) follows from Lemma 3.6.

Now we will prove (1) and (3). Both assertions follow easily from the following fact.

Observation 3.7 *Assume that $X_1, X_2, \ldots, X_5 \in \mathcal{C}(J)$ have the same $\underline{d}$-value $d < \infty$. Let $X = X_1X_2X_3X_4X_5$. If $\underline{d}(X) = d$ then $Y = X_2X_3X_4$ is a regular element of $\mathcal{C}(J)$.*

Proof We begin with the following lemma.

Lemma 3.8 *Assume that $X \in \mathcal{C}(J)$ and $\underline{d}(X) = d$. Then every component of an $\mathcal{H}$-class of X has dimension at least $\underline{d}(X)$.*

PROOF Recall from Lemma 2.27 that $U(J) = \{a \in J \mid a^2 \in J\}$ is an irreducible variety and that there exists a dominant (onto) morphism:

$$\theta_J : U(J) \to E(J). \tag{3.6}$$

Let $V(X)$ denote the union of all $\mathcal{H}$-classes of X with dimension equal to $\underline{d}(X)$. Let $a \in V(X)$. We know that $ag = e$, for some $g \in G = \mathrm{Gl_n}(\mathbb{K})$ and $e = e^2 \in E(J)$. Consider the map:

$$\theta : (Xg \cap U(J))g^{-1} \to E(J)$$

given by

$$\theta(x) = \theta_J(xg).$$

Then the set $\theta^{-1}(\theta(a))$ is equal to $\mathcal{H}_a \cap X$, thus it is the $\mathcal{H}$-class of X containing a. Thus the maximum dimension of any component of $\theta^{-1}(\theta(a))$ is equal to $\underline{d}(X)$. Therefore, it follows from Theorem 2.24 (iv), that the union of $\mathcal{H}$-classes of X contained in the domain of θ having dimension $\underline{d}(X)$ forms an open set O (since it is a complement of a closed set). Since $a \in O \subseteq V(X)$, it follows that $V(X)$ is an open subset of X. According to Theorem 2.24 (iii), every component of an $\mathcal{H}$-class of X has dimension at least $\underline{d}(X)$. ■

We follow [8, 2.1] to prove the entire Observation. Let $V = V(X)$, as in the proof of Lemma 3.8. Since V is open in X, we have $X = \overline{X_1X_2X_3X_4X_5}$, and there exist $a_i \in X_i$, $i = 1, 2, 3, 4, 5$, such that

$$c = a_1a_2a_3a_4a_5 \in V.$$

Replacing X_3 by X_3g and X_4 by $g^{-1}X_4$, for a suitable $g \in G = \mathrm{Gl_n}(\mathbb{K})$ we may assume that $a_3 = e \in E(J)$. Let H denote the $\mathcal{H}$-class of e in X_3. From the assumption on X_3 we know that $\dim(H) \geq d$. Let L denote the $\mathcal{H}$-class of c in X. Then $\dim(L) = d$, because $c \in V$. Now

$$a_1a_2Ha_4a_5 \subseteq a_1a_2X_3a_4a_5 \cap J \subseteq L. \tag{3.7}$$

Since X_3 is irreducible, $a_1a_2Xa_4a_5 \cap J$ is also irreducible and Lemma 3.8 yields $a_1a_2X_3a_4a_5 \cap J = a_1a_2Ha_4a_5$. This implies that H is irreducible and that $\dim(H) = d$. Let H' denote the $\mathcal{H}$-class of a_2 in X_2. Then, as above, H' is irreducible and $\dim(H') = d$. Now $H'H$ is contained in the $\mathcal{H}$-class of a_2e in X_2X_3. Again as above, $\overline{H'H}$ has dimension d. For some $g \in G, ga_2e = e$. So $H, gH'e$ are closed irreducible subsets of $\overline{gH'H}$. It follows that

$$H = gH'e = \overline{gH'H}.$$

Then $H = \overline{(gH'e)H} = \overline{HH}$, whence H is a connected group. Let

$$O_2 = \{x \in X_2 \mid a_1x \neq 0, xe \neq 0\},$$

$$O_4 = \{x \in X_4 \mid xa_5 \neq 0, ex \neq 0\}.$$

Then O_2, O_4 are open subsets of X_2, X_4 respectively, and $a_2 \in O_2$, $a_4 \in O_4$. So:

