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Three types of the wavelet decompositions

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THREE TYPES OF THE WAVELET DECOMPOSITIONS

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Abstract. The wavelet decomposition is an useful tool applicable in the areas related to signal analysis. We will present continuous, wavelet series and discrete wavelet transform (CWT, WST and DWT). DWT is often used to approximate others transformations, since it can be effectively computed numerically. Accuracy of such approximation will be discussed. We will also derive very efficient Mallat's algorithm for finding coefficients in decomposition and show how it can be quickly transferred from L^2 space (signals in time domain) to l^2 space (discrete signals).

1. Introduction

Wavelet transform ([2], [3], [12]) have become well known and very useful tool for various signal processing applications. It allows to achieve the time-frequency analysis of functions representing the real-world signals what is very important in fields such as speech analysis, noise reduction and data processing.

There are three types of wavelet transforms discussed in the literature: the continuous wavelet transform (CWT), the wavelet series transform (WST) and the discrete wavelet transform (DWT). The construction of each of them based on the existence the pair of wavelet functions, ψ and φ, which are linked together. CWT of function s is defined by

$$\tilde{s}_\psi(a,b) = \int_{\mathbb{R}} s(t)\psi\Big(\frac{t-b}{a}\Big)\,dt,$$

while WST is given as

$$d_{m,k} = \int_{\mathbb{R}} s(t)2^{\frac{m}{2}}\psi(2^m t-k)\,dt, \text{ for } m,k\in\mathbb{Z}.$$

Function s could be generally recovered from $\tilde{s}_\psi$ (appropriate theorem will be presented in section 3), in applications however the approximation

$$s(t) \approx \sum_{k\in\mathbb{Z}} c_{m,k}2^{\frac{m}{2}}\varphi(2^m t-k)$$

is usually used, where

$$c_{m,k} = \big\langle s, 2^{\frac{m}{2}}\varphi(2^m t-k)\big\rangle \tag{1.1}$$

and $m\in\mathbb{Z}$ determine a so-called resolution level and have influence on the approximation accuracy.

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CWT and WST are defined for time domain signals and constructed based on $L^2(\mathbb{R})$ inner products what can be also treated as the measurement of the similarities between signal and wavelet functions. In digital signal processing it is common practice that only sampled values of signals are available. In such situations more appropriate approach than CWT or WST is DWT which is adopted to sequences. WST enables to represent given signal $s \in L^2(\mathbb{R})$ (with high precision) as series in basis containing time domain functions, while DWT allows to do the same with discrete signal from $l^2(\mathbb{Z})$ and some basis consisting of sequences.

For both approaches exist very efficient algorithm for finding basis coefficients known as the Mallat's pyramid algorithm [6]. It involves determining the coefficients for lower resolution levels basing on values from higher levels. This means that coefficients for some starting resolution level have to be calculated first, i.e. the initialization step has to be done. For DWT it is reduced to taking sampled values of signal, while initial coefficients for WST have to be approximated by DWT. Common practice is to use sampled values of signal as the initial coefficients in WST. In many situations such procedure may be justified, however it was charged by Strang and Nguyen to be a "wavelet crime" [11]. Several alternative solutions to the problem have been proposed by authors of [11]. One of them is approach based on the Nyquist-Shannon sampling theorem which states that band limited function can be perfectly reconstructed from a countable sequence of samples. DWT can be then employed to reconstructed function defined on some interval. This approach has been analyzed and tested by Abry and Flandrin [1]. They proposed a simple algorithm for performing an initialization of DWT however their procedure is not always effective. The approach through the Nyquist-Shannon theorem was also explored by Zhang et al. in [13] where several numerical examples were given by authors to demonstrate the performance of their two initialization algorithms. Some useful informations can be found also in [10].

One may wonders what is the optimal procedure that may be applied to sampled signal in order to obtain the most accurate WST approximation. This more general approach assumed that some prefiltering on the sampled signal $s[n]$ is first performed instead of putting it in pyramid algorithm directly. It means that the new discrete signal $s_1[n]$, such that $s_1[n] = \sum_m s[n-m]h[m]$ for some sequence (filter) h, is considered. The task is to choose h in such way that WST approximation error would be as small as it is possible. Pu and Francis [8] have shown that starting directly from sampled values can lead to a large error. They gave an optimal filter for initialization, however the sequence obtained as a result is not easily realizable. The initialization problem in that form was considered also in [9], [4] and [5].

