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Various g-functions for Laguerre function expansions of Hermite type

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VARIOUS g-FUNCTIONS FOR LAGUERRE FUNCTION EXPANSIONS OF HERMITE TYPE

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Abstract

We examine weighted L^p boundedness of g-functions based on semigroups related to multi-dimensional Laguerre function expansions of Hermite type. A technique of vector valued Calderón-Zygmund operators is used.

1 Introduction

The study of g-functions (alias square functions) began in the twenties of the previous century. Since then they have been investigated by many authors in various contexts and forms, see [9] for a historical background. In our paper we study four types of g-functions connected with multi-dimensional Laguerre function expansions of Hermite type (see the next section for definitions). Some earlier, one-dimensional, results concerning g-functions in other Laguerre contexts may be found in [12] and [14]. A recent paper treating multi-dimensional g-functions in the context of Laguerre function expansions of convolution type is [13]. Certain one-dimensional results concerning g-functions in our setting are contained in [1] and [2]. The main result of our paper is Theorem 2.1. We prove it under the assumption $\alpha \in (\{-1/2\} \cup [1/2, \infty))^d$. It is however noteworthy, that with the exceptions of the smoothness estimates all the other partial results of our paper are valid under the weaker assumption $\alpha \in [-1/2, \infty)^d$. In the paper we use methods developed in [7]. The

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technicalities are similar to those from the previous paper of the author, see [15], but much more complicated.

The paper is organized as follows. Section 2 contains the setup, definitions, basic lemmas and the statement of the main theorem. In particular, the definitions of the Calderón-Zygmund kernels for the linearized versions of the considered g-functions are given. Section 3 is devoted to the proofs of the L^2 boundedness of the g-functions under examination and association conditions for their Calderón-Zygmund kernels. In Section 4 we show the growth and smoothness conditions for all considered kernels. Due to the complexity of the estimates Section 4 is split onto three subsections.

Throughout the paper we use a fairly standard notation with essentially all symbols referring to (if not stated otherwise) $\mathbb{R}^d_+ = (0, \infty)^d$. Thus Δ denotes the Laplacian restricted to $\mathbb{R}^d_+$, $A_p = A_p(\mathbb{R}^d_+)$ stands for the Muckenhoupt class of A_p weights, $L^p(w) = L^p(\mathbb{R}^d_+, w(x)dx)$ denotes the weighted L^p space (w being a non-negative weight on $\mathbb{R}^d_+$); we simply write L^p if $w \equiv 1$. By $\langle f, g\rangle$ we mostly mean the canonical inner product in L^2. The other meanings of the symbol $\langle \cdot, \cdot \rangle$ should be clear from the context. The symbol ∇_x represents the gradient operator with respect to the x variable. The notation $X \lesssim Y$ will be used to indicate that $X \le CY$ with a positive constant C independent of significant quantities. We will write $X \approx Y$ when $X \lesssim Y$ and $Y \lesssim X$. We shall also make a frequent use, most often without mentioning it in relevant places, of the fact that for any $A > 0$ and $a \ge 0$,

$$\sup_{t>0} t^a e^{-At} = C_{a,A} < \infty.$$

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2 Preliminaries

Let $\varphi^\alpha_k(x) = \varphi^{\alpha_1}_{k_1}(x_1) \cdots \varphi^{\alpha_d}_{k_d}(x_d)$ be the system of d-dimensional Laguerre functions,

$$\varphi^{\alpha_i}_{k_i}(x_i) = \left(\frac{2\Gamma(k_i+1)}{\Gamma(k_i+\alpha_i+1)}\right)^{1/2} L^{\alpha_i}_{k_i}(x_i^2) x_i^{\alpha_i+1/2} e^{-x_i^2/2}, \qquad x_i > 0, \quad i = 1, \dots, d,$$

where $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in (-1, \infty)^d$, and $L^{\alpha_i}_{k_i}$ denotes the Laguerre polynomial of degree k_i and order α_i. It is known that each φ^α_k is an eigenfunction of the operator

$$L_\alpha = -\Delta + |x|^2 + \sum_{i=1}^{d} \frac{1}{x_i^2}\left(\alpha_i^2 - \frac{1}{4}\right)$$

corresponding to the eigenvalue $\lambda^\alpha_{|k|} = 4|k| + 2|\alpha| + 2d$; here $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $|k| = k_1 + \ldots + k_d$ is the length of k. Moreover, $\{\varphi^\alpha_k : k \in \mathbb{N}^d\}$ is an orthonormal basis in L^2 and the following estimates hold (see [6, Section 5])

$$|\varphi^\alpha_k(x)| \lesssim 1, \quad x \in \mathbb{R}^d_+, \qquad |\langle f, \varphi^\alpha_k\rangle| \lesssim (2|k| + |\alpha| + d)^d \|f\|_{L^2}, \quad f \in L^2. \tag{2.1}$$

Let δ_j be the jth partial derivative associated with L_α (Laguerre-type partial derivative), given by

$$\delta_j = \frac{\partial}{\partial x_j} + x_j - \frac{1}{x_j}(\alpha_j + 1/2).$$

Using $\frac{d}{dx} L^\alpha_k = -L^{\alpha+1}_{k-1}$, $\alpha > -1$, $k \in \mathbb{N}$, cf. [4, (4.18.6)], it can be easily seen that

$$\delta_j \varphi^\alpha_k = -2\sqrt{k_j}\,\varphi^{\alpha+e_j}_{k-e_j}, \qquad k_j \geq 0, \tag{2.2}$$

where e_j is the j-th coordinate vector; here, by convention, $\varphi^{\alpha+e_j}_{k-e_j} = 0$ if $k_j < 1$.

Consider the operator

$$\mathcal{L}_\alpha f = \sum_{k \in \mathbb{N}^d} \lambda^\alpha_{|k|} \langle f, \varphi^\alpha_k\rangle \varphi^\alpha_k$$

on the domain

$$\mathrm{Dom}\,(\mathcal{L}_\alpha) = \left\{ f \in L^2 : \sum_{k \in \mathbb{N}^d} \left| \lambda^\alpha_{|k|} \langle f, \varphi^\alpha_k\rangle \right|^2 < \infty \right\}.$$

It is not hard to see that: $C^\infty_c\left(\mathbb{R}^d_+\right) \subseteq \mathrm{Dom}\,(\mathcal{L}_\alpha)$, $\mathcal{L}_\alpha$ is a self-adjoint extension of L_α, $\mathcal{L}_\alpha$ has the discrete spectrum $\{\lambda^\alpha_n : n \in \mathbb{N}\}$ and

$$\mathcal{L}_\alpha f = \sum_{n=0}^{\infty} \lambda^\alpha_n \mathcal{P}^\alpha_n f, \quad f \in \mathrm{Dom}\,(\mathcal{L}_\alpha),$$

is the spectral decomposition of $\mathcal{L}_\alpha$, where

$$\mathcal{P}^\alpha_n f = \sum_{|k|=n} \langle f, \varphi^\alpha_k\rangle \varphi^\alpha_k$$

are the spectral projections.

The heat-diffusion semigroup defined via the spectral theorem as

$$T^\alpha_t f = e^{-t\mathcal{L}_\alpha} f = \sum_{n=0}^{\infty} e^{-t\lambda^\alpha_n} \mathcal{P}^\alpha_n f, \qquad f \in L^2, \tag{2.3}$$

is a strongly continuous semigroup of contractions on L^2, having $\mathcal{L}_\alpha$ as its infinitesimal generator. The Poisson semigroup, given by

$$P_t^\alpha f = e^{-t(\mathcal{L}_\alpha)^{1/2}} f = \sum_{n=0}^{\infty} e^{-t(\lambda_n^\alpha)^{1/2}} \mathcal{P}_n^\alpha f, \qquad f \in L^2, \tag{2.4}$$

is also a strongly continuous semigroup of contractions on L^2, that has $(\mathcal{L}_\alpha)^{1/2}$ as its infinitesimal generator. By the principle of subordination,

$$P_t^\alpha f(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/4s}\, T_s f(x)\, ds = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x)\, du.$$

We also have the following integral representation of T_t^α :

$$T_t^\alpha f(x) = \int_{\mathbb{R}_+^d} \mathcal{G}_t^\alpha(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d, \quad f \in L^2,$$

where

$$\mathcal{G}_t^\alpha(x,y) = \sum_{n=0}^{\infty} e^{-t\lambda_n^\alpha} \sum_{|k|=n} \varphi_k^\alpha(x) \varphi_k^\alpha(y).$$

The operator P_t^α also has an integral representation via

$$P_t^\alpha f(x) = \int_{\mathbb{R}_+^d} P_t^\alpha(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d, \quad f \in L^2,$$

where

$$P_t^\alpha(x,y) = \sum_{n=0}^{\infty} e^{-t\sqrt{\lambda_n^\alpha}} \sum_{|k|=n} \varphi_k^\alpha(x) \varphi_k^\alpha(y).$$

The oscillating series defining $\mathcal{G}_t^\alpha(x,y)$ can be summed producing

$$\mathcal{G}_t^\alpha(x,y) = (\sinh 2t)^{-d} \exp\left(-\frac{1}{2}\coth 2t \left(|x|^2 + |y|^2\right)\right) \prod_{i=1}^{d} \sqrt{x_i y_i}\, I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right), \tag{2.5}$$

where I_ν, $\nu > -1$, is the modified Bessel function of the first kind and order ν. We also have a variant of the subordination principle on the level of integral kernels, namely

$$P_t^\alpha(x,y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/4s}\, \mathcal{G}_s^\alpha(x,y)\, ds = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{G}_{t^2/4u}^\alpha(x,y)\, du. \tag{2.6}$$

It is known, that $I_\nu(z)$, as a function of $z > 0$, is real, positive, smooth and satisfies

$$\frac{d}{dz} I_\nu(z) = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z). \tag{2.7}$$

Further, the following estimates hold.

Lemma 2.1 *(See [5]). Let $\nu \geq -1/2$. Then*

$$0 < I_\nu(z) - I_{\nu+1}(z) < \frac{2\,(\nu+1)}{z}\, I_{\nu+1}(z), \qquad z > 0.$$

For our purpose the standard asymptotics,

$$I_\nu(x) = O\left(x^\nu\right), \quad x \to 0^+, \qquad I_\nu(x) = O\left(x^{-\frac{1}{2}} e^x\right), \quad x \to \infty, \tag{2.8}$$

shall most frequently suffice. However, in several cases we will need more.

Lemma 2.2 *Let $\nu \geq -1/2$, then*

$$x\left[2I_{\nu+1}(x) - I_{\nu+2}(x) - I_\nu(x)\right] - I_{\nu+1}(x) = O(x^{-3/2} e^x), \quad x \to \infty.$$

Proof. Using the second order asymptotics at infinity for the Bessel function I_ν (see for instance [4, 5.11.10]), for large x we may write

$$I_\nu(x) = e^x (2\pi x)^{-1/2} \left(1 - \frac{4\nu^2 - 1}{8}\, x^{-1} + O(x^{-2})\right).$$

Thus, performing a simple calculation we see that

$$x\left[2I_{\nu+1}(x) - I_{\nu+2}(x) - I_\nu(x)\right] = e^x (2\pi x)^{-1/2} + O(x^{-3/2} e^x), \quad x \to \infty.$$

Since also $I_{\nu+1}(x) = e^x (2\pi x)^{-1/2} + O(x^{-3/2} e^x)$, $x \to \infty$, the lemma is proved. ∎

One of the key ingredients in proving our main result is the following simple observation.

Lemma 2.3 *(See [10].) Given $a>1$, and $x, y \in \mathbb{R}^d_+$, we have*

$$\int_0^1 u^{-a} \exp\left(-\frac{C|x-y|^2}{u}\right) du \lesssim |x-y|^{-2a+2}.$$

Let E be a Banach space and let $\Gamma(x,y)$ be a kernel defined on $\mathbb{R}^d_+ \times \mathbb{R}^d_+ \backslash \{(x,y) : x = y\}$ and taking values in E. We say that $\Gamma(x,y)$ is a standard kernel in the sense of the homogenous space $(\mathbb{R}^d_+, |\cdot|, dx)$, if it satisfies the growth estimate

$$\left\|\Gamma(x,y)\right\|_E \lesssim |x-y|^{-d} \tag{2.9}$$

and the smoothness estimates

$$\left\|\Gamma(x,y) - \Gamma(x',y)\right\|_E \lesssim |x-x'|\,|x-y|^{-d-1}, \qquad 2|x-x'| \leq |x-y|, \tag{2.10}$$

$$\left\|\Gamma(x,y) - \Gamma(x,y')\right\|_E \lesssim |y-y'|\,|x-y|^{-d-1}, \qquad 2|y-y'| \leq |x-y|. \tag{2.11}$$

A linear operator $T : L^2(\mathbb{R}^d_+) \to L^2(\mathbb{R}^d_+, E)$ is called a vector-valued Calderón-Zygmund operator in the sense of the space $(\mathbb{R}^d_+, dx)$ if:

a) T is bounded from $L^2(\mathbb{R}^d_+)$ to $L^2(\mathbb{R}^d_+, E)$,

b) there exists a standard $E-$ valued kernel $\Gamma(x,y)$ such that

$$Tf(x) = \int_{\mathbb{R}^d_+} \Gamma(x,y) f(y)\, dy, \qquad \text{a.e. on supp} f,$$

for every $f \in L^2(\mathbb{R}^d_+)$ vanishing outside a compact set contained in $\mathbb{R}^d_+$ (we will write $T \sim \Gamma(x,y)$ for such association).

It is known that a large part of the classical theory of Calderón-Zygmund operators remains valid, with appropriate adjustments, when the underlying space is $\mathbb{R}^d_+$ and the associated kernels are vector-valued, see [3] or comments in [8].

