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Maximal operators and their applications to spaces of holomorphic functions

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MAXIMAL OPERATORS AND THEIR APPLICATIONS TO SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. We give a brief exposition of the most important applications of maximal operators to spaces of holomorphic functions. After some background information on the Hardy-Littlewood maximal theorem, results concerning harmonic functions are presented. Then we proceed with applications to Hardy spaces on the upper half plane and on the unit disk.

1. Introduction

Maximal operators were first introduced in 1930 by G. H. Hardy and J. E. Littlewood in their paper [3]. One of the maximal functions defined in [3] was of the form

$$\Theta(x) = \Theta(x; f) = \sup_{0 \le \zeta < x} \frac{1}{x - \zeta} \int_{\zeta}^{x} f(t)dt,$$

f being nonnegative and integrable over an interval $[0, a]$.

In their main theorem autors used the concept of rearrangements of functions: Let f be a nonnegative function integrable over an interval $[0, a]$. Let $m_f(\lambda)$ denote the measure of the set in which $f(t) > \lambda$. Then the decreasing rearrangement of f is the function

$$f^{\star}(t) = \inf\{\lambda > 0 \colon m_f(\lambda) < t\}.$$

The Hardy-Littlewood maximal theorem was stated as follows

Theorem 1.1 ([3], Thm. 5)**.** *If $s(y)$ is any nondecreasing function defined for $y \ge 0$, then*

$$\int_0^a s(\Theta(x; f))dx \le \int_0^a s(\Theta(x; f^{\star}))dx.$$

In the case of $s(y) = |y|^p$, $p > 1$, Hardy and Littlewood used this theorem to deduce some interesting inequalities. Although they pointed out applications to function theory, it took some time for the importance of their theorem to be widely recognized.

The aim of this paper is to give a brief exposition of the most important applications of maximal operators to the spaces of holomorphic functions. We are mostly interested in the Hardy spaces H^p for the upper half plane $\mathscr{H} = \{z \in \mathbb{C} \colon \mathrm{Im} z > 0\}$. However, the theory for the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ will also be mentioned, and the results will be stated for both cases. Most of the presented results are from the book [2] by J. B. Garnett (chapters I - IV). We also refer the reader to [5] by P. Koosis (chapter VIII).

The next section is devoted to the definition and properties of the Hardy-Littlewood maximal operator $\mathcal{M}$ for functions on the real line $\mathbb{R}$ and on the unit circle $\mathbb{T} = \partial\mathbb{D}$. We present the Marcinkiewicz interpolation theorem, which is then used to prove a version of the Hardy-Littlewood maximal theorem. Section 3 contains some basic facts concerning Poisson kernels and harmonic functions. The notion of nontangential maximal function is introduced. After having compiled a number of auxiliary lemmas we present the proof of the Fatou's theorem for harmonic functions. In Section 4 we proceed with the standard definitions of Hardy spaces for the upper half plane and the unit disk. Then theorems from Sections 2 and 3 are applied. Namely, we present the proof of existence of the boundary values. The conjugation operator and the Riesz theorem are mentioned. Finally, we briefly sketch the proof of the fact that Hardy spaces can be defined only with the use of nontangential maximal function.

Let (X, μ) be a measure space. $L^p(X, \mu)$ will denote the space of p-integrable functions on X with

$$\|f\|_{L^p} = \|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$

For simplicity of notation we will also write $L^p(X)$ or $L^p(\mu)$. In particular, $L^p(\mathbb{R}, dx)$ will stand for p-integrable functions over the real line $\mathbb{R}$ with respect to the Lebesgue measure and $L^p(\mathbb{T}, dt/2\pi)$ will denote functions on the unit circle $\mathbb{T}$, p-integrable with respect to the normalized Lebesgue measure. If $f \in L^p(\mathbb{T})$, we will write simply $f(t)$ instead of $f(e^{it})$ when no confusion can arise.

For a subset I of the real line $\mathbb{R}$, $|I|$ will denote the Lebesgue measure of I.

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2. Hardy-Littlewood Maximal Function

The paper [3] contained several definitions of maximal operators. In this section we will be especially interested in the following one.

Definition 2.1. Let f be a locally integrable function on $\mathbb{R}$. The Hardy-Littlewood maximal function of f is

$$\mathcal{M}f(t) = \sup_{t \in I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f(s)| ds, \quad t \in \mathbb{R},$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ such that $t \in I$. If f is a function integrable on $\mathbb{T}$, we only take $t \in [-\pi, \pi]$ and the supremum is restricted to intervals $I \subset \mathbb{R}$ such that $|I| \leq 2\pi$.

The importance of the Hardy-Littlewood maximal function comes from the fact, that $\mathcal{M}f$ majorizes many other functions associated with f.

We will not focus our attention on the concept of rearrangements of functions. However, there is an idea connected with that concept, which will be extremely useful. Namely, it is the the distribution function m_f, mentioned in the introduction.

Definition 2.2. Let (X, μ) be a measure space and f be a measurable function on X. Function

$$m_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}),$$

defined for $\lambda > 0$ is called the distribution function of f.

Notice that the distribution function m_f is nonnegative and nonincreasing. If $f \in L^\infty$, then

$$\|f\|_\infty = \sup\{\lambda : m_f(\lambda) > 0\} = \inf\{\lambda : m_f(\lambda) = 0\}.$$

What is more, the distribution function determines all of the L^p norms of f.

Lemma 2.3 ([2], Lem. I.4.1, [5], p. 171)**.** *Let (X, μ) be a measure space and $0 < p < \infty$. For every measurable function f,*

$$\int_X |f|^p d\mu = \int_0^\infty p\lambda^{p-1} m_f(\lambda) d\lambda.$$

We will show, that if $f \in L^p$, $p \geq 1$ then $\mathcal{M}f$ is finite almost everywhere. A crucial thing concenring the operator $\mathcal{M}$ is the Hardy-Littlewood maximal theorem. It states that the operator $\mathcal{M}$ is (strong) type (p,p) for every $p > 1$ and weak type $(1,1)$.

Definition 2.4. Let (X,μ) and (Y,ν) be measure spaces and let T be a mapping form $L^p(X,\mu)$ to $L^q(Y,\nu)$, $0 < p, q \leq \infty$. We say that

a) T is of (strong) type (p,q) if

$$\|Tf\|_q \leq A\|f\|_p, \quad f \in L^p,$$

where $A > 0$ does not depend on f.

b) T is of weak type (p,q), $q < \infty$, if for every $\alpha > 0$,

$$\nu(\{y \in Y : |Tf(y)| > \alpha\}) \leq A\frac{\|f\|_p^q}{\alpha^q}, \quad f \in L^p,$$

where $A > 0$ does not depend on f or α.

If $q = \infty$ we say that T is of weak type (p,q) if it is of type (p,q).

Note, that Chebychev's inequality

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{\|f\|_p^p}{\lambda^p},$$

true for every $f \in L^p(X,\mu)$, $0 < p < \infty$ and $\lambda > 0$, implies that the operator $f \mapsto m_f$ is of weak type (p,p). It also yields that every operator of type (p,q) is of weak type (p,q).

To prove the weak type inequality for the operator $\mathcal{M}$ we will need the following lemma

Lemma 2.5 ([2], Lem. I.4.4)**.** *Let μ be a positive Borel measure on $\mathbb{R}$ and let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a finite family of open intervals in $\mathbb{R}$. There exists a pairwise disjoint subfamily $\mathcal{J} = \{J_1, \ldots, J_m\} \subset \mathcal{I}$, such that*

$$\sum_{i=1}^{m} \mu(J_i) \geq \frac{1}{2}\mu\left(\bigcup_{j=1}^{n} I_j\right).$$

Proof. Without loss of generality we can assume that no element of $\mathcal{I}$ is contained in the union of others. Otherwise it could be removed form $\mathcal{I}$ without changing the sum of $\mathcal{I}$.