2. Hilbert spaces and the Fourier Transform

In this paper we will focus our attention on the real valued functions from $L^2(\mathbb{R})$ and sequences from $l^2(\mathbb{Z})$ space (which we will call the analog and the discrete signal respectively). The notation $\langle \cdot, - \rangle$ will always correspond to the inner product from the first of these Hilbert spaces, i.e.

$$\langle s_1, s_2 \rangle = \int_{\mathbb{R}} s_1(x) s_2(x)\, dx, \quad s_1, s_2 \in L^2(\mathbb{R}).$$

We will also use the notation $\|\cdot\|$ or $\|\cdot\|_{L^2}$ for the norm

$$\|s\| = \|s\|_{L^2} = \sqrt{\int_{\mathbb{R}} s^2(x)\, dx},$$

given by this inner product, and the notation $\|s\|_{l^2}$ for the norm in $l^2(\mathbb{Z})$, i.e.

$$\|s\|_{l^2} = \sqrt{\sum_{n\in\mathbb{Z}} \big(s[n]\big)^2}, \quad \text{for } \{s[n]\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}).$$

Very important in the context of this article is the idea of Riesz basis, especially the orthonormal Riesz basis. In the interests of consistency we will post the relevant definitions here:

Definition 1 (Riesz basis)**.** *Family $\{e_k\}_{k\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is a Riesz basis of $L^2(\mathbb{R})$ if:*

(1) *for all $s \in L^2(\mathbb{R})$ there exist a unique $\alpha \in l^2(\mathbb{Z})$ such that $s = \sum_{k\in\mathbb{Z}} \alpha_k e_k$,*
(2) *there exist $0 < A \le B < \infty$ such that for all $s \in L^2(\mathbb{R})$:*

$$A\|\alpha\|_{l^2} \le \|s\|_{L^2} \le B\|\alpha\|_{l^2}.$$

Definition 2 (Orthonormal Riesz basis)**.** *Family $\{e_k\}_{k\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is an orthonormal Riesz basis of $L^2(\mathbb{R})$ if:*

(1) *$\{e_k\}_{k\in\mathbb{Z}}$ is a Riesz basis*
(2) *for all $i, j \in \mathbb{Z}$ we have $\langle e_i, e_j\rangle = \delta_{i,j}$.*

For the given Riesz basis $\{e_k\}_{k\in\mathbb{Z}}$ and $s \in L^2(\mathbb{R})$ the crucial issue is to find the coefficients $\alpha \in l^2(\mathbb{Z})$ such that $s = \sum_{k\in\mathbb{Z}} \alpha_k e_k$. If $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal Riesz basis then we have simple formula

$$\alpha_k = \langle s, e_k\rangle\,, \ \textit{for all } k \in \mathbb{Z}$$

and following Plancherel identity:

$$\|s\|^2_{L^2} = \|\alpha\|^2_{l^2} = \sum_{k\in\mathbb{Z}} \langle s, e_k\rangle^2\,.$$

This means that we can define isometric isomorphism $\Phi : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ by assigning to $s \in L^2(\mathbb{R})$ its coefficients in basis $\{e_k\}_{k\in\mathbb{Z}}$. One can show that Φ preserves also the inner product, i.e.:

$$\Big\{\Phi(s_1) = \{\alpha_n\}_{n\in\mathbb{Z}},\ \Phi(s_2) = \{\beta_n\}_{n\in\mathbb{Z}}\Big\} \Rightarrow \sum_{n\in\mathbb{Z}} \alpha_n\beta_n = \langle s_1, s_2\rangle\,,$$

thus an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ is mapped to an orthonormal basis $\{\Phi(e_k)\}_{k\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$. In fact all infinite dimensional separable Hilbert spaces are isometrically isomorphic to $l^2(\mathbb{Z})$.

