ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Błażej Wróbel

Uniwersytet Wrocławski

On Laplace transform type multipliers in the context of Laguerre function expansion of Hermite type

Praca semestralna nr 2

(semestr zimowy 2011/12)

Opiekun pracy: prof. dr hab. Krzysztof Stempak

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

ON LAPLACE TRANSFORM TYPE MULTIPLIERS IN THE CONTEXT OF LAGUERRE FUNCTION EXPANSIONS OF HERMITE TYPE

Błażej Wróbel

Institute of Mathematics of University of Wrocław
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: blazej.wrobel@math.uni.wroc.pl

Abstract

We investigate Laplace type multipliers associated to the multi–dimensional Laguerre function expansions of Hermite type. We prove that, under the assumption $\alpha_i \geq -1/2$, $\alpha_i \notin (-1/2, 1/2)$, these operators are Calderón-Zygmund operators. Consequently, their mapping properties follow by the general theory.

1 Introduction

The study of multipliers for various Laguerre systems began with the paper of Długosz [2]. In [17] Stempak and Trebels studied multipliers for Laguerre expansions of convolution type. A recent paper dealing with Laplace type multipliers for the same Laguerre system is the article by Drelichman, Durán and de Nápoli [3]. In [10] Sasso treated the topic in the Laguerre polynomials setting. Laplace type multipliers have been also considered for continuous orthogonal systems, see for instance Betancor, Martínez and Rodríguez-Mesa [1]. In this article we study Laplace type multipliers associated with Laguerre function expansions of Hermite type (see Section 2 for the definitions). We use two definitions (distinct, but not far from each other) of Laplace type multipliers. The first, $m_\kappa(x) = x\int_0^\infty e^{-xt}\kappa(t)\,dt$

Key words and phrases: Laguerre expansions of Hermite type, Calderón-Zygmund operator, Laplace type multiplier.

has its roots in Stein's monograph [11]. The second, $m_\mu(x) = \int_0^\infty e^{-xt}\, d\mu(t)$, follows that of [3]. When in need for more precision, we shall call a multiplier from the first definition 'Laplace type multiplier of the first kind', while from the second 'Laplace type multiplier of the second kind' (see the next section). To treat Laplace type multipliers of the first kind we use methods developed in [8], supported by an adaptation of the technicalities from [17]. In the context of this definition the paper is a generalization of the results obtained by the author in [16]. Also, if we know that the heat-diffusion semigroup $e^{-t\mathcal{L}_\alpha}$ is a contraction semigroup on all the L^p spaces, with $1 < p < \infty$, then from the general Littlewood-Paley theory for semigroups it follows that the multiplier operator $m_\kappa(\mathcal{L}_\alpha)$ is bounded on those L^p spaces (see [11]). However, the L^p contractiveness of $e^{-t\mathcal{L}_\alpha}$ is so far only known to hold for $\alpha \in [1/2, \infty)^d$. To treat Laplace type multipliers of the second kind we use some pointwise estimates for the heat-semigroup kernel, see Lemma 2.4. The main result of our paper is Theorem 2.5. We prove it assuming $\alpha \in (\{-1/2\} \cup [1/2, \infty))^d$. However, with the sole exception of the smoothness conditions, all the partial results of our paper are valid under the weaker assumption $\alpha \in [-1/2, \infty)^d$. By using the general Calderón-Zygmund theory, we also obtain weak type for $p = 1$, and allow weighted L^p spaces (with weights from the A_p Muckenhoupt class); see Corollary 2.7.

The paper is organized as follows. Section 2 contains the setup, the definitions of both kinds of Laplace type multipliers, basic lemmas and the statement of the main theorem. In particular we give the definitions of the Calderón-Zygmund kernels $K^\alpha_\kappa(x, y)$ and $K^\alpha_\mu(x, y)$ associated to $m_\kappa(\mathcal{L}_\alpha)$ and $m_\mu(\mathcal{L}_\alpha)$, respectively, in the sense of the Calderón-Zygmund theory, see Proposition 2.6. Since the notation we use follows that of [17], we are brief. Section 3 is devoted to the proof of the main theorem. In subsection 3.1 we justify the growth and smoothness conditions for the kernel $K^\alpha_\kappa(x, y)$, by referring to analogous proofs from [17]. Therefore we omit most of details. In subsection 3.2 we show the growth and smoothness estimates for the kernel $K^\alpha_\mu(x, y)$.

Throughout the paper we use a fairly standard notation with all symbols referring to $\mathbb{R}^d_+ = (0, \infty)^d$. Thus $A_p = A_p(\mathbb{R}^d_+)$ stands for the Muckenhoupt class of A_p weights, $L^p(w) = L^p(\mathbb{R}^d_+, w(x)dx)$ denotes the weighted L^p space (w being a non-negative weight on $\mathbb{R}^d_+$); we simply write L^p if $w \equiv 1$. By $\langle f, g\rangle$ we mean the canonical inner product in L^2. The symbol ∇_x represents the gradient operator with respect to the x variable. The notation $X \lesssim Y$ will be used to indicate that $X \leq CY$ with a positive constant C independent of significant quantities. We write $X \approx Y$ when $X \lesssim Y$ and $Y \lesssim X$. We shall also make a frequent use, often without mentioning it in relevant places, of the fact that for any $A > 0$ and $a \geq 0$,

$$\sup_{t>0} t^a e^{-At} = C_{a,A} < \infty.$$

Acknowledgements. The author would like to express his gratitude to Professor Krzysztof Stempak for suggesting the topic and many useful remarks during the preparation of the paper.

2 Preliminaries

Since the setting and majority of the notation we use are the same as in [16] and [17] we shall be brief. Let $\varphi_k^\alpha(x) = \varphi_{k_1}^{\alpha_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}^{\alpha_d}(x_d)$ be the system of d-dimensional Laguerre functions of Hermite type (as in [15, 6.4.12]), with $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in (-1, \infty)^d$. Each φ_k^α is an eigenfunction of the operator

$$L_\alpha = -\Delta + |x|^2 + \sum_{i=1}^{d} \frac{1}{x_i^2}\left(\alpha_i^2 - \frac{1}{4}\right)$$

corresponding to the eigenvalue $\lambda_{|k|}^\alpha = 4|k| + 2|\alpha| + 2d$; here Δ is the Laplacian restricted to $\mathbb{R}_+^d$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $|k| = k_1 + \ldots + k_d$ is the length of k. Moreover, $\{\varphi_k^\alpha : k \in \mathbb{N}^d\}$ is an orthonormal basis in L^2.