Denote $I_j = (\alpha_j, \beta_j)$, indexing the intervals so that $\alpha_j \leq \alpha_{j+1}$ for every $1 \leq j \leq n-1$.

Take an arbitrary $1 < k < n$. We have $\beta_{k+1} > \beta_k$. If it were not, then $\alpha_k \leq \alpha_{k+1}$ would imply $I_{k+1} \subset I_k$. On the other hand if $\alpha_{k+1} \leq \beta_{k-1}$, then $I_k \subset I_{k-1} \cup I_{k+1}$. Hence $\alpha_{k+1} > \beta_{k-1}$, and

$$I_{k-1} \cap I_{k+1} = \varnothing, \quad \text{for all } 1 < k < n.$$

This means that the subfamily $\mathcal{I}_1$ of odd-numbered intervals is pairwise disjoint. The same is true for the subfamily $\mathcal{I}_2$ of even-numbered intervals. Therefore

$$\mu\left(\bigcup_{k=1}^{n} I_k\right) \leq \sum_{k \text{ even}} \mu(I_k) + \sum_{k \text{ odd}} \mu(I_k),$$

and the desired subfamily is either $\mathcal{I}_1$ or $\mathcal{I}_2$. $\square$

In proving the strong type inequalities for the Hardy-Littlewood maximal operator we will make use of the Marcinkiewicz theorem. But first a remark is in order.

Remark 2.6. Let (X,μ) be a measurable space. If μ is a finite measure, then $L^p(X)\subset L^1(X)$ for all $p>1$. Therefore, if an operator T is defined on $L^1(X)$, it can be simultaneously treated as an operator on all $L^p(X)$, $p>1$.

This however is not the case if μ is infinite. To assure that T is well defined on all L^p spaces for $1\le p\le r$, we assume its domain to be L^1+L^r, the space of all functions $f=f_1+f_2$ with $f_1\in L^1$, $f_2\in L^r$.

If $f\in L^p$, $1\le p\le r$, denote $f_1=f\chi_{\{|f|>\gamma\}}$, $f_2=f\chi_{\{|f|\le\gamma\}}$, where γ is a fixed positive constant. Then, since $1-p\le 0$ and $r-p\ge 0$, we have

$$\|f_1\|_1=\int_{|f|>\gamma}|f|d\mu=\int_{|f|>\gamma}|f|^{1-p}|f|^pd\mu\le\gamma^{1-p}\int_X|f|^pd\mu=\gamma^{1-p}\|f\|_p^p<\infty,$$

and

$$\|f_2\|_r^r=\int_{|f|\le\gamma}|f|^rd\mu=\int_{|f|\le\gamma}|f|^{r-p}|f|^pd\mu\le\gamma^{r-p}\int_X|f|^pd\mu=\gamma^{r-p}\|f\|_p^p<\infty.$$

Hence, $f=f_1+f_2\in L^1+L^r$ and $L^p\subset L^1+L^r$.

Theorem 2.7 (Marcinkiewicz interpolation theorem, [7], Thm. 5, p. 21, [2], Thm. I.4.5)**.** *Let (X,μ) and (Y,ν) be two measure spaces, and let $1<r\le\infty$. Suppose that T is a mapping from $L^1(X,\mu)+L^r(X,\mu)$ to ν-measurable functions on Y, such that*

(i) $|T(f+g)(y)|\le|Tf(y)|+|Tg(y)|$;
(ii) $\nu(\{y\in Y\colon |Tf(y)|>\lambda\})\le(A_0/\lambda)\|f\|_1,\quad f\in L^1(X,\mu)$;
(iii) $\nu(\{y\in Y\colon |Tf(y)|>\lambda\})\le((A_1/\lambda)\|f\|_r)^r,\quad f\in L^r(X,\mu)$;

(when $r=\infty$ we assume that

$$\|Tf\|_\infty\le A_1\|f\|_\infty).$$

Then for $1<p<r$,

$$\|Tf\|_p\le A_p\|f\|_p,\quad f\in L^p(X,\mu),\tag{2.1}$$

where A_p depends only on A_0, A_1, p, and r.

Proof. Let $f\in L^p(X)$, $1<p<r$. We want to establish (2.1). To this end, we shall use Lemma 2.3:

$$\int_Y|Tf|^pd\nu=\int_0^\infty p\lambda^{p-1}m_{Tf}(\lambda)d\lambda,\tag{2.2}$$

and estimate the value of $m_{Tf}(\lambda)=\nu(\{y\in Y\colon |Tf(y)|>\lambda\})$.

Fix $\lambda>0$ and split f as in the Remark 2.6, with $\gamma=\frac{\lambda}{2A_1}$:

$$f_1=f\chi_{\{|f|>\frac{\lambda}{2A_1}\}},\quad f_2=f\chi_{\{|f|\le\frac{\lambda}{2A_1}\}}.$$

Assumption (i) gives

$$\{y\in Y\colon |Tf(y)|>\lambda\}\subset\{y\in Y\colon |Tf_1(y)|>\frac{\lambda}{2}\}\cup\{y\in Y\colon |Tf_2(y)|>\frac{\lambda}{2}\}.$$

Hence

$$m_{Tf}(\lambda)\le m_{Tf_1}(\frac{\lambda}{2})+m_{Tf_2}(\frac{\lambda}{2}),$$

and (2.2) yields

$$\|Tf\|_p^p\le\int_0^\infty p\lambda^{p-1}m_{Tf_1}(\lambda)d\lambda+\int_0^\infty p\lambda^{p-1}m_{Tf_2}(\lambda)d\lambda.\tag{2.3}$$

Since $f_1 \in L^1(X)$, by (ii) we have

$$m_{Tf_1}(\frac{\lambda}{2}) = \nu(\{y \in Y \colon |Tf_1(y)| > \frac{\lambda}{2}\}) \leq \frac{2A_0}{\lambda} \int_{|f|>\frac{\lambda}{2A_1}} |f| d\mu. \tag{2.4}$$

If $r = \infty$, then $\|f_2\|_\infty \leq \frac{\lambda}{2A_1}$, which along with (iii) implies that $\|Tf_2\|_\infty \leq \frac{\lambda}{2}$. Consequently, $m_{Tf_1}(\frac{\lambda}{2}) = 0$. From (2.4) it follows that

$$\begin{aligned}
\|Tf\|_p^p &\leq \int_0^\infty p\lambda^{p-1} \left(\frac{2A_0}{\lambda} \int_{|f|>\frac{\lambda}{2A_1}} |f| d\mu \right) d\lambda \\
&= 2A_0 p \int_X |f| \int_0^{2A_1|f|} \lambda^{p-2} d\lambda d\mu \\
&= \frac{2A_0 p}{p-1} \int_X |f| (2A_1|f|)^{p-1} d\mu = \frac{2^p A_1^{p-1} A_0 p}{p-1} \|f\|_p^p,
\end{aligned}$$

because $p - 2 > -1$. Thus, (2.1) is proved for $r = \infty$.

