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De Branges-Rovnyak Spaces

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Abstract. The de Branges-Rovnyak spaces are Hilbert spaces that "live" inside the Hardy space H^2. This spaces are defined in terms of Toeplitz operators. Hence we first discuss properties of bounded Toeplitz operators on H^2. In particular, we give another proof of the fact that the only non-zero Toeplitz operators that are partial isometries are those of the form T_φ, $T_{\bar{\varphi}}$ where φ is an inner function. Section 3, based on the book [13], describes Hilbert spaces that are ranges of contractions from one Hilbert space into another. Finally, in section 4 we introduce the definition of de Branges-Rovnyak spaces and describe their structure in some special cases.

1. Positive operators, projections and square roots

Let $\mathscr{H}$ be a complex Hilbert space. Its scalar product and norm will be denoted by $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}}$, respectively. If $\mathscr{H}_1$ is another Hilbert space, then $\mathscr{L}(\mathscr{H}_1,\mathscr{H})$ will stand for the space of all bounded linear operators from $\mathscr{H}_1$ into $\mathscr{H}$. For simplicity $\mathscr{L}(\mathscr{H},\mathscr{H})=\mathscr{L}(\mathscr{H})$. For an arbitrary $T\in\mathscr{L}(\mathscr{H}_1,\mathscr{H})$ let $\ker T$ and $\mathrm{Im} T$ denote the kernel and the range of the operator T, respectively.

Definition 1.1. If $P\in\mathscr{L}(\mathscr{H})$ and $P^2=P$, then P is called a projection. If in addition $P^*=P$, then P is called an orthogonal projection.

Remark 1.2. If P is a projection, then $Q=I-P$ is also a projection. Moreover, $QPx=Px-P^2x=0$ and if $Qx=0$, then $Px=x$. Hence, $\ker Q=\mathrm{Im}P$. Similarly, $\mathrm{Im}Q=\ker P$. Hence the range of a projection is always a closed subspace on which P acts like the identity.

If P is an orthogonal projection, then $(\mathrm{Im}P)^\perp=\ker P^*=\ker P$ and by the projection theorem

$$\mathscr{H}=\mathrm{Im}P\oplus\ker P.$$

On the other hand, for any closed subspace $\mathscr{M}\subset\mathscr{H}$ we have $\mathscr{H}=\mathscr{M}\oplus\mathscr{M}^\perp$. If for every $x\in\mathscr{H}$ we define $P_{\mathscr{M}}(x)$ by

$$P_{\mathscr{M}}(x)=P_{\mathscr{M}}(x_{\mathscr{M}}+x_{\mathscr{M}^\perp})=x_{\mathscr{M}},$$

then $P_{\mathscr{M}}$ is an ortogonal projection onto $\mathscr{M}$.

Definition 1.3. Let $\mathscr{H}$ be a Hilbert space. An operator $A\in\mathscr{L}(\mathscr{H})$ is called positive if

$$\langle Ax,x\rangle_{\mathscr{H}}\geq 0$$

for every $x\in\mathscr{H}$. If A is positive, we write $A\geq 0$. For $B\in\mathscr{L}(\mathscr{H})$ we say that $B\geq A$, provided $B-A\geq 0$.

We first note that in a complex setting $A\geq 0$ implies $A=A^*$. Indeed, from the positivity of A one gets

$$\langle Ax-A^*x,x\rangle_{\mathscr{H}}=0 \tag{1.1}$$

for every $x\in\mathscr{H}$. If we put $T=A-A^*$, then (1.1) gives

$$\langle T(x+y),x+y\rangle_{\mathscr{H}}=0,$$

for all $x, y \in \mathscr{H}$. This yields

$$\langle Tx, y\rangle_{\mathscr{H}} + \langle Ty, x\rangle_{\mathscr{H}} = 0.$$

Similarly,

$$\langle T(x+iy), x+iy\rangle_{\mathscr{H}} = 0$$

implies (multiplied by i) that

$$\langle Tx, y\rangle_{\mathscr{H}} - \langle Ty, x\rangle_{\mathscr{H}} = 0.$$

Hence, $\langle Tx, y\rangle_{\mathscr{H}} = 0$ for every $x, y \in \mathscr{H}$ and $T \equiv 0$.

Consequently,

Lemma 1.4 ([6], Prop. 2.13)**.** *If A is a positive operator, then*

$$\|A\| = \sup_{\|x\|_{\mathscr{H}}=1} |\langle Ax, x\rangle_{\mathscr{H}}| = \sup_{\|x\|_{\mathscr{H}}=1} \langle Ax, x\rangle_{\mathscr{H}}.$$

Proof. Since a positive operator is also self-adjoint, we get

$$\begin{aligned}\langle A(x+y), x+y\rangle_{\mathscr{H}} &= \langle Ax, x\rangle_{\mathscr{H}} + \langle Ax, y\rangle_{\mathscr{H}} + \langle Ay, x\rangle_{\mathscr{H}} + \langle Ay, y\rangle_{\mathscr{H}} \\ &= \langle Ax, x\rangle_{\mathscr{H}} + \langle Ax, y\rangle_{\mathscr{H}} + \overline{\langle Ax, y\rangle_{\mathscr{H}}} + \langle Ay, y\rangle_{\mathscr{H}} \\ &= \langle Ax, x\rangle_{\mathscr{H}} + 2\mathrm{Re}\langle Ax, y\rangle_{\mathscr{H}} + \langle Ay, y\rangle_{\mathscr{H}}.\end{aligned}$$

Similarly,

$$\langle A(x-y), x-y\rangle_{\mathscr{H}} = \langle Ax, x\rangle_{\mathscr{H}} - 2\mathrm{Re}\langle Ax, y\rangle_{\mathscr{H}} + \langle Ay, y\rangle_{\mathscr{H}}.$$

It follows that

$$4\mathrm{Re}\langle Ax, y\rangle_{\mathscr{H}} = \langle A(x+y), x+y\rangle_{\mathscr{H}} - \langle A(x-y), x-y\rangle_{\mathscr{H}}. \tag{1.2}$$

Put $C = \sup_{\|x\|_{\mathscr{H}}=1} |\langle Ax, x\rangle_{\mathscr{H}}|$. Then we have

$$C = \sup_{\|x\|_{\mathscr{H}}=1} |\langle Ax, x\rangle_{\mathscr{H}}| \leq \sup_{\|x\|_{\mathscr{H}}=1} \|A\|\,\|x\|^2_{\mathscr{H}} = \|A\|.$$

Moreover, by (1.2) and the parallelogram law one gets

$$4\mathrm{Re}\langle Ax, y\rangle_{\mathscr{H}} \leq C\left(\|x+y\|^2_{\mathscr{H}} + \|x-y\|^2_{\mathscr{H}}\right) = 2C\left(\|x\|^2_{\mathscr{H}} + \|y\|^2_{\mathscr{H}}\right)$$

If x and y are unit vectors, and $|\lambda| = 1$ is such that $\lambda\langle Ax, y\rangle_{\mathscr{H}} = |\langle Ax, y\rangle_{\mathscr{H}}|$, then

$$|\langle Ax, y\rangle_{\mathscr{H}}| = \mathrm{Re}\langle Ax, \bar{\lambda}y\rangle_{\mathscr{H}} \leq \frac{C}{2}\left(\|x\|^2_{\mathscr{H}} + \|\bar{\lambda}y\|^2_{\mathscr{H}}\right) = C.$$

Thus

$$\|Ax\|_{\mathscr{H}} = \sup_{\|y\|_{\mathscr{H}}=1} |\langle Ax, y\rangle_{\mathscr{H}}| \leq C$$

for every unit vector x, which implies $\|A\| \leq C$. This completes the proof. □

Lemma 1.5. *If $A \geq 0$ and $\|A\| \leq 1$, then $I - A \geq 0$ and $\|I - A\| \leq 1$.*

Proof. For $x \in \mathscr{H}$, using the Schwarz inequality we have

$$\langle (I-A)x, x\rangle_{\mathscr{H}} = \|x\|^2_{\mathscr{H}} - \langle Ax, x\rangle_{\mathscr{H}} \geq \|x\|^2_{\mathscr{H}} - \|Ax\|_{\mathscr{H}}\|x\|_{\mathscr{H}} \geq (1 - \|A\|)\|x\|^2_{\mathscr{H}} \geq 0.$$

Hence $I - A$ is a positive operator and

$$\|I - A\| = \sup_{\|x\|=1} \langle (I-A)x, x\rangle_{\mathscr{H}} = \sup_{\|x\|=1} (1 - \langle Ax, x\rangle_{\mathscr{H}}) \leq 1.$$

□

Definition 1.6. Let A be a positive operator. A positive operator B is called positive square root of A if $B^2 = A$. Positive square root of A is denoted $A^{1/2}$.

Theorem 1.7 ([1], Prop. 3.2.6)**.** *Let $A \in \mathscr{L}(\mathscr{H})$ and $A \geq 0$. Then there is a unique, positive $B \in \mathscr{L}(\mathscr{H})$, such that $B^2 = A$. Furthermore, B commutes with every bounded operator which commutes with A.*

Lemma 1.8. *For a positive operator $A \in \mathscr{L}(\mathscr{H})$ we have $\ker A^{1/2} = \ker A$ and $\|A^{1/2}\| = \|A\|^{1/2}$.*

Proof. Since $Ax = A^{1/2}A^{1/2}x$ then

$$\ker A^{1/2} \subset \ker A.$$

The other inclusion follows from the equalities

$$\|A^{1/2}x\|^2_{\mathscr{H}} = \langle A^{1/2}x, A^{1/2}x\rangle_{\mathscr{H}} = \langle Ax, x\rangle_{\mathscr{H}}. \tag{1.3}$$

By (1.3) and Lemma 1.4 we have the norm equality. □

Definition 1.9. Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces.

(i) An operator $U \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ is called an isometry if $\|Ux\|_{\mathscr{H}} = \|x\|_{\mathscr{H}_1}$ for every $x \in \mathscr{H}_1$.