Let us now define the main objects of our study, that is the g-function for the heat diffusion semigroup

$$g_H(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} T_t^\alpha f(x) \right|^2 t\, dt \right)^{1/2}, \tag{2.12}$$

the Laguerre-type partial derivative g-functions for the heat diffusion semigroup

$$g_H^j(f)(x) = \left(\int_0^\infty \left| \delta_j T_t^\alpha f(x) \right|^2 \, dt \right)^{1/2}, \quad j = 1, \dots, d \tag{2.13}$$

and their analogues for the Poisson semigroup

$$g_P(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} P_t^\alpha f(x) \right|^2 t\, dt \right)^{1/2}, \tag{2.14}$$

$$g_P^j(f)(x) = \left(\int_0^\infty \left| \delta_j P_t^\alpha f(x) \right|^2 t\, dt \right)^{1/2}, \quad j = 1, \dots, d. \tag{2.15}$$

Obviously, the square functions listed above are not linear, but by the technique of linearization they can be treated as linear operators. Namely, we associate with g_H the (linear) operator $L^2 \ni f \to G_H(f) \in L^2(\mathbb{R}^d_+, L^2(t\,dt))$,

$$G_H(f)(x) = \left\{ \frac{d}{dt} T_t^\alpha f(x) \right\}_{t>0}$$

and with g_H^j the operator $L^2 \ni f \to G_H^j(f) \in L^2(\mathbb{R}^d_+, L^2(dt))$,

$$G_H^j(f)(x) = \left\{ \delta_j T_t^\alpha f(x) \right\}_{t>0}.$$

Similarly, we associate with g_P the operator $L^2 \ni f \to G_P(f) \in L^2(\mathbb{R}^d_+, L^2(t\,dt))$,

$$G_P(f)(x) = \left\{ \frac{d}{dt} P_t^\alpha f(x) \right\}_{t>0}$$

and with g_P^j the operator $L^2 \ni f \to G_P(f) \in L^2(\mathbb{R}^d_+, L^2(t\,dt))$,

$$G_P^j = \{\delta_j P_t^\alpha f(x)\}_{t>0}.$$

Proofs that $f \in L^2$ is mapped by G_H, G_H^j, G_P, G_P^j into appropriate spaces are implicitly contained in Section 3, when we show L^2 boundedness of considered g-functions.

Differentiating formally under the integral sign we obtain

$$\frac{d}{dt} T_t^\alpha f(x) = \int_{\mathbb{R}^d_+} \frac{d}{dt} \mathcal{G}_t^\alpha(x,y) f(y)\,dy.$$

The above suggests that the operator $f \to G_H(f)$ is associated with the vector valued kernel

$$K(x,y) = \left\{ \frac{d}{dt} \mathcal{G}_t^\alpha(x,y) \right\}_{t>0}, \qquad x, y \in \mathbb{R}^d_+,$$

taking values, as it will be implicitly shown by estimates below, in $L^2(\mathbb{R}^d_+, t\,dt)$. Similar formal arguments suggests to connect $G_H^j(x,y)$ with the kernel $F_j(x,y) = \{\delta_{j,x}\,\mathcal{G}_t^\alpha(x,y)\}_{t>0}$. Next we connect $G_P(x,y)$ with the kernel $L(x,y) = \left\{\frac{d}{dt} P_t^\alpha(x,y)\right\}_{t>0}$ and $G_P^j(x,y)$ with the kernel $L_j(x,y) = \{\delta_{j,x}\,P_t^\alpha(x,y)\}_{t>0}$. By the use of (2.6) and differentiation under the integral sign we may also write

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{d}{ds} \mathcal{G}_s^\alpha(x,y) \Big|_{s=t^2/4u} \frac{t}{2u}\,du$$

and

$$L_j(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \delta_{j,x} \mathcal{G}_{t^2/4u}^\alpha\,du,$$

which, after changing the variable $s = t^2/4u$ can be also written as

$$L_j(x,y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty \delta_{j,x}\,\mathcal{G}_s^\alpha(x,y)\; s^{-3/2} e^{-t^2/4s}\,ds. \tag{2.16}$$

The main result of our paper is the following.

Theorem 2.1 *Let $\alpha \in (\{-1/2\} \cup [1/2,\infty))^d$. Then each of the operators G_H, G_H^j, G_P, G_P^j is a vector valued Calderón-Zygmund operator taking values in $L^2(dt)$ (in the case of G_H^j) or in $L^2(t\,dt)$ (in the remaining cases). Moreover $G_H \sim K(x,y)$, $G_H^j \sim F_j(x,y)$, $G_P \sim L(x,y)$, $G_P^j \sim L_j(x,y)$.*

The proof of the theorem naturally splits into showing the following three propositions.

Proposition 2.2 *Let $\alpha \in [-1/2,\infty)^d$. Then each of the operators G_H, G_H^j, G_P, G_P^j is bounded from L^2 to $L^2(\mathbb{R}^d_+, E)$, where $E = L^2(dt)$ in the case of G_H^j and $E = L^2(t\,dt)$ in the remaining cases. Consequently, the g-functions defined by* (2.12) – (2.15) *are bounded on L^2.*

Proposition 2.3 *Let $\alpha \in [-1/2,\infty)^d$. Then $G_H \sim K(x,y)$, $G_H^j \sim F_j(x,y)$, $G_P \sim L(x,y)$, $G_P^j \sim L_j(x,y)$.*

Proposition 2.4 *Let $\alpha \in (\{-1/2\} \cup [1/2,\infty))^d$. Let $\Gamma(x,y)$ be any of the vector-valued kernels $K(x,y)$, $F_j(x,y)$, $L(x,y)$, $L_j(x,y)$. Then $\Gamma(x,y)$ satisfies the standard estimates* (2.9), (2.10), (2.11), *with $E = L^2(dt)$ in the cases of $F_j(x,y)$ and $E = L^2(t\,dt)$ in the remaining cases.*

By referring to general vector valued Calderón-Zygmund theory, see for instance [3], we also get the following.

Corollary 2.5 *Let $\alpha \in (\{-1/2\} \cup [1/2,\infty))^d$. Then each of the g-functions g_H, g_H^j, g_P, g_P^j extends uniquely to a bounded operator on $L^p(w)$, $1 < p < \infty$, $w \in A_p$, and to a bounded operator from $L^1(w)$ to $L^{1,\infty}(w)$, $w \in A_1$.*

The proofs of the Propositions 2.2 and 2.3 are given in Section 3. The proof of Proposition 2.4, which is the most technical part, is located in Section 4. The restrictions imposed on α are to some extent natural, see [7, p. 407] for additional comments.

3 L^2 boundedness and kernel associations

First we will show the L^2 boundedness of the operators G_H, G_P, G_H^j and G_P^j. The g-functions g_H and g_P are essentially isometries on L^2. Namely, the following holds.

Proposition 3.1 *Let* $\alpha \in [-1/2,\infty)^d$*. Then* $\|g_H(f)\|_{L^2} = \|g_P(f)\|_{L^2} = \frac{1}{2}\|f\|_{L^2}$*, for* $f \in L^2$*. Consequently, the opeators* G_H *and* G_P *are bounded from* L^2 *to* $L^2(\mathbb{R}^d_+,\, L^2(t\,dt))$*.*

Proof. Since showing both equalities above is similar, we will only prove the first one. Differentiating term by term the series in (2.3) (or in (2.4) in the case of G_P), which is legitimate in view of (2.1), we get

$$\frac{d}{dt}T_t^\alpha f(x) = -\sum_{n=0}^{\infty} \lambda_n^\alpha e^{-t\lambda_n^\alpha} \sum_{|k|=n} \langle f, \varphi_k^\alpha\rangle\, \varphi_k^\alpha.$$

Applying the Fubini-Tonelli theorem and Parseval's equality we obtain

$$\|g_H(f)\|_{L^2}^2 = \int_0^\infty t \int_{\mathbb{R}^d_+} \left|\frac{d}{dt}T_t^\alpha f(x)\right|^2 dx\,dt = \sum_{n=0}^{\infty}\left(\int_0^\infty t\,e^{-2t\lambda_n^\alpha}\,dt\right)(\lambda_n^\alpha)^2 \sum_{|k|=n} |\langle f,\varphi_k^\alpha\rangle|^2 = \frac{1}{4}\|f\|_{L^2}^2.$$

■

For the g-functions g_H^j and g_P^j we only have L^2 boundedness.

Proposition 3.2 *Let* $\alpha \in [-1/2,\infty)^d$*. Then the g-functions* g_H^j *and* g_P^j *are bounded on* L^2*. Consequently, the operators* G_H^j *and* G_P^j *are bounded from* L^2 *to* $L^2(\mathbb{R}^d_+,\, L^2(dt))$ *or* $L^2(\mathbb{R}^d_+,\, L^2(t\,dt))$*, respectively.*

Proof. Once again, showing boundedness of both g-functions is similar; this time we will only handle the second one. Applying (2.2) to the series in (2.4) (or in (2.3) in the case of G_H^j), which is permitted since (2.1) hold, we get

$$\delta_j P_t^\alpha f(x) = -2\sum_{n=0}^{\infty} e^{-t\sqrt{\lambda_n^\alpha}} \sum_{|k|=n} \langle f,\varphi_k^\alpha\rangle\,\sqrt{k_j}\,\varphi_{k-e_j}^{\alpha+e_j}.$$

Using the Fubini-Tonelli theorem, Parseval's equality and the fact that $\{\varphi_k^{\alpha+e_j} \,:\, k\in\mathbb{N}^d\}$ is also an orthonormal basis in L^2, we get

$$\begin{aligned}\|g_P^j(f)\|_{L^2}^2 &= \int_0^\infty t\int_{\mathbb{R}^d_+} |\delta_j P_t^\alpha f(x)|^2\,dx\,dt \le 4\sum_{n=0}^{\infty}\left(\int_0^\infty t\,e^{-2t\sqrt{\lambda_n^\alpha}}\,dt\right)\sum_{|k|=n}|k|\,|\langle f,\varphi_k^\alpha\rangle|^2\\ &= \sum_{n=0}^{\infty}\frac{n}{\lambda_n^\alpha}\sum_{|k|=n}|\langle f,\varphi_k^\alpha\rangle|^2 \le \frac{1}{4}\|f\|_{L^2}^2. \quad ■\end{aligned}$$

Combining Propositions 3.1 and 3.2, we get Proposition 2.2.

Now we prove Proposition 2.3. Since the proofs of all four instances of Proposition 2.3 are similar, we will only show one of them, namely the case of G_P. The argumentation presented in our paper is essentially an adaptation of the reasoning from [11, Section 2], in the setting of Hermite function expansions.

Proof of Proposition 2.3, the case of G_P. Density arguments reduce our task to showing that

$$\left\langle \left\{ \frac{d}{dt} P_t^\alpha f(x) \right\}_{t>0}, h \right\rangle_{L^2(\mathbb{R}^d_+, L^2(t\,dt))} = \left\langle \int_{\mathbb{R}^d_+} L(x,y) f(y)\, dy,\, h \right\rangle_{L^2(\mathbb{R}^d_+,\, L^2(t\,dt))}, \tag{3.1}$$

for every $f \in C_c^\infty(\mathbb{R}^d_+)$ and $h(x,t) = h_1(x)h_2(t)$, with $h_1 \in C_c^\infty(\mathbb{R}^d_+)$, $h_2 \in C_c^\infty(\mathbb{R}_+)$ and $\operatorname{supp} f \cap \operatorname{supp} h_1 = \emptyset$. First we consider the left hand side of (3.1),

$$\left\langle \left\{ \frac{d}{dt} P_t^\alpha f(x) \right\}_{t>0}, h \right\rangle_{L^2(\mathbb{R}^d_+, L^2(t\,dt))} = \int_0^\infty t\overline{h_2(t)} \int_{\mathbb{R}^d_+} \frac{d}{dt} P_t^\alpha f(x) \overline{h_1(x)}\, dx\, dt$$

$$= \int_0^\infty t\overline{h_2(t)} \int_{\mathbb{R}^d_+} \Big(- \sum_{n=0}^\infty \sqrt{\lambda_n^\alpha} e^{-t\lambda_n^\alpha} \sum_{|k|=n} \langle f, \varphi_k^\alpha \rangle \varphi_k^\alpha \Big) \overline{h_1(x)}\, dx\, dt$$

$$= - \int_0^\infty t\overline{h_2(t)} \sum_{n=0}^\infty \sqrt{\lambda_n^\alpha} e^{-t\lambda_n^\alpha} \sum_{|k|=n} \langle f, \varphi_k^\alpha \rangle \langle \varphi_k^\alpha, h_1 \rangle\, dt.$$

Changing the order of integration in the first equality above is legitimate in view of Fubini's theorem, since

$$\int_{\mathbb{R}^d_+} \int_0^\infty t \left| \frac{d}{dt} P_t^\alpha f(x) \right| |h_1(x)| |h_2(t)|\, dt\, dx \le \left\| \frac{d}{dt} P_t^\alpha f(x) \right\|_{L^2(dx\,dt)} \|h_1\|_{L^2} \|h_2\|_{L^2(t\,dt)}$$

and the right hand side here is finite because $f \to \frac{d}{dt} P_t^\alpha f(x)$ is bounded from L^2 to $L^2(\mathbb{R}^d_+, L^2(t\,dt))$, see Proposition 3.1. The second equality above is obtained by exchanging $\frac{d}{dt}$ and $\sum$ in the spectral decomposition of $P_t^\alpha f(x)$, which is justified by (2.1). The third identity is also a consequence of Fubini's theorem, which is applicable, because for $t > 0$,

$$\int_{\mathbb{R}^d_+} \Big(\sum_{n=0}^\infty \sqrt{\lambda_n^\alpha} e^{-t\lambda_n^\alpha} \sum_{|k|=n} |\langle f, \varphi_k^\alpha \rangle| \varphi_k^\alpha(x) h_1(x)| \Big)\, dx$$

$$\lesssim \|f\|_{L^2} \sum_{n=0}^\infty \sqrt{\lambda_n^\alpha} e^{-t\lambda_n^\alpha} \sum_{|k|=n} \int_{\mathbb{R}^d_+} |\varphi_k^\alpha(x) h_1(x)|\, dx \lesssim \|f\|_{L^2} \|h_1\|_{L^2} \sum_{n=0}^\infty \sqrt{\lambda_n^\alpha} e^{-t\lambda_n^\alpha} (n+1)^d < \infty.$$

Next we consider the right hand side of (3.1). Changing the order of integration, we get

$$\left\langle \int_{\mathbb{R}^d_+} L(x,y)f(y)\,dy,\, h\right\rangle_{L^2(\mathbb{R}^d_+,\, L^2(t\,dt))} = \int_0^\infty t\overline{h_2(t)} \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} L(x,y)f(y)\overline{h_1(x)}\,dy\,dx\,dt.$$

