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Bartosz Łanucha

Uniwersytet M. Curie-Skłodowskiej w Lublinie

Hilbert matrix on spaces of analytic functions

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Opiekun pracy: Maria Nowak

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

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HILBERT MATRIX ON SPACES OF ANALYTIC FUNCTIONS

BARTOSZ ŁANUCHA

Abstract. We discuss the action of the Hilbert matrix operator, H, on Hardy spaces, Bergman spaces and the Bloch space on the unit disk. In particular, we give another proof of boundedness of the operator H on the Bergman spaces A^p for $2 < p < \infty$. For H^1 we get results analogous to that obtained for the Libera operator in [10]. Moreover, we introduce the A^2-space with logarythmic weight $\log^{\alpha}(2/(1-|z|^2))$ and prove that the Hilbert matrix operator maps this space boundedly into the ordinary Bergman space A^2 if $\alpha > 3$. Similarily, the logarythmically weighted Bloch space is boundedly mapped by H into the ordinary Bloch space. Finally, some estimates on $|(Hf)'(z)|$ for functions f from the Bloch space and Besov spaces are given.

1. Introduction

The Hilbert matrix is an infinite matrix H whose entries are $a_{n,k} = (n+k+1)^{-1}$. This matrix induces a linear operator on sequences:

$$H\colon (a_k)_{k\in\mathbb{N}_0} \quad \longmapsto \quad \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right)_{n\in\mathbb{N}_0}.$$

The following Hilbert's inequality implies that this operator is well defined and bounded on the space l^p of all p-summable sequences ($p > 1$).

Theorem 1.1 (Hilbert's inequality, [7], Chapter IX)**.** *Suppose $1 < p < \infty$. If $(a_k)_{k\in\mathbb{N}_0} \in l^p$, then*

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{\frac{1}{p}} \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{k=0}^{\infty}|a_k|^p\right)^{\frac{1}{p}}. \tag{1.1}$$

Moreover, the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is best possible.

Apart from sequence spaces, the Hilbert matrix can be viewed as an operator on spaces of analytic functions by its action on their Taylor coefficients. If

$$f(z) = \sum_{k=0}^{\infty}\hat{f}(k)z^k$$

is a function holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$, then we define a transformation H by

$$Hf(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\hat{f}(k)}{n+k+1}z^n. \tag{1.2}$$

Let $H(\mathbb{D})$ be the algebra of holomorphic functions in $\mathbb{D}$.

For $0 < p \le \infty$ the Hardy space H^p is the space of all holomorphic functions $f \in H(\mathbb{D})$ for which

$$\|f\|_p = \sup_{0\le r<1} M_p(r,f) < \infty,$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \qquad 0 < p < \infty;$$
$$M_\infty(r,f) = \sup_{0\le\theta\le 2\pi} |f(re^{i\theta})|.$$

It follows from the Hardy's inequality ([4], p. 48)

$$\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} \le \pi \|f\|_1$$

that H is well defined for each $f \in H^p$, $p \ge 1$. It was shown in [2] that for $f \in H^1$ (and as a consequence for every $f \in H^p$, $p \ge 1$) the Hilbert matrix operator can also be expressed in the integral form

$$Hf(z) = \int_0^1 \frac{f(r)}{1-rz} dr. \tag{1.3}$$

It is known that the operator H is bounded on H^p, $1 < p < \infty$ but not bounded on H^1 and H^∞ (see, e.g., [2]). Here we show that H acts as a bounded operator from H^∞ into BMOA and describe polynomials that are mapped into VMOA. Moreover, we show that if $f \in H^1$ is such that

$$\int_{-\pi}^{\pi} |f(e^{it})| \log \frac{\pi}{|t|} dt < \infty,$$

then $Hf \in H^1$.

We also investigate the action of the operator H on Bergman spaces A^p and give another proof of boundedness of Hilbert matrix operator on A^p for $2 < p < \infty$. Furthermore, we introduce the notion of logarithmically weighted space

$$A^2_{\log^\alpha} = \{f \in H(\mathbb{D}) \colon \|f\|^2_{\log,\alpha} = \int_{\mathbb{D}} |f(z)|^2 \left(\log\frac{2}{1-|z|^2}\right)^\alpha dA(z) < \infty\}, \quad \alpha > 0$$

and show that H maps $A^2_{\log^\alpha}$ into the ordinary Bergman space A^2 provided $\alpha > 3$.

Finally, we consider the operator H acting on the Bloch space with logarithmic weight. In particular, we prove that this space is mapped by H into the ordinary Bloch space and we give some estimates on $|(Hf)'(z)|$ for $f \in \mathcal{B}$ and $f \in \mathcal{B}_p$.