$$Y = \overline{O_2 X_3 O_4} \cap J. \tag{3.8}$$

Let $a \in O_2 X_3 O_4 \cap J$. Then $a \in x X_3 y$ for some $x \in O_2$, $y \in O_4$. Let H_0 denote the $\mathcal{H}$-class of a in Y. Then:

$$a_1 H_0 a_5 \subseteq a_1 U a_5 \cap L \subseteq L. \tag{3.9}$$

Since Y is irreducible, arguing as in (3.7), we get that H_0 is irreducible and $\dim(H_0) = d$. Comparing dimensions, we come to:

$$a \in H_0 = xHy.$$

Hence, in view of (3.8),

$$Y = \overline{O_2 H O_4}. \tag{3.10}$$

Proceeding as in (3.7), (3.9), we conclude that:

$$a_1 O_2 H = a_1 a_2 H, \quad H O_4 a_5 = H a_4 a_5. \tag{3.11}$$

There exist $g_1, g_2 \in G$ such that:

$$g_1 a_1 a_2 e = e, \quad e a_4 a_5 g_2 = e. \tag{3.12}$$

By (3.11), (3.12),

$$O_2 H O_4 a_5 g_2 g_1 a_1 O_2 H O_4 = O_2 H O_4 \tag{3.13}$$

By (3.10), (3.13), $YbY = Y$, where $b = a_5 g_2 g_1 a_1 \in J$. Therefore Y is regular in $\mathcal{C}(J)$.

■

The proof of (1) and (3) follows.

■

We have therefore proved Theorems 3.4 and 3.2. An important corollary follows.

Collorary 3.9 *Suppose that a semigroup S has an ideal chain*

$$S = S_0 \supset S_1 \supset \ldots \supset S_m \supset S_{m+1} = \{0\} \tag{3.14}$$

with each factor either a nilpotent semigroup or a 0-disjoint union of completely 0-simple semigroups. Then S is a $s\pi r$ semigroup. In particular, both $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ and $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ are $s\pi r$ semigroups.

Proof We proceed by induction on the indices of S_i. Take $s \in S_m$. If S_m is nilpotent, then s^n belongs to the zero subgroup of S. If S_m is a 0-disjoint union of completely 0-simple semigroups $S_{mt}, t \in T$, then s belongs to one of them, say S_{mt}. Since being a completely 0-simple semigroup requires $S_{mt} = \mathcal{J}_s \cup \{\theta\}$ to be $s\pi r$, it follows that there exists $n > 0$ such that s^n belongs to some subgroup G of $\mathcal{J}_s \cup \{\theta\} \subseteq S$. Clearly G is also a subgroup of S. The first step of induction follows.

Assume that, for some i, for every $s \in S_i$ there exists a natural number $n > 0$ such that s^n belongs to the subgroup of S. Take $s \in S_i/S_{i+1}$. If S_i/S_{i+1} is nilpotent then $s^n \in S_{i+1}$, for some $n > 0$. From the induction hypothesis there exists such $m > 0$ that $(s^n)^m = s^{nm}$ belongs to a subgroup of S. If, however, S_i/S_{i+1} is a 0-disjoint union of completely 0-simple semigroups $S_{iw}, w \in W$, then s belongs to one of them and the argument is as in the first case. The result follows.

■

4 Unitary subalgebras of $\mathrm{M_n}(\mathbb{K})$

In this last Section we consider certain properties of unitary subalgebras that can be recovered from the structure of the subspace semigroup $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. We explain the reasons for $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ to have regular factors with many (infinitely many, if $n > 3$) 0-disjoint completely 0-simple summands. This property follows from the connection between the regular $\mathcal{J}$-classes and the conjugacy classes of basic subalgebras of $\mathrm{M_n}(\mathbb{K})$.

First, we will recall few notions from the representation theory of finite dimensional algebras. As stated in the beginning, we assume the field $\mathbb{K}$ is algebraically closed. The finite dimensional algebra A with identity 1 is called **semisimple** if and only if for every submodule N of an A-module M there exists a submodule N' of M such that $M = N \oplus N'$.

Theorem 4.1 (Wedderburn) *Assume A is a finite dimensional algebra with identity over an algebraically closed field $\mathbb{K}$ such that the largest nilpotent ideal of A is equal to zero. Then A is semisimple and there exist positive integers $s_1, s_2, \dots, s_n$ such that:*

$$A \simeq \mathrm{M_{s_1}}(\mathbb{K}) \oplus \mathrm{M_{s_2}}(\mathbb{K}) \oplus \ldots \oplus \mathrm{M_{s_n}}(\mathbb{K}) \,.$$

The largest nilpotent ideal of A will be denoted by $J(A)$. Recall that an element $e \in A$ is called an **idempotent** if $e^2 = e$. An idempotent e is called **primitive** if Ae is an **indecomposable** A**-module**, that is a non-zero A-module such that there exist no non-zero A-modules N, N' with $Ae = N \oplus N'$.