If $r < \infty$, then by (iii),

$$m_{Tf_2}(\frac{\lambda}{2}) = \nu(\{y \in Y \colon |Tf_2(y)| > \frac{\lambda}{2}\}) \leq \left(\frac{2A_1}{\lambda} \|f_2\|_r \right)^r = \frac{2^r A_1^r}{\lambda^r} \int_{|f| \leq \frac{\lambda}{2A_1}} |f|^r d\mu,$$

and

$$\begin{aligned}
\int_0^\infty p\lambda^{p-1} m_{Tf_2}(\frac{\lambda}{2}) d\lambda &\leq \int_0^\infty p\lambda^{p-1} \frac{2^r A_1^r}{\lambda^r} \int_{|f| \leq \frac{\lambda}{2A_1}} |f|^r d\mu d\lambda \\
&= (2A_1)^r p \int_X |f|^r \int_{2A_1|f|}^\infty \lambda^{p-r-1} d\lambda d\mu \\
&= \frac{(2A_1)^r p}{r-p} \int_X |f|^r (2A_1|f|)^{p-r} d\mu = \frac{(2A_1)^p p}{r-p} \int_X |f|^p d\mu,
\end{aligned}$$

because $p - r - 1 < -1$. By (2.3) and the first part of the proof, this yields the desired conclusion. □

In other words, an operator which is both of weak type $(1,1)$ and weak type (r,r) is automatically of strong type (p,p) for all $1 < p < r$. For even more general version of this theorem see [7] (p. 272).

We are now in a position to show the following version of the Hardy-Littlewood maximal theorem

Theorem 2.8 ([2], Thm. I.4.3)**.** *If $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, then $\mathcal{M}f(t)$ is finite almost everywhere.*

(a) *If $f \in L^1(\mathbb{R})$, then*

$$|\{t \in \mathbb{R} \colon \mathcal{M}f(t) > \lambda\}| \leq (2/\lambda)\|f\|_1, \quad \lambda > 0.$$

(b) *If $f \in L^p(\mathbb{R})$, with $1 < p \leq \infty$, then*

$$\|\mathcal{M}f\|_p \leq A_p \|f\|_p,$$

where A_p depends only on p.

Proof. (a) Take $f \in L^1(\mathbb{R})$ and $\lambda > 0$. If t is taken from the measurable set $\{t \in \mathbb{R} \colon \mathcal{M}f(t) > \lambda\}$, then there exists an open interval $I \subset \mathbb{R}$, $t \in I$, satisfying:

$$|I| < \frac{1}{\lambda} \int_I |f(s)| ds. \tag{2.5}$$

Thus for every compact subset $K \subset \{t \in \mathbb{R} \colon \mathcal{M}f(t) > \lambda\}$ one can find a finite covering $\mathcal{I} = \{I_1, \ldots, I_n\}$, such that every element of $\mathcal{I}$ satisfies (2.5).

Lemma 2.5 gives a pairwise disjoint subfamily $\{J_1, \ldots, J_m\} \subset \mathcal{I}$,

$$\left| \bigcup_{j=1}^{n} I_j \right| \leq 2 \sum_{j=1}^{m} |J_j|.$$

Hence

$$|K| \leq 2 \sum_{j=1}^{m} |J_j| \leq 2 \sum_{j=1}^{m} \frac{1}{\lambda} \int_{J_j} |f(s)| ds \leq \frac{2}{\lambda} \int_{\mathbb{R}} |f(s)| ds.$$

Since K was chosen arbitrarily,

$$|\{t \in \mathbb{R} \colon \mathcal{M}f(t) > \lambda\}| \leq \frac{2}{\lambda} \int_{\mathbb{R}} |f(s)| ds.$$

(b) The Hardy-Littlewood maximal operator $\mathcal{M}$ is subadditive and as seen in the proof of (a), $\mathcal{M}$ is of weak type $(1, 1)$. It is also easy to see, that

$$\|\mathcal{M}f\|_\infty \leq \|f\|_\infty, \quad f \in L^\infty.$$

Therefore, the Marcinkiewicz interpolation theorem implies, that for all $1 < p \leq \infty$,

$$\|\mathcal{M}f\|_p \leq A_p \|f\|_p, \quad f \in L^p.$$

□

Similar arguments apply to the case of $L^p(\mathbb{T})$:

Theorem 2.9 (see [5], p. 172)**.** *If $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$, then $\mathcal{M}f(t)$ is finite almost everywhere.*

(a) *If $f \in L^1(\mathbb{T})$, then*

$$|\{t \in (-\pi, \pi) \colon \mathcal{M}f(t) > \lambda\}| \leq (6/\lambda)\|f\|_1, \quad \lambda > 0.$$

(b) *If $f \in L^p(\mathbb{T})$, with $1 < p \leq \infty$, then*

$$\|\mathcal{M}f\|_p \leq A_p \|f\|_p,$$

where A_p depends only on p.

3. Harmonic Functions

Recall that the Poisson kernel for the point $z \in \mathbb{D}$ is the function

$$P_z(s) = \mathrm{Re} \frac{e^{is} + z}{e^{is} - z} = \frac{1 - |z|^2}{|e^{is} - z|^2}, \quad s \in [-\pi, \pi].$$

If $f \in L^1(\mathbb{T})$ then the Poisson integral of f,

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_z(s) f(s) ds$$

is a harmonic function on the unit disk. A simple computation shows that

$$P_z(s) = \frac{1-r^2}{1-2r\cos(\theta-s)+r^2} = P_r(\theta-s), \quad z = re^{i\theta},$$

and the Poisson integral of f can be treated as a convolution

$$u(re^{i\theta}) = \frac{1}{2\pi} P_r * f(\theta).$$

For the upper half plane the Poisson kernel has the form

$$P_z(s) = \frac{1}{\pi}\mathrm{Im}\left(\frac{1}{s-z}\right), \quad s \in \mathbb{R},$$

and

$$P_z(s) = P_y(x-s) = \frac{y}{(x-s)^2+y^2}, \quad z = x+iy \in \mathscr{H}.$$

Similarly, the Poisson integral

$$u(z) = u(x+iy) = \int_{\mathbb{R}} P_z(s)f(s)ds = P_y * f(x)$$

defines a function harmonic on $\mathscr{H}$ for $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$.

In both cases $f(s)ds$ can be replaced by $d\mu$, μ being a finite measure (on $\mathbb{T}$ or $\mathbb{R}$, respectively).

For more details concerning Poisson kernels and Poisson integrals see [2] and [4].

Connections between functions and their Poisson integrals are described in the following lemmas:

Lemma 3.1 ([2], Thm. I.3.1)**.** (a) *If $1 \le p < \infty$ and if $f \in L^p(\mathbb{R})$, then*

$$\|P_y * f - f\|_p \overset{y\to 0}{\longrightarrow} 0.$$

(b) *When $f \in L^\infty(\mathbb{R})$, $P_y * f$ converges weak-star to f, i.e.,*

$$\int_{\mathbb{R}} (P_y * f)(x)g(x)dx \quad \overset{y\to 0}{\longrightarrow} \quad \int_{\mathbb{R}} f(x)g(x)dx$$

for all $g \in L^1(\mathbb{R})$.

(c) *If $d\mu$ is a finite measure on $\mathbb{R}$, the measures $(P_y * \mu)(x)dx$ converge weak-star to $d\mu$, i.e.,*

$$\int_{\mathbb{R}} (P_y * \mu)(x)g(x)dx \quad \overset{y\to 0}{\longrightarrow} \quad \int_{\mathbb{R}} g(x)d\mu(x)$$

for all g continuous on $\mathbb{R}$ and vanishing at infinity.

(d) *When f is bounded and uniformly continuous on $\mathbb{R}$, $P_y * f$ converges uniformly to f.*

Lemma 3.2 ([4], p. 32-33)**.** (a) *If $1 \le p < \infty$ and if $f \in L^p(\mathbb{T})$, then*

$$\|P_r * f - f\|_p \overset{r\to 1}{\longrightarrow} 0.$$

(b) *When $f \in L^\infty(\mathbb{T})$, $P_r * f$ converges weak-star to f, i.e.,*

$$\int_{-\pi}^{\pi} (P_r * f)(t)g(t)dt \quad \overset{r\to 1}{\longrightarrow} \quad \int_{-\pi}^{\pi} f(t)g(t)dt$$

for all $g \in L^1(\mathbb{T})$.