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2. Hilbert matrix as a Hankel operator on H^p

An infinite matrix whose entries $a_{n,k}$ are of the form $a_{n,k} = c_{n+k}$, where $(c_n)_{n\in\mathbb{N}_0}$ is a sequence of numbers, is called a Hankel matrix. In other words, it is a matrix whose entries depend only on the sum of the coordinates. A Hankel matrix considered as an operator on the space H^2 is called a Hankel operator. The Nehari's theorem (see [11], Thm. 3.2 or [9], Thm. 4.1.13) states that a Hankel operator is bounded on H^2 if and only if its matrix is given by a sequence $(\hat{g}(n))_{n\in\mathbb{N}_0}$ of the Fourier coefficients with non-negative indices of some function $g \in L^\infty(\mathbb{T})$, $\mathbb{T} = \partial\mathbb{D}$. Function g is called a symbol of a Hankel operator. Let us denote by H_g the operator corresponding to the function g. Then H_g is given by the formula

$$H_g f(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \hat{g}(n+k)\hat{f}(k)\right) z^n, \quad f \in H^2. \tag{2.1}$$

It is known (see [3]) that formula (2.1) can be extended to $f \in H^p$, $p > 1$. Moreover, we have

$$H_g f = P_+(M_g C f), \quad f \in H^p, \quad p > 1,$$

where $Cf(e^{it}) = f(e^{-it})$ is an isometry from H^p into $L^p(\mathbb{T})$, $M_g(u) = gu$ and P_+ is the Szegö projection given by

$$P_+ u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(t)}{1 - ze^{-it}} dt, \quad z \in \mathbb{D}.$$

Consider the function $b(t) = ie^{-it}(\pi - t)$, $0 \le t < 2\pi$. Of course b is bounded on the unit circle $\mathbb{T}$ and a trivial verification shows that its Fourier coefficients with non-negative indices are

$$\hat{b}(n) = \frac{1}{2\pi} \int_0^{2\pi} b(t) e^{-int} dt = \frac{1}{n+1}, \quad n \ge 0.$$

Therefore the Hilbert matrix operator H can be written in the following form

$$H = H_b = P_+ M_b C. \tag{2.2}$$

2.1. **Hilbert matrix operator acting on H^∞.** It is known that the Hilbert matrix operator does not act as a bounded operator form H^∞ into H^∞. To see this it is enough to notice that the constant function 1 is mapped to

$$H(1)(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1} = \frac{1}{z} \log \frac{1}{1-z},$$

which is not a bounded function.

Recall that the space BMOA consists of the functions $f \in H^1$ whose boundary values $f(e^{it})$ are of bounded mean oscillation on $\mathbb{T}$, that is

$$\sup_I \int_I |f(e^{it}) - I(f)| dt < \infty,$$

where supremum is taken over all intervals $I \subset \mathbb{T}$ and

$$I(f) = \frac{1}{|I|} \int_I f(e^{it}) dt.$$

If

$$\lim_{|I| \to 0} \int_I |f(e^{it}) - I(f)| dt = 0,$$

then we say that $f \in$ VMOA.

Since

$$H^\infty \subset \mathrm{BMOA} \subset \bigcap_{1<p<\infty} H^p,$$

the space BMOA is in a sense a limit space of H^p as $p \to \infty$. The famous Fefferman duality theorem (Thm. 7.1, [6]) states that the dual of H^1 can be identified with BMOA. What is more, VMOA can be identified with predual of H^1. It is also known that the space BMOA is the Szegö projection of $L^\infty(\mathbb{T})$ (see, e.g., [6], Thm. 7.2). This last fact along with (2.2) yields

Theorem 2.1. *The Hilbert matrix operator H acts as a bounded operator from H^∞ into* BMOA.

The example given at the begining of this subsection shows that not all polynomials are mapped by H into VMOA. However, we have

Theorem 2.2. *Let w be a polynomial of degree at least* 1*. Then $Hw \in$* VMOA *if and only if* $w(1) = 0$.

Proof. Formula (2.2) implies that Hw is the Szegö projection of the function $w(e^{-i\theta})b(\theta)$, where $b(\theta) = ie^{-i\theta}(\pi - \theta)$ for $0 \leq \theta < 2\pi$. The function b is continuous on the unit circle $\mathbb{T}$ except for 1. If $w(1) = 0$, then the function $w(e^{-i\theta})b(\theta)$ can be continuously extended on the whole unit circle. We know (see [13], Thm. 8.4.7) that the space VMOA is the Szegö projection of the space of functions continuous on $\mathbb{T}$. Hence $Hw \in$ VMOA.

It is also clear that if the function $w(e^{-i\theta})b(\theta)$ is continuous on $\mathbb{T}$ then $w(1) = 0$.