(ii) An operator $U \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ is called a partial isometry if there exists a closed subspace $\mathscr{M}$ of $\mathscr{H}_1$ such that

$$\|Ux\|_{\mathscr{H}} = \|x\|_{\mathscr{H}_1} \text{ for } x \in \mathscr{M}, \qquad Ux = 0 \text{ for } x \in \mathscr{M}^{\perp}.$$

The closed subspaces $\mathscr{M}$ and $\mathrm{Im}U$ are called the initial and final domain of U, respectively.

Remark 1.10. (*a*) The final domain of a partial isometry U is always closed. Indeed, take $\{y_n\}_{n\in\mathbb{N}} \subset \mathrm{Im}U$, $y_n \to y$ in $\mathscr{H}$. We can assume that $y_n = Ux_n$ where $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{M}$. Since $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and $\|y_n - y_m\|_{\mathscr{H}} = \|x_n - x_m\|_{\mathscr{H}_1}$ we see that $\{x_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Therefore $x_n \to x$ for some $x \in \mathscr{H}_1$. Closedness of $\mathscr{M}$ gives $x \in \mathscr{M}$. Continuity of U yields $y = Ux$, which means that $\mathrm{Im}U$ is closed.

(*b*) By definition $\mathscr{M}^{\perp} \subset \ker U$. On the other hand, if $x \in \ker U$, then $Ux = Ux_{\mathscr{M}} + Ux_{\mathscr{M}^{\perp}} = Ux_{\mathscr{M}} = 0$, where $x_{\mathscr{M}}$ and $x_{\mathscr{M}^{\perp}}$ are in $\mathscr{M}$ and $\mathscr{M}^{\perp}$, respectively. But then $\|Ux_{\mathscr{M}}\|_{\mathscr{H}} = \|x_{\mathscr{M}}\|_{\mathscr{H}_1} = 0$. Hence $x = x_{\mathscr{M}^{\perp}}$. Thus

$$\ker U = \mathscr{M}^{\perp}.$$

So we have the following

Proposition 1.11. *A Hilbert space operator $U \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ is a partial isometry if and only if*

$$\|Ux\|_{\mathscr{H}} = \|x\|_{\mathscr{H}_1} \quad \text{for} \quad x \in (\ker U)^{\perp}.$$

Theorem 1.12 ([15], Thm. 4.34)**.** *Let $\mathscr{H}_1$ and $\mathscr{H}$ be Hilbert spaces and let $U \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$.*

(a) *The following assertions are equivalent:*
   - (i) *U is a partial isometry with initial domain $\mathscr{M}$ and final domain $\mathscr{N}$,*
   - (ii) *$\mathrm{Im}U = \mathscr{N}$ and $\langle Ux, Uy\rangle_{\mathscr{H}} = \langle P_{\mathscr{M}}x, y\rangle_{\mathscr{H}_1}$ for all $x, y \in \mathscr{H}_1$,*
   - (iii) *$U^*U = P_{\mathscr{M}}$ and $UU^* = P_{\mathscr{N}}$,*
   - (iv) *U^* is a partial isometry with initial domain $\mathscr{N}$ and final domain $\mathscr{M}$.*

(b) *The following assertions are equivalent:*
   - (i) *U is unitary (isometric isomorphism),*
   - (ii) *$\mathrm{Im}U = \mathscr{H}$ and $\langle Ux, Uy\rangle_{\mathscr{H}} = \langle x, y\rangle_{\mathscr{H}_1}$ for all $x, y \in \mathscr{H}_1$,*
   - (iii) *$U^*U = I_{\mathscr{H}_1}$ and $UU^* = I_{\mathscr{H}}$, i.e., $U^* = U^{-1}$,*
   - (iv) *U^* is unitary.*

Proof. We will only prove part (a) of the Theorem, since part (b) is a consequence of (a).

(i) $\Rightarrow$ (ii)

Equality Im$U = \mathscr{N}$ is just the definition of the final domain of U.

For every $x \in \mathscr{H}_1$,

$$\langle UP_{\mathscr{M}}x, UP_{\mathscr{M}}x\rangle_{\mathscr{H}} = \langle P_{\mathscr{M}}x, P_{\mathscr{M}}x\rangle_{\mathscr{H}_1}.$$

Polarization identity

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

gives that for arbitrary $x, y \in \mathscr{H}_1$,

$$\langle UP_{\mathscr{M}}x, UP_{\mathscr{M}}y\rangle_{\mathscr{H}} = \langle P_{\mathscr{M}}x, P_{\mathscr{M}}y\rangle_{\mathscr{H}_1},$$

and therefore

$$\langle Ux, Uy\rangle_{\mathscr{H}} = \langle UP_{\mathscr{M}}x, UP_{\mathscr{M}}y\rangle_{\mathscr{H}} = \langle P_{\mathscr{M}}x, y\rangle_{\mathscr{H}_1}.$$

(ii) $\Rightarrow$ (i)

From (ii) we see, that $\|Ux\|^2_{\mathscr{H}} = \langle P_{\mathscr{M}}x, x\rangle_{\mathscr{H}_1}$ for all $x \in \mathscr{H}_1$. Hence,

$$\|Ux\|_{\mathscr{H}} = \|x\|_{\mathscr{H}_1} \quad \text{for} \quad x \in \mathscr{M}, \quad \text{and} \quad Ux = 0 \quad \text{for} \quad x \in \mathscr{M}^{\perp}.$$

(i) $\Rightarrow$ (iv)

Since (i) implies (ii),

$$\langle U^*Ux, U^*Ux\rangle_{\mathscr{H}_1} = \langle Ux, UU^*Ux\rangle_{\mathscr{H}} = \langle P_{\mathscr{M}}x, U^*Ux\rangle_{\mathscr{H}_1} = \langle UP_{\mathscr{M}}x, Ux\rangle_{\mathscr{H}} = \langle Ux, Ux\rangle_{\mathscr{H}}.$$

Therefore

$$\|U^*y\|_{\mathscr{H}_1} = \|y\|_{\mathscr{H}} \quad \text{for all } y \in \mathscr{N}.$$

But

$$\ker U^* = (\mathrm{Im}U)^{\perp} = \mathscr{N}^{\perp},$$

which means that U^* is a partial isometry with the initial domain $\mathscr{N}$. Form Remark 1.10 the set ImU^* is closed, and

$$\mathrm{Im}U^* = (\ker U)^{\perp} = \mathscr{M}^{\perp\perp} = \mathscr{M}.$$

(iv) $\Rightarrow$ (i)

Repeating the proof of implication (i) $\Rightarrow$ (iv) with U replaced by U^* we can show that U^{**} is a partial isometry. Then (i) follows, since $U = U^{**}$.

(i) $\Rightarrow$ (iii)

Again, (i) implies (ii), and by (ii)

$$\langle U^*Ux, y\rangle_{\mathscr{H}_1} = \langle Ux, Uy\rangle_{\mathscr{H}} = \langle P_{\mathscr{M}}x, y\rangle_{\mathscr{H}_1}$$

for all $x, y \in \mathscr{H}_1$. This means that $U^*U = P_{\mathscr{M}}$.

As already shown, (i) also implies (iv), and so

$$\langle UU^*x, y\rangle_{\mathscr{H}} = \langle U^*x, U^*y\rangle_{\mathscr{H}_1} = \langle P_{\mathscr{N}}x, y\rangle_{\mathscr{H}},$$

for all $x, y \in \mathscr{H}$, which means $UU^* = P_{\mathscr{N}}$.

(iii) $\Rightarrow$ (ii)

Obviously

$$\langle Ux, Uy\rangle_{\mathscr{H}} = \langle U^*Ux, y\rangle_{\mathscr{H}_1} = \langle P_{\mathscr{M}}x, y\rangle_{\mathscr{H}_1}$$

for all $x, y \in \mathscr{H}_1$. Hence U is a partial isometry with initial domain $\mathscr{M}$, and ImU is closed.

For every $x \in \mathscr{H}$,

$$\|U^*x\|^2_{\mathscr{H}_1} = \langle U^*x, U^*x\rangle_{\mathscr{H}_1} = \langle UU^*x, x\rangle_{\mathscr{H}} = \langle P_{\mathscr{N}}x, x\rangle_{\mathscr{H}} = \|P_{\mathscr{N}}x\|^2_{\mathscr{H}}.$$

Therefore $\ker U^* = \mathscr{N}^\perp$, and

$$\mathrm{Im}U = \overline{\mathrm{Im}U} = (\ker U^*)^\perp = \mathscr{N}^{\perp\perp} = \mathscr{N}.$$

□

2. Bounded Toeplitz operators

Definition 2.1. For $\varphi \in L^\infty(\mathbb{T})$ the Toeplitz operator $T_\varphi \colon H^2 \to H^2$ is given by

$$T_\varphi f = P_+(\varphi f),$$

where P_+ is the Szegö projection,

$$P_+ f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1 - ze^{-it}} dt.$$

Note that the corespondence $\varphi \mapsto T_\varphi$ is one-to-one. Indeed, if $\varphi = 0$, then obviously $T_\varphi = 0$. On the other hand, let $\varphi = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) e_k$, $e_k(t) = e^{ikt}$, and assume that $T_\varphi = 0$. Then $P_+(\varphi e_n) = 0$ for every $n \geq 0$. From this $\hat{\varphi}(k) = 0$ whenever $k + n \geq 0$, $n \geq 0$. Since for each $k \in \mathbb{Z}$ such n can be found, we see that $\varphi = 0$.

It is easy to see that $T_\varphi^* = T_{\overline{\varphi}}$. Indeed, for $f, g \in H^2$ we have

$$\langle T_\varphi f, g\rangle_2 = \langle P_+(\varphi f), g\rangle_2 = \langle \varphi f, g\rangle_2 = \langle f, \overline{\varphi} g\rangle_2 = \langle f, P_+(\overline{\varphi} g)\rangle_2 = \langle f, T_{\overline{\varphi}} g\rangle_2.$$

Lemma 2.2 ([2], Prop. 2.2)**.** *Let $\varphi \in L^\infty(\mathbb{T})$. If L_φ denotes the multiplication operator on $L^2(\mathbb{T})$, $L_\varphi \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$, $L_\varphi f = \varphi f$, then*

$$\|L_\varphi\| = \|\varphi\|_\infty$$

Proof. It is easy to see that $\|L_\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2$ for every $f \in L^2(\mathbb{T})$, hence $\|L_\varphi\| \leq \|\varphi\|_\infty$.