Let $\mathcal{L}_\alpha$ be the self–adjoint extension of L_α for which the spectral decomposition is given by φ_k^α and let $\{T_t^\alpha\}$ denote the heat-diffusion semigroup $T_t^\alpha = e^{-t\mathcal{L}_\alpha}$. Then for $f \in L^2$

$$T_t^\alpha f(x) = \int_{\mathbb{R}_+^d} \mathcal{G}_t^\alpha(x,y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d,$$

with

$$\begin{aligned} \mathcal{G}_t^\alpha(x,y) &= \sum_{n=0}^{\infty} e^{-t\lambda_n^\alpha} \sum_{|k|=n} \varphi_k^\alpha(x)\varphi_k^\alpha(y) \qquad (2.1)\\ &= (\sinh 2t)^{-d} \exp\left(-\frac{1}{2}\coth 2t\left(|x|^2 + |y|^2\right)\right) \prod_{i=1}^{d} \sqrt{x_i y_i}\, I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right), \end{aligned}$$

where I_ν, $\nu > -1$, is the modified Bessel function of the first kind and order ν. It is known, that $I_\nu(z)$, as a function of $z > 0$, is real, positive, smooth and satisfies

$$\frac{d}{dz} I_\nu(z) = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z). \qquad (2.2)$$

We shall use the standard asymptotics,

$$I_\nu(z) \approx z^\nu, \quad z \to 0^+, \qquad I_\nu(z) \approx z^{-\frac{1}{2}} e^z, \quad z \to \infty, \tag{2.3}$$

and the following lemmata.

Lemma 2.1 *(See [6]). Let $\nu \geq -1/2$. Then*

$$0 < I_\nu(z) - I_{\nu+1}(z) < \frac{2(\nu+1)}{z} I_{\nu+1}(z), \qquad z > 0.$$

Lemma 2.2 *(See [17]). Let $\nu \geq -1/2$, then*

$$z\left[2I_{\nu+1}(z) - I_{\nu+2}(z) - I_\nu(z)\right] - I_{\nu+1}(z) = O(z^{-3/2}e^z), \quad z \to \infty.$$

Lemma 2.3 *(See [12].) Given $a{>}1$, and $x, y \in \mathbb{R}^d_+$, we have*

$$\int_0^1 u^{-a} \exp\left(-\frac{C|x-y|^2}{u}\right) du \lesssim |x-y|^{-2a+2}.$$

The following Lemma shows that the heat kernel $\mathcal{G}^\alpha_t(x, y)$ is dominated up to a multiplicative constant by the heat kernel

$$G_t(x, y) = (\sinh 2t)^{-d/2} \exp\left(-\frac{1}{4\tanh t}|x-y|^2 - \frac{\tanh t}{4}|x+y|^2\right)$$

corresponding to the harmonic oscillator.

Lemma 2.4 *For $\alpha \in [-1/2, \infty)^d$ and $x, y \in \mathbb{R}^d_+$, we have*

$$\mathcal{G}^\alpha_t(x, y) \lesssim G_t(x, y).$$

Proof. The standard asymptotics imply the inequalities

$$I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right) \lesssim (x_i y_i)^{-1/2} (\sinh 2t)^{1/2} \exp\left(\frac{x_i y_i}{\sinh 2t}\right), \qquad i = 1, \ldots, d.$$

The above in turn implies $\mathcal{G}^\alpha_t(x, y) \lesssim (\sinh 2t)^{-d/2} e^{-T}$, where $T = \frac{1}{2}\coth 2t(|x|^2 + |y|^2) - \sum_{i=1}^d \frac{x_i y_i}{\sinh 2t}$. Now, an elementary computation shows that $T = \frac{1}{4\tanh t}|x-y|^2 + \frac{\tanh t}{4}|x+y|^2$, hence the lemma follows. ■

According to [11] we call a function $m : (0, \infty) \to \mathbb{C}$ a *Laplace type multiplier (of the first kind)* if it is of the form

$$m(x) = m_\kappa(x) = x \int_0^\infty e^{-xt} \kappa(t)\, dt, \tag{2.4}$$

with κ being a bounded measurable function on $(0,\infty)$. It should be noted, that m defined as above, satisfies Mihlin's conditions of any order, that is, $|x^j m^{(j)}(x)| \le C_j$, $j = 0,1,2,\dots$ (in particular m is bounded). Remarkable special cases of Laplace type multipliers of the first kind include the imaginary powers and functions of the form $m_\lambda(x) = \frac{x}{x+\lambda}$, for $\mathrm{Re}(\lambda) > 0$. The imaginary powers, $m_\gamma(x) = x^{-i\gamma}$, $\gamma \in \mathbb{R}$, $\gamma \neq 0$, are Laplace type multipliers corresponding to $\kappa(t) = \frac{1}{\Gamma(1+i\gamma)} t^{i\gamma}$. L^p boundedness properties of the imaginary power operators in the context of our paper have been studied by the author in [16]. Thus the present paper includes results of [16] as special cases. The other example, $m_\lambda(x)$, is a Laplace type multiplier corresponding to $\kappa_\lambda(t) = e^{-\lambda t}$. Since $m_\lambda(x) = 1 - \frac{\lambda}{x+\lambda}$, the resulting operator is a difference $I - \lambda(\lambda - (-\mathcal{L}_\alpha))^{-1}$, of the identity operator I and a constant λ times the resolvent operator $(\lambda - (-\mathcal{L}_\alpha))^{-1}$.