□

We remark that the last theorem also holds if w is a function analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$.

2.2. **Hilbert matrix operator acting on H^1.** As we have mentioned, the operator H is not bounded on H^1. To see this consider the example given in [1]. Namely, take the function

$$f_\varepsilon(z) = \frac{1}{(1-z)\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{1+\varepsilon}}.$$

Function f_ε belongs to H^1 for each $\varepsilon > 0$ and is positive on $[0, 1]$. Using formula (1.3) we find that

$$Hf_\varepsilon(z) = \sum_{n=0}^{\infty}\left(\int_0^1 r^n f_\varepsilon(r)dr\right)z^n.$$

If Hf_ε was in H^1, then the Hardy's inequality would imply the convergence of

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 r^n f_\varepsilon(r)dr &= \int_0^1 f_\varepsilon(r)\sum_{n=0}^{\infty}\frac{r^n}{n+1}dr\\ &= \int_0^1 f_\varepsilon(r)\left(\frac{1}{r}\log\frac{1}{1-r}\right)dr\\ &= \int_0^1 \frac{1}{(1-r)\left(\frac{1}{r}\log\frac{1}{1-r}\right)^{\varepsilon}}dr.\end{aligned}$$

This, however, is equal to ∞ for $\varepsilon \leq 1$.

In this section we are going to impose an additional assumption on $f \in H^1$, which guarantees that $Hf \in H^1$.

We start with the following auxiliary lemma.

Lemma 2.3. *If $f \in H^1$, then Hf extends to a continuous function on $\overline{\mathbb{D}}\backslash\{1\}$.*

Proof. By (1.2),

$$Hf(z) = \frac{1}{1-z}F_f(z),$$

where

$$F_f(z) = (1-z)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\hat{f}(k)}{n+k+1}z^n.$$

We will show that the function F_f can be continuously extended to $\overline{\mathbb{D}}$. To this end, we note that for $z \in \mathbb{D}$,

$$\begin{aligned} F_f(z) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^{n+1} \\ &= \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n - \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k} z^n \\ &= \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \left(\frac{1}{n+k+1} - \frac{1}{n+k} \right) \hat{f}(k) z^n \\ &= \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} - \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{(n+k)(n+k+1)} z^n. \end{aligned}$$

To see that the last double series converges absolutely and uniformly on $\overline{\mathbb{D}}$ it is enough to note that

$$\sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{(n+k)(n+k+1)} \right) |\hat{f}(k)| = \sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1}.$$

□

Since the function $z \mapsto (1-z)^{-1}$ is in H^p for all $0 < p < 1$, we have the following

Corollary 2.4. *The operator H acts as a bounded operator from H^1 into H^p, $0 < p < 1$.*

Now we will prove

Theorem 2.5. *If $f \in H^1$ is such that*

$$\int_{-\pi}^{\pi} |f(e^{it})| \log \frac{\pi}{|t|} dt < \infty, \tag{2.3}$$

then $Hf \in H^1$.

We shall use the following

Lemma 2.6. *If f is in H^1 and satisfies* (2.3) *then $g(z) = f(z) \log \frac{2}{1-z}$ is an element of H^1.*

Proof. Let us first observe that

$$\|g\|_{L^1} = \int_{-\pi}^{\pi} |f(e^{it})| \left| \log \frac{2}{1-e^{it}} \right| dt \le C\|f\|_{H^1} + \int_{-\pi}^{\pi} |f(e^{it})| \log \frac{2}{|1-e^{it}|} dt.$$

By the inequality

$$\left| \sin \frac{t}{2} \right| \ge \frac{|t|}{\pi} \ge 0, \quad |t| \le \pi,$$

we get

$$\int_{-\pi}^{\pi} |f(e^{it})| \log \frac{2}{|1-e^{it}|} dt = \int_{-\pi}^{\pi} |f(e^{it})| \log \frac{1}{\left|\sin \frac{t}{2}\right|} dt \le \int_{-\pi}^{\pi} |f(e^{it})| \log \frac{\pi}{|t|} dt. \tag{2.4}$$

From (2.4) and (2.3) we conclude that the boundary function $g(e^{it})$ is in $L^1(\mathbb{T})$. This, however, is still not enough to draw the conclusion that g belongs to H^1. We now show that $g \in H^p$,