Assume that $\|L_\varphi\| < \|\varphi\|_\infty$. From the definition of the norm $\|\cdot\|_\infty$ one gets the existence of a set $E \subset \mathbb{T}$ of positive measure, such that $|\varphi(e^{it})| > \|L_\varphi\|$ for every $t \in E$. Let χ_E denote the characteristic function of E. Then

$$\|L_\varphi \chi_E\|_2^2 = \frac{1}{2\pi}\int_E |\varphi(e^{it})|^2 dt > \|L_\varphi\|^2 \frac{1}{2\pi}\int_E dt = \|L_\varphi\|^2 \|\chi_E\|_2^2.$$

This however is a contradiction. Therefore $\|L_\varphi\| = \|\varphi\|_\infty$. □

Now we can prove

Proposition 2.3 ([2], Cor. 2.8)**.** *Let $\varphi \in L^\infty$. Then*

$$\|T_\varphi\| = \|\varphi\|_\infty.$$

Proof. Let $\mathscr{P}$ denote the set of all trigonometric polynomials and let $e_n(t) = e^{int}$ for $n \in \mathbb{Z}$.

If $f, g \in \mathscr{P}$, then it is easy to check that for n large enough

$$\langle L_\varphi f, g\rangle_2 = \langle L_{e_{-n}} T_\varphi L_{e_n} f, g\rangle_2.$$

(since $f \in \mathscr{P}$, $L_{e_n} f \in H^2$ for large n). This implies that

$$|\langle L_\varphi f, g\rangle_2| \leq \limsup_{n\to\infty} |\langle L_{e_{-n}} T_\varphi L_{e_n} f, g\rangle_2| \leq \|T_\varphi\| \|f\|_2 \|g\|_2.$$

Using the above inequality we get

$$\|L_\varphi f\|_2 = \sup\{|\langle L_\varphi f, g\rangle_2| \colon \; g \in \mathscr{P}, \; \|g\|_2 \leq 1\} \leq \|T_\varphi\| \|f\|_2$$

for every $f \in \mathscr{P}$. This way we showed that $\|L_\varphi\|_{L^2} \leq \|T_\varphi\|$, and in view of Lemma 2.2,

$$\|\varphi\|_\infty \leq \|T_\varphi\|.$$

On the other hand, from the definition of the Toeplitz operator

$$\|T_\varphi\| \le \|P_+\| \|L_\varphi\|_{L^2} = \|P_+\| \|\varphi\|_\infty.$$

Our claim follows from the fact, that the Szegö projection is an orthogonal projection from $L^2(\mathbb{T})$ onto H^2, hence $\|P_+\| = 1$. □

Definition 2.4. Let $S = T_{e_1}$. The operator S is called the unilateral shift,

$$Sf(z) = zf(z), \quad f \in H^2.$$

The adjoint of S, $S^* = T_{\overline{e_1}}$ is called the backward shift.

Note that for $f \in H^2$,

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

In what follows we will make use of the matrix representation of a Toeplitz operator.

Definition 2.5. Let A be a bounded linear operator on a Hilbert space $\mathscr{H}$. The matrix of A with respect to a given orthonormal basis $\{e_n\}_{n\in\mathcal{I}}$ is the matrix $M_A = [c_{i,j}]_{i,j\in\mathcal{I}}$, whose entries are given by

$$c_{i,j} = \langle Ae_j, e_i\rangle_{\mathscr{H}}.$$

Let us now take a look at the matrix of a Toeplitz operator T_φ with respect to the standard basis $\{z^n\}_{n\ge0}$. If $\varphi = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)e_k$, then

$$\langle T_\varphi e_n, e_m\rangle_2 = \langle \varphi z^n, z^m\rangle_2 = \langle \varphi, z^{m-n}\rangle_2 = \hat{\varphi}(m-n).$$

Hence

$$M_{T_\varphi} = [\hat{\varphi}(m-n)]_{m,n\ge0}\,.$$

So Toeplitz operators have matrices with constant diagonals.

Definition 2.6. A infinite matrix $M = [c_{i,j}]_{i,j\ge0}$ is called a Toeplitz matrix if its entries are constant along each diagonal.

Theorem 2.7 (see [5])**.** *Let T be a bounded linear operator on H^2. The following conditions are equivalent*

(a) $T = T_\varphi$ *for a function* $\varphi \in L^\infty(\mathbb{T})$,
(b) M_T *is a Toeplitz matrix,*
(c) $S^*TS = T$.

Proof. Let T be a bounded linear operator on H^2 and let $M_T = [c_{i,j}]_{i,j\ge0}$ be the corresponding matrix.

We have already seen that if $T = T_\varphi$ for a bounded function φ, then M_T is a Toeplitz matrix. If M_T is a Toeplitz matrix, then there exists a sequence $\{a_k\}_{k=-\infty}^{\infty}$ such that $c_{i,j} = a_{i-j}$. Put $\varphi(z) = \sum a_k z^k$. Boundedness of T gives

$$\sum_{k=-n}^{\infty} |a_k|^2 = \|Te_n\|_2^2 \le \|T\|^2\|e_n\|_2^2 = \|T\|^2,$$

for every $n \ge 0$. Hence $\varphi \in L^2(\mathbb{T})$ and by [16] (Prop. 9.1.2) we get that φ is bounded. This establishes the equivalence of (a) and (b).

If $S^*TS = T$, then

$$c_{i+1,j+1} = \langle Te_{j+1}, e_{i+1}\rangle_2 = \langle TSe_j, Se_i\rangle_2 = \langle S^*TSe_j, e_i\rangle_2 = \langle Te_j, e_i\rangle_2 = c_{i,j},$$

and M_T is a Toeplitz matrix. Similarly, if M_T is a Toeplitz matrix, then $c_{i+1,j+1} = c_{i,j}$, which implies

$$\langle S^*TSe_j, e_i\rangle_2 = \langle Te_j, e_i\rangle_2$$

for all $i, j \geq 0$. Hence $S^*TS = T$. This completes the proof of equivalence of (b) and (c). □

Corollary 2.8. *Let T be a bounded linear operator on H^2. If $TS = ST$, then $T = T_\varphi$, with $\varphi \in H^\infty$.*

Proof. Since $S^*S = I$, $TS = ST$ implies that $S^*TS = T$. By Theorem 2.7(a) there exists a function $\varphi \in L^\infty(\mathbb{T})$ such that $T = T_\varphi$. Moreover, for $k \geq 1$

$$\hat{\varphi}(-k) = \langle Te_k, e_0\rangle_2 = \langle TSe_{k-1}, e_0\rangle_2 = \langle Te_{k-1}, S^*e_0\rangle_2 = 0.$$

Hence $\varphi \in H^\infty$. □

Now we are going to derive an equation describing entries of a matrix of the product of two Toeplitz operators. Let φ, ψ be two bounded functions with Fourier coefficients $\{\alpha_k\}$ and $\{\beta_k\}$, respectively. If $T = T_\varphi T_\psi$, then

$$M_T = [c_{i,j}]_{i,j\geq 0} = M_{T_\varphi} M_{T_\psi} = [\alpha_{i-j}]_{i,j\geq 0}[\beta_{i-j}]_{i,j\geq 0}.$$

Thus, for every $i, j \geq 0$,

$$c_{i,j} = \sum_{k=0}^{\infty} \alpha_{i-k}\beta_{k-j},$$

and

$$\begin{aligned} c_{i+1,j+1} &= \sum_{k=0}^{\infty} \alpha_{i+1-k}\beta_{k-j-1} \\ &= \alpha_{i+1}\beta_{-j-1} + \sum_{k=1}^{\infty} \alpha_{i-(k-1)}\beta_{(k-1)-j} \\ &= \alpha_{i+1}\beta_{-j-1} + \sum_{k=0}^{\infty} \alpha_{i-k}\beta_{k-j}. \end{aligned}$$

So we arrive at the formula

$$c_{i+1,j+1} = \alpha_{i+1}\beta_{-j-1} + c_{i,j}, \quad i, j \geq 0. \tag{2.1}$$

Generally, the product of two Toeplitz matrices does not have to be a Toeplitz matrix, however we have the following

Theorem 2.9 ([5], Thm. 8)**.** *The product $T_\varphi T_\psi$ of two Toeplitz operators is a Toeplitz operator if and only if either $\overline{\varphi}$ or ψ is analytic. If the condition is satisfied, then $T_\varphi T_\psi = T_{\varphi\psi}$.*

Proof. ($\Leftarrow$)
If ψ is analytic, then for any $f \in H^2$,

$$T_\varphi T_\psi f = T_\varphi(\psi f) = P_+(\varphi\psi f) = T_{\varphi\psi} f.$$

If $\overline{\varphi}$ is analytic, then

$$T_\varphi T_\psi = (T_{\overline{\psi}} T_{\overline{\varphi}})^* = T^*_{\overline{\psi\varphi}} = T_{\varphi\psi}.$$

($\Rightarrow$)
If $T = T_\varphi T_\psi$ is a Toeplity operator, then by Theorem 2.7 the matrix $M_T = [c_{i,j}]_{i,j\geq 0}$ is a Toeplitz matrix, i.e., $c_{i+1,j+1} = c_{i,j}$ for all $i, j \geq 0$. Together with (2.1) this implies that

$$\alpha_{i+1}\beta_{-j-1} = 0 \quad \text{for all } i, j \geq 0.$$

If $\alpha_{i+1} = 0$ for each $i \geq 0$, then $\overline{\varphi}$ is analytic. If there exists $i_0 \geq 0$ such that $\alpha_{i_0+1} \neq 0$, we must have $\beta_{-j-1} = 0$ for all $j \geq 0$. Hence ψ is analytic.