Here, using Fubini's theorem is possible, since

$$\begin{aligned}
&\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\int_0^\infty t\,|L(x,y)f(y)||h_1(x)h_2(t)|\,dt\,dy\,dx \\
&\le \|f\|_\infty\|h_1\|_\infty\|h_2\|_{L^2(t\,dt)} \int_{\operatorname{supp} h_1}\int_{\operatorname{supp} f} \|L(x,y)\|_{L^2(t\,dt)}\,dy\,dx \\
&\lesssim \int_{\operatorname{supp} h_1}\int_{\operatorname{supp} f} |x-y|^{-d}\,dy\,dx < \infty.
\end{aligned}$$

Above we used the growth estimate for the kernel $L(x,y)$, proved in the next section, and the fact that the supports of f and h_1 are disjoint and bounded. Now, differentiating term by term the series defining $P_t^\alpha(x,y)$ (once again we need (2.1) to justify this step), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} L(x,y)f(y)\overline{h_1(x)}\,dy\,dx \\
&= \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} \Big(-\sum_{n=0}^\infty \sqrt{\lambda_n^\alpha}e^{-t\lambda_n^\alpha}\sum_{|k|=n}\varphi_k^\alpha(x)\varphi_k^\alpha(y)\Big) f(y)\overline{h_1(x)}\,dy\,dx \\
&= -\sum_{n=0}^\infty \sqrt{\lambda_n^\alpha}e^{-t\lambda_n^\alpha}\sum_{|k|=n}\langle f,\varphi_k^\alpha\rangle\langle\varphi_k^\alpha, h_1\rangle.
\end{aligned}$$

Using Fubini's theorem in the second identity above is possible since, with $t>0$ fixed,

$$\begin{aligned}
&\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} \Big(\sum_{n=0}^\infty \sqrt{\lambda_n^\alpha}e^{-t\lambda_n^\alpha}\sum_{|k|=n}|\varphi_k^\alpha(x)||\varphi_k^\alpha(y)|\Big)|f(y)h_1(x)|\,dy\,dx \\
&\lesssim \|f\|_{L^2}\|h_1\|_{L^2}\sum_{n=0}^\infty \sqrt{\lambda_n^\alpha}e^{-t\lambda_n^\alpha}(n+1)^d.
\end{aligned}$$

Integrating the last identities against $\overline{h_2(t)}\,t\,dt$ we see that both sides of (3.1) coincide.

4 Proof of Proposition 2.4

In this section we prove relevant kernel estimates for all previously considered operators. In fact, we will only prove estimates (2.9), (2.10) and (2.11), for the heat semigroup based kernels $K(x,y)$ and $F_j(x,y)$. From these (by virtue of the subordination principle), we shall easily obtain the desired estimates for the kernels $L(x,y)$ and $L_j(x,y)$, based on the Poisson semigroup (see [11] for a relevant discussion concerning more general contraction semigroups). Our methods are similar to those from [7]. Till the end of the paper, the following notation will be used:

$$A = A(u) = \frac{1+u^2}{4u}, \quad B = B(u) = \frac{1-u^2}{2u}, \quad Z_i = Z_i(u, x_i, y_i) = Bx_iy_i$$
$$\sqrt{xy} = \prod_{i=1}^{d} \sqrt{x_iy_i}, \qquad I_\alpha(Z) = \prod_{i=1}^{d} I_{\alpha_i}(Z_i),$$
$$D_\nu = D_\nu(x_1, y_1, u) = -2Ax_1I_\nu(Z_1) + By_1I_{\nu+1}(Z_1) + \frac{\nu+1/2}{x_1}I_\nu(Z_1),$$
$$W = \sum_{i=1}^{d} Z_i - A(|x|^2+|y|^2), \quad W_k = W - Z_k, \quad U_k = x_k^2 + y_k^2 + \sum_{i\neq k}(x_i-y_i)^2.$$

Further, in several places we will need to split D_ν; as it can be easily seen

$$D_\nu = M_1^\nu + M_2^\nu + M_3^\nu + \frac{\nu+1/2}{x_1}I_\nu(Z_1),$$

where

$$M_1^\nu = (-x_1+y_1)\frac{1}{2u}I_\nu(Z_1), \qquad M_2^\nu = -(x_1+y_1)\frac{u}{2}I_\nu(Z_1), \qquad M_3^\nu = By_1(I_{\nu+1}(Z_1) - I_\nu(Z_1)).$$

In the rest of the paper the following lemma will be extensively used.

Lemma 4.1 *Let $u \in (0,1)$, $\alpha \in [-1/2,\infty)^d$, $i = 1,\ldots,d$. We have*

- *(A)* $\log(\frac{1+u}{1-u}) \approx u$ *for* $u \in (0,1/2)$, $\log(\frac{1+u}{1-u}) \lesssim (1-u)^{-1/2}$ *for* $u \in (1/2,1)$; *consequently,* $\log\frac{1+u}{1-u} \lesssim u\,(1-u)^{-1/2}$.
- *(B)* $A \lesssim u^{-1}$, $\quad B \lesssim u^{-1}$; *moreover, if* $u \in (1/2,1)$, *then* $B \leq 3/4$ *and* $-A \leq -1/2$.
- *(C)* $Z_i \approx (1-u)u^{-1}x_iy_i$; *in particular for* $u \in (0,1/2)$, $Z_i \approx \frac{x_iy_i}{u}$, *while for* $u \in (1/2,1)$, $Z_i \approx (1-u)x_iy_i$.
- *(D)* $W = -\frac{1}{4u}|x-y|^2 - \frac{u}{4}|x+y|^2 \leq -\frac{1}{4u}|x-y|^2$.
- *(E) If* $u \in (1/2,1)$, *then* $W \leq -\frac{1}{8}|x+y|^2 - \frac{1}{4}|x-y|^2$.
- *(F)* $W_k \leq -\frac{1}{4u}U_k \leq -\frac{1}{4u}|x-y|^2$.
- *(G) If* $Z_1 < 1$, *then* $I_{\alpha_1}(Z_1) \lesssim u^{-\alpha_1}(1-u)^{\alpha_1}(x_1y_1)^{\alpha_1} \lesssim u^{1/2}(1-u)^{-1/2}(x_1y_1)^{-1/2}$.

(H) $I_{\alpha_i}(Z_i) \lesssim (x_iy_i)^{-1/2}u^{1/2}(1-u)^{-1/2}e^{Z_i}$.

(I) $|I_{\alpha_i+1}(Z_i) - I_{\alpha_i}(Z_i)| \lesssim (x_iy_i)^{-1}u(1-u)^{-1}I_{\alpha_i+1}(Z_i) \lesssim (x_iy_i)^{-3/2}u^{3/2}(1-u)^{-3/2}e^{Z_i}$.

Proof. Items (A)–(F) are trivial. Items (G), (H), (I) follow from the asymptotics (2.8) with the aid of Lemma 2.1. ■

Some of the items of the lemma are listed only for the sake of completeness, e.g. mostly we shall not write explicitly that we refer to the (trivial) items (A) or (B).

We will also frequently use the change of variable

$$t = t(u) = \frac{1}{2}\log\frac{1+u}{1-u}, \tag{4.1}$$

which was invented by Stefano Meda, in the context of the Ornstein-Uhlenbeck operator.

4.1 Estimates for the kernel $K(x,y)$

We will first show the growth estimate (2.9), under the assumption $\alpha \in [-1/2,\infty)^d$. Using (4.1) we see that it suffices to show that $\int_0^1 |K_u(x,y)|^2 \log\frac{1+u}{1-u}\,du \lesssim |x-y|^{-2d}$, where

$$K_u(x,y) = (1-u^2)^{-1/2}\frac{d}{dt}\mathcal{G}_t^\alpha(x,y)\bigg|_{t=\frac{1}{2}\log\frac{1+u}{1-u}}.$$

Differentiating (2.5), with the aid of (2.7), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_t^\alpha(x,y) &= -2d\cosh 2t(\sinh 2t)^{-d-1}e^{-1/2\coth 2t\left(|x|^2+|y|^2\right)}\prod_{i=1}^d \sqrt{x_iy_i}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &\quad+\left(|x|^2+|y|^2\right)(\sinh 2t)^{-d-2}e^{-1/2\coth 2t\left(|x|^2+|y|^2\right)}\prod_{i=1}^d \sqrt{x_iy_i}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &\quad-2\cosh 2t(\sinh 2t)^{-d-2}e^{-1/2\coth 2t\left(|x|^2+|y|^2\right)}S_t^\alpha(x,y)\prod_{i=1}^d\sqrt{x_iy_i},\end{aligned}$$

with

$$S_t^\alpha(x,y) = \sum_{j=1}^d\left[\alpha_j\sinh 2t\; I_{\alpha_j}\left(\frac{x_jy_j}{\sinh 2t}\right) + x_jy_jI_{\alpha_j+1}\left(\frac{x_jy_j}{\sinh 2t}\right)\right]\prod_{i\neq j}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right).$$

After some rearrangement of terms we see that

$$\begin{aligned}\cosh 2t\, S_t^\alpha(x,y) =& |\alpha| \cosh 2t \sinh 2t \prod_{i=1}^d I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right) + \langle x,y\rangle \prod_{i=1}^d I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right) \\ &+ \sum_{j=1}^d x_j y_j \left[I_{\alpha_j+1}\left(\frac{x_j y_j}{\sinh 2t}\right) - I_{\alpha_j}\left(\frac{x_j y_j}{\sinh 2t}\right)\right] \prod_{i\neq j} I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right) \\ &+ (\cosh 2t - 1) \sum_{j=1}^d x_j y_j I_{\alpha_j+1}\left(\frac{x_j y_j}{\sinh 2t}\right) \prod_{i\neq j} I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right).\end{aligned}$$

Using the above equality, with the fact that the change of variables (4.1) transforms $\sinh 2t$ into $\frac{2u}{1-u^2}$ and $\cosh 2t$ into $\frac{1-u^2}{1+u^2}$ in mind, we decompose $K_u(x,y) = \sum_{i=1}^4 K_i$, where

$$\begin{aligned} K_1 &= -2(d+|\alpha|)(2u)^{-d-1}(1-u^2)^{d-1/2}(1+u^2)\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\, I_\alpha(Z), \\ K_2 &= |x-y|^2(2u)^{-d-2}(1-u^2)^{d+3/2}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\, I_\alpha(Z), \\ K_3 &= 2(2u)^{-d-2}(1-u^2)^{d+3/2}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)} \sum_{j=1}^d x_j y_j \left[I_{\alpha_j}(Z_j) - I_{\alpha_j+1}(Z_j)\right] \prod_{i\neq j} I_{\alpha_i}(Z_i), \\ K_4 &= -(2u)^{-d}(1-u^2)^{d+1/2}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)} \sum_{j=1}^d x_j y_j I_{\alpha+e_j}(Z). \end{aligned}$$

Therefore, to prove the growth condition for the kernel $K(x,y)$ is suffices to show that

$$\int_0^1 |K_i|^2 \log\frac{1+u}{1-u}\, du \lesssim |x-y|^{-2d}, \qquad i=1,\dots,4. \tag{4.2}$$

Proof of (4.2)**.** If $i=1$, we use (A) and (H), to get $|K_1| \lesssim u^{-d/2-1}(1-u)^{d/2-1/2}e^W$. Since $W \le -\frac{1}{4u}|x-y|^2$ this leads to

$$\int_0^1 |K_1|^2 \log\frac{1+u}{1-u}\, du \lesssim \int_0^{1/2} u^{-d-1} e^{-\frac{1}{2u}|x-y|^2}\, du + \int_{1/2}^1 (1-u)^{d-3/2} e^W\, du.$$

The first integral in the sum above is easily estimated by $|x-y|^{-2d}$, see Lemma 2.3. For the second integral, applying (E) and (F), we get

$$\int_{1/2}^1 (1-u)^{d-3/2} e^W\, du \lesssim \int_{1/2}^1 (1-u)^{d-3/2}\, du\, e^{-\frac14|x+y|^2} e^{-\frac12|x-y|^2} \lesssim |x-y|^{-2d}.$$

If $i=2$, we apply (A), (D) and (H), getting $\int_0^1 |K_2|^2 \log \frac{1+u}{1-u}\, du \lesssim |x-y|^4 \int_0^1 u^{-d-3} e^{-\frac{1}{2u}|x-y|^2}\, du$. Now an application of Lemma 2.3 produces the desired bound. For $i=3$, an application of (H) and (I) to every term of the sum defining K_3 shows that the expression under consideration is bounded by a constant times $\int_0^1 u^{-d-1} e^{-\frac{1}{2u}|x-y|^2}\, du$. The desired bound follows by applying again Lemma 2.3. Finishing the proof of (4.2) we pass to the $i=4$ part. Splitting the sum defining K_4, and accordingly the integral under examination, we see that it suffices to bound by $|x-y|^{-2d}$ each of the resulting d integrals seperately. By the use of Lemma (D) and (H) we see, that the j-th integral is not greater than a constant times $(x_j y_j)^2 \int_0^1 u^{-d+1} e^{-\frac{1}{2u}|x-y|^2 - \frac{u}{2}|x+y|^2}\, du$. Since $(x_j y_j)^2 u^2 e^{-\frac{u}{2}|x+y|^2} \lesssim 1$, the j-th integral can be further bounded by $\int_0^1 u^{-d-1} e^{-\frac{1}{2u}|x-y|^2}\, du$, which is less than a constant times $|x-y|^{-2d}$, by Lemma 2.3. ■

Now we pass to the proofs of smoothness conditions (2.10), (2.11), for the kernel $K(x,y)$. By symmetry reasons it suffices to show (2.10) only. Using again the change of variable (4.1), we are reduced to proving the inequality

$$\int_0^1 |K_u(x,y) - K_u(x',y)|^2 \log \frac{1+u}{1-u}\, du \lesssim |x-y|^{-2d}\, |x-x'|^2, \quad 2|x-x'| \le |x-y|.$$