We call a function $m : (0,\infty) \to \mathbb{C}$ a *Laplace type multiplier of the second kind* if it is of the form

$$m(x) = m_\mu(x) = \int_0^\infty e^{-xt}\, d\mu(t), \tag{2.5}$$

where μ is a complex Borel measure with total variation $|\mu|$ satisfying the condtition $\int_0^\infty e^{-td} d|\mu|(t) < \infty$. The latter assumption is a technical one, well suited for our setting. It implies in particular that $m(x)$ is bounded on the halfline $[d,\infty)$ (hence also on the spectrum $\sigma(\mathcal{L}_\alpha) = \{\lambda_n^\alpha : n \in \mathbb{N}\}$). Here, remarkable special cases of Laplace type multiplier operators include the Laguerre fractional integral operators I^σ, $\sigma > 0$, and the resolvent operators $R_\lambda = (\lambda - (-\mathcal{L}_\alpha))^{-1}$, for $\mathrm{Re}(\lambda) > 0$. The operators $I^\sigma = (\mathcal{L}_\alpha)^{-\sigma}$, correspond to the choice of $m_\sigma(x) = x^{-\sigma}$, and $d\mu_\sigma(t) = \frac{1}{\Gamma(\sigma)} t^{\sigma-1}\, dt$. $L^p - L^q$ boundedness properties of these operators have been studied by Nowak and Stempak in [9]. Thus Theorem 2.5 of this paper contains an enhancement of [9, Theorem 3.1] in the case $p = q$. The resolvent operators correspond to the choice of $m_\lambda(x) = \frac{1}{x+\lambda}$ and $d\mu_\lambda(t) = e^{-\lambda t}\, dt$.

It should be noted that in many cases the two definitions are comparable up to a constant. Namely, if we assume for example that κ is bounded and continuously differentiable, $\lim_{t\to 0+} \kappa(t) = \kappa(0)$ exists, and κ' is integrable then

$$x \int_0^\infty e^{-xt} \kappa(t)\, dt = \kappa(0) + \int_0^\infty e^{-xt} \kappa'(t)\, dt.$$

The left hand side of the above equation is a Laplace type multiplier of the first kind m_κ of the function κ, while the right hand side is a constant plus a Laplace type multiplier of the second kind m_μ of the measure μ with the density $\kappa'(t)$. On the level of multiplier operators the above equation says that, $m_\kappa(\mathcal{L}_\alpha) - m_\mu(\mathcal{L}_\alpha) = \kappa(0)I$, where I is the identity operator.

For a Laplace type multiplier m by the spectral theorem we have

$$m(\mathcal{L}_\alpha)f = \sum_{k\in\mathbb{N}^d} m\big(4|k| + 2|\alpha| + 2d\big)\,\langle f, \varphi_k^\alpha\rangle\, \varphi_k^\alpha, \qquad f \in L^2. \tag{2.6}$$

Since in both definitions (2.4) and (2.5) the function m is bounded on the spectrum of $\mathcal{L}_\alpha$, $m(\mathcal{L}_\alpha)$ is a bounded operator on L^2. Motivated by the fact that (at least formally) we have

$$\begin{aligned} m_\kappa(\mathcal{L}_\alpha)f(x) &= \mathcal{L}_\alpha \int_0^\infty e^{-t\mathcal{L}_\alpha} f(x)\,\kappa(t)\,dt = \int_0^\infty -\frac{d}{dt} T_t^\alpha f(x)\,\kappa(t)\,dt \\ &= \int_{\mathbb{R}^d_+} \Big(- \int_0^\infty \frac{d}{dt}\mathcal{G}_t^\alpha(x,y)\kappa(t)\,dt \Big) f(y)\,dy, \end{aligned}$$

we *define* the kernel of the Laplace type multiplier m_κ as

$$K^\alpha(x,y) = K^\alpha_\kappa(x,y) = -\int_0^\infty \frac{d}{dt}\mathcal{G}_t^\alpha(x,y)\kappa(t)\,dt. \tag{2.7}$$

Analogously, the formal computations

$$m_\mu(\mathcal{L}_\alpha)f(x) = \int_0^\infty e^{-t\mathcal{L}_\alpha} f(x)\,d\mu(t) = \int_{\mathbb{R}^d_+} \Big(\int_0^\infty \mathcal{G}_t^\alpha(x,y)d\mu(t) \Big)\, f(y)\,dy,$$

lead us to *define* the kernel of the Laplace type multiplier m_μ as

$$K^\alpha(x,y) = K^\alpha_\mu(x,y) = \int_0^\infty \mathcal{G}_t^\alpha(x,y)d\mu(t). \tag{2.8}$$

From the estimates that will follow it can be deduced that the definitions (2.7), (2.8) are valid for $x \neq y$. The main result of our paper is the following.

Theorem 2.5 *Let $\alpha \in (\{-1/2\} \cup [1/2,\infty))^d$ and m be a Laplace type multiplier given either by* (2.4) *or by* (2.5)*. Then the kernel $K^\alpha(x,y)$ given by* (2.7) *or* (2.8)*, respectively, satisfies the growth condition*

$$|K^\alpha(x,y)| \lesssim |x-y|^{-d}, \qquad x,y \in \mathbb{R}^d_+, \tag{2.9}$$

and the smoothness condition

$$|\nabla_x K^\alpha(x,y)| + |\nabla_y K^\alpha(x,y)| \lesssim |x-y|^{-d-1}, \qquad x,y \in \mathbb{R}^d_+. \tag{2.10}$$

The restrictions imposed on α are to some extent natural, see [8, p. 407] for additional comments. Methods from [13] can be easily adapted to prove the following.