Equality $T_\varphi T_\psi = T_{\varphi\psi}$ follows from the first part of the proof. □

Theorem 2.10 ([5], Thm. 9)**.** *A necessary and sufficient condition that two Toeplitz operators commute is that either both be analytic, or both be co-analytic, or one is a linear function of the other.*

Proof. Let $\varphi = \sum \alpha_k e_k$, $\psi = \sum \beta_k e_k$ be two bounded functions and assume that $T_\varphi T_\psi = T_\psi T_\varphi$. This yields equality of matrices $M_{T_\varphi T_\psi}$ and $M_{T_\psi T_\varphi}$. Put $M = M_{T_\varphi T_\psi} = M_{T_\psi T_\varphi} = [c_{i,j}]_{i,j\geq 0}$. By (2.1) one gets

$$c_{i+1,j+1} = \alpha_{i+1}\beta_{-j-1} + c_{i,j},$$

and

$$c_{i+1,j+1} = \beta_{i+1}\alpha_{-j-1} + c_{i,j}$$

for all $i, j \geq 0$. From this

$$\alpha_{i+1}\beta_{-j-1} = \beta_{i+1}\alpha_{-j-1} \quad \text{for all } i, j \geq 0. \tag{2.2}$$

Now observe that (2.2) holds if and only if one of the following conditions is satisfied

(a) $\alpha_i = 0$ for all i,
(b) $\alpha_{-j-1} = 0 = \beta_{-j-1}$ for all $j \geq 0$,
(c) $\alpha_{i+1} = 0 = \beta_{i+1}$ for all $i \geq 0$,
(d) there exist $i_0, j_0 \geq 0$ such that $\alpha_{i_0+1} \neq 0$ and $\alpha_{-j_0-1} \neq 0$.

Case (a) is trivial. If (b) holds, then both φ and ψ are analytic. Similarly, if (c) holds both functions are co-analytic. If (d) holds, we can define

$$\lambda = \frac{\beta_{-j_0-1}}{\alpha_{-j_0-1}} = \frac{\beta_{i_0+1}}{\alpha_{i_0+1}}.$$

Since (2.2) holds for every $i \geq 0$ and every $j \geq 0$, we get $\alpha_{i+1}\beta_{-j_0-1} = \beta_{i+1}\alpha_{-j_0-1}$, which implies

$$\beta_{i+1} = \lambda\alpha_{i+1} \quad \text{for all } i \geq 0.$$

Similarly,

$$\beta_{-j-1} = \lambda\alpha_{-j-1} \quad \text{for all } j \geq 0.$$

In other words, $\beta_k = \lambda\alpha_k$ for all $k \neq 0$. From this

$$\begin{aligned}\psi = \sum \beta_k e_k &= \beta_0 e_0 + \sum_{k\neq 0} \lambda\alpha_k e_k \\ &= \beta_0 e_0 - \lambda\alpha_0 e_0 + \sum \lambda\alpha_k e_k = (\beta_0 - \lambda\alpha_0)e_0 + \lambda\varphi,\end{aligned}$$

and

$$T_\psi = \lambda T_\varphi + \beta_0 - \lambda\alpha_0.$$

The second part of the theorem follows from Theorem 2.9. □

Theorem 2.11 ([5], Cor. 3)**.** *A Toeplitz operator T_φ is an isometry if and only if φ is an inner function.*

Proof. $(\Leftarrow)$
Let φ be an inner function. Then $|\varphi| = 1$ almost everywhere on $\mathbb{T}$ and for every $f \in H^2$,

$$\|T_\varphi f\|_2^2 = \langle T_\varphi f, T_\varphi f\rangle_2 = \langle T_{\overline{\varphi}} T_\varphi f, f\rangle_2 = \langle T_{|\varphi|^2} f, f\rangle_2 = \langle f, f\rangle_2 = \|f\|_2.$$

Hence T_φ is an isometry.

($\Rightarrow$)
Assume T_φ is an isometry. It means that for each $f \in H^2$, $\|T_\varphi f\|_2^2 = \|f\|_2^2$. Then

$$\langle T_{\overline{\varphi}} T_\varphi f, f\rangle_2 = \langle f, f\rangle_2,$$

which means that $T_{\overline{\varphi}} T_\varphi = I = T_{e_0}$. So the product $T_{\overline{\varphi}} T_\varphi$ is again a Toeplitz operator. According to Theorem 2.9, either $\overline{\varphi}$ or φ is analytic and $T_{\overline{\varphi}\varphi} = T_{|\varphi|^2} = T_{e_0}$. In other words, φ is analytic and $|\varphi|^2 = 1$ almost everywhere on $\mathbb{T}$. □

It is a bit more difficult to find all the Toeplitz operators which are partial isometries. Obviously, since every isometry is a partial isometry, T_φ is a partial isometry for an inner function φ. It follows from Theorem 1.12 that $T_{\overline{\varphi}}$ is a partial isometry for any inner φ. We also have

Theorem 2.12 ([4])**.** *The only non-zero Toeplitz operators that are partial isometries are those of the form T_ϕ and $T_{\overline{\phi}}$, where ϕ is an inner function.*

The proof is based on the following lemma.

Lemma 2.13 ([4], Lem. 2)**.** *A non-zero Toeplitz operator T_f achieves its norm if and only if $T_f = \lambda T_{\overline{\varphi}} T_\psi$, where $\lambda > 0$ and φ, ψ are inner functions.*

Proof. First assume that $T_f = \lambda T_{\overline{\varphi}} T_\psi$ with φ, ψ inner. Then $\|T_f\| = \lambda$. Take $g \in \operatorname{Im} T_\varphi$ and let $g = T_\varphi h$, $h \in H^2$. Then we have

$$\begin{aligned}\|T_f g\|_2^2 &= \lambda^2 \langle T_{\overline{\varphi}} T_\psi g, T_{\overline{\varphi}} T_\psi g\rangle_2 = \lambda^2 \langle T_{\overline{\varphi}} T_\psi g, T_{\overline{\varphi}} T_\psi T_\varphi h\rangle_2 \\ &= \lambda^2 \langle T_{\overline{\varphi}} T_\psi g, T_{\overline{\varphi}} T_\varphi T_\psi h\rangle_2 = \lambda^2 \langle T_{\overline{\varphi}} T_\psi g, T_\psi h\rangle_2 = \lambda^2 \langle T_{\overline{\psi}} T_{\overline{\varphi}} T_\psi g, h\rangle_2 \\ &= \lambda^2 \langle T_{\overline{\varphi}} T_{\overline{\psi}} T_\psi g, h\rangle_2 = \lambda^2 \langle g, T_\varphi h\rangle_2 = \lambda^2 \langle g, g\rangle_2 = \lambda^2 \|g\|_2^2.\end{aligned}$$

Now take $T_f \neq 0$ and assume that $\|T_f\| = 1$. Define

$$G = \{g \in H^2 \colon \|T_f g\|_2 = \|g\|_2\}.$$

We will prove that G is actually a subspace of H^2. To this end we establish the following equivalence

$$g \in G \quad \Leftrightarrow \quad fg \in H^2. \tag{2.3}$$

Since T_f achieves its norm, we can find $g_0 \in G$, $g_0 \neq 0$. Notice that $\|f\|_\infty = \|T_f\| = 1$. Hence

$$\|g_0\|_2 = \|T_f g_0\|_2 = \|P_+(f g_0)\|_2 \leq \|f g_0\|_{L^2} \leq \|g_0\|_2,$$

and we in fact get a sequence of equalities. In particular, $\|P_+(f g_0)\|_2 = \|f g_0\|_{L^2}$, which means that $f g_0 \in H^2$. Similar reasoning holds for any $g \in G$, so we have shown one implication of (2.3).

Moreover, $\|f g_0\|_{L^2} = \|g_0\|_2$, or in other words,

$$\frac{1}{2\pi}\int_0^{2\pi} (1 - |f(e^{it})|^2)|g_0(e^{it})|^2 dt = 0.$$

Since $g_0 \neq 0$, it can not vanish on any subset of $\mathbb{T}$ of positive measure (see [8], Privalov's uniqueness theorem, p. 62), which implies that $|f| = 1$ almost everywhere on $\mathbb{T}$. Therefore, for every $g \in H^2$, $\|fg\|_{L^2} = \|g\|_2$. Thus if $g \in H^2$ and $fg \in H^2$, then

$$\|T_f g\|_2 = \|fg\|_2 = \|g\|_2,$$

i.e., $g \in G$. This ends the proof of (2.3).

Now, the fact that G is a subspace of H^2 is an easy consequence of (2.3). What is more, if $g \in G$ then $fg \in H^2$ and $S(fg) \in H^2$, where S stands for the shift operator. Since $S(fg) =$

$fS(g)$, it follows that $Sg \in G$. Thus, we have shown that the subspace G is shift-invariant. By the Beurling theorem ([7], p. 99) there exists an inner function φ, such that

$$G = \varphi H^2 = \mathrm{Im}T_\varphi.$$

In particular, $\varphi \in G$ and $\psi = \varphi f \in H^2$. It is easy to see, that ψ is also an inner function. Therefore

$$T_f = T_{\overline{\varphi}\psi}.$$

□

To prove Theorem 2.12, Brown and Douglas use the following fact.

Lemma 2.14 ([4], Lem. 1). *Let V be an isometry on a Hilbert space $\mathscr{H}$ and let T be a bounded operator that commutes with V. Suppose that the deficiency of V, i.e., the subspace $(\mathrm{Im}V)^\perp$, is invariant under T. Then T also commutes with V^*.*

However, it turns out that the proof can be simplified considerably.

2.1. **Proof of Theorem 2.12.** If T_f is a partial isometry then it achieves its norm ($\|T_f\| = 1$). Hence, by Lemma 2.13,

$$T_f = T_{\overline{\varphi}}T_\psi$$

for some inner functions φ and ψ. It follows from the proof of Lemma 2.13 that

$$G = \{g \in H^2 \colon \|T_f g\|_2 = \|g\|_2\} = \varphi H^2 = \mathrm{Im}T_\varphi.$$

Obviously, operators T_φ and T_ψ commute. If we prove that $T_{\overline{\varphi}}$ and T_ψ commute, then by Theorem 2.10 one of the following holds

(a) both $T_{\overline{\varphi}}$ and T_ψ are analytic,
(b) both $T_{\overline{\varphi}}$ and T_ψ are co-analytic,
(c) $T_\psi = \alpha T_{\overline{\varphi}} + \beta$ for some constants α, β.

If (a) holds, then φ is constant. Similarly if (b) holds, ψ is constant. If (c) is true, then

$$T_\psi T_\varphi = \alpha T_{\overline{\varphi}} T_\varphi + \beta T_\varphi \quad \text{and} \quad T_\varphi T_\psi = \alpha T_\varphi T_{\overline{\varphi}} + \beta T_\varphi.$$

This implies that $T_{\overline{\varphi}}T_\varphi = T_\varphi T_{\overline{\varphi}}$ and by Theorem 2.10 φ is constant.