By the mean value theorem $K_u(x,y) - K_u(x',y) = \left\langle \nabla_x K_u(x,y)\big|_{x=\theta(u)}, x - x' \right\rangle$, where, for any fixed u, $\theta(u)$ lies in the interval $[x,x']$. Since in the definition of $K_u(x,y)$ no variable is privileged, to prove (2.10) it is sufficient to show that

$$\int_0^1 \left(\partial_{x_1} K_u(x,y)\big|_{x=\theta(u)}\right)^2 \log \frac{1+u}{1-u}\, du \lesssim |x-y|^{-2d-2}, \quad \theta(u) \in [x,x'],\ 2|x-x'| \le |x-y|.$$

Using (2.7) we see that $\partial_{x_1}\left(I_\nu(Z_1) e^{-A(|x|^2+|y|^2)} \sqrt{x_1 y_1}\right) = D_\nu e^{-A(|x|^2+|y|^2)} \sqrt{x_1 y_1}$. Hence, $\partial_{x_1} K_u(x,y) = \sum_{k=1}^9 J_k^\alpha = \sum_{k=1}^9 J_k^\alpha(x,y,u)$, where

$$J_1^\alpha = -2(d+|\alpha|)(2u)^{-d-1}(1-u^2)^{d-1/2}(1+u^2)\sqrt{xy}\, e^{-A(|x|^2+|y|^2)} D_{\alpha_1} \prod_{i=2}^d I_{\alpha_i}(Z_i),$$

comes from differentiating K_1,

$$J_2^\alpha = 2(x_1 - y_1)(2u)^{-d-2}(1-u^2)^{d+3/2}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)} I_\alpha(Z),$$

$$J_3^\alpha = |x-y|^2 (2u)^{-d-2}(1-u^2)^{d+3/2}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)} D_{\alpha_1} \prod_{i=2}^d I_{\alpha_i}(Z_i),$$

come from differentiating K_2,

$$J_4^\alpha = 2(2u)^{-d-2}(1-u^2)^{d+3/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}\left(D_{\alpha_1}-D_{\alpha_1+1}\right)x_1y_1\prod_{i=2}^d I_{\alpha_i}(Z_i),$$

$$J_5^\alpha = 2(2u)^{-d-2}(1-u^2)^{d+3/2}y_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}\left(I_{\alpha_1}(Z_1)-I_{\alpha_1+1}(Z_1)\right)\prod_{i=2}^d I_{\alpha_i}(Z_i),$$

$$J_6^\alpha = 2(2u)^{-d-2}(1-u^2)^{d+3/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1}\sum_{j=2}^d x_jy_j(I_{\alpha_j}(Z_j)-I_{\alpha_j+1}(Z_j))\prod_{i\neq 1,j} I_{\alpha_i}(Z_i),$$

come from differentiating K_3, and finally

$$J_7^\alpha = -(2u)^{-d}(1-u^2)^{d+1/2}y_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}I_{\alpha+e_1}(Z),$$

$$J_8^\alpha = -(2u)^{-d}(1-u^2)^{d+1/2}x_1y_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1+1}\prod_{i=2}^d I_{\alpha_i}(Z_i),$$

$$J_9^\alpha = -(2u)^{-d}(1-u^2)^{d+1/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1+1}\sum_{j=2}^d x_jy_jI_{\alpha_j+1}(Z_j)\prod_{i\neq j,1}^d I_{\alpha_i}(Z_i),$$

come from differentiating K_4. Therefore, the proof of the smoothness condition for the kernel $K(x,y)$ will follow, if we show that, for $k=1,\ldots,9$,

$$\int_0^1 \left|J_k^\alpha(\theta(u),y,u)\right|^2 \log\frac{1+u}{1-u}\,du \lesssim |x-y|^{-2d-2}, \quad 2|x-x'|\leq|x-y|, \tag{4.3}$$

where, as above, for every $u\in[0,1]$, $\theta(u)$ is a point in the interval $[x,x']$.

Proof of (4.3)**.** Before proceeding to the proof a few remarks are in order. From now on to the end of the proof of (4.3) a tilde above an expression will denote, that it depends on x through $\theta(u)$. Thus, for example, $\widetilde{J}_k^\alpha = J_k^\alpha(\theta(u),y,u)$, $k=1,\ldots,9$. In the estimates that follow we will often use, sometimes without reference, the fact that if $2|x-x'|\leq|x-y|$ and $\theta(u)\in[x,x']$, then $|x-y|\approx|\theta(u)-y|$, uniformly in $u\in[0,1]$. From now on, to the end of the proof of (4.3), we assume that $2|x-x'|\leq|x-y|$ and $\theta(u)\in[x,x']$. We also write θ instead of $\theta(u)$, for short.

We start with showing (4.3) for $k=2,7$. An application of (D) and (H) gives

$$|J_2^\alpha| \lesssim |x_1-y_1|u^{-d/2-2}(1-u)^{d/2+3/2}e^{-\frac{1}{2u}|x-y|^2},$$

$$|J_7^\alpha| \lesssim y_1e^{-\frac{u}{4}|x+y|^2}u^{-d/2}(1-u)^{d/2+1/2}e^{-\frac{1}{4u}|x-y|^2}.$$

Hence $|\widetilde{J}_2^\alpha| \lesssim |x-y| u^{-d/2-2}(1-u)^{d/2+3/2} e^{-\frac{C}{u}|x-y|^2}$. Using (A) and then Lemma 2.3 gives

$$\int_0^1 |\widetilde{J}_2^\alpha|^2 \log \frac{1+u}{1-u}\, du \lesssim |x-y|^2 \int_0^1 u^{-d-3} e^{-\frac{C}{u}|x-y|^2}\, du \lesssim |x-y|^{-2d-2}.$$

Similarly, since $y_1 u^{1/2} e^{-\frac{u}{4}|x+y|^2} \lesssim 1$, using (A) and Lemma 2.3 we arrive at

$$\int_0^1 |\widetilde{J}_7^\alpha|^2 \log \frac{1+u}{1-u}\, du \lesssim \int_0^1 u^{-d} e^{-\frac{C}{u}|x-y|^2}\, du \lesssim \int_0^1 u^{-d-2} e^{-\frac{C}{u}|x-y|^2}\, du \lesssim |x-y|^{-2d-2}.$$

The remaining cases of k in the proof of (4.3) require splitting the range of integration in (4.3) onto (0,1/2) and (1/2,1). We start with integration over $(1/2, 1)$. If $k = 5$, then by using (D) and (H), we see that for $u \in (1/2, 1)$,

$$|J_5^\alpha| \lesssim (1-u)^{d/2+1/2} y_1 e^{-\frac{1}{8}|x+y|^2} e^{-\frac{1}{4}|x-y|^2}.$$

A routine application of the fact that $|\theta - y| \approx |x-y|$ and (A) gives the estimate

$$\int_{1/2}^1 |\widetilde{J}_5^\alpha|^2 \log \frac{1+u}{1-u}\, du \lesssim |x-y|^{-2d-2}.$$

The proofs of

$$\int_{1/2}^1 |\widetilde{J}_k^\alpha|^2 \log \frac{1+u}{1-u}\, du \lesssim |x-y|^{-2d-2}, \quad k = 1, 3, 4, 6, 8, 9. \tag{4.4}$$

are more involved. They require the following.

Lemma 4.2 *Let $\nu = -1/2$ or $\nu \in [1/2, \infty)$, and $b \geq 0$. Then*

$$E_{\nu,b} = \int_{1/2}^1 (1-u)^{d-1/2} \left(\widetilde{D}_\nu \sqrt{\theta_1 y_1} e^{\widetilde{W}_1}\right)^2 (|\theta| + |y|)^b\, du \lesssim |x-y|^{-2d-2}.$$

Proof. Using Lemmata 2.1 and (B), we see that $|-2Ax_1 I_\nu(Z_1) + By_1 I_{\nu+1}(Z_1)| \lesssim (x_1+y_1) I_\nu(Z_1)$. Hence, using again (H), and the fact that $|\theta - y| \approx |x-y|$,

$$\begin{aligned}
E^1 &= \int_{1/2}^{1} (1-u)^{d-1/2} \big([-2Ax_1 I_\nu(\widetilde{Z}_1) + By_1 I_{\nu+1}(\widetilde{Z}_1)] \sqrt{\theta_1 y_1} e^{\widetilde{W}_1} \big)^2 (|\theta| + |y|)^b \, du \\
&\lesssim \int_{1/2}^{1} (1-u)^{d-3/2} (|\theta| + |y|)^{b+2} e^{2\widetilde{W}} \, du \\
&\lesssim \int_{1/2}^{1} (1-u)^{d-3/2} (|\theta| + |y|)^{b+2} e^{-\frac{1}{4}|\theta+y|^2} e^{-\frac{1}{2}|\theta-y|^2} \, du \lesssim e^{-C|x-y|^2}.
\end{aligned}$$

To prove Lemma 4.2 it remains to demonstrate that

$$E^2 = \Big(\nu + \frac{1}{2}\Big)^2 \int_{1/2}^{1} (1-u)^{d-1/2} \big(\frac{1}{\theta_1} I_\nu(\widetilde{Z}_1) \sqrt{\theta_1 y_1} e^{\widetilde{W}_1} \big)^2 (|\theta| + |y|)^b \, du,$$

satisfies $E^2 \lesssim e^{-C|x-y|^2}$. If $\nu = -1/2$, then there is nothing to show. If $\nu \geq 1/2$, we consider two cases. In the case of $\widetilde{Z}_1 > 1$, because of (C), we have $(1-u)^{-1} \lesssim \theta_1 y_1$, therefore, applying (H) gives

$$\frac{1}{\theta_1} I_\nu(\widetilde{Z}_1) \sqrt{\theta_1 y_1} \lesssim \frac{1}{\theta_1} (1-u)^{-1/2} e^{\widetilde{Z}_1} \lesssim \frac{1}{\theta_1} (1-u)^{-1} e^{\widetilde{Z}_1} \lesssim y_1 e^{\widetilde{Z}_1}.$$

In the case of $\widetilde{Z}_1 \leq 1$, we simply use (H), to get

$$\frac{1}{\theta_1} I_\nu(\widetilde{Z}_1) \sqrt{\theta_1 y_1} \lesssim \frac{1}{\theta_1} (\widetilde{Z}_1)^\nu \sqrt{\theta_1 y_1} \lesssim \big((\theta_1 y_1)(1-u) \big)^{\nu - 1/2} y_1 (1-u)^{1/2}.$$

Since $\widetilde{Z}_1 \leq 1$ and $\nu \geq 1/2$, by (C), the last quantity above is less than a constant times $y_1 e^{\widetilde{Z}_1}$. From the consideration of both cases, (E), and the fact that $|\theta - y| \approx |x-y|$, we get

$$E^2 \lesssim \int_{1/2}^{1} (1-u)^{d-1/2} y_1^2 (|\theta| + |y|)^b e^{-\frac{1}{4}|\theta+y|^2} \, du \, e^{-C|x-y|^2} \lesssim e^{-C|x-y|^2}. \quad \blacksquare$$

We now return to the proof of (4.4). If $k=1$, then an application of (H) gives the estimate $|\widetilde{J}_1^\alpha| \lesssim (1-u)^{d/2} \widetilde{D}_{\alpha_1} \sqrt{\theta_1 y_1} e^{\widetilde{W}_1}$. Using (A), we obtain $\int_0^1 |\widetilde{J}_1^\alpha|^2 \log \frac{1+u}{1-u} \, du \lesssim E_{\alpha_1,0}$. If $k=3$, then it is easily seen that $\widetilde{J}_3^\alpha \lesssim |\theta - y|^2 \widetilde{J}_1^\alpha \lesssim (|\theta| + |y|)^2 \widetilde{J}_1^\alpha$, for $u \in (1/2, 1)$. Therefore, repeating the arguments of the $k=1$ case, we arrive at $\int_{1/2}^1 |\widetilde{J}_3^\alpha|^2 \log \frac{1+u}{1-u} \, du \lesssim E_{\alpha_1,4}$. If

$k = 4$, then from the observation that $|\widetilde{J}_4^\alpha| \lesssim \theta_1 y_1(|\widetilde{J}_1^\alpha| + |\widetilde{J}_3^{\alpha+1}|)$, and Lemma 4.2, we get $\int_{1/2}^1 |\widetilde{J}_4^\alpha|^2 \log\frac{1+u}{1-u}\, du \lesssim E_{\alpha_1,4} + E_{\alpha_1+1,4}$. If $k = 6$, we use (H) to get $\theta_j y_j |I_{\alpha_j}(\widetilde{Z}_j) - I_{\alpha_j+1}(\widetilde{Z}_j))| \lesssim (\theta_j y_j)^{-1/2}(1-u)^{-3/2}e^{\widetilde{Z}_j}$. It follows that $|\widetilde{J}_6^\alpha| \lesssim (1-u)^{d/2+1/2}\widetilde{D}_{\alpha_1}\sqrt{\theta_1 y_1}e^{\widetilde{W}_1}$ and consequently $\int_{1/2}^1 |\widetilde{J}_6^\alpha|^2 \log\frac{1+u}{1-u}\, du \lesssim E_{\alpha_1,0}$. Assume now $k = 8$. Obviously, for $u \in (0,1/2)$, $|\widetilde{J}_8^\alpha| \lesssim \theta_1 y_1 |J_1^{\alpha+e_1}|$. Hence, $\int_{1/2}^1 |\widetilde{J}_8^\alpha|^2 \log\frac{1+u}{1-u}\, du \lesssim E_{\alpha_1+1,4}$. If $k = 9$, then it is not hard to see, that the $j-$th term of the sum defining $\widetilde{J}_9^\alpha$ may be estimated by a constant times $\theta_j y_j \widetilde{J}_1^{\alpha+e_j}$. Thus $\int_{1/2}^1 |\widetilde{J}_9^\alpha|^2 \log\frac{1+u}{1-u}\, du \lesssim E_{\alpha_1,4}$. By virtue of Lemma 4.2, in each of the considered cases the bound by $|x-y|^{-2d-2}$ follows, hence the proof of (4.4) is finished.

We now continue the treatment of the remaining cases of k by considering integration over $(0,1/2)$. Since $\log\frac{1+u}{1-u} \approx u$, $u \in (0,1/2)$, it remains to demonstrate that

$$\int_0^{1/2} |\widetilde{J}_k^\alpha|^2 u\, du \lesssim |x-y|^{-2d-2}, \quad k = 1,3,4,5,6,8,9. \tag{4.5}$$

For $k = 1,3,6,8$, the above will be a consequence of the following lemma.