Proposition 2.6 *Let $\alpha \in [-1/2, \infty)^d$ and m be a Laplace type multiplier given either by (2.4) or by (2.5). Then the multiplier operator defined by (2.6), is associated with the kernel $K^\alpha(x,y)$ given by (2.7) or (2.8), respectively, in the sense that for any two functions $f, g \in C_c^\infty\left(\mathbb{R}_+^d\right)$ with disjoint supports we have*

$$\langle m(\mathcal{L}_\alpha) f, g\rangle = \int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d} K^\alpha(x,y) f(y)\overline{g(x)}\, dy\, dx.$$

Proof. In the case of the multiplier m_κ we need to slightly modify the reasoning from [16]. It is enough to observe that the relevant proof in [16] is also valid if we replace therein $t^{-i\gamma}$ by the bounded function $\kappa(t)$. In the case of the multiplier m_μ, looking in detail at the argument used in the proof of [13, Proposition 4.2], together with some pointwise estimates for the Laugerre functions φ_k^α, see for instance [7, Section 5], shows that in order to repeat that argument in the present situation it is enough to verify that

$$\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d}\int_0^\infty \left|\mathcal{G}_t^\alpha(x,y) f(y)\overline{g(x)}\right| \, d\mu(t)\, dy\, dx < \infty.$$

The above follows from the growth condition for the kernel $K_\mu^\alpha(x,y)$, see Section 3.2.■

By the general Calderón-Zygmund theory, see for instance [4], combining Theorem 2.5, Proposition 2.6 and the fact that $m(\mathcal{L}_\alpha)$ is bounded on L^2, we also get the following.

Corollary 2.7 *Let $\alpha \in (\{-1/2\} \cup [1/2, \infty))^d$ and m be a Laplace type multiplier given either by (2.4) or by (2.5). Then the multiplier operator $m(\mathcal{L}_\alpha)$, defined initially on L^2 by (2.6), extends uniquely to a bounded operator on $L^p(w)$, $1 < p < \infty$, $w \in A_p$, and to a bounded operator from $L^1(w)$ to $L^{1,\infty}(w)$, $w \in A_1$.*

3 Proof of Theorem 2.5

3.1 The case of $K_\kappa^\alpha(x,y)$

Since the structure of the proofs and the proofs themselves are similar to those from [17], we shall be brief. We use the change of variable

$$t = t(u) = \frac{1}{2}\log\frac{1+u}{1-u}, \tag{3.1}$$

invented by Stefano Meda, in the context of the Ornstein-Uhlenbeck operator. It seems that the proof would be less complicated without using the above, however since in [17] we relied deeply on (3.1) and here we concentrate on adapting the reasoning from the latter paper, for the sake of readers convenience, we maintain using (3.1). Till the end of this subsection, the following notation from [17] will be used:

$$A = A(u) = \frac{1+u^2}{4u}, \quad B = B(u) = \frac{1-u^2}{2u}, \quad Z_i = Z_i(u, x_i, y_i) = Bx_iy_i,$$

$$\sqrt{xy} = \prod_{i=1}^{d} \sqrt{x_iy_i}, \qquad I_\alpha(Z) = \prod_{i=1}^{d} I_{\alpha_i}(Z_i),$$

$$D_\nu = D_\nu(x_1, y_1, u) = -2Ax_1I_\nu(Z_1) + By_1I_{\nu+1}(Z_1) + \frac{\nu+1/2}{x_1}I_\nu(Z_1),$$

$$W = \sum_{i=1}^{d} Z_i - A(|x|^2 + |y|^2), \quad W_k = W - Z_k, \quad U_k = x_k^2 + y_k^2 + \sum_{i\neq k}(x_i - y_i)^2.$$

Further, in several places we will need to split D_ν; as it can be easily seen

$$D_\nu = M_1^\nu + M_2^\nu + M_3^\nu + \frac{\nu+1/2}{x_1}I_\nu(Z_1),$$

where

$$M_1^\nu = (-x_1 + y_1)\frac{1}{2u}I_\nu(Z_1), \quad M_2^\nu = -(x_1 + y_1)\frac{u}{2}I_\nu(Z_1), \quad M_3^\nu = By_1(I_{\nu+1}(Z_1) - I_\nu(Z_1)). \tag{3.2}$$

In the proof of the standard estimates for the kernel $K^\alpha(x, y)$ the following lemma justified in [17] will be used.

Lemma 3.1 *Let $u \in (0, 1)$, $\alpha \in [-1/2, \infty)^d$, $i = 1, \ldots, d$. We have*

(A) $\log(\frac{1+u}{1-u}) \approx u$ *for* $u \in (0, 1/2)$, $\log(\frac{1+u}{1-u}) \lesssim (1-u)^{-1/2}$ *for* $u \in (1/2, 1)$; *consequently,* $\log\frac{1+u}{1-u} \lesssim u\,(1-u)^{-1/2}$.

(B) $A \lesssim u^{-1}$, $\quad B \lesssim u^{-1}$; *moreover, if* $u \in (1/2, 1)$, *then* $B \leq 3/4$ *and* $-A \leq -1/2$.

(C) $Z_i \approx (1-u)u^{-1}x_iy_i$; *in particular for* $u \in (0, 1/2)$, $Z_i \approx \frac{x_iy_i}{u}$, *while for* $u \in (1/2, 1)$, $Z_i \approx (1-u)x_iy_i$.

(D) $W = -\frac{1}{4u}|x-y|^2 - \frac{u}{4}|x+y|^2 \leq -\frac{1}{4u}|x-y|^2$.

(E) *If* $u \in (1/2, 1)$, *then* $W \leq -\frac{1}{8}|x+y|^2 - \frac{1}{4}|x-y|^2$.

(F) $W_k \leq -\frac{1}{4u}U_k \leq -\frac{1}{4u}|x-y|^2$.

(G) If $Z_1 < 1$, then $I_{\alpha_1}(Z_1) \lesssim u^{-\alpha_1}(1-u)^{\alpha_1}(x_1y_1)^{\alpha_1} \lesssim u^{1/2}(1-u)^{-1/2}(x_1y_1)^{-1/2}$.

(H) $I_{\alpha_i}(Z_i) \lesssim (x_iy_i)^{-1/2}u^{1/2}(1-u)^{-1/2}e^{Z_i}$.

(I) $|I_{\alpha_i+1}(Z_i) - I_{\alpha_i}(Z_i)| \lesssim (x_iy_i)^{-1}u(1-u)^{-1}I_{\alpha_i+1}(Z_i) \lesssim (x_iy_i)^{-3/2}u^{3/2}(1-u)^{-3/2}e^{Z_i}$.