So our goal is to show that the operators $T_{\overline{\varphi}}$ and T_ψ commute. In other words, if we put $T = T_\psi T_{\overline{\varphi}}$, we need to prove that $T_f = T$. Since both T_f and T are linear and $H^2 = G \oplus G^\perp$, problem reduces to showing that $T_f = T$ on G and on $G^\perp$.

Let $g \in G$. Since $G = \mathrm{Im}T_\varphi$, $g = T_\varphi h$ for some $h \in H^2$. Therefore

$$T_f g = T_{\overline{\varphi}}T_\psi g = T_{\overline{\varphi}}T_\psi T_\varphi h = T_{\overline{\varphi}}T_\varphi T_\psi h = T_\psi h = T_\psi T_{\overline{\varphi}} T_\varphi h = T_\psi T_{\overline{\varphi}} g = Tg.$$

So $T_f = T$ on G.

Since T_f is a partial isometry with the initial domain G, $G = (\ker T_f)^\perp$ and $G^\perp = \ker T_f$. On the other hand, $G = \mathrm{Im}T_\varphi$ so

$$G^\perp = (\mathrm{Im}T_\varphi)^\perp = \ker T_{\overline{\varphi}}.$$

Consequently, if $g \in G^\perp$, then $Tg = T_\psi T_{\overline{\varphi}} g = 0$. Also it follows from the above that $T_f g = 0$ for $g \in G^\perp$.

This way we proved Theorem 2.12 without refering to Lemma 2.14. Moreover, now one can see that a reasoning similar to the one given above can be applied to prove the lemma itself.

2.2. **Proof of Lemma 2.14.** Let T be a bounded linear operator on $\mathscr{H}$ such that $TV = VT$ and $(\mathrm{Im}V)^{\perp}$ is invariant under T. It is enough to prove that $TV^* = V^*T$ on $\mathrm{Im}V$ and on $(\mathrm{Im}V)^{\perp}$.

Let $g \in \mathrm{Im}V$ and $g = Vh$ for some $h \in \mathscr{H}$. Since V is an isometry,

$$TV^*g = TV^*Vh = Th = V^*VTh = V^*TVh = V^*Tg.$$

Since $(\mathrm{Im}V)^{\perp} = \ker V^*$, we have $TV^*g = 0$ for $g \in (\mathrm{Im}V)^{\perp}$. By assumption $Tg \in \ker V^*$ for $g \in (\mathrm{Im}V)^{\perp}$. Consequently $V^*Tg = 0$, which proves $TV^* = V^*T$ on $(\mathrm{Im}V)^{\perp}$.

3. Operator ranges or Hilbert spaces in Hilbert spaces

Definition 3.1. Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. We say that $\mathscr{H}_1$ is contained boundedly in $\mathscr{H}$ if

(a) $\mathscr{H}_1$ is a vector subspace of $\mathscr{H}$,
(b) the inclusion map $J \colon \mathscr{H}_1 \hookrightarrow \mathscr{H}$ is bounded.

If J is a contraction, i.e., $\|J\|_{\mathscr{H}_1 \to \mathscr{H}} \leq 1$, we say that $\mathscr{H}_1$ is contained contractively in $\mathscr{H}$.

In other words, condition (b) can be expressed as

$$\|x\|_{\mathscr{H}} \leq C\|x\|_{\mathscr{H}_1}$$

for some $C > 0$ and every $x \in \mathscr{H}_1$ ($C = 1$ for spaces contained contractively). This means that the initial topology on $\mathscr{H}_1$ must be finer than the induced topology.

Examples.

(1) Every closed subspace $\mathscr{H}_1$ (with the induced topology) is contained contractively in $\mathscr{H}$. In this case the inclusion J is an isometry.
(2) Let $\mathscr{H}$ be a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$. If $\langle \cdot, \cdot \rangle_1$ is another scalar product on $\mathscr{H}$ giving an equivalent norm, that is there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\| \leq \|x\|_1 \leq C_2\|x\|$$

for every $x \in \mathscr{H}$, then $\mathscr{H}$ equipped with the topology given by $\langle \cdot, \cdot \rangle$ is contained boundedly in $\mathscr{H}$ equipped with the topology given by $\langle \cdot, \cdot \rangle_1$, and vice versa.

Definition 3.2. For a bounded operator $A \colon \mathscr{H}_1 \to \mathscr{H}$ let $\mathcal{M}(A)$ be the range of A equipped with a scalar product

$$\langle Ax, Ay \rangle_{\mathcal{M}(A)} = \langle x, y \rangle_{\mathscr{H}_1}$$

for every $x, y \in \mathscr{H}_1$ such that at least one of them is orthogonal to the kernel of A.

Remark 3.3. To see that the product $\langle \cdot, \cdot \rangle_{\mathcal{M}(A)}$ is well defined take arbitrary $x, y \in \mathcal{M}(A)$ and assume that $x = Ax_1 = Ax_2$, $y = Ay_1 = Ay_2$ for some $x_1, x_2, y_1, y_2 \in (\ker A)^{\perp}$. Then

$$\langle x, y \rangle_{\mathcal{M}(A)} = \langle Ax_1, Ay_1 \rangle_{\mathcal{M}(A)} = \langle x_1, y_1 \rangle_{\mathscr{H}_1},$$

and analogously

$$\langle x, y \rangle_{\mathcal{M}(A)} = \langle x_2, y_2 \rangle_{\mathscr{H}_1}.$$

Therefore it is enough to show that

$$\langle x_1, y_1 \rangle_{\mathscr{H}_1} = \langle x_2, y_2 \rangle_{\mathscr{H}_1}.$$

But this is an easy consequence of the fact that $x_1 - x_2 \in \ker A$ and $y_1 - y_2 \in \ker A$.

Remark 3.4. Consider the surjection $A\colon \mathscr{H}_1 \to \mathcal{M}(A)$. Then for each $x \in \mathscr{H}_1$ we have

$$\begin{aligned}\|Ax\|^2_{\mathcal{M}(A)} = \|Ax_\perp\|^2_{\mathcal{M}(A)} &= \langle Ax_\perp, Ax_\perp\rangle_{\mathcal{M}(A)} \\ &= \langle x_\perp, x_\perp\rangle_{\mathscr{H}_1} = \|x_\perp\|^2_{\mathscr{H}_1} \le \|x\|^2_{\mathscr{H}_1}.\end{aligned}$$

This means that A is a partial isometry. Also, one can see that

$$\|Ax\|_{\mathscr{H}} = \|Ax_\perp\|_{\mathscr{H}} \le \|A\|\|x_\perp\|_{\mathscr{H}_1} = \|A\|\|Ax_\perp\|_{\mathcal{M}(A)} = \|A\|\|Ax\|_{\mathcal{M}(A)}.$$

So $\mathcal{M}(A)$ is contained boundedly in $\mathscr{H}$.

Proposition 3.5 ([13], I-3)**.** *Suppose $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$. Let T_y be a bounded linear functional on $\mathscr{H}$ induced by $y \in \mathscr{H}$, i.e., for every $x \in \mathscr{H}$*

$$T_y x = \langle x, y\rangle_{\mathscr{H}}.$$

Then the restriction $T_{y|\mathcal{M}(A)}$ is a bounded linear functional on $\mathcal{M}(A)$ induced by AA^*y, that is

$$T_y(Ax) = \langle Ax, AA^*y\rangle_{\mathcal{M}(A)}.$$

Proof. Since $\mathcal{M}(A)$ is contained boundedly in $\mathscr{H}$, $T_{y|\mathcal{M}(A)}$ is a bounded linear functional on $\mathcal{M}(A)$. The Riesz representation theorem yields the existence of an element $z \in \mathcal{M}(A)$ such that

$$T_y(Ax) = \langle Ax, z\rangle_{\mathcal{M}(A)}$$

for every $x \in \mathscr{H}_1$. However,

$$\begin{aligned}T_y(Ax) &= \langle Ax, y\rangle_{\mathscr{H}} = \langle Ax_\perp, y\rangle_{\mathscr{H}} = \langle x_\perp, A^*y\rangle_{\mathscr{H}_1} \\ &= \langle Ax_\perp, AA^*y\rangle_{\mathcal{M}(A)} = \langle Ax, AA^*y\rangle_{\mathcal{M}(A)}.\end{aligned}$$

Thus $z = AA^*y$. □

Theorem 3.6 (Douglas's Criterion, [13], I-4)**.** *Let $\mathscr{H}$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, and let A and B be bounded operators from $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, into $\mathscr{H}$. Then the operator inequality $AA^* \le BB^*$ is necessary and sufficient for existence of a factorization $A = BR$ with R a contraction from $\mathscr{H}_1$ into $\mathscr{H}_2$.*

Proof. Assume that R is a contraction from $\mathscr{H}_1$ into $\mathscr{H}_2$, such that $A = BR$. We show that

$$\langle (BB^* - AA^*)x, x\rangle_{\mathscr{H}} \ge 0 \quad \text{for all } x \in \mathscr{H}.$$

Since $A = BR$,

$$\begin{aligned}\langle (BB^* - AA^*)x, x\rangle_{\mathscr{H}} &= \langle (BB^* - BRR^*B^*)x, x\rangle_{\mathscr{H}} \\ &= \langle (B^* - RR^*B^*)x, B^*x\rangle_{\mathscr{H}_2} = \langle (I - RR^*)B^*x, B^*x\rangle_{\mathscr{H}_2} \ge 0,\end{aligned}$$

where the last inequality follows from Lemma 1.5.

Assume now that $AA^* \le BB^*$ and define $Q\colon B^*(\mathscr{H}) \to A^*(\mathscr{H})$ by

$$QB^*x = A^*x, \quad x \in \mathscr{H}. \tag{3.1}$$

First of all we need to prove that Q is well defined. In other words we need to show that if $B^*x_1 = B^*x_2$ for some $x_1, x_2 \in \mathscr{H}$, then also $A^*x_1 = A^*x_2$. To this end, it is enought to prove that

$$\ker B^* \subset \ker A^*.$$

If $x \in \ker B^*$, then

$$-\|A^*x\|^2_{\mathscr{H}_1} = -\langle AA^*x, x\rangle_{\mathscr{H}} = \langle BB^*x - AA^*x, x\rangle_{\mathscr{H}} \ge 0,$$

which means $x \in \ker A^*$.