One of the most commonly used tools applicable to signal analysis is the Fourier transform. It can be used to convert a time-dependent function modeling real signal into a function whose argument is frequency. Lets suppose that $s \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (theory can be extended to all $L^2(\mathbb{R})$ space). We will denote the Fourier transform of s by $\hat{s}$ and define as:

$$\hat{s}(\omega) = \int_{\mathbb{R}} s(t)e^{-i\omega t}\, dt.$$

If $\hat{s}$ is also in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ than function s can be recovered from $\hat{s}$ by applying the inverse Fourier transform which can be written as follows:

$$s(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{s}(\omega) e^{i\omega t} \, d\omega.$$

The Fourier transform has a number of very desirable in applications properties which makes it useful in mathematics, physics and engineering. One of them is undoubtedly the inner product preservation property appearing in the literature as Parseval's relation:

$$\int_{\mathbb{R}} s_1(t) s_2(t) \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{s_1}(\omega) \overline{\hat{s_2}(\omega)} \, d\omega,$$

what immediately gives us a so called Plancherel theorem:

$$\int_{\mathbb{R}} s^2(t) \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \hat{s}(\omega) \right|^2 \, d\omega.$$

Lets take a rectangular function $R(t)$ which is equal to $\frac{1}{2}$ for arguments from $[-1, 1]$ and zero otherwise. Using the Fourier transform on function $R(t)$ we have:

$$\hat{R}(\omega) = \int_{\mathbb{R}} R(t) e^{-i\omega t} \, dt = \frac{1}{2} \int_{-1}^{1} e^{-i\omega t} \, dt = \frac{e^{i\omega} - e^{-i\omega}}{2i\omega} = \frac{\sin \omega}{\omega}.$$

Fourier transform of defined above function $R(t)$ appears very often in topics connected with signal analysis and is denoted as

$$\operatorname{sinc}(x) := \frac{\sin x}{x}.$$

In particular function *sinc* is fundamental in the concept of reconstructing the original continuous band limited signal from uniformly spaced samples what is associated with following theorem.

Theorem 1 (Shannon-Nyquist)**.** *Suppose that there exist B such that $\hat{s}(\omega) = 0$ for all ω greater than B and let $\Delta t = \frac{\pi}{B}$ be the sampling rate. Then s can be exactly reconstructed from samples and can represented as*

$$s(t) = \sum_{n \in \mathbb{Z}} f(n\Delta t) \cdot sinc\Big(\frac{\pi x}{B} - \pi n\Big).$$

Shannon-Nyquist theorem has a wide applications. It suggests how densely signal with known boundary of frequency spectrum should be sampled in order to not reduced the information contained in the whole signal.

3. Wavelets decomposition

Fourier transform of function s consists in integrating function s multiplied by $F(\omega) = e^{-i\omega t}$ which is called the kernel. The Fourier transform can be modified by considering kernels with compact supports. This leads to wavelets transforms where two parametric kernels are used allowing to localize given function both in time and frequency. Wavelets provides a new type of function representation enabling local signal analysis.

Kernel functions (wavelets) are constructed based on the function $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \psi(t) \, dt = 0. \tag{3.1}$$

We will also assume that the function ψ is L_2-normalized, i.e. $\int_{\mathbb{R}} \psi^2(t)\, dt = 1$.

By translation and dilation of the function ψ we define wavelets:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\Big(\frac{t-b}{a}\Big), \text{ for } a > 0, b \in \mathbb{R}. \tag{3.2}$$

For all $(a, b) \in (0,\infty) \times \mathbb{R}$ we simply have $\|\psi_{a,b}\|_2 = \|\psi\|_2 = 1$.

The continuous wavelet transform (CWT) of $s \in L(\mathbb{R})$ is the two-arguments function $\tilde{s}_\psi(a, b)$ defined by

$$\tilde{s}_\psi(a,b) := \int_{\mathbb{R}} s(t)\psi_{a,b}(t)\, dt, \text{ for } a > 0, b \in \mathbb{R}. \tag{3.3}$$

Theorem 2 (Daubechies, [2])**.** *Lets suppose that:*

$$C_\psi = 2\int_0^\infty \frac{|\widehat{\psi}(t)|^2}{t}\, dt < \infty. \tag{3.4}$$

Then it is possible to reconstruct function s from its wavelet transform using following formula:

$$s(t) = C_\psi^{-1}\int_{[0,\infty]\times\mathbb{R}} \frac{\tilde{s}_\psi(a,b)}{a^2}\psi_{a,b}(t)\, da\, db.$$

The condition 3.4 leads to $\widehat{\psi}(0) = 0$ which gives $\int_{\mathbb{R}} \psi(t)\, dt = 0$. This explains why it is important to choose ψ such that 3.1 is met.