We will first justify the growth estimate (2.9), under the assumption $\alpha \in [-1/2,\infty)^d$. We follow the outline of the proof of the growth condition for the kernel $K(x,y)$ from [17, pp. 13–15]. Using (3.1) we see that it suffices to show that $\int_0^1 |K_u(x,y)|\,du \lesssim |x-y|^{-d}$, where

$$K_u(x,y) = (1-u^2)^{-1}\kappa(t)\frac{d}{dt}\mathcal{G}_t^\alpha(x,y)\bigg|_{t=\frac{1}{2}\log\frac{1+u}{1-u}},$$

so that $K^\alpha(x,y) = \int_0^1 K_u(x,y)\,du$. Differentiating (2.1), with the aid of (2.2), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_t^\alpha(x,y) = &-2d\cosh 2t(\sinh 2t)^{-d-1}e^{-\frac{1}{2}\coth 2t\left(|x|^2+|y|^2\right)}\prod_{i=1}^d \sqrt{x_iy_i}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &+\left(|x|^2+|y|^2\right)(\sinh 2t)^{-d-2}e^{-\frac{1}{2}\coth 2t\left(|x|^2+|y|^2\right)}\prod_{i=1}^d \sqrt{x_iy_i}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &-2\cosh 2t(\sinh 2t)^{-d-2}e^{-\frac{1}{2}\coth 2t\left(|x|^2+|y|^2\right)}S_t^\alpha(x,y)\prod_{i=1}^d\sqrt{x_iy_i},\end{aligned}$$

with

$$S_t^\alpha(x,y) = \sum_{j=1}^d\left[\alpha_j\sinh 2t\; I_{\alpha_j}\left(\frac{x_jy_j}{\sinh 2t}\right) + x_jy_jI_{\alpha_j+1}\left(\frac{x_jy_j}{\sinh 2t}\right)\right]\prod_{i\neq j}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right).$$

After some rearrangement of terms we see that

$$\begin{aligned}\cosh 2t\,S_t^\alpha(x,y) = &|\alpha|\cosh 2t\sinh 2t\prod_{i=1}^d I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right) + \langle x,y\rangle\prod_{i=1}^d I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &+\sum_{j=1}^d x_jy_j\left[I_{\alpha_j+1}\left(\frac{x_jy_j}{\sinh 2t}\right) - I_{\alpha_j}\left(\frac{x_jy_j}{\sinh 2t}\right)\right]\prod_{i\neq j}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &+(\cosh 2t-1)\sum_{j=1}^d x_jy_jI_{\alpha_j+1}\left(\frac{x_jy_j}{\sinh 2t}\right)\prod_{i\neq j}I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right).\end{aligned}$$

Using the above equality, together with the fact that the change of variables (3.1) transforms $\sinh 2t$ into $\frac{2u}{1-u^2}$ and $\cosh 2t$ into $\frac{1-u^2}{1+u^2}$, we decompose $K_u(x,y) = \sum_{i=1}^4 K_i$, where

$$K_1 = -2(d+|\alpha|)\widetilde{\kappa}(u)(2u)^{-d-1}(1-u^2)^{d-1}(1+u^2)\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\, I_\alpha(Z),$$

$$K_2 = |x-y|^2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\, I_\alpha(Z),$$

$$K_3 = 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)} \sum_{j=1}^d x_j y_j \left[I_{\alpha_j}(Z_j) - I_{\alpha_j+1}(Z_j)\right] \prod_{i\neq j} I_{\alpha_i}(Z_i),$$

$$K_4 = -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^{d}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)} \sum_{j=1}^d x_j y_j I_{\alpha+e_j}(Z),$$

with $\widetilde{\kappa}(u) = \kappa(\tanh t)$. Therefore, to prove the growth condition for the kernel $K^\alpha(x,y)$ it suffices to show that

$$\int_0^1 |K_i|\, du \lesssim |x-y|^{-d}, \qquad i = 1,\ldots,4. \tag{3.3}$$

Now, to avoid collision of symbols, we denote the expressions K_i, $i=1,\ldots,4$, appearing in the proof of the growth condition for the kernel $K(x,y)$ in [17, pp. 14] by $\overline{K}_i$, $i=1,\ldots,4$. Then obviously, $K_i = \widetilde{\kappa}(u)(1-u^2)^{-1/2}\overline{K}_i$. Since $\widetilde{\kappa}$ is bounded we may focus on showing that $\int_0^1 |\overline{K}_i|(1-u^2)^{-1/2}\, du \lesssim |x-y|^{-d}$, $i=1,\ldots,4$. This may be accomplished by following the proof of the growth condition for the kernel $K(x,y)$ from [17], with the aid of Lemma 3.1. Details and appropriate modifications are left to the reader.

Now we pass to the justification of the smoothness condition (2.10) for the kernel $K^\alpha(x,y)$. From the symmetry in x and y of the kernel and the fact that none of the variables $x_1,\ldots,x_d$ is distinguished we see that in order to prove (2.10) it suffices to show that

$$\left|\int_0^1 \partial_{x_1} K_u(x,y)\, du\right| \lesssim |x-y|^{-d-1},$$

(differentiation under the integral sign is implicitly justified by the estimates that follow). Using (2.2) we see that $\partial_{x_1}\left(I_\nu(Z_1)e^{-A(|x|^2+|y|^2)}\sqrt{x_1y_1}\right) = D_\nu e^{-A(|x|^2+|y|^2)}\sqrt{x_1y_1}$. Hence, $\partial_{x_1}K_u(x,y) = \sum_{k=1}^9 J_k^\alpha = \sum_{k=1}^9 J_k^\alpha(x,y,u)$, where

$$J_1^\alpha = -2(d+|\alpha|)\widetilde{\kappa}(u)(2u)^{-d-1}(1-u^2)^{d-1}(1+u^2)\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1}\prod_{i=2}^d I_{\alpha_i}(Z_i),$$

comes from differentiating K_1,

$$J_2^\alpha = 2(x_1-y_1)\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)}I_\alpha(Z),$$
$$J_3^\alpha = |x-y|^2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1}\prod_{i=2}^d I_{\alpha_i}(Z_i),$$