Moreover, for $x \in \mathscr{H}$,

$$\begin{aligned} \|QB^*x\|^2_{\mathscr{H}_1} &= \langle QB^*x, QB^*x\rangle_{\mathscr{H}_1} = \langle A^*x, A^*x\rangle_{\mathscr{H}_1} \\ &= \langle AA^*x, x\rangle_{\mathscr{H}} \le \langle BB^*x, x\rangle_{\mathscr{H}} = \|B^*x\|^2_{\mathscr{H}_2}. \end{aligned}$$

Therefore Q can be extended to $Q\colon \overline{B^*(\mathscr{H})} \to \mathscr{H}_1$, $\|Q\| \le 1$. Setting $Qx = 0$ for $x \in \ker B = \overline{B^*(\mathscr{H})}^{\perp}$, we obtain a contraction $Q\colon \mathscr{H}_2 \to \mathscr{H}_1$.

Put $R = Q^*$. Obviously $R\colon \mathscr{H}_1 \to \mathscr{H}_2$, $\|R\| \le 1$. Moreover, by (3.1) $A = BR$. ☐

Corollary 3.7 ([13], I-5)**.** *Let $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$, A and B be as in Theorem 3.6.*

(i) *The space $\mathcal{M}(A)$ is contained contractively in the space $\mathcal{M}(B)$ if and only if $AA^* \le BB^*$.*
(ii) *The spaces $\mathcal{M}(A)$ and $\mathcal{M}(B)$ coincide as Hilbert spaces if and only if $AA^* = BB^*$. In particular $\mathcal{M}(A) = \mathcal{M}((AA^*)^{1/2})$ (as Hilbert spaces).*
(iii) *The space $\mathcal{M}(A)$ is an ordinary subspace of $\mathscr{H}$ if and only if A is a partial isometry.*

Proof. (i)
Assume $AA^* \le BB^*$. By the Douglas's criterion there exists a contraction R, such that $A = BR$. Consequently,

$$\mathcal{M}(A) = A(\mathscr{H}_1) = B(R(\mathscr{H}_1)) \subset B(\mathscr{H}_2) = \mathcal{M}(B).$$

Moreover, if we take $y \in \mathcal{M}(A)$, $y = Ax$, $x \in (\ker A)^{\perp}$, then

$$\begin{aligned} \|y\|^2_{\mathcal{M}(B)} &= \|Ax\|^2_{\mathcal{M}(B)} = \|BRx\|^2_{\mathcal{M}(B)} = \|B(Rx)_\perp\|^2_{\mathcal{M}(B)} \\ &= \|(Rx)_\perp\|^2_{\mathscr{H}_2} \le \|Rx\|^2_{\mathscr{H}_2} \le \|x\|^2_{\mathscr{H}_1} = \|Ax\|^2_{\mathcal{M}(A)} = \|y\|^2_{\mathcal{M}(A)}, \end{aligned}$$

where $(Rx)_\perp \in (\ker B)^{\perp}$. Hence $\mathcal{M}(A)$ is contained contractively in $\mathcal{M}(B)$.

If $\mathcal{M}(A)$ is contained contractively in $\mathcal{M}(B)$, then in particular $A(\mathscr{H}_1) \subset B(\mathscr{H}_2)$ as sets. Hence, for every $x \in \mathscr{H}_1$, $Ax \in B(\mathscr{H}_2)$ and there exists a unique $y_\perp \in (\ker B)^{\perp}$, such that $Ax = By_\perp$. Define a linear operator $R\colon \mathscr{H}_1 \to \mathscr{H}_2$ by

$$Rx = y_\perp.$$

Since $y_\perp$ is unique, R is well defined. Moreover,

$$\begin{aligned} \|Rx\|^2_{\mathscr{H}_2} &= \langle y_\perp, y_\perp\rangle_{\mathscr{H}_2} = \langle By_\perp, By_\perp\rangle_{\mathcal{M}(B)} \\ &= \|By_\perp\|^2_{\mathcal{M}(B)} \le \|By_\perp\|^2_{\mathcal{M}(A)} \\ &= \|Ax\|^2_{\mathcal{M}(A)} \le \|x\|^2_{\mathscr{H}_1}. \end{aligned}$$

In other words, R is a contraction, and obviously $A = BR$. Douglas's criterion implies that $AA^* \le BB^*$.

(ii)
If $\mathcal{M}(A) = \mathcal{M}(B)$ as Hilbert spaces, then in particular $\mathcal{M}(A)$ is contained contractively in $\mathcal{M}(B)$ and $\mathcal{M}(B)$ is contained contractively in $\mathcal{M}(A)$. Hence

$$AA^* \le BB^* \quad \text{and} \quad AA^* \ge BB^*.$$

Denote $T = AA^* - BB^*$. We need to show that $T \ge 0$ and $-T \ge 0$ imply $T = 0$. To this end, notice that

$$0 = \langle Tx, x\rangle_{\mathscr{H}} + \langle -Tx, x\rangle_{\mathscr{H}}$$

is a sum of two positive numbers for every $x \in \mathscr{H}$. As a consequence, $\langle Tx, x\rangle_{\mathscr{H}} = 0$ for all $x \in \mathscr{H}$, and by Lemma 1.4 $T = 0$.

Now the equality $AA^* = BB^*$ easily yields $\mathcal{M}(A) = \mathcal{M}(B)$ as sets and $\|\cdot\|_{\mathcal{M}(A)} = \|\cdot\|_{\mathcal{M}(B)}$. Hence $\mathcal{M}(A) = \mathcal{M}(B)$ as Hilbert spaces.

Take $B = (AA^*)^{1/2}$. Since the positive square root is self-adjoint, $AA^* = BB^*$. By what we have already proved, $\mathcal{M}(A) = \mathcal{M}((AA^*)^{1/2})$.

(iii)
If A is a partial isometry, by Theorem 1.12 AA^* is an orthogonal projection P on $\mathrm{Im}A$. Clearly $(AA^*)^{1/2}$ is the same projection and by (ii),

$$\mathcal{M}(A) = \mathcal{M}((AA^*)^{1/2}) = \mathcal{M}(P)$$

as Hilbert spaces. But $(\ker P)^{\perp} = \mathrm{Im}P$ and by definition

$$\langle x, y\rangle_{\mathcal{M}(P)} = \langle Px, Py\rangle_{\mathcal{M}(P)} = \langle x, y\rangle_{\mathscr{H}}$$

for all $x, y \in \mathcal{M}(P)$. Hence $\mathcal{M}(P) = \mathcal{M}(A)$ is an ordinary subspace of $\mathscr{H}$.

On the other hand, if $\mathcal{M}(A)$ is an ordinary subspace of $\mathscr{H}$, then $\langle\cdot,\cdot\rangle_{\mathcal{M}(A)} = \langle\cdot,\cdot\rangle_{\mathscr{H}}$ and

$$\langle Ax, Ay\rangle_{\mathscr{H}} = \langle Ax, Ay\rangle_{\mathcal{M}(A)} = \langle x_{\perp}, y\rangle_{\mathscr{H}_1}$$

for all $x, y \in \mathscr{H}_1$. Theorem 1.12 implies that A is a partial isometry. □

Definition 3.8. For a Hilbert space contraction A let $\mathcal{H}(A)$ be defined by

$$\mathcal{H}(A) = \mathcal{M}((I - AA^*)^{1/2}).$$

Remark 3.9. If A is a contraction, then so is $(I - AA^*)^{1/2}$ and $\mathcal{H}(A)$ is contained contractively in $\mathscr{H}$.

Lemma 3.10 (Intertwining Relation, [13], I-7)**.** *If $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H})$ is a Hilbert space contraction, then*

$$A(I - A^*A)^{1/2} = (I - AA^*)^{1/2}A.$$

Theorem 3.11 ([13], I-8)**.** *Let A be a contraction from the Hilbert space $\mathscr{H}_1$ into the Hilbert space $\mathscr{H}$. Then the vector $x \in \mathscr{H}$ belongs to $\mathcal{H}(A)$ if and only if A^*x belongs to $\mathcal{H}(A^*)$. If x_1 and x_2 are two vectors in $\mathcal{H}(A)$, then*

$$\langle x_1, x_2\rangle_{\mathcal{H}(A)} = \langle x_1, x_2\rangle_{\mathscr{H}} + \langle A^*x_1, A^*x_2\rangle_{\mathcal{H}(A^*)}. \tag{3.2}$$

Proof. If $x \in \mathcal{H}(A)$, $x = (I - AA^*)^{1/2}y$ for some $y \in \mathscr{H}$, the intertwining relation yields

$$A^*x = A^*(I - AA^*)^{1/2}y = (I - A^*A)^{1/2}A^*y,$$

which means that $A^*x \in \mathcal{H}(A^*)$.

Now, take $x \in \mathscr{H}$ such that $A^*x = (I - A^*A)^{1/2}y$ for $y \in \mathscr{H}_1$. We have

$$\begin{aligned} x &= (I - AA^*)x + AA^*x \\ &= (I - AA^*)x + A(I - A^*A)^{1/2}y \\ &= (I - AA^*)^{1/2}[(I - AA^*)^{1/2}x + Ay], \end{aligned}$$

where the last equality follows from the intertwining relation. Hence $x \in \mathcal{H}(A)$.