(c) *If $d\mu$ is a finite measure on $\mathbb{T}$, the measures $\frac{1}{2\pi}(P_r * \mu)(t)dt$ converge weak-star to $d\mu$, i.e.,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (P_r * \mu)(t)g(t)dt \quad \overset{r\to 1}{\longrightarrow} \quad \int_{-\pi}^{\pi} g(t)d\mu(t)$$

for all g continuous on $\mathbb{T}$.

(d) *When f is continuous on $\mathbb{T}$, $P_r * f$ converges uniformly to f.*

Poisson integrals define harmonic functions. But there is also a way to determine whether an arbitrary harmonic function is a Poisson integral (of its boundary values or of a measure):

Lemma 3.3 ([2], Thm. I.3.5)**.** *Let u be a harmonic function on the upper half plane $\mathscr{H}$. Then*

(a) *If $1 < p \leq \infty$, u is the Poisson integral of a function in $L^p(\mathbb{R})$ if and only if*

$$\sup_{y>0} \|u(x+iy)\|_{L^p(dx)} < \infty.$$

(b) *u is the Poisson integral of a finite measure on $\mathbb{R}$ if and only if*

$$\sup_{y>0} \|u(x+iy)\|_{L^1(dx)} < \infty.$$

Lemma 3.4 ([4], p. 33)**.** *Let u be a harmonic function in the unit disk $\mathbb{D}$ and $u_r(e^{i\theta}) = u(re^{i\theta})$. Then*

(a) *If $1 < p \leq \infty$, u is the Poisson integral of a function in $L^p(\mathbb{T})$ if and only if*

$$\sup_{r<1} \|u_r\|_{L^p(\mathbb{T})} < \infty.$$

(b) *u is the Poisson integral of a finite measure on $\mathbb{T}$ if and only if*

$$\sup_{r<1} \|u_r\|_{L^1(\mathbb{T})} < \infty.$$

(c) *u is the Poisson integral of a finite positive measure on $\mathbb{T}$ if and only if f is nonnegative.*

Fix $\alpha > 0$. For $t \in \mathbb{R}$ consider $\Gamma_\alpha(t)$, the cone in $\mathscr{H}$ with vertex t and angle $2\arctan\alpha$,

$$\Gamma_\alpha(t) = \{z = x + iy \in \mathscr{H} \colon |x - t| < \alpha y\}.$$

Theorem 3.5 ([2], Thm. I.4.2)**.** *Let $f \in L^1(dt/(1+t^2))$ and let u be the Poisson integral of f,*

$$u(x+iy) = \int_{\mathbb{R}} P_y(x-s)f(s)ds.$$

Then

$$\sup_{z\in\Gamma_\alpha(t)} |u(z)| \leq A_\alpha \mathcal{M}f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where A_α is a constant depending only on α.

Proof. Assume for a moment that $t = 0$.

Let $z \in \Gamma_\alpha(0)$ be such that $\mathrm{Re} z = 0$. Then

$$u(z) = u(iy) = \int_{\mathbb{R}} P_y(s)f(s)ds.$$

The Poisson kernel P_y is a nonnegative even function, decreasing on $s > 0$. Therefore we can find a sequence of nonnegative step functions $\{h_n\}$, such that $0 \leq h_1 \leq \ldots \leq h_n \leq \ldots \leq P_y$, h_n converges pointwise to P_y, and

$$h_n(s) = \sum_{k=1}^{n} a_k \chi_{(-x_k,x_k)}(s), \quad a_k \geq 0.$$

Since $\int P_y(s)ds = 1$,

$$\sum_{k=1}^{n} 2x_k a_k = \int_{\mathbb{R}} h_n(s)ds \leq \int_{\mathbb{R}} P_y(s)ds = 1.$$

Moreover,

$$\left|\int_{\mathbb{R}} h_n(s)f(s)ds\right| \le \int_{\mathbb{R}} h_n(s)|f(s)|ds \le \sum_{k=1}^{n} a_k \int_{-x_k}^{x_k} |f(s)|ds$$
$$= \sum_{k=1}^{n} 2x_k a_k \frac{1}{2x_k}\int_{-x_k}^{x_k} |f(s)|ds \le \sum_{k=1}^{n} 2x_k a_k \mathcal{M}f(0) \le \mathcal{M}f(0).$$

for every $n \in \mathbb{N}$. Monotone convergence theorem gives

$$|u(iy)| \le \int_{\mathbb{R}} P_y(s)|f(s)|ds \le \mathcal{M}f(0).$$

For an arbitrary $z = x + iy \in \Gamma_\alpha(0)$, $|x| < \alpha y$, define

$$\psi(s) = \sup_{|\delta|\ge|s|} P_y(x-\delta), \quad s \in \mathbb{R}.$$

Namely,

$$\psi(s) = \begin{cases} P_y(0) & \text{for } 0 \le s \le |x|, \\ P_y(x-s) & \text{for } s \ge |x| \end{cases}.$$

Hence

$$\int_{\mathbb{R}} \psi(s)ds \le \int_{\mathbb{R}} P_y(s)ds + \frac{2|x|}{\pi y} \le A_\alpha = 1 + \frac{2\alpha}{\pi}.$$

Since ψ is nonnegative, even and decreasing on $s > 0$, repeating the approximation argument we get

$$\int_{\mathbb{R}} \psi(s)|f(s)|ds \le A_\alpha \mathcal{M}f(0).$$

This yields

$$|u(x+iy)| \le \int_{\mathbb{R}} P_y(x-s)|f(s)|ds \le \int_{\mathbb{R}} \psi(s)|f(s)|ds \le A_\alpha \mathcal{M}f(0),$$

which establishes the formula

$$\sup_{z\in\Gamma_\alpha(0)} |u(z)| \le A_\alpha \mathcal{M}f(0). \tag{3.2}$$

To deduce (3.1) from (3.2) it is enough to notice that the operator $\mathcal{M}$ commutes with translations, i.e., $\mathcal{M}f(t) = \mathcal{M}f_t(0)$ where $f_t(s) = f(s+t)$.

□

In the case of the unit circle $\mathbb{T}$, instead of the cone $\Gamma_\alpha(t)$ consider the region

$$\Gamma_\alpha(e^{it}) = \{z \in \mathbb{D} \colon \frac{|e^{it} - z|}{1-|z|} < \alpha\}, \quad \alpha > 1.$$

We can now state the analogue of Theorem 3.5:

Theorem 3.6 ([5], p. 180)**.** *Let $f \in L^1(\mathbb{T})$ and let u be the Poisson integral of f,*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - s)f(s)ds.$$

Then

$$\sup_{z\in\Gamma_\alpha(e^{it})} |u(z)| \le A_\alpha \mathcal{M}f(t), \quad t \in [-\pi,\pi],$$

where A_α is a constant depending only on α.

The two preceding theorems point out another very important maximal function:

Definition 3.7. Let u be a harmonic function on $\mathscr{H}$. The nontangential maximal function of u at $t \in \mathbb{R}$ is

$$u^*(t) = \sup_{z \in \Gamma_\alpha(t)} |u(z)|.$$

If u is harmonic on $\mathbb{D}$, then

$$u^*(t) = \sup_{z \in \Gamma_\alpha(e^{it})} |u(z)|, \quad t \in [-\pi, \pi].$$

Note that the value of nontangential maximal function is dependent on the choice of α.