come from differentiating K_2,

$$J_4^\alpha = 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)}\left(D_{\alpha_1}-D_{\alpha_1+1}\right)x_1y_1\prod_{i=2}^d I_{\alpha_i}(Z_i),$$
$$J_5^\alpha = 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}y_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}\left(I_{\alpha_1}(Z_1)-I_{\alpha_1+1}(Z_1)\right)\prod_{i=2}^d I_{\alpha_i}(Z_i),$$
$$J_6^\alpha = 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1}\sum_{j=2}^d x_jy_j(I_{\alpha_j}(Z_j)-I_{\alpha_j+1}(Z_j))\prod_{i\neq 1,j} I_{\alpha_i}(Z_i),$$

come from differentiating K_3, and finally

$$J_7^\alpha = -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^d y_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}I_{\alpha+e_1}(Z),$$
$$J_8^\alpha = -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^d x_1y_1\sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1+1}\prod_{i=2}^d I_{\alpha_i}(Z_i),$$
$$J_9^\alpha = -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^d \sqrt{xy}e^{-A(|x|^2+|y|^2)}D_{\alpha_1+1}\sum_{j=2}^d x_jy_jI_{\alpha_j+1}(Z_j)\prod_{i\neq j,1}^d I_{\alpha_i}(Z_i),$$

come from differentiating K_4. Therefore, the proof of the smoothness condition for the kernel $K^\alpha(x,y)$ will follow, if we show that, for $k=1,\ldots,9$,

$$\int_0^1 \left|J_k^\alpha(x,y,u)\right| du \lesssim |x-y|^{-d-1}. \tag{3.4}$$

As previously, to avoid collision of symbols, we denote the expressions J_k^α, $k=1,\ldots,9$, appearing in the proof of the smoothness condition for the kernel $K(x,y)$ in [17] by $\overline{J_k^\alpha}$, $k=1,\ldots,9$. Then, clearly $J_k^\alpha = \kappa(u)(1-u^2)^{-1/2}\,\overline{J_k^\alpha}$ and since κ is bounded our task reduces to showing that $\int_0^1 \left(1-u^2\right)^{-1/2}\left|\overline{J_k^\alpha(x,y,u)}\right| du \lesssim |x-y|^{-d-1}$. This may be obtained by using Lemma 3.1 and following (step by step) the scheme of the analogous proof from [17, pp. 16–22]. As in the latter paper, when $Z_1>1$, we also need to use the splitting (3.2). In the present paper the whole task is a bit simpler (at least in notation) than in [17], since the kernel $K^\alpha(x,y)$ is not vector–valued, therefore we can use the gradient condition and do not need to use the the mean value theorem. The most subtle part is for $k=4$, when we also need to use Lemma 2.2. We omit the details, however for the sake of completeness, we state the analogues of [17, Lemmata 4.2, 4.3, 4.4] in our context.

Lemma 3.2 *Let $\nu=-1/2$ or $\nu\in[1/2,\infty)$, and $b\geq 0$. Then*

$$E_{\nu,b}=(|x|+|y|)^b\int_{1/2}^{1}(1-u)^{d/2-1/2}\big|D_\nu\sqrt{x_1y_1}e^{W_1}\big|\,du\lesssim|x-y|^{-d-1}.$$

Lemma 3.3 *Let $\nu=-1/2$ or $\nu\in[1/2,\infty)$, and $a\geq 0$, $b\geq 0$. Then,*

$$R=R_{\nu,a,b}=(x_1+y_1)^b\int_{0}^{1/2}u^{-d/2-a}\big|D_\nu\sqrt{x_1y_1}e^{W_1}\big|\,du\lesssim|x-y|^{-d-2a+2-b}.$$

Lemma 3.4 *Let $a\geq 0$, $b\geq 0$. Then,*

$$X=X_{a,b}=X_{a,b}(x,y)=\int_{(0,1/2)\cap\{Z_1>1\}}\frac{1}{(x_1)^b}u^{-d/2-a}e^{-\frac{C}{u}|x-y|^2}\,du\lesssim|x-y|^{-d-2a+2-b}.$$

The proofs of the above, are all similar to the proofs of the appropriate lemmata [17, Lemmata 4.2, 4.3, 4.4], therefore we omit them.

3.2 The case of $K_\mu^\alpha(x,y)$

This time we will not use the change of the variables (3.1). We employ the notation analogous to that of the previous section:

$$\sqrt{xy}=\prod_{i=1}^{d}\sqrt{x_iy_i},\quad \widetilde{Z}_i=\widetilde{Z}_i(t,x_i,y_i)=\frac{x_iy_i}{\sinh 2t},\quad I_\alpha(\widetilde{Z})=\prod_{i=1}^{d}I_{\alpha_i}(\widetilde{Z}_i),$$
$$F_\nu(x_1,y_1,t)=-\coth(2t)\,x_1I_\nu(\widetilde{Z}_1)+\frac{y_1}{\sinh 2t}I_{\nu+1}(\widetilde{Z}_1)+\frac{\nu+1/2}{x_1}I_\nu(\widetilde{Z}_1),$$
$$T=\frac{1}{4\tanh t}|x-y|^2+\frac{\tanh t}{4}|x+y|^2,\quad T_1=T+\widetilde{Z}_1.$$

Arguing as in the proof of Lemma 2.4 we see that $T_1=\frac{1}{2}\coth 2t\left(|x|^2+|y|^2\right)-\sum_{k=2}^{d}\widetilde{Z}_k$. In the reasoning that follows we will often use (sometimes without explicit mention) the fact (implied by (2.3)) that for $\alpha_i\geq-1/2$, $i=1,2,\ldots,d$,

$$I_{\alpha_i}(\widetilde{Z}_i)\lesssim(\sinh 2t)^{1/2}(x_iy_i)^{-1/2}e^{\widetilde{Z}_i},\qquad 0<\widetilde{Z}_i<\infty,\quad(3.5)$$
$$I_{\alpha_i}(\widetilde{Z}_i)\lesssim(\sinh 2t)^{-\alpha_i}(x_iy_i)^{\alpha_i}\leq(\sinh 2t)^{1/2}(x_iy_i)^{-1/2}\approx I_{-1/2}(\widetilde{Z}_i),\qquad 0<\widetilde{Z}_i<1.\quad(3.6)$$

As previously, we start with justifying the growth estimate (2.9). By the assumptions made on the measure μ we see that it is enough to check that $\mathcal{G}_t^\alpha(x,y) \lesssim e^{-td}|x-y|^{-d}$. Obviously, Lemma 2.4 implies

$$\mathcal{G}_t^\alpha(x,y) \lesssim |x-y|^{-d}\Big(\frac{\tanh t}{\sinh 2t}\Big)^{d/2}\Big(\frac{|x-y|^2}{\tanh t}\Big)^{d/2}\exp\Big(-\frac{1}{4\tanh t}|x-y|^2\Big).$$

From the above we easily get the desired bound.