Definition 4.2 *A ring R is called a* ***division ring*** *if and only if R satisfies all axioms of a field except the commutativity of the product. An algebra A is called* ***basic*** *if and only if $A/J(A)$ is a product of division algebras.*

If $\{e_1, e_2, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents in A, then A is basic iff $Ae_i \not\simeq Ae_j$ (as left A-modules), for all $i \neq j$.

By A-mod we denote the abelian category of all left finite dimensional A-modules. Two algebras A, B are said to be Morita equivalent if there exists an equivalence of the category A-mod of left modules over A and the category B-mod of left modules over B.

Definition 4.3 *We say that a basic algebra B is a* ***basic algebra associated to an algebra*** *A if and only if A and B are Morita equivalent.*

The basic algebra associated to a finite dimensional algebra A is unique up to isomorphism. Assume that A is a $\mathbb{K}$-algebra with a complete set $\{e_1, e_2, \ldots, e_n\}$ of primitive orthogonal idempotents. Consider the following subalgebra A^b of A:

$$A^b = e_A A e_A,$$

where $e_A = e_{j_1} + \ldots + e_{j_t}$, and $e_{j_1}, \ldots, e_{j_t}$ are chosen such that $Ae_{j_i} \not\simeq Ae_{j_s}$ for $i \neq s$ and each A-module Ae_j is isomorphic to one of the modules $Ae_{j_1}, \ldots, Ae_{j_t}$. The element e_A is called a **basic idempotent of** A. One checks that the algebra A^b does not depend on the choice of the sets $\{e_1, e_2, \ldots, e_n\}$ and $\{e_{j_1}, \ldots, e_{j_t}\}$, up to a $\mathbb{K}$-algebra isomorphism. The algebra A^b is basic and there is an equivalence of categories (of left modules) A-mod $\simeq A^b$-mod (see [2, I.6]).

As an example take $A = \mathrm{M_n}(\mathbb{K})$. Then as e_A we can take any of the diagonal idempotents of rank one in A and $A^b \simeq K$.

* * *

After the introductory part, we will demonstrate the connection between the basic subalgebras of $\mathrm{M_n}(\mathbb{K})$ and the regular $\mathcal{J}$-classes of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. We begin with the following lemma.

Lemma 4.4 *Every regular $\mathcal{J}$-class of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ contains a basic algebra.*

Proof Take a non-zero regular $\mathcal{J}$-class J of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. The set $E(J)$ of idempotents of J is clearly non-empty and every member of $E(J)$ is a subalgebra of $\mathrm{M_n}(\mathbb{K})$ with unity. Let $A \in J$ be such subalgebra. Let e be a basic idempotent of A. Observe that $Ae * A$, where $*$ is a semigroup operation in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ is a subalgebra of A. Indeed, $Ae * A = \mathrm{lin}_{\mathbb{K}}(AeA)$ is the principal ideal in A generated by the idempotent e. But it is a property of e that $\mathrm{lin}_{\mathbb{K}} AeA = A$. Therefore $Ae * A = A$. It is clear that in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ we have:

$$(Ae * A)\mathcal{R}Ae, \quad (Ae * A)\mathcal{L}A, \quad eA\mathcal{L}A, \quad Ae\mathcal{R}A.$$

Thus the $\mathcal{J}$-class of A contains the basic subalgebra eAe of A and the assertion follows. ■

Theorem 4.5 *Let $E, F \in \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ be two basic algebras which are in the same $\mathcal{J}$-class J. Then E, F are conjugate in $\mathrm{M_n}(\mathbb{K})$. The regular $\mathcal{J}$-classes of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ are in one-to-one correspondence with the conjugacy classes of the basic algebras contained in $\mathrm{M_n}(\mathbb{K})$. In particular, $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ has infinitely many regular $\mathcal{J}$-classes if $n \geq 4$.*

Proof We will once again use the map $\mu : \mathcal{C}(\mathrm{M_n}(\mathbb{K})) \to \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$, which was introduced in (3.1) in order to prove the main result of this paper. Apparently, μ preserves, to some degree, the structure and the partial ordering of the $\mathcal{J}$-classes of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ and $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$. To be more precise, the following lemma holds:

Lemma 4.6 *Let J be a regular class of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. Then there exists a unique minimal (with respect to the $\mathcal{J}$-order) $\mathcal{J}$-class J' of $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ such that $\mu(J') \subseteq J$. Moreover J' is regular an $\mu(J') = J$.*

Proof We follow [8, 3.3]. Since $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ has no infinite chains of regular $\mathcal{J}$-classes (see Theorem 3.2), there exists a minimal regular $\mathcal{J}$-class J' satisfying $\mu(J') \subseteq J$. We will say that J' is minimal over J. If $\mu(X) \in J$, for some $X \in \mathcal{C}(\mathrm{M_n}(\mathbb{K}))$, then for $I = \mathcal{C}(\mathrm{M_n}(\mathbb{K}))X\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ we must have: $J \subseteq \mu(I)$. It follows that I contains a regular element. Hence the entire I is regular (see Theorem 2.9 and Corollary 3.9). This implies that J' is a minimal $\mathcal{J}$-class of $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ over J (not only minimal regular $\mathcal{J}$-class).

Let $e = e^2 \in \mu(J') \subseteq J$. Since $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ is sπr, there exists $x \in \phi^{-1}(e)$ which lies in a maximal subgroup H of $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$. Then μ maps H into the maximal subgroup $G \subseteq \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ containing e, whence $\mu(f) = e$ for the unity f of H. Observe that $\mu(H) = G$. If $g \in G$, there exists $h \in \mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ such that $\mu(h) = g$. Thus $\mu(fhf) = ege = g$. But $fhf \in J'$ because fhf is in the ideal generated by J' and J' is minimal over J. Thus $fhf \in J' \cap fJ'f = H$. It follows that $g \in \mu(H)$, as claimed.

Let $x \in \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ be such that $x\mathcal{R}\mu(J')$ in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. Then $x\mathcal{R}e'$, for some $e' = e'^2 \in \mu(\mathcal{C}(\mathrm{M_n}(\mathbb{K})))$ and $\mu(a) = x$, for some $a \in \mathcal{C}(\mathrm{M_n}(\mathbb{K}))$. By the above there exists an idempotent $f' \in J'$ such that $\mu(f') = e'$. Then

$$\mu(a) = x = e'x = \mu(f')\mu(a) = \mu(f'a)$$

so by the minimality of J' over J it follows that $f'a \in J'$. Hence $x \in \mu(J')$. Similarly one shows that every element of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ that is $\mathcal{L}$-related to an element of $\mu(J')$ is contained in $\mu(J')$. As J is a regular $\mathcal{J}$-class and $\mathcal{J} = \mathcal{D}$ in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ it follows that $\mu(J') = J$.

It remains to show that J' is unique. If J'' is another $\mathcal{J}$-class of $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ which is minimal over J, then $\mu(J'J'') = \mu(J)\mu(J'') = JJ \subseteq J$. Clearly, this contradicts the minimality of J'. ■

We will say that the $\mathcal{J}$-class J' of $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ **covers** the $\mathcal{J}$-class J in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. The importance of covers appears in the next lemma.

Lemma 4.7 *Let $F \in \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ be a basic algebra. If J is the $\mathcal{J}$-class of F in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ and J' covers J in $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$, then $F \in J'$ in $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$.*

Proof Observe that every subalgebra of $\mathrm{M_n}(\mathbb{K})$ that has an identity is an idempotent in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. Therefore, there exists an idempotent $X \in J'$ such that $\mu(X) = F$, where $\mu : \mathcal{C}(\mathrm{M_n}(\mathbb{K})) \to \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ is defined by (3.1). By the proof of Observation 3.7 we deduce that $X \cdot e \cdot X = X$, for an idempotent $e \in X$. Thus $F = \mathrm{lin}_{\mathbb{K}}(X)$ implies $\mathrm{lin}_{\mathbb{K}}(FeF) = F$. Since F is basic it follows that e is the unity of F. So $X = eXe$. Since $X \in J'$ and $F \cdot X = F$, $F \in J'$. ■

We prove the entire Theorem. Assume that $E, F \in \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ are basic and $E \,\mathcal{J}\, F$. Let J' be the $\mathcal{J}$-class of $\mathcal{C}(\mathrm{M_n}(\mathbb{K}))$ that covers $J = \mathcal{J}_E = \mathcal{J}_F$. From the previous lemma we know that $E, F \in J'$. In the proof of Observation 3.7 we proved that any two $\mathcal{J}$-related idempotents are in fact conjugate (if each of them has an identity). It follows that E, F are conjugate in $\mathrm{M_n}(\mathbb{K})$. The rest is clear.