Lemma 4.3 *Let $\nu = -1/2$ or $\nu \in [1/2,\infty)$, and $a \geq 0$, $b \geq 0$. Then,*

$$R = R_{\nu,a,b} = \int_0^{1/2} u^{-d-a}\left(\widetilde{D}_\nu\sqrt{\theta_1 y_1}e^{\widetilde{W}_1}\right)^2(\theta_1+y_1)^b\, du \lesssim |x-y|^{-2d-2a+2-b}.$$

Proof. We split the integral defining R onto two integrals, according to the cases $\widetilde{Z}_1 \leq 1$ and $\widetilde{Z}_1 > 1$ and denote the resulting integrals by R_1 and R_2, respectively.

We start with bounding R_1. Using (B) and (H), we see that

$$\begin{aligned}\widetilde{D}_\nu &\leq |-2A\theta_1 I_\nu(\widetilde{Z}_1) + By_1 I_{\nu+1}(\widetilde{Z}_1)| + \frac{\nu+1/2}{\theta_1}I_\nu(\widetilde{Z}_1)\\ &\lesssim (\theta_1+y_1)(\theta_1 y_1)^{-1/2}u^{-1/2} + \frac{\nu+1/2}{\theta_1}I_\nu(\widetilde{Z}_1).\end{aligned}$$

Therefore, using (F), we get

$$\begin{aligned}R_1 \lesssim &\int_{(0,1/2)\cap\{\widetilde{Z}_1\leq 1\}} u^{-d-a-1}(\theta_1+y_1)^{b+2}e^{-\frac{1}{2u}\widetilde{U}_1}\, du\\ &+ (\nu+1/2)^2\int_{(0,1/2)\cap\{\widetilde{Z}_1\leq 1\}} \frac{1}{\theta_1^2}u^{-d-a}(\theta_1+y_1)^b\left(I_\nu(\widetilde{Z}_1)\sqrt{\theta_1 y_1}e^{\widetilde{W}_1}\right)^2 du.\end{aligned}$$

Because $(\frac{\theta_1+y_1}{u^{1/2}})^{b+2}e^{-\frac{1}{4u}\widetilde{U}_1} \lesssim 1$ and $|x-y|^2 \lesssim |\theta-y|^2 \le \widetilde{U}_1$, we see that the first integral above is less than a constant times $\int_0^{1/2} u^{-d-a+b/2}e^{-\frac{C}{u}|x-y|^2}\,du$. If $\nu = -1/2$, then the second integral vanishes. Otherwise, using (G), we can bound it by a constant times $\int_{(0,1/2)\cap\{\widetilde{Z}_1\le 1\}} u^{-d-a-2\nu}(y_1^2)(y_1\theta_1)^{2\nu-1}(\theta_1+y_1)^b e^{-\frac{1}{2u}\widetilde{U}_1}\,du$. Now, since $|x-y|^2 \lesssim |\theta-y|^2 \le \widetilde{U}_1$ and $(u^{-1}y_1^2)(u^{-1}y_1\theta_1)^{2\nu-1}(u^{-1/2}(\theta_1+y_1))^b e^{-\frac{1}{4u}\widetilde{U}_1} \lesssim 1$, the second integral can be further estimated by $\int_0^{1/2} u^{-d-a+b/2}e^{-\frac{C}{u}|x-y|^2}\,du$. Hence,

$$R_1 \lesssim \int_0^{1/2} u^{-d-a+b/2}e^{-\frac{C}{u}|x-y|^2}\,du \lesssim \int_0^{1/2} u^{-d-a-b/2}e^{-\frac{C}{u}|x-y|^2}\,du \lesssim |x-y|^{-2d-2a+2-b}.$$

To treat R_2 we decompose $\widetilde{D}_\nu$ onto $\widetilde{M}_1^\nu + \widetilde{M}_2^\nu + \widetilde{M}_3^\nu + \frac{\nu+1/2}{\theta_1}I_\nu(\widetilde{Z}_1)$. Using (I) we see that $\widetilde{M}_3^\nu \lesssim \frac{1}{\theta_1}I_\nu(\widetilde{Z}_1)$, therefore to bound R_2 it is enough to show that

$$R_{2,i} = \int_{(0,1/2)\cap\{\widetilde{Z}_1>1\}} u^{-d-a}\left(\widetilde{M}_i^\nu\sqrt{\theta_1 y_1}e^{\widetilde{W}_1}\right)^2(\theta_1+y_1)^b\,du, \quad i=1,2,$$

$$R_{2,3} = \int_{(0,1/2)\cap\{\widetilde{Z}_1>1\}} u^{-d-a}\left(\frac{1}{\theta_1}I_\nu(\widetilde{Z}_1)\sqrt{\theta_1 y_1}e^{\widetilde{W}_1}\right)^2(\theta_1+y_1)^b\,du,$$

satisfy $R_{2,i} \lesssim |x-y|^{-2d-2a+2-b}, \quad i=1,2,3$. To estimate $R_{2,1}$, we use (D), (H) and the fact that $(\theta_1-y_1)^2 \le |\theta-y|^2 \approx |x-y|^2$ to obtain

$$|x-y|^{-2}R_{2,1} \lesssim \int_0^{1/2} u^{-d-a-1}e^{-\frac{1}{2u}|\theta-y|^2}(\theta_1+y_1)^b e^{-\frac{u}{2}|\theta+y|^2}\,du \lesssim \int_0^{1/2} u^{-d-a-1-b/2}e^{-\frac{C}{u}|x-y|^2}\,du.$$

Applying Lemma 2.3, we get $R_{2,1} \lesssim |x-y|^{-2d-2a+2-b}$. Similar arguments easily lead to

$$\begin{aligned} R_{2,2} &\lesssim \int_0^{1/2} u^{-d-a+3}e^{-\frac{1}{2u}|\theta-y|^2}(\theta_1+y_1)^{b+2}e^{-\frac{u}{2}|\theta+y|^2}\,du \\ &\lesssim \int_0^{1/2} u^{-d-a+2-b/2}e^{-\frac{C}{u}|x-y|^2}\,du \lesssim |x-y|^{-2d-2a+2-b}. \end{aligned}$$

To prove the lemma we now need only to estimate $R_{2,3}$. To do this we further split the integral defining $R_{2,3}$ on two integrals: the local part $R_{2,3}^{loc}$, where $y_1/2 < \theta_1 < 2y_1$, and the global part $R_{2,3}^{glob}$, where $\theta_1 \le y_1/2$ or $\theta_1 \ge 2y_1$. If $\theta_1 \approx y_1$ and $\widetilde{Z}_1 > 1$, then $\frac{1}{\theta_1^2} = \frac{1}{\theta_1 y_1}\frac{y_1}{\theta_1} \lesssim u^{-1}$. Hence, using the fact that $|\theta-y| \approx |x-y|$ and (D), (H) (which, as we have seen above, implies in particular that $(\theta_1+y_1)^b e^{\frac{1}{2}\widetilde{W}} \lesssim u^{-b/2}$), we see that $R^{loc}_{2,3} \lesssim \int_0^{1/2} u^{-d-a-b/2} e^{-\frac{C}{u}|x-y|^2}\, du$. An application of Lemma 2.3 gives the desired bound for $R^{loc}_{2,3}$. Otherwise, if θ_1 is not comparable with y_1, then $(\theta_1+y_1)^2 \approx (\theta_1-y_1)^2$ and $\frac{1}{\theta_1^2} \leq \frac{(\theta_1+y_1)^2}{(\theta_1 y_1)^2} \lesssim (\theta_1-y_1)^2(\theta_1 y_1)^{-2} \lesssim (\theta_1-y_1)^2 u^{-2}$, for $\widetilde{Z}_1 > 1$. Hence, because $(\theta_1-y_1)^2 \leq |\theta-y|^2 \lesssim |x-y|^2$, we get $R^{loc}_{2,3} \lesssim |x-y|^2 \int_0^{1/2} u^{-d-a-1-b/2} e^{-\frac{C}{u}|x-y|^2}\, du$. The latter has already been bounded while showing the bound for $R_{2,1}$. ■

We now come back to the proof of (4.5) for $k = 1,3,6,8$. For $k=1$, using (A) and (H) (for $i = 2,\dots,d$), we see that $\int_0^{1/2} |\widetilde{J}^\alpha_1|^2 u\, du \lesssim R_{\alpha_1,2,0}$. If $k=3$, using (A) and (H) and the fact that $|\theta-y| \approx |x-y|$, we get $\int_0^{1/2} |\widetilde{J}^\alpha_3|^2 u\, du \lesssim |x-y|^4 R_{\alpha_1,4,0}$. If $k=6$, using (H) and (I), we see that each term defining $\widetilde{J}_6$ is estimated by $u^{-d/2-3/2}\widetilde{D}_{\alpha_1} e^{\widetilde{W}_1}$, therefore $\int_0^{1/2} |\widetilde{J}^\alpha_6|^2 u\, du \lesssim R_{\alpha_1,2,0}$. Finally, if $k=8$, then since $x_1 y_1 \leq (x_1+y_1)^2$, applying (H), we easily get $\int_0^{1/2} |\widetilde{J}^\alpha_8|^2 u\, du \lesssim R_{\alpha_1+1,0,4}$. In each of the considered cases an application of Lemma 4.3 gives the desired bound.

To prove (4.5) for $k=9$, we split the sum defining $\widetilde{J}_9$ and bound each of the resulting $d-1$ integrals seperately. The rest of the proof is actually only a repetition of the arguments used in the proof of Lemma 4.3, with appropriate changes (e.g killing in some places $u^2(\theta_j y_j)^2$ by the exponent $e^{-\frac{u}{2}|\theta+y|^2}$). We omit the details.

To prove (4.5) for $k = 4,5$ we need (among others) another technical lemma, which will be also useful in the smoothness estimates for the kernel F_j.

Lemma 4.4 *Let $a \geq 0$, $b \geq 0$. Then,*

$$X = X_{a,b} = X_{a,b}(x,y) = \int\limits_{(0,1/2)\cap\{Z_1>1\}} \frac{1}{(x_1)^b} u^{-d-a} e^{-\frac{C}{u}|x-y|^2}\, du \lesssim |x-y|^{-2d-2a+2-b}.$$

Moreover, the same inequality holds if we replace $X_{a,b}$ by $\widetilde{X}_{a,b}$.

Proof. We will only prove the first statement. The proof of the second is analogous, requiring additionaly only the fact that $|\theta-y| \approx |x-y|$. We split the integral defining X onto two integrals X^1, X^2, according to the cases $\{y_1/2 \leq x_1 \leq 2y_1\}$ and $\{x_1 < y_1/2\} \cup \{x_1 > 2y_1\}$, respectively. If $Z_1 > 1$ and $x_1 \approx y_1$, then $\frac{1}{(x_1)^b} = (\frac{y_1}{x_1}\frac{1}{x_1 y_1})^{b/2} \lesssim u^{-b/2}$. Hence, we get $X^1 \lesssim \int_0^1 u^{-d-a-b/2} e^{-\frac{C}{u}|x-y|^2}\, du$, which can be further estimated, via Lemma 2.3, by $|x-y|^{-2d-2a+2-b}$. If $Z_1 > 1$ and x_1, y_1 are not comparable, then $(x_1+y_1)^2 \approx (x_1-y_1)^2 \lesssim |x-y|^2$. Since $\frac{1}{x_1^2} \leq (x_1^2+y_1^2)(x_1 y_1)^{-2}$, it follows that $\frac{1}{(x_1)^b} \lesssim |x-y|^b (x_1 y_1)^{-b} \lesssim |x-y|^b u^{-b}$. Now applying Lemma 2.3 we obtain $X^2 \lesssim |x-y|^b |x-y|^{-2d-2a+2-2b} = |x-y|^{-2d-2a+2-b}$. ■

We will now justify (4.5) for $k=5$. We split the integral $\int_0^{1/2}|\widetilde{J}_5^\alpha|^2 u\,du$ onto two integrals, according to the cases $\widetilde{Z}_1>1$ and $\widetilde{Z}_1\le 1$. Using (D) and (I), the first of these integrals is easily estimated by $\widetilde{X}_{1,2}$, and the desired estimate follows by Lemma 4.4. As for the second integral, by (D) and (G) it is estimated by $\int_{(0,1/2)\cap\{\widetilde{Z}_1\le 1\}} u^{-d-3}y_1^2 e^{-\frac{1}{2u}\widetilde{U_1}}\,du$. Since $\frac{y_1^2}{u}e^{-\frac{1}{2u}\widetilde{U_1}}\lesssim e^{-\frac{C}{u}|\theta-y|^2}$, the second integral can be further estimated by $\int_0^1 u^{-d-2}e^{-\frac{C}{u}|x-y|^2}\,du$, which is less than a constant times $|x-y|^{-2d-2}$, because of Lemma 2.3.

It remains to show (4.5) for $k=4$. Again, we split the integral under examination onto two integrals q_1 and q_2, corresponding to the cases $\widetilde{Z}_1\le 1$ and $\widetilde{Z}_1>1$ respectively.

To estimate q_1 we write $\widetilde{D}_v=\Big[-2A\theta_1 I_\nu(\widetilde{Z}_1)+By_1 I_{\nu+1}(\widetilde{Z}_1)\Big]+\Big[\frac{\nu+1/2}{\theta_1}I_\nu(\widetilde{Z}_1)\Big]$. By the use of (B) and (G) we may estimate the expression in the first square bracket by $(\theta_1+y_1)(\theta_1y_1)^{-1/2}u^{-1/2}$, while in the second by $\frac{1}{\theta_1}(\theta_1y_1)^{-1/2}u^{1/2}$. Hence also,

$$(\theta_1y_1)^{3/2}\left|\widetilde{D}_{\alpha_1}-\widetilde{D}_{\alpha_1+1}\right|\lesssim(\theta_1+y_1)(\theta_1y_1)u^{-1/2}+y_1u^{1/2}. \tag{4.6}$$

Using the fact that for $u\in(0,1/2)$, the inequality $\widetilde{Z}_1\le 1$ implies

$$(\theta_1y_1)u^{-1/2}=(\theta_1y_1)^{1/2}\Big(\frac{\theta_1y_1}{u}\Big)^{1/2}\lesssim u^{1/2},$$

we can further estimate $(\theta_1y_1)^{3/2}\left|\widetilde{D}_{\alpha_1}-\widetilde{D}_{\alpha_1+1}\right|\lesssim(\theta_1+y_1)u^{1/2}$. Applying (F), we get

$$(\theta_1+y_1)^2\exp(-2\widetilde{W_1})\lesssim(\theta_1+y_1)^2e^{-\frac{1}{4u}\widetilde{U}_1}e^{-\frac{1}{4u}\widetilde{U}_1}\lesssim ue^{-\frac{1}{4u}|\theta-y|^2}.$$

Now, by the use of (H) and the fact that $|\theta-y|\approx|x-y|$, it follows that

$$q_1\lesssim\int_0^{1/2}(\theta_1+y_1)^2u^{-d-3}e^{-2\widetilde{W_1}}\,du\lesssim\int_0^{1/2}u^{-d-2}e^{-\frac{C}{u}|x-y|^2}\,du.$$

Applying Lemma 2.3 we obtain $q_1\lesssim|x-y|^{-2d-2}$.