$0 < p < 1$. For an arbitrary $0 < p < 1$ there exists q such that $1 < q < \frac{1}{p}$. Let q' be defined by $\frac{1}{q} + \frac{1}{q'} = 1$. Using the Hölder's inequality we obtain

$$\begin{aligned} M_p(r,g) &= \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \left| \log \frac{2}{1-re^{it}} \right|^p dt \right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^{qp} dt \right)^{\frac{1}{qp}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \log \frac{2}{1-re^{it}} \right|^{q'p} dt \right)^{\frac{1}{q'p}} \\ &\leq \|f\|_{H^{qp}} \left\| \log \frac{2}{1-z} \right\|_{H^{q'p}} \end{aligned}$$

for any $0 < r < 1$. Since $qp < 1$ and $H^1 \subset H^{qp}$, we get

$$\|g\|_{H^p} \leq \|f\|_{H^{qp}} \left\| \log \frac{2}{1-z} \right\|_{H^{q'p}} < \infty.$$

Now the Smirnov theorem ([8], p.74) implies that $g \in H^1$. □

Proof of Theorem 2.5. Take $f \in H^1$. Using the Fubini theorem and Lemma 8.2.3 in [12], we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |Hf(e^{it})| dt &\leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{|f(r)| dr}{|1-re^{it}|} dt \\ &= \int_0^1 |f(r)| \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1-re^{it}|} dr \leq C \int_0^1 |f(r)| \log \frac{2}{1-r} dr. \end{aligned}$$

If f satisfies (2.3) then, by Lemma 2.6, we are allowed to apply the Fejér-Riesz inequality ([4], Thm. 3.13) to the function $g(z) = f(z) \log \frac{2}{1-z}$. This yields

$$\frac{1}{2\pi} \int_0^{2\pi} |Hf(e^{it})| dt \leq C \|g\|_{H^1},$$

which means that $Hf \in L^1(\mathbb{T})$. Since, according to Corollary 2.4, Hf is in H^p for $0 < p < 1$, the Smirnov theorem ([8], p.74) implies that Hf is in H^1.

□

3. Hilbert matrix operator acting on A^p

Recall that for $0 < p < \infty$, the Bergman space A^p is the space consisting of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where $dA(z)$ is the area measure on $\mathbb{D}$ normalized so that $\int_{\mathbb{D}} dA(z) = 1$.

In [1], Diamantopoulos proved that the Hilbert matrix operator H acts as a bounded operator from A^p into A^p for $2 < p < \infty$. His proof is based on a representation of H in terms of weighted composition operators. We start this section with an alternative proof of this fact. We shall use the formula

$$Hf(z) = \int_{\mathbb{D}} \frac{f(\overline{w})}{(1-w)(1-\overline{w}z)} dA(w), \tag{3.1}$$

obtained in [3].

Theorem 3.1. *The operator H acts as a bounded operator from A^p into A^p, $2 < p < \infty$.*

Proof. From (3.1) we have that

$$(Hf)'(z) = \int_{\mathbb{D}} \frac{f(\overline{w})\overline{w}}{(1-w)(1-\overline{w}z)^2} dA(w).$$

Consequently, for a sufficiently small positive number α we can write

$$\begin{aligned} |(Hf)'(z)| &\le \int_{\mathbb{D}} \frac{|f(\overline{w})||\overline{w}|(1-|w|)^{\alpha}(1-|w|)^{-\alpha}}{|1-w||1-\overline{w}z|^2} dA(w) \\ &\le \left(\int_{\mathbb{D}} \frac{|f(\overline{w})|^p(1-|w|)^{\alpha p}}{|1-\overline{w}z|^2} dA(w) \right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} \frac{(1-|w|)^{-\alpha q}}{|1-w|^q|1-\overline{w}z|^2} dA(w) \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Of course $q<2$, since $p>2$. Now, using Lemma 8.2.3 in [12], we estimate the last integral.

$$\begin{aligned} \int_{\mathbb{D}} \frac{(1-|w|)^{-\alpha q}}{|1-w|^q|1-\overline{w}z|^2} dA(w) &\le C \int_0^1 \frac{r(1-r)^{-\alpha q}}{(1-r|z|)^2} \int_0^{2\pi} \frac{dt}{|1-re^{it}|^q} dr \\ &\le C \int_0^1 \frac{r(1-r)^{-\alpha q}}{(1-r|z|)^2(1-r)^{q-1}} dr \\ &= C \sum_{n=1}^{\infty} n|z|^{n-1} \int_0^1 r^n(1-r)^{1-q-\alpha q} dr \\ &= C \sum_{n=1}^{\infty} \frac{n\Gamma(n+1)\Gamma(2-q-\alpha q)}{\Gamma(n+3-q-\alpha q)} |z|^{n-1} \sim \frac{1}{(1-|z|)^{q+\alpha q}} \end{aligned}$$