It turns out (due to fact that L^2 is separable space) that in many cases a countable subsets of wavelets are sufficient to represent signal $s(t)$. For concreteness if we take $a = 2^{-m}$ and $b = 2^{-m}k$ for $m, k \in \mathbb{Z}$ we will get a countable family of wavelets in form:

$$\psi_{m,k}(t) = 2^{\frac{m}{2}}\psi(2^m t - k) \text{ for } m, k \in \mathbb{Z}. \tag{3.5}$$

Lets now suppose for the moment that ψ is a function such that $\{\psi_{m,k}\}_{m,k\in\mathbb{Z}}$ is an orthonormal Riesz base of $L^2(\mathbb{R})$. This means that every function $s \in L^2(\mathbb{R})$ can be represented in the form:

$$s(t) = \sum_{m,k\in\mathbb{Z}} d_{m,k}\psi_{m,k}(t).$$

We will call this form of s as the wavelet series. Coefficients in the wavelet series, given by

$$d_{m,k} := \int_{\mathbb{R}} s(t)\psi_{m,k}(t)\, dt = 2^{\frac{m}{2}}\int_{\mathbb{R}} s(t)\psi(2^m t - k)\, dt \tag{3.6}$$

or equivalently

$$d_{m,k} := \tilde{s}_\psi(2^{-m}, 2^{-m}k),$$

are called the Wavelet Series Transform (WST).

The question of existence orthonormal Riesz base consisting of functions $\{\psi_{m,k}\}_{m,k\in\mathbb{Z}}$ is very important at the context of applications connected with wavelets. This problem, leading to concept of multi-resolution analysis, will be discuss below.

Definition 3 (Multi-resolution analysis)**.** *A multi-resolution analysis of $L^2(\mathbb{R})$ is a family $S = \{S_j\}_{j\in\mathbb{Z}}$ of closed vectorial subspaces, called approximation spaces, which satisfy following conditions:*

(1) $S_j \subset S_{j+1}$ *for* $j \in \mathbb{Z}$,

(2) $\overline{\bigcup_{j\in\mathbb{Z}} S_j} = L^2(\mathbb{R})$,

(3) $\bigcap_{j\in\mathbb{Z}} S_j = \{0\}$,
(4) $s(\cdot) \in S_j \Leftrightarrow s(2\cdot) \in S_{j+1}$,
(5) *there exist* $g \in S_0$ *such that* $\{g(t-k)\}_{k\in\mathbb{Z}}$ *is a Riesz base of* S_0.

We construct also a second family of subspaces, $\{W_j\}_{j\in\mathbb{Z}}$. For each $j \in \mathbb{Z}$ we define W_j as the orthogonal complement of S_j in S_{j+1}, i.e.

$$S_{j+1} = S_j \oplus W_j \text{ and } S_j \perp W_j.$$

We will say that $\{W_j\}_{j\in\mathbb{Z}}$ are detail spaces. The below theorem tells us about connection between the multi-resolutions analysis and issue of orthogonal wavelet bases of $L^2(\mathbb{R})$.

Theorem 3. *Lets suppose that* S *is a multi-resolution analysis of* $L^2(\mathbb{R})$ *and* $g \in L^2(\mathbb{R})$ *is such that* $\{g(t-k)\}_{k\in\mathbb{Z}}$ *is a Riesz base of* S_0. *Then we can construct functions* $\varphi \in L^2(\mathbb{R})$ *and* $\psi \in L^2(\mathbb{R})$ *such that for* $\varphi_{m,k}(t) := 2^{\frac{m}{2}}\varphi(2^m t - k)$ *and* $\psi_{m,k}(t) := 2^{\frac{m}{2}}\psi(2^m t - k)$ *following properties are met:*

(1) $\{\psi_{m,k}\}_{m,k\in\mathbb{Z}}$ *is an orthonormal base of* $L^2(\mathbb{R})$,
(2) *for all* $M \in \mathbb{Z}$, $\Big\{\{\varphi_{M,k}\}_{k\in\mathbb{Z}}, \{\psi_{m,k}\}_{m,k\in\mathbb{Z},m\geq M}\Big\}$ *is an orthonormal base of* $L^2(\mathbb{R})$.