To get the estimate $q_2\lesssim|x-y|^{-2d-2}$ (and thus to finish the proof of (4.5), hence also the proof of (4.3)), we use asymptotics provided by Lemma 2.2 (note that this is the most delicate part of the proof of (4.3)).

$$\widetilde{D}_{\alpha_1}-\widetilde{D}_{\alpha_1+1}=\sum_{i=1}^4\widetilde{V}_i^{\alpha_1}, \tag{4.7}$$

with

$$\begin{aligned}
V_1^{\alpha_1} &= M_1^{\alpha_1} - M_1^{\alpha_1+1} = (-x_1+y_1)\frac{1}{2u}\big[I_{\alpha_1}(Z_1) - I_{\alpha_1+1}(Z_1)\big],\\
V_2^{\alpha_1} &= M_2^{\alpha_1} - M_2^{\alpha_1+1} = -(x_1+y_1)\frac{u}{2}\big[I_{\alpha_1}(Z_1) - I_{\alpha_1+1}(Z_1)\big],\\
V_3^{\alpha_1} &= By_1\big[2I_{\alpha_1+1}(Z_1) - I_{\alpha_1}(Z_1) - I_{\alpha_1+2}(Z_1)\big] - \frac{1}{x_1}I_{\alpha_1+1}(Z_1),\\
V_4^{\alpha_1} &= \frac{\alpha_1+1/2}{x_1}\big[I_{\alpha_1}(Z_1) - I_{\alpha_1+1}(Z_1)\big].
\end{aligned}$$

Thus, to bound q_2 it suffices to demonstrate that for $j=1,2,3,4$

$$Y_j = \int\limits_{(0,1/2)\cap \widetilde{Z}_1>1} \left(u^{-d-2}\,\theta_1 y_1 \widetilde{V}_j^{\alpha_1}\,\sqrt{\theta y}\prod_{i=2}^{d} I_{\alpha_i}(\widetilde{Z}_i)e^{-A(|\theta|^2+|y|^2)}\right)^2 u\,du \lesssim |x-y|^{-2d-2}.$$

Let $j=1$. Using (I) we get $(\theta_1 y_1)^{3/2}\big|\widetilde{V}_1^{\alpha_1}\big| \lesssim |\theta_1 - y_1| u^{1/2} e^{\widetilde{Z}_1}$. Then, applying (D), (H) and the fact that $|\theta_1 - y_1| \lesssim |\theta - y| \approx |x-y|$ leads to $Y_1 \lesssim |x-y|^2 \int_0^{1/2} u^{-d-3} e^{-\frac{C}{u}|x-y|^2}\,du$. An application of Lemma 2.3 gives the desired bound. To bound Y_2 once again we apply (I) to get $(\theta_1 y_1)^{3/2}\big|\widetilde{V}_2^{\alpha_1}\big| \lesssim (\theta_1+y_1)u^{5/2}e^{\widetilde{Z}_1}$. Then, another application of (D) and (H) produces $Y_2 \lesssim \int_0^{1/2} u^{-d+1}(\theta_1+y_1)^2 e^{-\frac{u}{2}|\theta+y|^2} e^{-\frac{1}{2u}|\theta-y|^2}\,du$. Since $|\theta - y| \approx |x-y|$ and $(\theta_1+y_1)^2 u\, e^{-\frac{u}{2}|\theta+y|^2} \lesssim 1$ we can further write

$$Y_2 \lesssim \int\limits_0^1 u^{-d} e^{-\frac{C}{u}|\theta-y|^2}\,du \lesssim \int\limits_0^1 u^{-d-2} e^{-\frac{C}{u}|\theta-y|^2}\,du.$$

The latter integral is less than or equal to a constant times $|x-y|^{-2d-2}$, because of Lemma 2.3. We will now bound Y_4. Using (I) we see that $(\theta_1 y_1)^{3/2}\big|\widetilde{V}_4^{\alpha_1}\big| \lesssim \frac{1}{\theta_1}u^{3/2}e^{\widetilde{Z}_1}$. Now, with the aid of (D) and (H) it follows that $Y_4 \lesssim \widetilde{X}_{1,2}$, where $\widetilde{X}_{1,2}$ is the quantity from Lemma 4.4. Applying that Lemma we get $Y_4 \lesssim |x-y|^{-2d-2}$. It remains to bound Y_3. The key observation is that $(\theta_1 y_1)^{3/2}\widetilde{V}_3^{\alpha_1} = y_1(\theta_1 y_1)^{1/2} \times \theta_1 \widetilde{V}_3^{\alpha_1}$ and

$$\theta_1 \widetilde{V}_3^{\alpha_1} = \widetilde{Z}_1\big[2I_{\alpha_1+1}(\widetilde{Z}_1) - I_{\alpha_1}(\widetilde{Z}_1) - I_{\alpha_1+2}(\widetilde{Z}_1)\big] - I_{\alpha_1+1}(\widetilde{Z}_1). \tag{4.8}$$

From the accurate asymptotics of Lemma 2.2, we see that, for $u \in (0,1/2)$ and $\widetilde{Z}_1 > 1$, $\theta_1\widetilde{V}_3^{\alpha_1}$ may be estimated by $(\theta_1 y_1)^{-3/2}u^{3/2}e^{\widetilde{Z}_1}$. Consequently, $(\theta_1 y_1)^{3/2}\big|\widetilde{V}_3^{\alpha_1}\big| \lesssim \frac{1}{\theta_1}u^{3/2}e^{\widetilde{Z}_1}$. We have already performed the rest of needed estimates while showing the bound for Y_4.

The proof of (4.5) and thus (4.3) is finished. ■

4.2 Estimates for the kernels $F_j(x,y)$

Since (by symmetry reasons) the proofs of (2.9), (2.10) and (2.11), for $\Gamma = F_j$, are completely analogous no matter $j = 1,\dots,d$ is, we will only consider F_1. We start with proving the growth estimate for the kernel $F_1(x,y)$, assuming $\alpha \in [-1/2,\infty)^d$. That is, we show that $\|F_1(x,y)\|_{L^2(dt)} \lesssim |x-y|^{-d}$. With the aid of (2.7) we compute that

$$\delta_{j,x}\,\mathcal{G}_t^\alpha(x,y) = (\sinh 2t)^{-d-1}\Big[y_1 I_{\alpha_1+1}\big(\frac{x_1y_1}{\sinh 2t}\big) - x_1 e^{-2t} I_{\alpha_1}\big(\frac{x_1y_1}{\sinh 2t}\big)\Big]\prod_{i=2}^d I_{\alpha_i}\big(\frac{x_iy_i}{\sinh 2t}\big)\sqrt{xy}$$
$$\times \exp\Big(-\frac{1}{2}\coth 2t\,\big(|x|^2+|y|^2\big)\Big).$$

Therefore, by the change of variable (4.1),

$$\|F_1(x,y)\|^2_{L^2(dt)} = 2^{-d-1}\int_0^1 \Big(u^{-d-1}(1-u^2)^{d+1/2}\Big[y_1 I_{\alpha_1+1}(Z_1) - x_1\frac{1-u}{1+u}I_{\alpha_1}(Z_1)\Big]$$
$$\times \prod_{i=2}^d I_{\alpha_i}(Z_i)\sqrt{xy}e^{-A(|x|^2+|y|^2)}\Big)^2\,du.$$

Since

$$y_1 I_{\alpha_1+1}(Z_1) - x_1\frac{1-u}{1+u}I_{\alpha_1}(Z_1) = (y_1-x_1)I_{\alpha_1+1}(Z_1) + x_1\big(I_{\alpha_1+1}(Z_1)-I_{\alpha_1}(Z_1)\big) + x_1\frac{2u}{1+u}I_{\alpha_1}(Z_1), \tag{4.9}$$

in order to prove (2.9) for $F_1(x,y)$, it is sufficient to show that

$$\int_0^1 |N_i|^2\,du \lesssim |x-y|^{-2d}, \qquad i=1,2,3, \tag{4.10}$$

where,

$$N_1 = (y_1-x_1)u^{-d-1}(1-u)^{d+1/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}I_{\alpha+e_1}(Z),$$
$$N_2 = u^{-d-1}(1-u)^{d+1/2}x_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}\big[I_{\alpha_1+1}(Z_1) - I_{\alpha_1}(Z_1)\big]\prod_{i=2}^d I_{\alpha_i}(Z_i),$$
$$N_3 = u^{-d}(1-u)^{d+1/2}x_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}I_\alpha(Z).$$

Proof of (4.10)**.** If $i=1$, a routine application of (D) and (H) produces $|N_1| \lesssim |x_1-y_1|u^{-d/2-1}(1-u)^{d/2+1/2}e^{-\frac{1}{4u}|x-y|^2}$. Thus

$$\int_0^1 |N_1|^2\,du \lesssim |x-y|^2\int_0^1 u^{-d-2}e^{-\frac{1}{2u}|x-y|^2}\,du \lesssim |x-y|^{-2d}.$$

If $i=2$, we use the splitting $\int_0^1 |N_2|^2\,du = \int_0^{1/2} |N_2|^2\,du + \int_{1/2}^1 |N_2|^2\,du$. Since, by Lemma 2.1, $|I_{\alpha_1}(Z_1) - I_{\alpha_1+1}(Z_1)| \lesssim I_{\alpha_1}(Z_1)$, applying (E) and (H), for $u \in (0,1/2)$, we get $|N_2| \lesssim x_1 e^{-\frac{1}{8}|x+y|^2} e^{-\frac{1}{4}|x-y|^2}$. It follows that $\int_{1/2}^1 |N_2|^2\,du \lesssim e^{-\frac{1}{2}|x-y|^2} \lesssim |x-y|^{-2d}$. To treat $\int_0^{1/2} |N_2|^2\,du$ we further split the integral in question onto two integrals, according to the cases $Z_1 \le 1$, $Z_1 > 1$. Applying the inequality $|I_{\alpha_1}(Z_1) - I_{\alpha_1+1}(Z_1)| \lesssim I_{\alpha_1}(Z_1)$, (H), and the fact that $x_1^2 e^{-\frac{1}{2u}U_1} \lesssim u\,e^{-\frac{1}{4u}U_1} \lesssim u\,e^{-\frac{1}{4u}|x-y|^2}$, we see that the first of these integrals is estimated by $\int_0^{1/2} u^{-d-1} e^{-\frac{1}{4u}|x-y|^2}\,du \lesssim |x-y|^{-2d}$ (the last inequality comes from Lemma 2.3). Next, using (H) and (I) we see that the second integral is estimated by $\frac{1}{y_1^2}\int_{(0,1/2)\cap\{Z_1>1\}} u^{-d} e^{-\frac{1}{2u}|x-y|^2}\,du$. The desired bound for the integral $\int_{(0,1/2)\cap\{Z_1>1\}} |N_2|^2\,du$ now follows from Lemma 4.4. If $i=3$, then using (D) and (H) we easily arrive at

$$|N_3|^2 \lesssim u^{-d} x_1^2 e^{-\frac{u}{2}|x+y|^2} e^{-\frac{1}{2u}|x-y|^2} \lesssim u^{-d-1} e^{-\frac{1}{2u}|x-y|^2},$$

and an application of Lemma 2.3 gives the desired estimate $\int_0^1 |N_3|^2\,du \lesssim |x-y|^{-2d}$. ■

We pass to the proofs of the smoothness conditions (2.10), (2.11), for the kernel $F_1(x,y)$. This time, since $\delta_{1,x}\,\mathcal{G}_t^\alpha(x,y)$ is not symmetric we need to show both of them. Using once again the change of variable (4.1), we see that it is sufficient to show that

$$\int_0^1 |F_u(x,y) - F_u(x',y)|^2\,du \lesssim |x-x'|^2 |x-y|^{-2d-2}, \qquad 2|x-x'| \le |x-y|, \tag{4.11}$$

and

$$\int_0^1 |F_u(x,y) - F_u(x,y')|^2\,du \lesssim |y-y'|^2 |x-y|^{-2d-2}, \quad 2|y-y'| \le |x-y|, \tag{4.12}$$

where $F_u(x,y) = (1-u^2)^{-1/2} (\delta_{1,x}\mathcal{G}_t^\alpha(x,y))\big|_{t=\log\frac{1+u}{1-u}}$.

Proof of (4.11)**.** As in the proof of the smoothness estimate for the kernel $K(x,y)$, we apply the mean value theorem, to get $F_u(x,y) - F_u(x',y) = \left\langle \nabla_x F_u(x,y)\big|_{x=\beta(u)}, x-x' \right\rangle$, where, for any fixed u, $\beta(u)$ lies in the interval $[x,x']$. Since x_1 is the only distinguished place in the x variable in $F_u(x,y)$ the proof of (4.11) reduces to showing that

$$\int_0^1 \left(\partial_{x_1} F_u(x,y)\big|_{x=\beta(u)}\right)^2 du \lesssim |x-y|^{-2d-2}, \quad \beta(u) \in [x,x'], \quad 2|x-x'| \le |x-y|, \tag{4.13}$$

$$\int_0^1 \left(\partial_{x_2} F_u(x,y)\big|_{x=\beta(u)}\right)^2 du \lesssim |x-y|^{-2d-2}, \quad \beta(u) \in [x,x'], \quad 2|x-x'| \le |x-y|. \tag{4.14}$$

From now on, till the end of the proof of (4.11), remarks similar to those made in the beginning of the proof of (4.3) (with θ replaced by β) will be in force.