To obtain formula (3.2) let $x_1, x_2 \in \mathcal{H}(A)$ be such that

$$A^*x_j = (I - A^*A)^{1/2}y_j, \qquad j = 1, 2,$$

and both y_1 and y_2 are orthogonal to the kernel of $(I - A^*A)^{1/2}$. From the first part of the proof we have

$$x_j = (I - AA^*)^{1/2}[(I - AA^*)^{1/2}x_j + Ay_j], \qquad j = 1, 2.$$

It is easy to see that $(I - AA^*)^{1/2}x_j$, $j = 1, 2$ are orthogonal to $\ker(I - AA^*)^{1/2}$. If $z \in \ker(I - AA^*)^{1/2}$, then in view of the intertwining relation,

$$(I - A^*A)^{1/2}A^*z = A^*(I - AA^*)^{1/2}z = 0$$

and $A^*z \in \ker(I - A^*A)^{1/2}$, and from our choice of y_j,

$$\langle Ay_j, z\rangle_{\mathscr{H}} = \langle y_j, A^*z\rangle_{\mathscr{H}_1} = 0.$$

Hence $(I - AA^*)^{1/2}x_j + Ay_j$, $j = 1, 2$, are both orthogonal to the kernel of $(I - AA^*)^{1/2}$ and consequently,

$$\langle x_1, x_2\rangle_{\mathcal{H}(A)} = \langle (I - AA^*)^{1/2}x_1 + Ay_1, (I - AA^*)^{1/2}x_2 + Ay_2\rangle_{\mathscr{H}}.$$

The inner product on the right side expands into a sum of four terms. One of them is

$$\begin{aligned}\langle (I - AA^*)^{1/2}x_1, (I - AA^*)^{1/2}x_2\rangle_{\mathscr{H}} &= \langle x_1, (I - AA^*)x_2\rangle_{\mathscr{H}}\\ &= \langle x_1, x_2\rangle_{\mathscr{H}} - \langle A^*x_1, A^*x_2\rangle_{\mathscr{H}_1}.\end{aligned}$$

Using the intertwining relation the other two terms can be rewritten as

$$\begin{aligned}\langle (I - AA^*)^{1/2}x_1, Ay_2\rangle_{\mathscr{H}} &= \langle x_1, (I - AA^*)^{1/2}Ay_2\rangle_{\mathscr{H}}\\ &= \langle x_1, A(I - A^*A)^{1/2}y_2\rangle_{\mathscr{H}} = \langle A^*x_1, A^*x_2\rangle_{\mathscr{H}_1},\end{aligned}$$

and similarily

$$\langle Ay_1, (I - AA^*)^{1/2}x_2\rangle_{\mathscr{H}} = \langle A^*x_1, A^*x_2\rangle_{\mathscr{H}_1}.$$

Finally, there is $\langle Ay_1, Ay_2\rangle_{\mathscr{H}}$. We have

$$\begin{aligned}\langle x_1, x_2\rangle_{\mathcal{H}(A)} &= \langle x_1, x_2\rangle_{\mathscr{H}} + \langle A^*x_1, A^*x_2\rangle_{\mathscr{H}_1} + \langle Ay_1, Ay_2\rangle_{\mathscr{H}}\\ &= \langle x_1, x_2\rangle_{\mathscr{H}} + \langle (I - A^*A)^{1/2}y_1, (I - A^*A)^{1/2}y_2\rangle_{\mathscr{H}_1} + \langle Ay_1, Ay_2\rangle_{\mathscr{H}}\\ &= \langle x_1, x_2\rangle_{\mathscr{H}} + \langle y_1, (I - A^*A)y_2\rangle_{\mathscr{H}_1} + \langle Ay_1, Ay_2\rangle_{\mathscr{H}}\\ &= \langle x_1, x_2\rangle_{\mathscr{H}} + \langle y_1, y_2\rangle_{\mathscr{H}_1}\\ &= \langle x_1, x_2\rangle_{\mathscr{H}} + \langle (I - A^*A)^{1/2}y_1, (I - A^*A)^{1/2}y_2\rangle_{\mathcal{H}(A^*)}\\ &= \langle x_1, x_2\rangle_{\mathscr{H}} + \langle A^*x_1, A^*x_2\rangle_{\mathcal{H}(A^*)}.\end{aligned}$$

□

Remark 3.12 ([13], I-9)**.** Theorem 3.11 states that

$$\mathcal{H}(A) = (A^*)^{-1}(\mathcal{H}(A^*)),$$

which is equivalent to

$$A^*(\mathcal{H}(A)) = \mathcal{H}(A^*) \cap \mathcal{M}(A^*).$$

If we reverse the roles of A and A^* in the above formula, we get

$$\mathcal{H}(A) \cap \mathcal{M}(A) = A(\mathcal{H}(A^*)).$$

The space $\mathcal{H}(A) \cap \mathcal{M}(A)$ is called the overlapping space.

Proposition 3.13 ([13], I-9)**.** *The overlapping space is trivial if and only if A is a partial isometry.*

Proof. If A is a partial isometry, then AA^* is a projection (Theorem 1.12). Since AA^* and $I - AA^*$ are orthogonal projections, then $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are ordinary subspaces of $\mathscr{H}$, orthogonal complements of each other. Obviously in this case the overlapping space is trivial.

On the other hand, if the overlapping space is trivial, then $A(I - A^*A)^{1/2} = 0$. Therefore $A(I - A^*A) = 0$ and $A = AA^*A$. Hence AA^* and A^*A are projections and A is a partial isometry (also $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are orthogonal complements). □

Remark 3.14. From what has been shown one sees that the overlapping space is trivial if and only if $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are ordinary subspaces of $\mathscr{H}$, orthogonal complements of each other.

4. De Branges-Rovnyak Spaces

The spaces now called de Branges-Rovnyak spaces were introduced by de Branges and Rovnyak in 1966 ([3]). Since then the spaces have been studied in numerous papers. Thanks in large part to the work of Sarason ([11] - [14]) it was also realized that these spaces have many connections with other topics in complex analysis and operator theory.

Before we give the definition of the de Branges-Rovnyak spaces we introduce some notations. According to the previous section, $\mathcal{M}(T_\varphi)$ is the range of the Toeplitz operator T_φ equipped with the scalar product as in Definition 3.2. If $\|\varphi\|_\infty \leq 1$, then it also makes sense to consider the space $\mathcal{H}(T_\varphi)$ (see Definition 3.8). From now on, we will write $\mathcal{M}(\varphi)$ and $\mathcal{H}(\varphi)$ instead of $\mathcal{M}(T_\varphi)$ and $\mathcal{H}(T_\varphi)$, respectively, and $\langle\cdot,\cdot\rangle_\varphi$ and $\|\cdot\|_\varphi$ will denote the inner product and norm for $\mathcal{H}(\varphi)$.

Definition 4.1. If b is a function from the unit ball of H^∞, then $\mathcal{H}(b)$ is called the de Branges-Rovnyak space.

From Section 3 we get the following properties of de Branges-Rovnyak spaces.

We first note that the space $\mathcal{H}(b)$ has a reproducing kernel given by

$$k_w^b = (1 - \overline{b(w)}b)k_w,$$

where $k_w(z) = (1-\bar{w}z)^{-1}$ is a reproducing kernel for H^2. Indeed, it follows from Proposition 3.5 that $k_w^b = (I - T_bT_{\bar{b}})k_w$. Now, for $z \in \mathbb{D}$,

$$T_{\bar{b}}k_w(z) = \langle T_{\bar{b}}k_w, k_z\rangle_2 = \langle k_w, bk_z\rangle_2 = \overline{\langle bk_z, k_w\rangle_2} = \overline{b(w)k_z(w)} = \overline{b(w)}k_w(z),$$

and our claim follows.

We remark that the equality

$$T_{\bar{b}}k_w(z) = \overline{b(w)}k_w(z) \tag{4.1}$$

holds for any $b \in H^\infty$.

It is also worth noting here that if $b(w) = 0$, then $k_w \in \mathcal{H}(b)$.

From Theorem 3.11 applied to $A = T_b$ we get the following.

Corollary 4.2 ([13], II-4)**.** *A function $f \in H^2$ belongs to $\mathcal{H}(b)$ if and only if $T_{\bar{b}}f$ belongs to $\mathcal{H}(\bar{b})$. What is more*

$$\langle f, g\rangle_b = \langle f, g\rangle_2 + \langle T_{\bar{b}}f, T_{\bar{b}}g\rangle_{\bar{b}}$$

for every $f, g \in \mathcal{H}(b)$.

As a consequence of Remark 3.12 we get

Remark 4.3 ([13], II-5)**.** $\mathcal{H}(b) \cap \mathcal{M}(b) = T_b\mathcal{H}(\bar{b})$.

Now we describe the spaces $\mathcal{H}(b)$ in simple cases where b is either a constant function or $\|b\|_\infty < 1$.

We first note that if $b(z) = \lambda$, $|\lambda| \leq 1$, then there are two cases.

If $|\lambda| = 1$, then the operator $(I - T_bT_{\bar{b}})^{1/2}$ is the zero operator and $\mathcal{H}(b) = \{0\}$.

If $|\lambda| < 1$, then $(I - T_bT_{\bar{b}})^{1/2}$ is the operator of multiplication by $(1-|\lambda|^2)^{1/2}$. In this case $\mathcal{H}(b) = H^2$ and

$$\|f\|_b = \|(1-|\lambda|^2)^{-1/2}f\|_2 = (1-|\lambda|^2)^{-1/2}\|f\|_2$$

for every $f \in H^2$.

From now on we will assume that b is a nonconstant function from the unit ball of H^∞.

Proposition 4.4 ([9], Prop. 3)**.** *If $\|b\|_\infty < 1$, then $\mathcal{H}(b) = H^2$ as vector spaces and for $f \in H^2$ we have that $\|f\|_b = \|(I - T_bT_{\bar{b}})^{-1/2}f\|_2$.*

Proof. If $\|b\|_\infty < 1$, then $\|T_b T_{\bar{b}}\| < 1$ and the operator $I - T_b T_{\bar{b}}$ is surjective and invertible (see [1], Lem. 1.7.2). From the inclusion

$$(I - T_b T_{\bar{b}})(H^2) \subset (I - T_b T_{\bar{b}})^{1/2}(H^2)$$

and Lemma 1.8, $(I - T_b T_{\bar{b}})^{1/2}$ is also surjective and invertible. In particular $\mathcal{H}(b) = H^2$.

Since $(I - T_b T_{\bar{b}})^{1/2}$ has a trivial kernel, then $(I - T_b T_{\bar{b}})^{-1/2} f \in (\ker(I - T_b T_{\bar{b}})^{1/2})^\perp$ for an arbitrary $f \in H^2$. Therefore $\|f\|_b = \|(I - T_b T_{\bar{b}})^{-1/2} f\|_2$. □

Next we are going to describe the structure of the space $\mathcal{H}(b)$ when b is an inner function.