The nontangential maximal function will prove extremly useful in the proof of the Fatou theorem. Namely, one of the main ingredients of that proof is the following

Theorem 3.8 ([2], Thm. I.5.1)**.** *Take $1 \leq p < \infty$. Let u be a function harmonic on $\mathscr{H}$, such that*

$$\sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|^p dx < \infty. \tag{3.3}$$

If $p > 1$, then $u^*(t) \in L^p(\mathbb{R})$, and

$$\|u^*\|_p^p \leq B_p \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|^p dx.$$

If $p = 1$, then

$$|\{t \in \mathbb{R} \colon u^*(t) > \lambda\}| \leq \frac{B_1}{\lambda} \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)| dx.$$

The constants B_p depend only on p and α.

Proof. Let u be a harmonic function on $\mathscr{H}$, such that (3.3) holds.

If $p > 1$, Theorem 3.3 implies that u is a Poisson integral of a function $f \in L^p(\mathbb{R})$. Since by Theorem 3.5

$$u^*(t) \leq A_\alpha \mathcal{M} f(t), \quad t \in \mathbb{R},$$

the Hardy-Littlewood maximal theorem gives

$$\|u^*\|_p^p \leq A_\alpha^p \|\mathcal{M} f\|_p^p \leq (A_\alpha A_p)^p \|f\|_p^p.$$

By Theorem 3.1,

$$\|f\|_p^p \leq \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|^p dx.$$

Hence

$$\|u^*\|_p^p \leq B_p \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|^p dx,$$

which is the desired conclusion for $p > 1$.

If $p = 1$, Theorem 3.3 gives only that u is a Poisson integral of a finite measure μ on $\mathbb{R}$. The same reasoning as in the proofs of Theorems 3.5 and 2.8(a) applied to the maximal function

$$\mathcal{M}(d\mu)(t) = \sup_{t \in I} \frac{1}{|I|} |\mu|(I)$$

yields

$$u^*(t) \leq A_\alpha \mathcal{M}(d\mu)(t), \quad t \in \mathbb{R},$$

and

$$|\{t \in \mathbb{R} \colon \mathcal{M}(d\mu)(t) > \lambda\}| \leq \frac{2}{\lambda} \int_{\mathbb{R}} d|\mu|,$$

for every $\lambda > 0$. Since Theorem 3.1 shows that μ is a weak-star limit of the measures $u(x+iy)dx$, $y \to 0$, we conclude that

$$\int_{\mathbb{R}} d|\mu| \le \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|dx.$$

Hence

$$|\{t \in \mathbb{R} \colon u^*(t) > \lambda\}| \le |\{t \in \mathbb{R} \colon \mathcal{M}(d\mu)(t) > \frac{\lambda}{A_\alpha}\}| \le \frac{2A_\alpha}{\lambda} \int_{\mathbb{R}} d|\mu| \le \frac{2A_\alpha}{\lambda} \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|dx.$$

□

Similar arguments apply to the case of the unit disk. Using Theorems 3.4, 3.2 and 3.6, one gets

Theorem 3.9 (see [5], p. 180)**.** *Take $1 \le p < \infty$. Let u be a function harmonic on $\mathbb{D}$, such that*

$$\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{is})|^p ds < \infty. \tag{3.4}$$

If $p > 1$, then $u^*(t) \in L^p(\mathbb{T})$, and

$$\|u^*\|_p^p \le B_p \sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{is})|^p ds.$$

If $p = 1$, then

$$|\{t \in [-\pi, \pi] \colon u^*(t) > \lambda\}| \le \frac{B_1}{\lambda} \sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{is})| ds.$$

The constants B_p depend only on p and α.

Finally, we proceed with

Theorem 3.10 (Fatou, [2], Thm. I.5.3)**.** *Take $1 \le p \le \infty$. Let $u(z)$ be a function harmonic on $\mathscr{H}$, such that*

$$\sup_{y>0} \|u(x+iy)\|_{L^p(dx)} < \infty.$$

Then for almost all $t \in \mathbb{R}$ the nontangential limit

$$\lim_{\Gamma_\alpha(t) \ni z \to t} u(z) = f(t)$$

exists.

If $p > 1$, $u(z)$ is the Poisson integral of the boundary value function $f(t)$, and if $1 < p < \infty$,

$$\|u(x+iy) - f(x)\|_{L^p(dx)} \xrightarrow{y \to 0} 0.$$

If $p = 1$, then $u(z)$ is the Poisson integral of a finite measure μ on $\mathbb{R}$, and μ is related to the boundary value function $f(t)$ by

$$d\mu = f(t)dt + d\nu,$$

where $d\nu$ is singular with respect to the Lebesgue measure.

Proof. Let $p < \infty$. Assume that u is a Poisson integral of a real-valued function $f \in L^p(\mathbb{R})$ and define

$$\Omega_f(t) = \limsup_{\Gamma_\alpha(t) \ni z \to t} u(z) - \liminf_{\Gamma_\alpha(t) \ni z \to t} u(z).$$

Note that

$$\Omega_f(t) \le 2u^*(t), \tag{3.5}$$

and Theorem 3.9 implies that $\Omega_f(t)$ is finite almost everywhere. Our goal is to show that $\Omega_f(t) = 0$ almost everywhere. To this end, take $\varepsilon > 0$. By (3.5),

$$|\{t \in \mathbb{R} \colon \Omega_f(t) > \varepsilon\}| \leq |\{t \in \mathbb{R} \colon u^*(t) > \frac{\varepsilon}{2}\}|,$$

and Theorem 3.8, along with Chebychev's inequality if $p > 1$, gives

$$|\{t \in \mathbb{R} \colon \Omega_f(t) > \varepsilon\}| \leq B_p \left(\frac{2}{\varepsilon}\|f\|_p\right)^p. \tag{3.6}$$

Density of $C_0(\mathbb{R})$ in $L^p(\mathbb{R})$ implies the existence of $g \in C_0(\mathbb{R})$, such that $\|f + g\|_p \leq \varepsilon^2$. Theorem 3.1(d) states that $P_y * g$ converges uniformly to g. Consequently, $\Omega_f(t) = 0$ for all $t \in \mathbb{R}$ and $\Omega_f(t) = \Omega_{f+g}(t)$. Therefore by (3.6),

$$|\{t \in \mathbb{R} \colon \Omega_f(t) > \varepsilon\}| = |\{t \in \mathbb{R} \colon \Omega_{f+g}(t) > \varepsilon\}| \leq B_p \left(\frac{2}{\varepsilon}\|f + g\|_p\right)^p \leq C_p \varepsilon^p.$$

Since $\varepsilon > 0$ was chosen arbitrarily, $\Omega_f(t) = 0$ almost everywhere and the nontangential limit exists. Theorem 3.1 gives that $u(x + iy)$ tends to $f(x)$ in the $L^p(dx)$ norm when $y \to 0$, so the nontangential limit of u at t must be $f(t)$ almost everywhere.

If $p = 1$,

$$\sup_{y>0} \|u(x + iy)\|_{L^1(dx)} < \infty.$$

Then u is the Poisson integral of a finite measure μ on $\mathbb{R}$. Define $u_1(x + iy) = P_y * f(x)$ and $u_2(x + iy) = P_y * \nu(x)$, where $d\mu = f(t)dt + d\nu$ and ν is singular to the Lebesgue measure dt.

As a Poisson integral of an L^1 function, u_1 has nontangential limit almost everywhere (see the first part of the proof). But u_2 has nontangential limit zero almost everywhere as a Poisson integral of a singular measure ([2], Lem. I.5.4). Hence u has nontangential limit f almost everywhere.

If $p = \infty$, then for $A > 0$ define $f_1 = f\chi_{[-A,A]}$, $f_2 = f\chi_{\mathbb{R}\setminus[-A,A]}$. Of course $f = f_1 + f_2$ and $f_1 \in L^1(\mathbb{R})$. Moreover, $u = u_1 + u_2$, where $u_1(x+iy) = P_y * f_1(x)$, $u_2(x+iy) = P_y * f_2(x)$. Since f_1 is integrable, u_1 has nontangential limit almost everywhere. Since f_2 equals zero on $(-A, A)$, u_2 has limit zero everywhere on $(-A, A)$ ([2], Lem. I.3.3). Letting $A \to \infty$ we get the desired result for $p = \infty$. □

The following result for the disk may be proved in much the same way as Theorem 3.10.