Starting with g we can therefore get (in the constructive way) the pair of functions φ and ψ which generate an orthogonal bases. These functions are called respectively father wavelet (or scaling function) and mother wavelet. Theorem 3 yields another possibility for expression function s as a wavelet series. Indeed, having functions φ and ψ, for $M \in \mathbb{Z}$ we can expand s as:

$$s(t) = \sum_{k\in\mathbb{Z}} c_{M,k}\varphi_{M,k}(t) + \sum_{m\geq M, k\in\mathbb{Z}} d_{m,k}\psi_{m,k}(t) \tag{3.7}$$

for $c_{M,k} = \langle s, \varphi_{M,k}\rangle$ and $d_{m,k} = \langle s, \psi_{m,k}\rangle$ (see 3.6).

Evaluations of coefficients $c_{m,k}$ and $d_{m,k}$ using inners products for wide set of resolution levels are computationally inefficient. We will see however that there exist a simple algorithm for finding values $c_{m,k}$ and $d_{m,k}$ knowing coefficients for higher resolution level (i.e. making a decomposition).

Lets S be a multi-resolution analysis of $L^2(\mathbb{R})$. Since $\varphi(x) \in S_0 \subset S_1 = lin\{\varphi_{1,j} : j \in \mathbb{Z}\}$, there exist $h \in l^2(\mathbb{Z})$ such that:

$$\varphi(x) = \sum_{j\in\mathbb{Z}} h_j \varphi_{1,j}(x) = \sum_{j\in\mathbb{Z}} \sqrt{2} h_j \varphi(2x - j). \tag{3.8}$$

Formula 3.8 is called twin-scale relation (dilation equation or refinement equation). Some basic properties of h can easily be derived. The set $\{\varphi_{1,k} : k \in \mathbb{Z}\}$ is an orthonormal basis of Hilbert space S_1. Thus, thanks to Plancherel identity, we have $\|h\|_{l_2} = \|\varphi\|_{L_2} = 1$. Moreover $\int_{\mathbb{R}} \varphi(t)dt = \frac{\sqrt{2}}{2}\sum_{j\in\mathbb{Z}} h_j \int_{\mathbb{R}} \varphi(y)dy$ which gives $\sum_{j\in\mathbb{Z}} h_j = \sqrt{2}$.

Due to the fact that also $W_0 \subset S_1$ we can expand $\psi(t)$ for some $g \in l^2(\mathbb{Z})$ as:

$$\psi(x) = \sum_{j\in\mathbb{Z}} g_j \varphi_{1,j}(x) = \sum_{j\in\mathbb{Z}} \sqrt{2} g_j \varphi(2x - j). \tag{3.9}$$

For concreteness one can show that $g_j = (-1)^{j-1} h_{-j-1}$ so ψ can be built after straightforward modification of coefficients h_j.

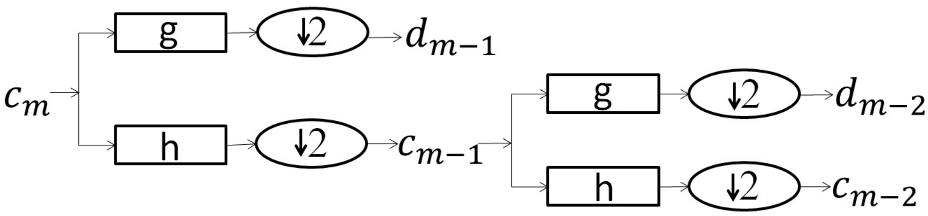

FIGURE 1. Mallat's pyramid algorithm

Using twin-scale relation we get:

$$\begin{aligned}\varphi_{m,k}(t) &= 2^{\frac{m}{2}}\varphi\big(2^m t - k\big) = 2^{\frac{m}{2}}\sum_{j\in\mathbb{Z}} h_j\sqrt{2}\varphi\big(2(2^m t - k) - j\big)\\ &= 2^{\frac{m+1}{2}}\sum_{j\in\mathbb{Z}} h_j\varphi\big(2^{m+1}t - 2k - j\big)\\ &= \sum_j h_j\varphi_{m+1,2k+j}(t) = \sum_j h_{j-2k}\varphi_{m+1,j}(t).\end{aligned} \tag{3.10}$$