■

Theorem 4.5 can be proved without methods of algebraic geometry, see [7, Prop. 3]. In fact, if we replace the algebra $\mathrm{M_n}(\mathbb{K})$ with any finite dimensional algebra A over an algebraically closed field, the assertion remains the same. However, restricting to $\mathrm{M_n}(\mathbb{K})$ only, allows us to prove even more. We will show that any two subalgebras with identities that fall into the same regular $\mathcal{J}$-class of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ are, in fact, Morita-equivalent. This extends, by far, the result of Theorem 4.5. In order to prove this, we must relate the semigroup operation on $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ and the Morita equivalence of unitary subalgebras of $\mathrm{M_n}(\mathbb{K})$.

Lemma 4.8 *Let $A, B \subseteq \mathrm{M_n}(\mathbb{K})$ be subalgebras with identities. Consider the bimodules ${}_AP_B \subseteq A$, ${}_BQ_A \subseteq B$. Assume that the following identities hold in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$:*

$$P * Q = A, \quad Q * P = B. \tag{4.1}$$

Then the algebras A, B are Morita equivalent.

Proof Consider the maps $f : P \otimes_B Q \to A$ and $g : Q \otimes_A P \to B$ given by:

$$f(p \otimes q) = pq, \quad g(q \otimes p) = qp. \tag{4.2}$$

From the definition (1.1) of the semigroup operation $*$ on $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ the ordered sextuple (A, B, P, Q, f, g) forms the so-called pre-equivalence set, see [4, §12.7]. From (4.1) it follows that f is an epimorphism. As a consequence, Proposition 12.7 in [4] yields that f, g are both isomorphisms. The assertion is an easy consequence of Proposition 12.13 in [4]. ■

We are ready to prove the following, interesting fact.

Theorem 4.9 *Assume that A, B are subalgebras with identities. Then if $A \,\mathcal{J}\, B$ in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$, then A, B are Morita equivalent.*

Proof We follow [8, 3.6]. Suppose that the algebras A, B belong to the same $\mathcal{J}$-class of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. According to Definition 2.14 there exist $V, W \in \mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ such that $A = V * B * W$. Since A, B are unitary subalgebras, $A^2 = A$ and $B^2 = B$

in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ and we can assume that $V = A * V * B$ and $W = B * W * A$. Thus the subspace V possesses a natural structure of an A-B-bimodule. Analogously, a structure of a B-A-bimodule may be introduced on W. Observe that:

$$A = A * A = V * B * W * A = V * W. \tag{4.3}$$

On the other hand,

$$W * V * W * V = W * A * V = W * V,$$

so that $W * V$ is an idempotent in $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. Moreover, it follows that $W * V$ and A are in the same $\mathcal{J}$-class of $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$. Thus $(W * V)\,\mathcal{J}\,B$. Corollary 3.9 yields that $\mathcal{S}(\mathrm{M_n}(\mathbb{K}))$ is strongly π-regular. It follows from Corollary 2.8 that $W * V = B$. Indeed:

$$B * (W * V) = W * A * V = W * V = W * A * V = W * A * V * B = (W * V) * B.$$

According to (4.3), the assertion follows from Lemma 4.8. ■

Observe that if A, B are Morita equivalent, then there exist subalgebras A', B' of A, B, respectively, that

$$A\,\mathcal{J}\,A',\ B\,\mathcal{J}\,B' \text{ and } A' \simeq B'.$$

Indeed, observe that if A, B are Morita equivalent then it follows from the proof of Lemma 4.4 that $A\,\mathcal{J}\,A'$ and $B\,\mathcal{J}\,B'$, for any some basic algebras A', B' associated to A, B respectively. Obviously, A', B' are Morita equivalent. It is known that $A' \simeq eM_r(B')e$ for some $r \geq 1$ and an idempotent $e \in M_r(B')$ such that $\mathrm{lin}_{\mathbb{K}}(M_r(B')eM_r(B')) = M_r(B')$, see [1, 22.7]. Both A' and B' are therefore isomorphic to basic subalgebras of $M_r(B')$. Thus $A' \simeq B'$.

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