Theorem 3.11 ([4], p. 38). *Take $1 \leq p \leq \infty$. Let u be a function harmonic on $\mathbb{D}$, such that*

$$\sup_{r<1} \|u_r\|_p < \infty.$$

Then almost everywhere on $\mathbb{T}$ the nontangential limit

$$\lim_{\Gamma_\alpha(e^{it}) \ni z \to t} u(z) = f(t)$$

exists.

If $p > 1$, $u(z)$ is the Poisson integral of the boundary value function $f(t)$, and if $1 < p < \infty$,

$$\|u_r - f\|_p \overset{r\to 1}{\longrightarrow} 0.$$

If $p = 1$, then $u(z)$ is the Poisson integral of a finite measure μ on $\mathbb{T}$, and μ is related to the boundary value function $f(t)$ by

$$d\mu = \frac{1}{2\pi} f(t)dt + d\nu,$$

where $d\nu$ is singular with respect to the Lebesgue measure.

4. H^p Spaces

With the preparation concerning harmonic functions we are now ready to apply the maximal operator techniques to spaces of holomorphic functions. Our main concern will be Hardy spaces on the unit disk $\mathbb{D}$ and on the upper half plane $\mathscr{H}$.

Let $0 < p < \infty$. If f is an analytic function on the unit disk $\mathbb{D}$, then $f \in H^p(\mathbb{D})$ if

$$\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p dt = \|f\|_{H^p}^p < \infty.$$

For $p = \infty$ the space $H^\infty(\mathbb{D})$ is defined as the space of bounded analytic functions on $\mathbb{D}$ with norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

In the case of the upper half plane we say that a function f analytic on $\mathscr{H}$ is in $H^p(dt)$, $0 < p < \infty$, if

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p dx = \|f\|_{H^p}^p < \infty.$$

Again, $H^\infty(dt)$ is the space of bounded analytic functions on $\mathscr{H}$, and

$$\|f\|_\infty = \sup_{z\in\mathscr{H}} |f(z)|.$$

For more details see [2], [5], [4] or [1].

4.1. **Boundary Values.** We start with applying results from Section 3 to show that if $f \in H^p$, then f has a nontangential limit almost everywhere on the boundary (of $\mathbb{D}$ or $\mathscr{H}$). Moreover, the boundary value function is in L^p. For $1 < p < \infty$ this is a consequence of the Fatou's theorem. That it is true for $0 < p \leq 1$ follows from the fact that each nonzero function $f \in H^p$ can be factored in the form $f = Bg$, where $g \in H^p$ and B is a Blaschke product formed from the zeros of f (see [2], Thm. II.2.3 or [1], Thm. 2.5).

Theorem 4.1 ([2], Thm. II.3.1). *Let $0 < p < \infty$ and let $f(z)$ be a function in $H^p(dt)$. Then for any $\alpha > 0$, the nontangential maximal function*

$$f^*(t) = \sup_{z\in\Gamma_\alpha(t)} |f(z)|$$

is in $L^p(\mathbb{R})$ and

$$\|f^*\|_p^p \leq A_\alpha \|f\|_{H^p}^p, \tag{4.1}$$

where the constant A_α depends only on α. Moreover, for almost all $t \in \mathbb{R}$, $f(z)$ has a nontangential limit $f(t) \in L^p(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} |f(t)|^p dt = \|f\|_p^p = \|f\|_{H^p}^p \tag{4.2}$$

and

$$\lim_{y\to 0} \|f(t+iy) - f(t)\|_p^p = 0. \tag{4.3}$$

Proof. Let $f \in H^p(dt)$.

If $1 < p < \infty$, then Theorem 3.10 states that the nontangential limit $f(t)$ exists almost everywhere, $f \in L^p(\mathbb{R})$ and

$$\lim_{y\to 0} \|f(t+iy) - f(t)\|_p^p = 0.$$

By Theorem 3.8 the nontangential maximal function f^* is in $L^p(\mathbb{R})$ and

$$\|f^*\|_p^p \leq A_{p,\alpha}\|f\|_{H^p}^p. \tag{4.4}$$

Therefore the dominated convergence theorem implies

$$\|f\|_{H^p}^p = \int_{\mathbb{R}} |f(t)|^p dt.$$

If $p \leq 1$ and $f \not\equiv 0$, write $f = Bg$, where B is the Blaschke product formed from the zeros of f, $\|g\|_{H^p} = \|f\|_{H^p}$, and g has no zeros in $\mathscr{H}$. It follows that for any $p_1 > 1$, g^{p/p_1} is analytic in $\mathscr{H}$ and $g^{p/p_1} \in H^{p_1}(dt)$. Theorem 3.8 gives $(g^*)^{p/p_1} = (g^{p/p_1})^* \in L^{p_1}$ and

$$\|g^*\|_p^p = \|(g^{p/p_1})^*\|_{p_1}^{p_1} \leq B_{p_1}\|g^{p/p_1}\|_{H^{p_1}}^{p_1} = B_{p_1}\|g\|_{H^{p_1}}^{p_1}. \tag{4.5}$$

Since $|B| \leq 1$, $|f| \leq |g|$ and $f^* \leq g^*$. Therefore inequality (4.5) with $p_1 = 2$ yields

$$\|f^*\|_p^p \leq B_2\|f\|_{H^p}^p, \quad 0 < p \leq 1,$$

with a constant that does not depend on p. This implies, that constant in the inequality (4.4) also does not depend on p, $0 < p < \infty$.

We have seen that the function $G(z) = (g(z))^{\frac{p}{p_1}}$ has nontangential limit $G(t)$ almost everywhere. For each $p < 1$, p_1 can be chosen so that p/p_1 is a positive integer m. So $g(z) = G(z)^m$ also has nontangential limit and the same is true for f (Blaschke products have boundary values almost everywhere).

Again, (4.1) with the dominated convergence theorem gives (4.2). Similarly one gets (4.3). □

The same reasoning, along with Theorems 3.9 and 3.11, implies that Theorem 4.1 is true for the unit disk:

Theorem 4.2 ([2], p. 56)**.** *Let $0 < p < \infty$ and let $f(z)$ be a function in $H^p(\mathbb{D})$. Then for any $\alpha > 0$, the nontangential maximal function*

$$f^*(t) = \sup_{z\in\Gamma_\alpha(e^{it})} |f(z)|$$

is in $L^p(\mathbb{T})$ and

$$\|f^*\|_p^p \leq A_\alpha\|f\|_{H^p}^p,$$

where the constant A_α depends only on α. Moreover, for almost all $t \in [-\pi, \pi]$, $f(z)$ has a nontangential limit $f(t) \in L^p(\mathbb{T})$ satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p dt = \|f\|_p^p = \|f\|_{H^p}^p$$

and

$$\lim_{r\to 1} \|f_r - f\|_p^p = 0,$$

where $f_r(t) = f(re^{it})$.

In particular we see, that $H^p(dt)$ is isometric with a closed subspace of $L^p(\mathbb{R})$, and $H^p(\mathbb{D})$ is isometric with a closed subspace of $L^p(\mathbb{T})$.