Definition 4.5. If b is an inner function, let P_b denote the orthogonal projection of H^2 onto its closed subspace bH^2, and let $P_{\bar{b}}$ denote the orthogonal projection of H^2 onto $(bH^2)^\perp$.

Proposition 4.6 ([9], Prop. 5)**.** *If b is an inner function, then $\mathcal{H}(b) = \mathcal{M}(P_{\bar{b}}) = (bH^2)^\perp$, and $\|f\|_b = \|f\|_2$ for every $f \in \mathcal{H}(b)$.*

Proof. Recall that if b is an inner function, T_b is an isometry. By Theorem 1.12, $P_b = T_b T_{\bar{b}}$. Therefore $P_{\bar{b}} = I - T_b T_{\bar{b}}$ and as a consequence of the Douglas's criterion, $\mathcal{H}(b) = \mathcal{M}(P_{\bar{b}})$.

If $f \in \mathcal{H}(b)$, then

$$\|f\|_b = \|f\|_{\mathcal{M}(P_{\bar{b}})} = \|P_{\bar{b}} f\|_{\mathcal{M}(P_{\bar{b}})} = \|f\|_2.$$

□

The next two examples describe $\mathcal{H}(b)$ in the case when $b(z) = z^n$, $n \geq 0$, and b is a disk automorphism.

Example 4.7. Let $b(z) = e_n(z) = z^n$, $n \geq 1$. Then

$$bH^2 = \{f \in H^2 \colon f^{(k)}(0) = 0,\ 0 \leq k < n\}.$$

This impies that $g \in \mathcal{H}(b) = (bH^2)^\perp$ if and only if

$$\langle g, e_m \rangle_2 = 0 \quad \text{for every } m \geq n.$$

Hence

$$\mathcal{H}(e_n) = \left\{ \sum_{k=0}^{n-1} \alpha_k e_k \colon \alpha_k \in \mathbb{C} \right\}.$$

Example 4.8. Take $\alpha \in \mathbb{D} \backslash \{0\}$ and let

$$b(z) = \varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

This time

$$bH^2 = \{f \in H^2 \colon f(\alpha) = 0\} = \{f(z) = (\alpha - z) g(z) \colon g \in H^2\}.$$

In particular, $\alpha e_m - e_{m+1} \in bH^2$ for every $m \geq 0$. This means that for every $g \in \mathcal{H}(b)$,

$$\langle g, \alpha e_m - e_{m+1} \rangle_2 = 0, \quad m \geq 0.$$

From this $\langle g, e_{m+1} \rangle_2 = \bar{\alpha} \langle g, e_m \rangle_2$ for $m \geq 0$, which gives $\hat{g}(m) = \bar{\alpha}^m g(0)$, $m \geq 0$, and consequently

$$g(z) = \sum_{m=0}^{\infty} \hat{g}(m) e_m(z) = g(0) \sum_{m=0}^{\infty} \bar{\alpha}^m z^m = \frac{g(0)}{1 - \bar{\alpha} z}.$$

Therefore,

$$\mathcal{H}(\varphi_\alpha) = \{\lambda k_\alpha \colon \lambda \in \mathbb{C}\},$$

where $k_\alpha(z) = (1 - \bar{\alpha} z)^{-1}$.

The projection $P_{\bar{b}}$ is of the form $P_{\bar{b}}f = \beta(f)k_\alpha$. Since $k_\alpha \in \mathcal{H}(b)$, for $f \in H^2$,

$$f(\alpha) = \langle f, k_\alpha\rangle_2 = \langle P_{\bar{b}}f, k_\alpha\rangle_2 = \langle \beta(f)k_\alpha, k_\alpha\rangle_2 = \beta(f)\|k_\alpha\|_2^2.$$

Hence $\beta(f) = f(\alpha)/\|k_\alpha\|_2^2$ and

$$P_{\bar{b}}f = \frac{f(\alpha)}{\|k_\alpha\|_2^2}k_\alpha.$$

It follows from (4.1) that k_α is an eigenvector of the backward shift operator, more exactly we have

$$S^*k_\alpha = T_{\bar{z}}k_\alpha = \overline{\alpha}k_\alpha,$$

and the space $\mathcal{H}(\varphi_\alpha)$ from Example 4.8 is backward shift invariant. The next proposition shows that the spaces $\mathcal{H}(b)$ are always S^*-invariant. Moreover, $\mathcal{H}(b)$ is invariant under $T_{\overline{\varphi}}$ for an arbitrary $\varphi \in H^\infty$.

Proposition 4.9 ([13], II-7). *If φ is in H^∞, then $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are both invariant under $T_{\overline{\varphi}}$ and the norm of $T_{\overline{\varphi}}$ as an operator in each of these spaces does not exceed $\|\varphi\|_\infty$.*

Proof. Assume that $\|\varphi\|_\infty = 1$.

(1) The case of $\mathcal{H}(\bar{b})$:

Our goal is to show that

$$T_{\overline{\varphi}}(\mathcal{H}(\bar{b})) \subset \mathcal{H}(\bar{b}) \quad \text{and} \quad \|T_{\overline{\varphi}}\|_{\mathcal{H}(\bar{b})\to\mathcal{H}(\bar{b})} \leq 1.$$

The desired inclusion is

$$T_{\overline{\varphi}}(I - T_{\bar{b}}T_b)^{1/2}(H^2) \subset (I - T_{\bar{b}}T_b)^{1/2}(H^2),$$

and can also be written as $\mathcal{M}(A) \subset \mathcal{M}(B)$, where

$$A = T_{\overline{\varphi}}(I - T_{\bar{b}}T_b)^{1/2} \quad \text{and} \quad B = (I - T_{\bar{b}}T_b)^{1/2}.$$

Now

$$\begin{aligned} BB^* - AA^* &= I - T_{\bar{b}}T_b - T_{\overline{\varphi}}(I - T_{\bar{b}}T_b)T_\varphi \\ &= I - T_{\bar{b}}T_b - T_{\overline{\varphi}}T_\varphi + T_{\overline{\varphi}}T_{\bar{b}}T_bT_\varphi \\ &= I - T_{|b|^2} - T_{|\varphi|^2} + T_{|\varphi b|^2} = T_{(1-|\varphi|^2)(1-|b|^2)}, \end{aligned}$$

which is a positive operator. Therefore the Douglas's criterion implies that the inclusion holds. Moreover, the inclusion map $\mathcal{J}\colon \mathcal{M}(A) \hookrightarrow \mathcal{M}(B)$ is bounded, $\|\mathcal{J}\| \leq 1$. In other words

$$\|T_{\overline{\varphi}}(I - T_{\bar{b}}T_b)^{1/2}f\|_{\mathcal{M}(B)}^2 \leq \|T_{\overline{\varphi}}(I - T_{\bar{b}}T_b)^{1/2}f\|_{\mathcal{M}(A)}^2 \quad \text{for all } f \in H^2.$$

Since $A = T_{\overline{\varphi}}B$, then

$$\ker B \subset \ker A \quad \text{and} \quad (\ker A)^\perp \subset (\ker B)^\perp.$$

Take $f \in H^2$ and write $f = f_0 + f_\perp$, where $f_0 \in \ker B$ and $f_\perp \in (\ker B)^\perp$. Similarly, for $f_\perp$ write $f_\perp = f_{\perp,0} + f_{\perp,\perp}$, $f_{\perp,0} \in \ker A$, $f_{\perp,\perp} \in (\ker A)^\perp$. This allows us to write

$$\begin{aligned} \|Bf\|_{\mathcal{M}(B)}^2 &= \|Bf_\perp\|_{\mathcal{M}(B)}^2 = \|f_\perp\|_2^2 \\ &= \|f_{\perp,0}\|_2^2 + \|f_{\perp,\perp}\|_2^2 \geq \|f_{\perp,\perp}\|_2^2 \\ &= \|Af_{\perp,\perp}\|_{\mathcal{M}(A)}^2 = \|Af_\perp\|_{\mathcal{M}(A)}^2 \\ &= \|Af\|_{\mathcal{M}(A)}^2 \geq \|Af\|_{\mathcal{M}(B)}^2 = \|T_{\overline{\varphi}}Bf\|_{\mathcal{M}(B)}^2, \end{aligned}$$

or in other words, $\|T_{\overline{\varphi}}g\|_{\bar{b}}^2 \leq \|g\|_{\bar{b}}^2$ for every $g \in \mathcal{H}(\bar{b})$, $g = Bf = (I - T_{\bar{b}}T_b)^{1/2}f$. This is precisely our assertion for $T_{\overline{\varphi}}$.

(2) The case of $\mathcal{H}(b)$:

We need to prove

$$T_{\overline{\varphi}}(\mathcal{H}(b)) \subset \mathcal{H}(b) \quad \text{and} \quad \|T_{\overline{\varphi}}\|_{\mathcal{H}(b)\to\mathcal{H}(b)} \leq 1.$$

Recall that Corrolary 4.2 states that $f \in \mathcal{H}(b)$ if and only if $T_{\overline{b}}f \in \mathcal{H}(\overline{b})$. Since $\mathcal{H}(\overline{b})$ is invariant under $T_{\overline{\varphi}}$, then $T_{\overline{\varphi}}T_{\overline{b}}f \in \mathcal{H}(\overline{b})$ for every $f \in \mathcal{H}(b)$. However,

$$T_{\overline{\varphi}}T_{\overline{b}}f = T_{\overline{b}}T_{\overline{\varphi}}f,$$

and Corrolary 4.2 applied one more time gives $T_{\overline{\varphi}}f \in \mathcal{H}(b)$. Hence $\mathcal{H}(b)$ is invariant under $T_{\overline{\varphi}}$.

What is more, form Corrolary 4.2 and from what has been proved for $\mathcal{H}(\overline{b})$,

$$\|T_{\overline{\varphi}}f\|_b^2 = \|T_{\overline{\varphi}}f\|_2^2 + \|T_{\overline{b}}T_{\overline{\varphi}}f\|_b^2 = \|T_{\overline{\varphi}}f\|_2^2 + \|T_{\overline{\varphi}}T_{\overline{b}}f\|_b^2 \leq \|f\|_2^2 + \|T_{\overline{b}}f\|_b^2 = \|f\|_b^2.$$

This concludes the proof. □

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