`Proof of` (4.13). Since $\partial_{x_1}\Big(I_\nu(Z_1)\sqrt{x_1y_1}e^{-A(|x|^2+|y|^2)}\Big) = D_\nu\sqrt{x_1y_1}e^{-A(|x|^2+|y|^2)}$, we compute that

$$\partial_{x_1}\bigg(\Big[y_1I_{\alpha_1+1}(Z_1)-x_1\frac{1-u}{1+u}I_{\alpha_1}(Z_1)\Big]\sqrt{x_1y_1}e^{-A(|x|^2+|y|^2)}\bigg)$$
$$=\bigg((y_1-x_1)D_{\alpha_1+1}-\frac{1-u}{1+u}I_{\alpha_1}(Z_1)+x_1(D_{\alpha_1+1}-D_{\alpha_1})+2x_1\frac{u}{1+u}D_{\alpha_1}\bigg)\sqrt{x_1y_1}e^{-A(|x|^2+|y|^2)}.$$

Therefore, in order to prove (4.13), it is sufficient to show that

$$\int_0^1|\widetilde{H}_i^\alpha|^2\,du\lesssim|x-y|^{-2d},\quad i=1,2,3,4, \tag{4.15}$$

where, $H_i^\alpha=H_i^\alpha(x,y,u)$, and

$$H_1^\alpha=(y_1-x_1)u^{-d-1}(1-u)^{d+1/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1+1}\prod_{i=2}^dI_{\alpha_i}(Z_i),$$
$$H_2^\alpha=u^{-d-1}(1-u)^{d+1/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}I_\alpha(Z),$$
$$H_3^\alpha=u^{-d-1}(1-u)^{d+1/2}x_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}(D_{\alpha_1+1}-D_{\alpha_1})\prod_{i=2}^dI_{\alpha_i}(Z_i),$$
$$H_4^\alpha=u^{-d}(1-u)^{d+1/2}x_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1}\prod_{i=2}^dI_{\alpha_i}(Z_i).$$

We start with showing (4.15) for $i=2$. Using (D) and (H) we easily get $|H_2^\alpha|\lesssim u^{-d/2-1}(1-u)^{d/2+1/2}e^{-\frac{1}{4u}|x-y|^2}$. Since $|\beta-y|\approx|x-y|$, it follows that $\int_0^1(\widetilde{H}_2^\alpha)^2\,du\lesssim\int_0^1u^{-d-2}e^{-\frac{C}{u}|x-y|^2}\,du\lesssim|x-y|^{-2d-2}$.

The inequalities $\int_{1/2}^1|\widetilde{H}_i^\alpha|^2\,du\lesssim|x-y|^{-2d-2}$, $i=1,3,4$, follow easily from Lemma 4.2 and (H). In fact we may get an even better bound $\int_{1/2}^1|\widetilde{H}_i^\alpha|^2\,du\lesssim e^{-C|x-y|^2}$, $i=1,3,4$.

To prove that $\int_0^{1/2}|\widetilde{H}_i^\alpha|^2\,du\lesssim|x-y|^{-2d-2}$, $i=1,4$, we shall use Lemma 4.3. Using (H) we see that $|H_1^\alpha|^2\lesssim|x-y|^2u^{-d-3}(D_{\alpha_1+1}\sqrt{x_1y_1}e^{W_1})^2$. Since $|x-y|\approx|\beta-y|$, applying Lemma 4.3 produces $\int_0^{1/2}|\widetilde{H}_1^\alpha|^2\,du\lesssim|x-y|^2R_{\alpha_1,3,0}\lesssim|x-y|^{-2d-2}$. Similar arguments also lead to $\int_0^{1/2}|\widetilde{H}_4^\alpha|^2\,du\lesssim R_{\alpha_1,1,2}\lesssim|x-y|^{-2d-2}$.

To finish the proof of (4.15) it remains to show that $\int_0^{1/2}|\widetilde{H}_3^\alpha|^2\,du\lesssim|x-y|^{-2d-2}$. Once again we split the integral in question onto two integrals r_1, r_2 according to the cases $\widetilde{Z}_1\le1$ and $\widetilde{Z}_1>1$, respectively. To estimate r_1, modifying (4.6) slightly, we see that

$\beta_1(\beta_1 y_1)^{1/2}|\widetilde{D}_{\alpha_1} - \widetilde{D}_{\alpha_1+1}| \lesssim \beta_1(y_1+\beta_1)u^{-1/2} + u^{1/2}$. From these, by the use of (F) and (H), it follows that for $\widetilde{Z}_1 \le 1$, $|\widetilde{H}_3^{\alpha}| \lesssim (\beta_1(y_1+\beta_1)u^{-1/2} + u^{1/2})u^{-d/2-3/2}e^{-\frac{1}{4u}\widetilde{U}_1}$. Next, since $|x-y|^2 \lesssim U_1$, $\beta_1(y_1+\beta_1)u^{-1}e^{-\frac{1}{8u}\widetilde{U}_1} \lesssim 1$ and $|\beta - y| \approx |x-y|$, we can further estimate $|\widetilde{H}_3^{\alpha}| \lesssim u^{-d/2-1}e^{-\frac{1}{8u}\widetilde{U}_1} \lesssim u^{-d/2-1}e^{-\frac{C}{u}|x-y|^2}$. Hence, applying Lemma 2.3, we get $r_1 \lesssim \int_0^{1/2} u^{-d-2}e^{-\frac{C}{u}|x-y|^2}\,du \lesssim |x-y|^{-2d-2}$. It remains to bound r_2 (this the most subtle part of the whole proof of (4.15)). Splitting $\widetilde{D}_{\alpha_1} - \widetilde{D}_{\alpha_1+1}$ as in (4.7), we see that it is sufficient to demonstrate that

$$s_j = \int\limits_{(0,1/2)\cap\{\widetilde{Z}_1>1\}} \left(u^{-d-1}\,\beta_1\widetilde{V}_j^{\alpha_1}\,\sqrt{\beta y}\prod_{i=2}^{d} I_{\alpha_i}(\widetilde{Z}_i)e^{-A(|\beta|^2+|y|^2)}\right)^2 du,$$

satisfies $s_j \lesssim |x-y|^{-2d-2}$, $\quad j = 1,2,3,4$. We proceed analogically to the proof of the bound for q_2 in the $k=4$ case of (4.5). First we show the bound for s_1. Using (I) we get $\beta_1(\beta_1 y_1)^{1/2}\big|\widetilde{V}_1^{\alpha_1}\big| \lesssim |\beta_1 - y_1|\frac{1}{y_1}u^{1/2}e^{\widetilde{Z}_1}$. Applying (D), (H) and the fact that $|\beta_1 - y_1| \lesssim |\beta - y| \approx |x-y|$, we arrive at $s_1 \lesssim |x-y|^2\frac{1}{y_1^2}\int_{(0,1/2)\cap\{\widetilde{Z}_1>1\}} u^{-d-2}e^{-\frac{C}{u}|\beta-y|^2}\,du$. From this, proceeding as in the proof of Lemma 4.4, we get the desired estimate. To bound s_2 it is enough to apply (H), getting, $\beta_1(\beta_1 y_1)^{1/2}\big|\widetilde{V}_2^{\alpha_1}\big| \lesssim (\beta_1+y_1)^2u^{3/2}e^{\widetilde{Z}_1}$. Then, another application of (D) and (H) produces $s_2 \lesssim \int_0^{1/2} u^{-d}(\beta_1+y_1)^4e^{-\frac{u}{2}|\beta+y|^2}e^{-\frac{1}{2u}|\beta-y|^2}\,du$. Since $|\beta-y| \approx |x-y|$ and $(\beta_1+y_1)^4u^2e^{-\frac{u}{2}|\beta+y|^2} \lesssim 1$, we can further write $s_2 \lesssim \int_0^1 u^{-d-2}e^{-\frac{C}{u}|x-y|^2}\,du$. The latter integral is less than or equal to a constant times $|x-y|^{-2d-2}$, because of Lemma 2.3. We will now bound s_4. Using (I) we see that $\beta_1(\beta_1 y_1)^{1/2}\big|\widetilde{V}_4^{\alpha_1}\big| \lesssim \frac{u}{\beta_1 y_1}u^{1/2}e^{\widetilde{Z}_1}$. Now, since if $\widetilde{Z}_1 > 1$, then $\frac{u}{\beta_1 y_1} < 1$, with the aid of (D), (H) and the fact that $|\beta - y| \approx |x-y|$, we obtain $s_4 \lesssim \int_{(0,1/2)\cap\{\widetilde{Z}_1>1\}} u^{-d-2}e^{-\frac{C}{u}|x-y|^2}\,du$. The desired bound follows by invoking Lemma 2.3. We are left with bounding s_3. Similarly to the estimation of Y_3, the key observation is (4.8). From the Lemma 2.2, we see that, for $u \in (0,1/2)$ and $\widetilde{Z}_1 > 1$, $\beta_1\widetilde{V}_3^{\alpha_1}$ may be estimated by $(\beta_1 y_1)^{-3/2}u^{3/2}e^{\widetilde{Z}_1}$. Consequently, $\beta_1(\beta_1 y_1)^{1/2}\big|\widetilde{V}_3^{\alpha_1}\big| \lesssim \frac{u}{\beta_1 y_1}u^{1/2}e^{\widetilde{Z}_1}$. We have already performed the rest of needed estimates while showing the bound for s_4. The proof of (4.15) and thus also of (4.13) is finished.

`Proof of` (4.14). With the aid of the identity

$$\partial_{x_2}\left(I_\nu(Z_2)\sqrt{x_2 y_2}e^{-A(|x|^2+|y|^2)}\right) = D_\nu(x_2,y_2,u)\sqrt{x_2 y_2}e^{-A(|x|^2+|y|^2)},$$

we easily compute that

$$\partial_{x_2}F_u(x,y) = \big[y_1 I_{\alpha_1+1}(Z_1) - x_1\frac{1-u}{1+u}I_{\alpha_1}(Z_1)\big]D_{\alpha_2}^{two}\prod_{i=3}^{d} I_{\alpha_i}(Z_i)\sqrt{xy}e^{-A(|x|^2+|y|^2)},$$

where $D^{two}_{\alpha_2} = D_{\alpha_2}(x_2, y_2, u)$ (the latter expression being defined in the beginning of Section 4). Now, from the decomposition (4.9) we see that in order to prove (4.14) it suffices to show that

$$\int_0^1 |\widetilde{T}_i^{\alpha}|^2\, du \lesssim |x-y|^{-2d-2}, \qquad i = 1, 2, 3, \tag{4.16}$$

where, $T_i^{\alpha} = T_i^{\alpha}(x, y, u)$, and

$$T_1^{\alpha} = (y_1 - x_1) u^{-d-1} (1-u)^{d+1/2} \sqrt{xy}\, e^{-A(|x|^2+|y|^2)} I_{\alpha_1+1}(Z_1) D^{two}_{\alpha_2} \prod_{i=3}^{d} I_{\alpha_i}(Z_i),$$

$$T_2^{\alpha} = u^{-d-1} (1-u)^{d+1/2} x_1 \sqrt{xy}\, e^{-A(|x|^2+|y|^2)} (I_{\alpha_1+1}(Z_1) - I_{\alpha_1}(Z_1)) D^{two}_{\alpha_2} \prod_{i=3}^{d} I_{\alpha_i}(Z_i),$$

$$T_3^{\alpha} = u^{-d} (1-u)^{d+1/2} x_1 \sqrt{xy}\, e^{-A(|x|^2+|y|^2)} D^{two}_{\alpha_2} \prod_{i\neq 2} I_{\alpha_i}(Z_i).$$

The proof of (4.16) for $i = 1$ is essentially a repetition of the proof of (4.15) for $k = 1$, with the roles of the variables (x_1, y_1) and (x_2, y_2) switched in some places. The bounds $\int_{1/2}^1 |\widetilde{T}_i^{\alpha}|^2\, du \lesssim |x-y|^{-2d-2}$, $i = 2, 3$, can be deduced from the (x_2, y_2) variable version of Lemma 4.2 (with $\widetilde{D}_{\nu}$, $\theta_1 y_1$ and $\widetilde{W}_1$ replaced by $\widetilde{D}^{two}_{\nu}$, $\beta_1 y_1$, and $\widetilde{W}_2$, correspondingly). The estimate $\int_0^{1/2} |\widetilde{T}_3^{\alpha}|^2\, du \lesssim |x-y|^{-2d-2}$ follows from an observation that $\beta_1^2 e^{-\frac{u}{2}|\beta+y|^2} \lesssim u^{-1}$ and the (omitted) proof of the following (x_2, y_2) variable privileged version of Lemma 4.3 (with $a = 2$).