4.2. **The Riesz Theorem.** Let $f \in L^1(\mathbb{T})$ and let u be the Poisson integral of f. Then the harmonic conjugate function of u, denoted $\widetilde{u}$ is given by the formula

$$\widetilde{u}(z) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} Q_z(s) f(s) ds,$$

where $-Q_z(s)$ is the conjugate Poisson kernel,

$$-Q_z(s) = \operatorname{Im}\left(\frac{e^{is}+z}{e^{is}-z}\right) = \frac{2r\sin(\theta-s)}{1-2r\cos(\theta-s)+r^2}, \quad z = re^{i\theta}.$$

For the upper half plane the conjugate Poisson kernel is defined by

$$Q_y(t) = \frac{1}{\pi}\frac{t}{t^2+y^2},$$

and the harmonic conjugate $\widetilde{u}$ has the form

$$\widetilde{u}(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-t}{(x-t)^2+y^2} f(t)dt = Q_y * f(x), \quad z = x+iy$$

provided $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, and

$$\widetilde{u}(z) = \int_{\mathbb{R}} \left(Q_y(x-t) + \frac{1}{\pi}\frac{t}{1+t^2}\right) f(t)dt$$

if $p = \infty$ (the difference comes from the fact that $Q_y \notin L^1(\mathbb{R})$ and we want to assure the convergence of the integral defining $\widetilde{u}$).

In both cases, f being a function on the unit circle or the upper half plane, it can be shown that $\widetilde{u}$ has nontangential limit $\widetilde{f}$ almost everywhere. The linear mapping

$$f \longmapsto \widetilde{f}$$

is called the conjugation operator. A question arises, whether the conjugation operator is L^p-bounded or not. This is strictly connected to the question about boundedness of the Szegö projection from L^p to H^p.

If $p = 2$, then it is not hard to obtain the following results

Theorem 4.3 ([2], Thm. III.1.5)**.**

a) *If $f \in L^2(\mathbb{T})$, then $\widetilde{f} \in L^2(\mathbb{T})$ and*

$$\|\widetilde{f}\|_2^2 = \|f\|_2^2 - |a_0|^2,$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$.

b) *If $f \in L^2(\mathbb{R})$, then $\widetilde{f} \in L^2(\mathbb{R})$ and*

$$\|\widetilde{f}\|_2 = \|f\|_2.$$

The second part of the theorem can be proved using the Hilbert transform.

The question of boundedness of the conjugation operator was answered by M. Riesz in 1927 (for $1 < p < \infty$).

Theorem 4.4 ([2], Thm. III.2.3)**.** *If $1 < p < \infty$, there is a constant A_p such that*

$$\|\widetilde{f}\|_p \le A_p \|f\|_p \tag{4.6}$$

if $f \in L^p(\mathbb{R})$ or $f \in L^p(\mathbb{T})$.

Among numerous proofs of the Riesz theorem there is one that makes use of the maximal conjugate function:

Definition 4.5. Fix $\alpha > 1$ and let $f \in L^1$. The maximal conjugate function of f is

$$(\widetilde{f})^*(t) = \sup_{z \in \Gamma_\alpha(t)} |\widetilde{u}(z)|,$$

where Γ_α stands for $\Gamma_\alpha(t) \subset \mathscr{H}$ or $\Gamma_\alpha(e^{it}) \subset \mathbb{D}$ if f is in $L^1(\mathbb{R})$ or $L^1(\mathbb{T})$, respectively.

The proof is based on the following theorem.

Theorem 4.6 ([2], Thm. III.2.1). *There is a constant A_α depending only on α such that*

$$|\{t\colon (\widetilde{f})^*(t) > \lambda\}| \leq \frac{A_\alpha}{\lambda} \|f\|_1 \tag{4.7}$$

if $f \in L^1(\mathbb{R})$ or $f \in L^1(\mathbb{T})$.

An immediate consequence of the Theorem 4.6 is the fact that the conjugation operator is of weak type $(1, 1)$. Moreover, by Theorem 4.3 it is also of strong type $(2, 2)$. Hence by the Marcinkiewicz interpolation theorem we conclude, that (4.6) holds for $1 < p \leq 2$. That this is true for $2 < p < \infty$ follows from a standard duality argument.

4.2.1. *Thm. 4.6 - proof for the unit disk.* Suppose (4.7) holds for nonnegative functions. An arbitrary f can be written in the form $f = f_+ - f_-$, where $f_+ \geq 0$, $f_- \geq 0$, and

$$\begin{aligned} |\{t\colon (\widetilde{f})^*(t) > \lambda\}| &\leq |\{t\colon (\widetilde{f_+})^*(t) > \frac{\lambda}{2}\}| + |\{t\colon (\widetilde{f_-})^*(t) > \frac{\lambda}{2}\}| \\ &\leq \frac{2A_\alpha}{\lambda}\|f_+\|_1 + \frac{2A_\alpha}{\lambda}\|f_-\|_1 \leq \frac{8A_\alpha}{\lambda}\|f\|_1. \end{aligned}$$

Therefore without loss of generality we can assume that $f \geq 0$.

For $\lambda > 0$ consider the harmonic function

$$h(x + iy) = \frac{1}{\pi} \int_{|s|>\lambda} \frac{x}{(y - s)^2 + x^2} ds. \tag{4.8}$$

For all $x + iy \in \mathbb{C}$, $h(x + iy) \geq 0$ and $h(x + iy) \geq \frac{1}{2}$ whenever $y \geq \lambda$. Moreover, on the positive real axis

$$h(x) = \frac{2}{\pi} \int_\lambda^\infty \frac{x}{s^2 + x^2} ds \leq \frac{2x}{\pi\lambda}. \tag{4.9}$$

Let $F(z) = F(re^{it}) = (P_r + iQ_r) * f(t)$. Function F is analytic in $\mathbb{D}$, $F(0) = \|f\|_1$ and $\mathrm{Re}F > 0$. Therefore $g(z) = h(F(z))$ is a positive harmonic function on $\mathbb{D}$. By Theorem 3.4, g is a Poisson integral of a positive measure μ on $\mathbb{D}$ with

$$\int_{\mathbb{D}} d|\mu| = g(0) = h(F(0)) = h(\|f\|_1).$$

If $|\mathrm{Im}F(z)| > \lambda$, then $g(z) > \frac{1}{2}$ and

$$\{t\colon (\widetilde{f})^*(t) > \lambda\} \subset \{t\colon g^*(t) > \frac{1}{2}\}. \tag{4.10}$$

But, as seen in the proof of Theorem 3.8,

$$|\{t\colon g^*(t) > \lambda\}| \leq 2A_\alpha \int_{\mathbb{D}} d|\mu| = 2A_\alpha h(\|f\|_1).$$

Together with (4.9) and (4.10) this yields

$$|\{t\colon (\widetilde{f})^*(t) > \lambda\}| \leq \frac{4A_\alpha}{\pi\lambda}\|f\|_1,$$

which proves the theorem for the unit disk.

4.2.2. *Thm. 4.6 - proof for the upper half plane.* Proof for the upper half plane requires an additional lemma:

Lemma 4.7 ([2] Lem. III.2.2)**.** *If μ is a finite measure on $\mathbb{R}$ with Poisson integral u, then*

$$\int_{\mathbb{R}} d\mu = \lim_{y\to\infty} \int_{\mathbb{R}} \frac{y^2}{t^2+y^2} d\mu(t) = \lim_{y\to\infty} \pi y u(iy).$$

Take $f \in L^1(\mathbb{R})$. As before we can assume that $f \geq 0$.

Define $g(z) = h(F(z))$, where h is given by (4.8) and $F(x+iy) = (u+i\widetilde{u})(x+iy) = (P_y + iQ_y) * f(x)$. Function F is now analytic on $\mathscr{H}$ and $\mathrm{Re}F > 0$. Hence, g is harmonic and positive. Also $0 \leq g \leq 1$, which by Theorem 3.3 gives that g is the Poisson integral of its boundary values $g(t)$.