Now we pass to the proof of the smoothness estimate (2.10) for the kernel $K_\mu^\alpha(x,y)$. Once again symmetry reasons and the fact that that none of the variables $x_1,\ldots,x_d$ is distinguished reduce our task to showing that

$$\Big|\int_0^\infty \partial_{x_1}\mathcal{G}_t^\alpha(x,y)\,d\mu(t)\Big| \lesssim |x-y|^{-d-1}$$

(differentiation under the integral sign is implicitly justified by the estimates that follow). The assumptions made on the measure μ imply, that to prove the above estimate it is enough to show that

$$|\partial_{x_1}\mathcal{G}_t^\alpha(x,y)| \lesssim e^{-td}|x-y|^{-d-1}, \qquad t>0, \quad x,y\in\mathbb{R}_+^d. \tag{3.7}$$

With the aid of (2.2) we compute that

$$\partial_{x_1}\mathcal{G}_t^\alpha(x,y) = (\sinh 2t)^{-d}\sqrt{xy}\exp\big(-\frac{1}{2}\coth 2t(|x|^2+|y|^2)\big)F_{\alpha_1}(x_1,y_1,t)\prod_{i=2}^d I_{\alpha_i}(\widetilde{Z}_i).$$

We start with estimating $\partial_{x_1}\mathcal{G}_t^\alpha(x,y)$ for $t\in[1,\infty)$. Using Lemma 2.1 we get

$$|F_{\alpha_1}(x_1,y_1,t)| \lesssim (x_1+y_1)\coth(2t)\,I_{\alpha_1}(\widetilde{Z}_1) + \frac{\alpha_1+1/2}{x_1}I_{\alpha_1}(\widetilde{Z}_1), \qquad t>0. \tag{3.8}$$

Note, that for $t\geq 1$, $\tanh t \approx \coth 2t \approx 1$ and $\sinh 2t \approx e^{2t}$. Consequently, if $\alpha_1=-1/2$, from Lemma 2.4 it follows that for $t\geq 1$,

$$|\partial_{x_1}\mathcal{G}_t^\alpha(x,y)| \lesssim (x_1+y_1)\exp\big(-C|x+y|^2\big)\exp\big(-C|x-y|^2\big)e^{-td} \lesssim e^{-td}|x-y|^{-d-1}.$$

Assume now $\alpha_1\geq 1/2$. We consider two cases. First, if $\widetilde{Z}_1<1$, then using (3.6) we see that

$$\frac{1}{x_1}\sqrt{x_1y_1}I_{\alpha_1}(\widetilde{Z}_1) \lesssim y_1(\sinh 2t)^{-1/2} \approx y_1(\sinh 2t)^{-1}\sqrt{x_1y_1}I_{-1/2}(\widetilde{Z}_1).$$

On the other hand, if $\widetilde{Z}_1 \ge 1$, then $\frac{1}{x_1} \le \frac{y_1}{\sinh 2t}$, and consequently,

$$\frac{1}{x_1}\sqrt{x_1 y_1} I_{\alpha_1}(\widetilde{Z}_1) \le y_1 (\sinh 2t)^{-1} \sqrt{x_1 y_1} I_{\alpha_1}(\widetilde{Z}_1).$$

It follows that in both cases $\frac{1}{x_1}\sqrt{x_1 y_1} I_{\alpha_1}(\widetilde{Z}_1) \lesssim (x_1 + y_1)\sqrt{x_1 y_1} I_{-1/2}(\widetilde{Z}_1)$, for $t \ge 1$. Hence the case $\alpha_1 \ge 1/2$ is reduced to the case $\alpha_1 = -1/2$.

Now we shall show the bound (3.7) for $t \in (0,1)$. Note first, that for $t \in (0,1)$ each of the quantities: $\coth 2t$, $(\sinh 2t)^{-1}$, $(\tanh t)^{-1}$ is comparable with t^{-1}, while $e^{-td} \approx 1$.

We consider two cases. First assume $\widetilde{Z}_1 < 1$. From (3.8) we see that to prove (3.7) in this case it suffices to estimate each of the two quantities,

$$P_1 = (x_1 + y_1) t^{-1} \mathcal{G}_t^\alpha(x,y) \qquad \text{and} \qquad P_2 = \frac{\alpha_1 + 1/2}{x_1} \mathcal{G}_t^\alpha(x,y)$$

by $C|x-y|^{-d-1}$. From (3.6) we obtain $P_1 \lesssim (x_1 + y_1) t^{-d/2-1} e^{-T_1}$. Since for $t \in (0,1)$, $t^{-1} \approx \coth 2t$, we get $(x_1 + y_1) t^{-1/2} e^{-\frac{1}{2}T_1} \lesssim 1$. Consequently,

$$P_1 \lesssim t^{-d/2-1/2} e^{-\frac{1}{2}T_1} \le t^{-d/2-1/2} e^{-\frac{1}{2}T} \lesssim |x-y|^{-d-1}.$$

The last inequality above comes from the observation that $\big(|x-y|^2 \, t^{-1}\big)^{d/2+1/2} e^{-\frac{1}{2}T} \lesssim 1$, for $t \in (0,1)$. It remains to bound P_2. Clearly, if $\alpha_1 = -1/2$, then there is nothing to estimate. If $\alpha_1 \ge 1/2$, we use again (3.6), to get $\frac{1}{x_1}\sqrt{x_1 y_1} I_{\alpha_1}(\widetilde{Z}_1) \lesssim y_1 t^{-1/2}$. It follows that $P_2 \lesssim (x_1 + y_1) t^{-d/2-1} e^{-T_1} \lesssim |x-y|^{-d-1}$, as in the case of P_1.