(we have to assume that $\alpha < \frac{2-q}{q}$). Hence

$$\int_{\mathbb{D}} |(Hf)'(z)|^p(1-|z|)^p dA(z) \le C \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(\overline{w})|^p(1-|w|)^{\alpha p}(1-|z|)^p}{|1-\overline{w}z|^2(1-|z|)^{p+\alpha p}} dA(w)dA(z).$$

Next, using the Fubini theorem we get

$$\begin{aligned} \int_{\mathbb{D}} |(Hf)'(z)|^p(1-|z|)^p dA(z) &\le C \int_{\mathbb{D}} |f(\overline{w})|^p(1-|w|)^{\alpha p} \int_{\mathbb{D}} \frac{(1-|z|)^{-\alpha p}}{|1-\overline{w}z|^2} dA(z)dA(w) \\ &\le C \int_{\mathbb{D}} |f(\overline{w})|^p(1-|w|)^{\alpha p} \frac{1}{(1-|w|)^{\alpha p}} dA(w) = C\|f\|^p_{A^p}, \end{aligned}$$

where the last inequality is true for $\alpha < \frac{1}{p}$ (we used Lemma 4.2.2 in [13]). Therefore our estimates are true for any α such that $0<\alpha<\min\left\{\frac{1}{p}, \frac{2-q}{q}\right\}$.

Finally, from the inequality

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \le \|f\|_{A^p} \tag{3.2}$$

(see [10]) and Theorem 4.2.9 in [13] it follows that

$$\|Hf\|_{A^p} \le |Hf(0)| + \|(1-|z|^2)(Hf)'(z)\|_{A^p} \le C\|f\|_{A^p},$$

which ends our proof. □

3.1. **Logarithmically weighted Bergman spaces.** It is known that the Hilbert matrix operator does not act as a bounded operator on A^2. The proof of this fact was also given by Diamantopoulos in [1]. But one can consider the function f defined by

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{\log(n+1)} z^n.$$

Obviously $f \in A^2$ since

$$\|f\|_{A^2}^2 = \sum_{n=1}^{\infty} \frac{1}{(n+1)\log^2(n+1)} < \infty.$$

However,

$$Hf(0) = \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} = \infty.$$

This example, presented in [3], shows not only that H does not act on A^2, but also provides us with a function $f \in A^2$, for which even the series defining $Hf(0)$ is divergent.

In what follows we shall conctruct a subspace of A^2, which is mapped by the Hilbert matrix operator into A^2. For $\alpha > 0$ we define the weighted Bergman space $A^2_{\log^\alpha} \subset A^2$ as the space of all analytic functions $f \in H(\mathbb{D})$, such that

$$\|f\|_{\log^\alpha}^2 = \int_{\mathbb{D}} |f(z)|^2 \left(\log \frac{2}{1-|z|^2}\right)^\alpha dA(z) < \infty.$$

We claim that H is well defined on $A^2_{\log^\alpha}$ for $\alpha > 2$. This is a consequence of the following

Lemma 3.2. *Let $\alpha > 2$. If $f \in A^2_{\log^\alpha}$, then*

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \leq C\|f\|_{\log^\alpha},$$

where $C > 0$ is a constant independent of f.

Proof. Our proof is analogous to the proof of Lemma 4.1 given in [10].

Take an arbitrary element $f \in A^2_{\log^\alpha}$. It is a well known fact that the function $s \mapsto M_2(s, f)$ is increasing on $[0, 1)$. Since $s \mapsto \left(\log \frac{2}{1-s^2}\right)^\alpha$ is also an increasing function, using the Chebyshev inequality, we obtain

$$\begin{aligned}
\|f\|_{\log^\alpha}^2 &= \int_0^1 M_2^2(s, f) \left(\log \frac{2}{1-s^2}\right)^\alpha sds \\
&\geq \int_0^1 M_2^2(s, f) \left(\log \frac{2}{1-s^2}\right)^\alpha ds \cdot \int_0^1 sds \\
&\geq \frac{1}{2} \int_r^1 M_2^2(s, f) \left(\log \frac{2}{1-s^2}\right)^\alpha ds \\
&\geq \frac{1}{2}(1-r) \left(\log \frac{2}{1-r^2}\right)^\alpha M_2^2(r, f) \\
&\geq \frac{1}{2^{\alpha+1}}(1-r) \left(\log \frac{2}{1-r}\right)^\alpha M_2^2(r, f).
\end{aligned}$$

This means that for every $r \in [0,1)$,

$$\sum_{k=0}^{\infty} |\hat{f}(k)|^2 r^{2k} = M_2^2(r,f) \le C\|f\|_{\log^\alpha}^2 (1-r)^{-1}\left(\log\frac{2}{1-r}\right)^{-\alpha}.$$