For $\lambda > 0$,

$$|\{t \in \mathbb{R}\colon (\widetilde{f})^*(t) > \lambda\}| \leq |\{t \in \mathbb{R}\colon g^*(t) > \frac{1}{2}\}| \leq 2B_\alpha \int_{\mathbb{R}} g(t)dt.$$

By Lemma 4.7:

$$\int_{\mathbb{R}} g(t)dt = \lim_{y\to\infty} \pi y h(F(iy)) = \lim_{y\to\infty} y \int_{\mathbb{R}} \frac{u(iy)}{(u(iy))^2 + (\widetilde{u}(iy)-s)^2} d\mu(t).$$

Since $\lim_{y\to\infty} \widetilde{u}(iy) = 0$, another application of Lemma 4.7 gives

$$\int_{\mathbb{R}} g(t)dt = \frac{1}{2\pi} \int_{\mathbb{R}} f(t)dt \int_{|s|>\lambda} \frac{ds}{s^2} = \frac{2\|f\|_1}{\pi\lambda}.$$

Finally,

$$|\{t \in \mathbb{R}\colon (\widetilde{f})^*(t) > \lambda\}| \leq \frac{4B_\alpha}{\pi\lambda}\|f\|_1,$$

which completes the proof.

The proof of Theorem 4.6 presented here was given by P. Koosis (see [5]).

4.3. H^p **Spaces and Maximal Functions.** The Hardy spaces can also be defined in a conformally invariant way. Namely, $f \in H^p$ if and only if the subharmonic function $|f|^p$ has a harmonic majorant. However, one can give a definition of the H^p spaces, $p < \infty$, only with use of the maximal functions. It turns out that an analytic function f is in H^p, $0 < p < \infty$ if and only if the nontangential maximal function of $\mathrm{Re}f$ belongs to L^p. More precisely,

Theorem 4.8 ([2], Cor. III.3.4)**.** *Let $0 < p < \infty$ and let $u(z) = \mathrm{Re}f(z)$, where f is an analytic function. Then $f \in H^p$ if and only if $u^* \in L^p$. There are constants c_1 and c_2, depending only on p, such that*

$$c_1\|u^*\|_p \leq \|f\|_{H^p} \leq c_2\|u^*\|_p.$$

This theorem holds for Hardy spaces on the upper half plane as well as on the unit disk.

Note that the first inequality was already proved in the first part of this section. The second one is an immediate consequence of the following theorem, due to Burkholder, Gundy and Silverstein:

Theorem 4.9 ([2], Thm. III.3.1)**.** *Let $0 < p < \infty$.*

a) *If u is a real-valued harmonic function on $\mathscr{H}$ such that $u^* \in L^p(\mathbb{R})$, then there is a harmonic conjugate function $\widetilde{u}$ for u such that*

$$\sup_{y>0} \int_{\mathbb{R}} |\widetilde{u}(x+iy)|^p dx \leq C_p \int_{\mathbb{R}} |u^*(t)|^p dt.$$

b) *If u is a real-valued harmonic function on $\mathbb{D}$ such that $u^* \in L^p(\mathbb{T})$, then*

$$\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widetilde{u}(re^{it})|^p dt \leq C_p \frac{1}{2\pi} \int_{-\pi}^{\pi} |u^*(t)|^p dt,$$

where $\widetilde{u}(0) = 0$.

There is a strong connection between the preceeding theorem and the Riesz theorem. In fact, Theorem 4.9 together with the Hardy-Littlewood maximal theorem implies the Theorem 4.4. On the other hand, for $p > 1$ Theorem 4.9 is a consequence of the Riesz theorem. Therefore we will sketch the proof of Theorem 4.9 only for $p < 2$.

Although Thm 4.9 is true for the nontangential maximal function defined for an arbitrary $\alpha > 0$, we are going to give the proof under an additional assumption that

$$u^*(t) = \sup_{z\in\Gamma_1(t)} |u(z)|$$

in the case of $\mathscr{H}$, and

$$u^*(t) = \sup_{z\in\Gamma_2(e^{it})} |u(z)|$$

in the case of $\mathbb{D}$. The fact that Theorem 4.9 is true for cones of any angle stems from the properties of the vertical maximal function (see [2], p. 116).

4.3.1. *Proof of the Theorem 4.9.* The proof is based on the following two lemmas (proofs can be found in [2]).

Lemma 4.10 ([2], Lem. III.3.2). *If $F(z) = u(z) + iv(z)$ is of class H^2, then*

$$m_v(\lambda) \leq 2m_{u^*}(\lambda) + \frac{2}{\lambda^2} \int_0^{\lambda} sm_{u^*}(s)ds. \tag{4.11}$$

Lemma 4.11 ([2], Lem. III.3.3). *If $0 < p \leq 2$, and if $u^* \in L^p$, then*

$$\left(\int_{\mathbb{R}} |u(x+iy)|^2 dx \right)^{1/2} \leq 2y^{1/2-1/p} \left(\int_{\mathbb{R}} |u^*(s)|^p ds \right)^{1/p}, \quad y > 0.$$

Assume that u is a real-valued harmonic function on $\mathscr{H}$ and $u^* \in L^p$, $0 < p < 2$.

For every $y_0 > 0$, using Lemma 4.11 one can find a unique conjugate function v, defined on $y > y_0$, such that $f = u + iv$ satisfies

$$\sup_{y>y_0} \int_{\mathbb{R}} |f(x+iy)|^2 dx < \infty.$$

Therefore v does not depend on y_0.

Lemma 4.10 applied to $F(z) = u_0(z) + v_0(z)$, $u_0(z) = u(z+iy_0)$, $v_0(z) = v(z+iy_0)$, and the fact that $u_0^* \leq u^*$, give

$$m_{v_0}(\lambda) \leq 2m_{u^*}(\lambda) + \frac{2}{\lambda^2} \int_0^{\lambda} sm_{u^*}(s)ds.$$

Hence, by Lemma 2.3

$$\begin{aligned}
\int_{\mathbb{R}} |v(x+iy_0)|^p dx &= \int_0^{\infty} p\lambda^{p-1} m_{v_0}(\lambda)d\lambda \\
&\leq 2\int_0^{\infty} p\lambda^{p-1} m_{u^*}(\lambda)d\lambda + 2\int_0^{\infty} psm_{u^*}(s) \int_s^{\infty} \lambda^{p-3} d\lambda ds \\
&= 2\|u^*\|_p^p + \frac{2}{2-p} \int_0^{\infty} ps^{p-1} m_{u^*}(s)ds = 2(1 + \frac{1}{2-p})\|u^*\|_p^p.
\end{aligned}$$

This concludes the proof, since the right side does not depend on $y_0 > 0$.

The same reasoning and the following version of Lemma 4.10 give the proof for the disk.

Lemma 4.12. *If* $F(z) = u(z) + i\widetilde{u}(z)$ *is of class* $H^2(\mathbb{D})$, $\widetilde{u}(0) = 0$, *then*

$$m_{\widetilde{u}}(\lambda) \leq C m_{u^*}(\lambda) + \frac{1}{\lambda^2} \int_0^\lambda s m_{u^*}(s) ds. \tag{4.12}$$

References

[1] P. L. Duren, *Theory of H^p spaces,* Pure Appl. Math., vol. 38, Academic Press, New York, 1970.

[2] J. B. Garnett, *Bounded analytic functions,* Springer, New York, 2010.

[3] G. H. Hardy, J. E. Littlewood, *A maximal theorem with function theoretic application,* Acta Math. 54 (1930), 81-116.

[4] K. Hoffman, *Banach spaces of analytic functions,* Dover, New York, 1988.

[5] P. Koosis, *Introduction to H^p spaces,* Cambridge University Press, Cambridge, 1998.

[6] W. Rudin, *Functional analysis,* McGraw-Hill Book Company, New York, 1987.

[7] E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, 1970.

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