Assume now $\widetilde{Z}_1 \ge 1$. By analogy to (3.2), we use the splitting

$$\begin{aligned} F_{\alpha_1}(x_1, y_1, t) =& (-x_1 + y_1) \frac{1}{2\tanh t} I_{\alpha_1}(\widetilde{Z}_1) - (x_1 + y_1) \frac{\tanh t}{2} I_{\alpha_1}(\widetilde{Z}_1) \\ &+ \frac{y_1}{\sinh 2t}\big[I_{\alpha_1+1}(\widetilde{Z}_1) - I_{\alpha_1}(\widetilde{Z}_1)\big] + \frac{\alpha_1 + 1/2}{x_1} I_{\alpha_1}(\widetilde{Z}_1) \equiv \sum_{k=1}^{4} H_k(x_1, y_1, t). \end{aligned}$$

Thus, to prove the estimate for $\partial_{x_1}\mathcal{G}_t^\alpha(x,y)$ in this case it is enough to show that for $t \in (0,1)$, $\widetilde{Z}_1 \ge 1$, and $k = 1,2,3,4$, we have

$$Y_k \equiv t^{-d/2}\sqrt{xy}\exp\big(-\frac{1}{2}\coth 2t(|x|^2 + |y|^2)\big)|H_k(x_1,y_1,t)| \prod_{i=2}^{d} I_{\alpha_i}(\widetilde{Z}_i) \lesssim |x-y|^{-d-1}.$$

If $k=1$, using (3.5) we get $Y_1 \lesssim |x-y| t^{-d/2-1} e^{-T}$. Since $t^{-d/2-1} e^{-T} \lesssim |x-y|^{-d-2}$, we obtain the desired estimate. If $k=2$, we use (3.5) again to get $Y_2 \lesssim (x_1 + y_1) t^{-d/2+1} e^{-T}$. Observing that $(x_1 + y_1) t^{-1/2} e^{-\frac{1}{2}T} \lesssim 1$ and $t^{-d/2-1/2} e^{-\frac{1}{2}T} \lesssim |x-y|^{-d-1}$, we arrive at

$Y_2 \lesssim |x-y|^{-d-1}$. To bound Y_3, we use Lemma (2.1) and (3.5), getting $Y_3 \lesssim \frac{1}{x_1} t^{-d/2} e^{-T}$. Now we use the simple splitting $\frac{1}{x_1} = \left(\frac{y_1}{x_1}\right)^{1/2} (x_1 y_1)^{-1/2}$, and consider two cases. If $\frac{1}{2} y_1 < x_1 < 2y_1$, then since $\widetilde{Z}_1 \geq 1$ and $t \in (0,1)$, it follows from the above splitting, that $Y_3 \lesssim t^{-d/2-1/2} e^{-T} \lesssim |x-y|^{-d-1}$. On the other hand, if x_1 and y_1 are not comparable, then $(x_1+y_1) \approx |x_1 - y_1|$. Since $\widetilde{Z}_1 \geq 1$, we get $\frac{1}{x_1} \leq (x_1+y_1)(x_1 y_1)^{-1} \leq (x_1+y_1)(\sinh 2t)^{-1}$. Consequently, $Y_3 \lesssim |x-y|\, t^{-d/2-1} e^{-T} \lesssim |x-y|^{-d-1}$. The estimate for Y_4 was already obtained while showing the bound for Y_3. This finishes proving (3.7) for $t \in (0,1)$ in the case when $\widetilde{Z}_1 \geq 1$ and thus completes the proof of (3.7).

References

[1] J. J. Betancor, T. Martínez and L. Rodríguez-Mesa, *Laplace Transform Type Multipliers for Hankel Transforms*, Canad. Math. Bull. Vol. 51 (4) (2008) 487–496.

[2] J. Długosz, *L^p–multipliers for the Laguerre expansions*, Colloq. Math. 54 (1987) 287–293.

[3] I. Drelichman, R. G. Durán, P. L. de Nápoli, *Multipliers of Laplace transform type for Laguerre and Hermite expansions*, http://arxiv.org/abs/1009.2518, preprint 2010.

[4] J. Duoandikoetxea, *Fourier Analysis*, Amer. Math. Soc., Providence, RI, 2001.

[5] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, 1972.

[6] I. Nåsell, *Rational bounds for ratios of modified Bessel functions*, SIAM J. Math. Anal. 9 (1978) 1–11.

[7] A. Nowak, *Heat-diffusion and Poisson integrals for Laguerre and special Hermite expansions on weighted L^p spaces*, Studia Math. 158 (2003), 239–268.

[8] A. Nowak, K. Stempak, *Riesz transforms and conjugacy for Laguerre function expansions of Hermite type*, J. Funct. Anal. 244 (2007) 399–433.

[9] A. Nowak, K. Stempak, *Negative powers of Laguerre operators*, http://arxiv.org/abs/0912.0038, preprint 2009.

[10] E. Sasso, *Spectral multipliers of Laplace transform type for the Laguerre operator*, Bull. Austral. Math. Soc. Vol. 69 (2004) 255-266.

[11] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*, Annals Math. Studies, Princeton Univ. Press, 1970.

[12] K. Stempak, J. L. Torrea, *Poisson integrals and Riesz transforms for Hermite function expansions with weights*, J. Funct. Anal. 202 (2003) 443–472.

[13] K. Stempak, J.L. Torrea, *Higher Riesz transforms and imaginary powers associated to the harmonic oscillator*, Acta Math. Hungar. 111 (2006) 43–64.

[14] K. Stempak, W. Trebels, *On weighted transplantation and multipliers for Laguerre expansions*, Math. Ann. 300 (1994), 203–219.

[15] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Math. Notes 42, Princeton Univ. Press, Princeton, 1993.

[16] B. Wróbel, *Imaginary powers of a Laguerre differential operator*, Acta Math. Hungar. 124 (2009), 333-351.

[17] B. Wróbel, *On g-functions for Laguerre function expansions of Hermite type*, to appear in Proc. Ind. Acad. Sc. (Math. Sc.), preprint 2010.