Taking $r = 1 - 1/m$, we get

$$\sum_{k=m}^{2m} |\hat{f}(k)|^2 \left(1-\frac{1}{m}\right)^{4m} \le \sum_{k=m}^{2m} |\hat{f}(k)|^2 \left(1-\frac{1}{m}\right)^{2k} \le C\|f\|_{\log^\alpha}^2 m\left(\log 2m\right)^{-\alpha}.$$

But it is not too difficult to notice that this implies

$$\sum_{k=m}^{2m} |\hat{f}(k)|^2 \le C\|f\|_{\log^\alpha}^2 m\left(\log 2m\right)^{-\alpha}.$$

Consequently,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{|\hat{f}(k)|}{k+1} &= |\hat{f}(0)| + \sum_{k=1}^{\infty}\sum_{j=2^{k-1}}^{2^k-1}\frac{|\hat{f}(j)|}{j+1} \le |\hat{f}(0)| + \sum_{k=1}^{\infty} 2^{1-k}\sum_{j=2^{k-1}}^{2^k-1}|\hat{f}(j)| \\ &\le |\hat{f}(0)| + \sum_{k=1}^{\infty} 2^{1-k}\left(\sum_{j=2^{k-1}}^{2^k-1} 1\right)^{\frac{1}{2}}\left(\sum_{j=2^{k-1}}^{2^k-1}|\hat{f}(j)|^2\right)^{\frac{1}{2}} \\ &\le |\hat{f}(0)| + \sum_{k=1}^{\infty} 2^{1-k}2^{\frac{k-1}{2}}\left(C\|f\|_{\log^\alpha}^2 2^{k-1}\left(\log 2^k\right)^{-\alpha}\right)^{\frac{1}{2}} \\ &= |\hat{f}(0)| + C\|f\|_{\log^\alpha}\sum_{k=1}^{\infty}\frac{1}{k^{\frac{\alpha}{2}}} \le C\|f\|_{\log^\alpha},\end{aligned}$$

where the last inequality follows from the fact that $\alpha > 2$. □

The next lemma is a result similar to that obtained in [5], p. 36 (it should be also compared with Lemma 5.1.1 in [12]).

Lemma 3.3. *If $f \in A^2_{\log^\alpha}$, $\alpha > 0$, then there exists a constant $C > 0$ such that*

$$|f(z)| \le \frac{C\|f\|_{\log^\alpha}}{(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)^{\frac{\alpha}{2}}}$$

for every $z \in \mathbb{D}$.

Proof. Let $F \in H(\mathbb{D})$. Subharmonicity of $|F|^2$ implies the existance of $C > 0$ such that

$$|F(0)|^2 \le C\int_{\mathbb{D}} |F(\omega)|^2 dA(\omega). \tag{3.3}$$

Applying this inequality to F of the form

$$F(\omega) = f(\varphi_z(\omega))\varphi_z'(\omega)\left(\log\frac{2}{1-\varphi_z(\omega)\overline{z}}\right)^{\frac{\alpha}{2}},$$

where $\varphi_z(\omega) = \frac{z-\omega}{1-\overline{z}\omega}$, $z \in \mathbb{D}$ - fixed, we get

$$|f(z)|^2|\varphi_z'(0)|^2\left(\log\frac{2}{1-|z|^2}\right)^{\alpha} \le C\int_D |f(\varphi_z(\omega))|^2\left|\log\frac{2}{1-\varphi_z(\omega)\overline{z}}\right|^{\alpha}|\varphi_z'(\omega)|^2 dA(\omega).$$

Since

$$\varphi_z'(\omega) = \frac{|z|^2 - 1}{(1 - \overline{z}\omega)^2},$$

the change of variable $\zeta = \varphi_z(\omega)$ gives

$$\begin{aligned}
|f(z)|^2(1-|z|^2)^2 \left(\log \frac{2}{1-|z|^2}\right)^\alpha &\le C \int_D |f(\zeta)|^2 \left|\log \frac{2}{1-\zeta\overline{z}}\right|^\alpha dA(\zeta) \\
&\le C\|f\|_{A^2}^2 + C \int_D |f(\zeta)|^2 \left(\log \frac{2}{|1-\zeta\overline{z}|}\right)^\alpha dA(\zeta) \\
&\le C\|f\|_{A^2}^2 + C \int_D |f(\zeta)|^2 \left(\log \frac{2}{1-|z|^2}\right)^\alpha dA(\zeta) \\
&\le C\|f\|_{\log^\alpha}^2,
\end{aligned}$$

which proves our claim. □

We remark that Lemma 3.3 implies that the Hilbert matrix operator acting on $A^2_{\log^\alpha}$, $\alpha > 2$, can also be expressed in integral form (1.3).

Now we are ready to prove the main result of this section.