Lemma 4.5 *Let $\nu = -1/2$ or $\nu \in [1/2, \infty)$, and $a \geq 0$. Then,*

$$R^{two} = R^{two}_{\nu,a} = \int_0^{1/2} u^{-d-a} \left(\widetilde{D}^{two}_{\nu} \sqrt{\beta_2 y_2}\, e^{\widetilde{W}_2} \right)^2 du \lesssim |x-y|^{-2d-2a+2}.$$

In order to estimate the remaining integral $\int_0^{1/2} (\widetilde{T}_2^{\alpha})^2\, du$ (and thus finish the proof of (4.16)), we split the integral in question onto two integrals p_1, p_2, corresponding to the cases $\widetilde{Z}_1 \leq 1, \widetilde{Z}_1 > 1$, respectively. Since $\beta_1 e^{-A(\beta_1^2+y_1^2)} \lesssim u^{1/2} e^{-\frac{C}{u}|\beta_1 - y_1|^2}$, from (H), we get $|\beta_1 (I_{\alpha_1+1}(\widetilde{Z}_1) - I_{\alpha_1}(\widetilde{Z}_1))| \lesssim |\beta_1 I_{\alpha_1}(\widetilde{Z}_1)| \lesssim \beta_1 (\beta_1 y_1)^{-1/2} u^{1/2}$, and consequently

$$|\widetilde{T}_2| \lesssim \beta_1 u^{-d/2-3/2} e^{-\frac{1}{4u}\sum_{i=3}^{d} |\beta_i - y_i|^2} e^{-A(\beta_1^2+y_1^2)} \lesssim u^{-d/2-1} e^{-\frac{C}{u}\sum_{i\neq 2} |\beta_i - y_i|^2}.$$

From the above estimate, slightly modifying the proof of Lemma 4.5, with $\nu = \alpha_2$, $a = 2$, (by changing the expressions in the exponents from $-\frac{1}{4u}$ to $-\frac{C}{u}$ in some places) we easily derive the desired estimate for p_1. To bound p_2 we split it further onto two integrals p_2^{loc}, p_2^{glob} according to the cases $\frac{1}{2}\beta_1 < y_1 < 2\beta_1$ and $\{\beta_1 \le \frac{1}{2}y_1\} \cup \{\beta_1 \ge 2y_1\}$, respectively. Using (I) we see that

$$\beta_1(\beta_1 y_1)^{1/2} |I_{\alpha_1+1}(\widetilde{Z}_1) - I_{\alpha_1}(\widetilde{Z}_1)| \lesssim \frac{u^{3/2}}{y_1}. \tag{4.17}$$

Moreover, if $\widetilde{Z}_1 > 1$, then $\frac{u}{y_1} = \frac{u}{\beta_1 y_1}\theta_1 \lesssim (\frac{u}{\beta_1 y_1})^{1/2}\beta_1 \lesssim u^{1/2}(\frac{\beta_1}{y_1})^{1/2}$. Hence, with the aid of (H) we arrive at $p_2^{loc} \lesssim R^{two}_{\alpha_2,2}$. An application of Lemma 4.5 produces $p_2^{loc} \lesssim |x-y|^{-2d-2}$. Let us now assume that β_1 is not comparable with y_1. Then $(\beta_1 - y_1)^2 \approx (\beta_1^2 + y_1^2)$, and consequently, if $\widetilde{Z}_1 > 1$, then $\frac{\beta_1}{y_1} \lesssim \frac{\beta_1^2+y_1^2}{\beta_1 y_1} \lesssim (\beta_1 - y_1)^2 \frac{1}{\beta_1 y_1} \lesssim u^{-1}|\beta - y|^2$. Since $|\beta - y| \approx |x-y|$, it follows that $\frac{u}{y_1} \lesssim |x-y|$. Hence, using (4.17) and (H), we arrive at $p_2^{glob} \lesssim |x-y|^2 R^{two}_{\alpha_2,3}$. The desired bound follows from Lemma 4.5. The proof of (4.16) and thus also of (4.11) is finished. ■

Proof of (4.12)**.** Following the scheme of the proof of (4.11), we use the mean value theorem to get $F_u(x,y) - F_u(x,y') = \left\langle \nabla_y F_u(x,y)\big|_{y=\gamma(u)}, y - y' \right\rangle$, where, for any fixed u, $\gamma(u)$ lies in the interval $[y,y']$. As previously, since y_1 is the only distinguished place in the y variable in $F_u(x,y)$ the proof of (4.12) reduces to showing that

$$\int_0^1 \left(\partial_{y_1} F_u(x,y)\big|_{y=\gamma(u)}\right)^2 du \lesssim |x-y|^{-2d-2}, \quad \gamma(u) \in [y,y'], \quad 2|y-y'| \le |x-y|, \tag{4.18}$$

$$\int_0^1 \left(\partial_{y_2} F_u(x,y)\big|_{y=\gamma(u)}\right)^2 du \lesssim |x-y|^{-2d-2}, \quad \gamma(u) \in [y,y'], \quad 2|y-y'| \le |x-y|. \tag{4.19}$$

From now on, to the end of the proof of (4.12) a tilde above an expression will mean, that it depends on y through $\gamma(u)$. In the estimates that follow we will use the fact that if $2|y-y'| \le |x-y|$ and $\gamma(u) \in [y,y']$, then $|x-y| \approx |\gamma(u) - y|$, uniformly in $u \in [0,1]$. We will also write γ instead of $\gamma(u)$ for short.

`Proof of` (4.19). The proof is completely analogous to the proof of (4.14) (we need only to change places of the x_1 and y_1 variable), therefore we omit it.

`Proof of` (4.18). Denote $D_\nu^r = D_\nu^r(x_1,y_1,u) = D_\nu(y_1,x_1,u)$. Then obviously,

$$\partial_{y_1}\left(I_\nu(Z_1)e^{-A(|x|^2+|y|^2)}\sqrt{x_1 y_1}\right) = D_\nu^r e^{-A(|x|^2+|y|^2)}\sqrt{x_1 y_1}.$$

Consequently,

$$\begin{aligned}
&\partial_{y_1}\big(\big[y_1 I_{\alpha_1+1}(Z_1) - x_1\frac{1-u}{1+u}I_{\alpha_1}(Z_1)\big]\sqrt{x_1y_1}e^{-A(|x|^2+|y|^2)}\big)\\
&= \big(I_{\alpha_1+1}(Z_1) + y_1 D^r_{\alpha_1+1} - x_1\frac{1-u}{1+u}D^r_{\alpha_1}\big)\sqrt{x_1y_1}e^{-A(|x|^2+|y|^2)}\\
&= \big((y_1-x_1)D^r_{\alpha_1} + I_{\alpha_1+1}(Z_1) + y_1(D^r_{\alpha_1+1} - D^r_{\alpha_1}) + 2x_1\frac{u}{1+u}D^r_{\alpha_1}\big)\sqrt{x_1y_1}e^{-A(|x|^2+|y|^2)}.
\end{aligned}$$

From the above decomposition we see, that in order to prove (4.18) it suffices to show that

$$\int_0^1 |\widetilde{Q}_i^\alpha|^2\,du \lesssim |x-y|^{-2d-2}, \qquad i=1,2,3,4, \tag{4.20}$$

where, $Q_i^\alpha = Q_i^\alpha(x,y,u)$, and

$$\begin{aligned}
Q_1^\alpha &= (y_1-x_1)u^{-d-1}(1-u)^{d+1/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}D^r_{\alpha_1}\prod_{i=2}^d I_{\alpha_i}(Z_i),\\
Q_2^\alpha &= u^{-d-1}(1-u)^{d+1/2}\sqrt{xy}e^{-A(|x|^2+|y|^2)}I_{\alpha+e_1}(Z),\\
Q_3^\alpha &= u^{-d-1}(1-u)^{d+1/2}y_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}(D^r_{\alpha_1+1} - D^r_{\alpha_1})\prod_{i=2}^d I_{\alpha_i}(Z_i),\\
Q_4^\alpha &= u^{-d}(1-u)^{d+1/2}x_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}D^r_{\alpha_1}\prod_{i=2}^d I_{\alpha_i}(Z_i).
\end{aligned}$$

Symmetry reasons easily reduce showing (4.20) for $i=1,2,3$, to (4.15) for $i=1,2,3$. Namely, the desired estimates can be easily obtained once we observe that $Q_1^\alpha(x,y,u) = -H_1^{\alpha-e_1}(y,x,u)$, $Q_2^\alpha(x,y,u) = H_2^{\alpha+e_1}(y,x,u)$ and $Q_3^\alpha(x,y,u) = H_3^\alpha(y,x,u)$.

It remains to show (4.20) for $k=4$. The bound $\int_{1/2}^1 |\widetilde{Q}_4^\alpha|^2\,du \lesssim |x-y|^{-2d-2}$ follows by modifying slightly Lemma 4.2, replacing $\widetilde{D}_\nu$ with $\widetilde{D}^r_\nu$, x with γ, and θ with y (the proof of such a version of the lemma is analogous). The bound $\int_0^{1/2} |\widetilde{Q}_4^\alpha|^2\,du \lesssim |x-y|^{-2d-2}$ is a consequence of

$$\int_0^{1/2} (\widetilde{Q}_4^\alpha)^2\,du \lesssim \int_0^{1/2} (x_1)^2 u^{-d-1}\big(\widetilde{D}^r_{\alpha_1}\sqrt{x_1\gamma_1}e^{\widetilde{W}_1}\big)^2\,du,$$

and a modified version of Lemma 4.3 (with $\widetilde{D}_\nu$ replaced by $\widetilde{D}^r_\nu$, x by γ, and θ by y, $a=1$, $b=2$, $\nu=\alpha_1$). The proof of (4.12) and thus also of the smoothness condition for the kernel $F_1(x,y)$ is finished. ■

4.3 Estimates for the kernels $L(x,y)$ and $L_j(x,y)$.

In this section we shall achieve the desired estimates for the kernels of both g-functions based on the Poisson semigroup. To do this we shall use their definitions (obtained by the subordination principle) and the already proved bounds for $K(x,y)$ and $F_j(x,y)$.

Proof of Proposition 2.4, the case of $L(x,y)$. We will handle the growth estimate first. Using Minkowski's integral inequality and the change of variable $s = t^2/4u$ gives

$$\begin{aligned}\|L(x,y)\|_{L^2(t\,dt)} &= \Big(\int_0^\infty t \Big(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \partial_s \mathcal{G}_s^\alpha(x,y) \Big|_{s=t^2/4u} \frac{t}{2u}\, du \Big)^2 dt \Big)^{1/2} \\ &\le \int_0^\infty \Big(\int_0^\infty t \frac{e^{-2u}}{u} \Big(\frac{t}{2u} \partial_s \mathcal{G}_s^\alpha(x,y) \Big|_{s=t^2/4u} \Big)^2 dt \Big)^{1/2} du \\ &= \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \Big(\int_0^\infty 2s \big(\partial_s \mathcal{G}_s^\alpha(x,y) \big)^2 ds \Big)^{1/2} du.\end{aligned}$$

Combining the above with the growth estimate from Proposition 2.4 (the case of $K(x,y)$) we get the growth estimate for the kernel $L(x,y)$.

It remains to prove the smoothness estimate. Symmetry reasons reduce our task to showing that

$$\|L(x,y) - L(x',y)\|_{L^2(t\,dt)} \lesssim |x-x'| |x-y|^{-d-1}, \quad 2|x-x'| \le |x-y|.$$

Using Minkowski's integral inequality and then changing the variable $s = t^2/4u$ we get

$$\begin{aligned}&\|L(x,y) - L(x',y)\|_{L^2(t\,dt)} \\ &= \Big(\int_0^\infty t \Big(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \big(\partial_s \mathcal{G}_s^\alpha(x,y) - \partial_s \mathcal{G}_s^\alpha(x',y) \big) \Big|_{s=t^2/4u} \frac{t}{2u}\, du \Big)^2 dt \Big)^{1/2} \\ &\le \int_0^\infty \Big(\int_0^\infty t \frac{e^{-2u}}{u} \Big(\frac{t}{2u} \big(\partial_s \mathcal{G}_s^\alpha(x,y) - \partial_s \mathcal{G}_s^\alpha(x',y) \big) \Big|_{s=t^2/4u} \Big)^2 dt \Big)^{1/2} du \\ &= \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \Big(\int_0^\infty 2s \big(\partial_s \mathcal{G}_s^\alpha(x,y) - \partial_s \mathcal{G}_s^\alpha(x',y) \big)^2 ds \Big)^{1/2} du.\end{aligned}$$

Now the smoothness estimate from Proposition 2.4 (the case of $K(x,y)$) comes into play, finishing the proof of the case of $L(x,y)$ in Proposition 2.4.

Proof of Proposition 2.4, the case of $L_j(x,y)$. As usually, we start with showing the growth condition. We shall be brief, since the proof is similar to the proof of the $L(x,y)$ case. Minkowski's integral inequality and then the change of variable $s=t^2/4u$ lead to

$$\|L_j(x,y)\|_{L^2(t\,dt)} \le \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \Big(\int_0^\infty 2u\big(\delta_{j,x}\,\mathcal{G}_s^\alpha(x,y)\big)^2\,ds\Big)^{1/2}\,du,$$

which by referring to Proposition 2.4 (the case of $F_j(x,y)$) gives the desired estimate.

To prove the smoothness estimates it is enough to check that

$$\|L_j(x,y)\|_{L^2(t\,dt)} \lesssim |x-x'||x-y|^{-d-1}, \quad 2|x-x'| \le |x-y|;$$

parallel arguments show that the other smoothness estimate is true. Using Minkowski's integral inequality and then changing the variable as before we arrive at

$$\|L_j(x,y)-L_j(x',y)\|_{L^2(t\,dt)} \le \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \Big(\int_0^\infty 2u\big(\delta_{j,x}\,\mathcal{G}_s^\alpha(x,y)-\delta_{j,x}\,\mathcal{G}_s^\alpha(x',y)\big)^2\,ds\Big)^{1/2}\,du.$$

The smoothness bound in the case of $L_j(x,y)$ follows now from the already proved case of $K_j(x,y)$ in Proposition 2.4.

Remark. We can obtain the growth estimate for the kernel $L_j(x,y)$ in other way, by referring to [7], where the growth estimate for the Calderón-Zygmund kernel $R_j^\alpha(x,y)$ (see [7, p. 406] for definition) of the Laguerre-Riesz transforms was proved. In the latter paper the authors in fact showed that under the assumption $\alpha\in[-1/2,\infty)^d$, we have the bound $\int_0^\infty |\delta_{j,x}\,\mathcal{G}_t^\alpha(x,y)|t^{-1/2}\,dt \lesssim |x-y|^{-d}$. Now, using the (2.16) version of the definition of $L_j(x,y)$, Minkowski's integral inequality and then the change of variable $u=t^2/4s$, we get

$$\begin{aligned}\|L_j(x,y)\|_{L^2(t\,dt)} &\le \frac{1}{\sqrt{4\pi}} \int_0^\infty \Big(\int_0^\infty \Big(\frac{t}{s}\Big)^3 e^{-t^2/4s}\big(\delta_{j,x}\,\mathcal{G}_t^\alpha(x,y)\big)^2\,dt\Big)^{1/2}\,ds \\ &= \frac{1}{2\sqrt{2\pi}} \int_0^\infty |\delta_{j,x}\,\mathcal{G}_s^\alpha(x,y)| s^{-1/2}\Big(\int_0^\infty u\,e^{-u}\,du\Big)^{1/2}\,ds \lesssim |x-y|^{-d}.\end{aligned}$$

The smoothness estimate however cannot be derived from [7], since we need its difference form, which is not proved in that paper.

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