Theorem 3.4. *If $\alpha > 3$, then H acts as a bounded operator from $A^2_{\log^\alpha}$ to A^2.*

Proof. From (1.3) and the integral form of Minkowski's inequality we obtain

$$\begin{aligned}
\|Hf\|_{A^2} &= \left(\int_{\mathbb{D}} |Hf(z)|^2 dA(z)\right)^{\frac{1}{2}} \\
&\le \left(\int_{\mathbb{D}} \left(\int_0^1 \frac{|f(r)|}{|1-rz|} dr\right)^2 dA(z)\right)^{\frac{1}{2}} \\
&\le \int_0^1 |f(r)| \left(\int_{\mathbb{D}} \frac{dA(z)}{|1-rz|^2}\right)^{\frac{1}{2}} dr.
\end{aligned}$$

By Lemma 8.2.3 in [12],

$$\|Hf\|_{A^2} \le C \int_0^1 |f(r)| \left(\log \frac{2}{1-r^2}\right)^{\frac{1}{2}} dr,$$

which along with Lemma 3.3 implies that

$$\|Hf\|_{A^2} \le C \int_0^1 \frac{dr}{(1-r^2)\left(\log \frac{2}{1-r^2}\right)^{\frac{\alpha-1}{2}}} \|f\|_{\log^\alpha},$$

and a simple computation shows that the last integral converges provided $\alpha > 3$. □

4. The Bloch and Besov space

For $1 < p \le \infty$ let $\mathcal{B}_p$ denote the analytic Besov space consisting of functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_p} := |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \, dA(z)\right)^{1/p} < \infty.$$

In the case $p = \infty$ this is understood as

$$\|f\|_{\mathcal{B}_\infty} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty,$$

and hence $\mathcal{B}_\infty = \mathcal{B}$ is the Bloch space. The reader is referred to, e.g., [13] for results on the Besov and Bloch spaces.

It is easy to check that if $f(z) = \log \frac{1}{1-z}$, then

$$Hf(z) = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right) z^n.$$

This shows that the Bloch space $\mathcal{B}$ is not mapped into itself.

The following lemma describes a space of analytic functions in $\mathbb{D}$ that are mapped by H into the Bloch space.

Lemma 4.1. *If $f \in H(\mathbb{D})$ satisfies the condition*

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)\left(\log\frac{2}{1-|z|}\right)^{1+\varepsilon} < \infty \tag{4.1}$$

for an $\varepsilon > 0$, then $Hf \in \mathcal{B}$.

Proof. Assume that $f \in H(\mathbb{D})$ satisfies (4.1) and set

$$F(z) = f(z) - f(0).$$

Since, as mentioned before

$$H(1)(z) = \frac{1}{z}\log\frac{1}{1-z} \in \mathcal{B},$$

it is enough to show that $HF \in \mathcal{B}$. By Lemma 4.2.8 in [13] we can write

$$F(z) = \int_{\mathbb{D}} \frac{F'(w)(1-|w|^2)}{\bar{w}(1-\bar{w}z)^2} dA(w). \tag{4.2}$$

Consequently,

$$\begin{aligned}
|(HF)'(z)| &\le \int_0^1 \int_{\mathbb{D}} \frac{r|F'(w)|(1-|w|^2)}{|\bar{w}||1-\bar{w}r|^2|1-rz|^2} dA(w)dr \\
&\le C\int_0^1 \int_{\mathbb{D}} \frac{\log^{-1-\varepsilon}\left(\frac{2}{1-|w|}\right)}{|\bar{w}||1-\bar{w}r|^2|1-rz|^2} dA(w)dr \\
&= C\int_0^1 \int_0^1 \frac{\log^{-1-\varepsilon}\left(\frac{2}{1-s}\right)}{|1-rz|^2} \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1-sre^{-i\theta}|^2} dsdr \\
&\le C\int_0^1 \int_0^1 \log^{-1-\varepsilon}\left(\frac{2}{1-s}\right) \frac{dsdr}{(1-sr)(1-r|z|)^2} \\
&\le \frac{C}{1-|z|}\int_0^1 \log^{-1-\varepsilon}\left(\frac{2}{1-s}\right)\frac{1}{1-s}ds.
\end{aligned}$$

Since the last integral is finite for an $\varepsilon > 0$, our claim is proved. □

On the other hand, we have

Lemma 4.2. *If $f \in \mathcal{B}$, then*

$$|(Hf)'(z)| \le C\frac{1}{1-|z|}\log\frac{2}{1-|z|}.$$

Proof. For $f \in \mathcal{B}$ set

$$A_n(f) = \sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} = \int_{\mathbb{D}} \frac{f(z)}{1-\bar{z}} |z|^{2n} dA(z).$$

Using the Fubini theorem and formula (4.2), we obtain, under the additional assumption that $f(0) = 0$,

$$\begin{aligned}
|A_n(f)| &\leq \left| \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\bar{w}(1-\bar{w}z)^2} dA(w) \frac{|z|^{2n}}{1-\bar{z}} dA(z) \right| \\
&= \left| \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\bar{w}} \int_0^1 \left(\frac{1}{\pi} \int_0^{2\pi} \frac{d\theta}{(1-\bar{w}re^{i\theta})^2(1-re^{-i\theta})} \right) r^{2n+1} dr dA(w) \right| \\
&\leq C \int_0^1 r^{2n+1} \int_{\mathbb{D}} \frac{dA(w)}{|1-r^2\bar{w}|^2} dr \leq C \int_0^1 r^{2n+1} \log \frac{2}{(1-r^2)} dr \\
&\leq C \int_0^1 r^n \left(\log 2 + \sum_{k=1}^{\infty} \frac{r^k}{k} \right) dr \leq C \frac{1}{n+1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1} \right) \\
&\leq C \frac{1}{n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right),
\end{aligned}$$

for $n \in \mathbb{N}$. Hence

$$\begin{aligned}
|(Hf)'(z)| &= \left| \sum_{n=1}^{\infty} n A_n(f) z^{n-1} \right| \leq \sum_{n=1}^{\infty} n |A_n(f)| |z|^{n-1} \\
&\leq C \sum_{n=0}^{\infty} \left(1 + \frac{1}{2} + \dots \frac{1}{n+1} \right) |z|^n \\
&\leq C \frac{1}{1-|z|} \log \frac{1}{1-|z|} + C \frac{1}{|z|} \log \frac{1}{1-|z|} \\
&\leq C \frac{1}{1-|z|} \log \frac{2}{1-|z|}.
\end{aligned}$$

□

The example of $f(z) = \log \frac{1}{1-z}$ shows that the inequality in the last lemma cannot be improved.

Similar but a little bit more complicated calculations give the following

Lemma 4.3. *If* $f \in \mathcal{B}_p$, $1 < p < \infty$, *then*

$$|(Hf)'(z)| \leq C \frac{1}{1-|z|} \left(\log \frac{2}{1-|z|} \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the notation from the proof of Lemma 4.2 and under the assumption that $f(0) = 0$ we get, in much the same way as above,

$$|A_n(f)| \leq C \int_0^1 r^{2n+1} \int_{\mathbb{D}} \frac{|f'(w)|(1-|w|^2)}{|1-r^2\bar{w}|^2} dA(w) dr.$$

Applying the Hölder's inequality and Lemma 4.2.2 in [13] we have

$$\begin{aligned}|A_n(f)| &\le C\|f\|_{\mathcal{B}_p}\int_0^1 r^{2n+1}\left(\int_{\mathbb{D}}\frac{(1-|w|^2)^{2q-2}}{|1-r^2\bar{w}|^{2q}}dA(w)\right)^{\frac{1}{q}}dr\\ &\le C\|f\|_{\mathcal{B}_p}\int_0^1 r^{2n+1}\left(\log\frac{1}{1-r^2}\right)^{\frac{1}{q}}dr\end{aligned}$$

Hence, using the Hölder's inequality again,

$$\begin{aligned}|A_n(f)|^q &\le C\|f\|^q_{\mathcal{B}_p}\left(\int_0^1 r^{2n+1}dr\right)^{q-1}\int_0^1 r^{2n+1}\log\frac{1}{1-r^2}dr\\ &\le C\|f\|^q_{\mathcal{B}_p}\left(\frac{1}{n+1}\right)^q\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)\\ &\le C\|f\|^q_{\mathcal{B}_p}\left(\frac{1}{n}\right)^q\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}|(Hf)'(z)|^q &\le \left(\sum_{n=0}^{\infty}(n+1)|A_{n+1}(f)||z|^n\right)^q\\ &\le \left(\sum_{n=0}^{\infty}(n+1)^q|A_{n+1}(f)|^q|z|^n\right)\left(\sum_{n=0}^{\infty}|z|^n\right)^{q-1}\\ &\le C\|f\|^q_{\mathcal{B}_p}\left(\frac{1}{1-|z|}\right)^{q-1}\left(\sum_{n=0}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)|z|^n\right)\\ &\le C\|f\|^q_{\mathcal{B}_p}\left(\frac{1}{1-|z|}\right)^q\log\frac{2}{1-|z|},\end{aligned}$$

which proves our claim. □

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Instytut Matematyki UMCS, pl. M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland
E-mail address: bartosz.lanucha@gmail.com