Taking M equal to $m+1$ we can expand s in orthogonal basis using (3.7). Keeping in mind that $c_{m,k} = \langle s, \varphi_{m,k}\rangle$, we immediately have

$$c_{m,k} = \sum_j h_j c_{m+1,2k+j}. \tag{3.11}$$

Since (3.9) takes we can similarly get the following relationships:

$$\psi_{m,k} = \sum_j g_{j-2k}\varphi_{m+1,j}, \tag{3.12}$$

$$d_{m,k} = \sum_j g_j c_{m+1,2k+j}. \tag{3.13}$$

In the sequences spaces we can define the operation of decimation $dec(x)$ as $dec(\{x_n\}_{n\in\mathbb{Z}}) = \{x_{2n}\}_{n\in\mathbb{Z}}$. This operator is often denoted by the symbol $\downarrow 2$ and can be used to write formulas (3.11) and (3.13) in the form of convolution [7]:

$$\begin{cases}\{c_{m-1,k}\}_{k\in\mathbb{Z}} = dec\big(\{c_{m,n}\}_{n\in\mathbb{Z}} * \{h_{-n}\}_{n\in\mathbb{Z}}\big),\\ \{d_{m-1,k}\}_{k\in\mathbb{Z}} = dec\big(\{c_{m,n}\}_{n\in\mathbb{Z}} * \{g_{-n}\}_{n\in\mathbb{Z}}\big).\end{cases}$$

Above formulas have significant meaning in the applications. Mallat's pyramidal procedure was illustrated in Figure 1.

We have to mention about few important problems occurring in this moment. First, the initialization step has to be provided in the iterative procedure. This means that we have to approximate $\{c_{M,k}\}$ for some $M \in \mathbb{Z}$ and k in some index set spacious enough (and these computations are time extensive due to integrals occurring in formulas) to get coefficients $c_{m,k}$ and $d_{m,k}$ from lower resolutions level. Taking appropriate M is actually saying that function which we analyzed is in S_M. Such approach is justified since $S_j \subset S_{j+1} \subset \ldots$ and $\overline{\bigcup_{j\in\mathbb{Z}} S_j} = L^2(\mathbb{R})$ but the choice of M is not obvious and has to be dictated by the specific type of examined problem. Moreover we have to work with sampled function s, i.e. we know values of s only on some discrete set of arguments (signal measurements made at specific moments). More adequately is therefore to consider sequences:

$$\{s[n]\}_{n\in\mathbb{Z}} := \{s(n\Delta t)\}_{n\in\mathbb{Z}} \tag{3.14}$$

instead of real domain functions. Above observation raise the issue of discrete signal decompositions in $l^2(\mathbb{Z})$ space.

It turns out that in $l^2(\mathbb{Z})$ the multi-resolution analysis could be constructed in an analogous way as in $L^2(\mathbb{R})$ [3]. We will however derive the required results using the natural connections between both spaces. Lets take $M \in \mathbb{Z}$ such that approximation of s by S_M is sufficient and denote $B_{M,M} := \{\varphi_{M,k}\}_{k\in\mathbb{Z}}$. Since $B_{M,M}$ is the orthonormal basis of S_M, for all $s \in S_M$ we can find exactly one $\alpha^s \in l^2(\mathbb{Z})$ such that $s = \sum_{k\in\mathbb{Z}} \alpha^s_k \varphi_{M,k}$. We can define isometric isomorphism Φ between S_M and $l^2(\mathbb{Z})$:

$$S_M \ni s \longmapsto \Phi(s) = \alpha^s \in l^2(\mathbb{Z}). \tag{3.15}$$

Isomorphism Φ simply gives the basis of $l^2(\mathbb{Z})$ corresponding to $B_{M,M}$. Indeed, we have

$$\varphi_{[M,k]}[n] := \big(\Phi(\varphi_{M,k})\big)[n] = \delta_{k,n}.$$

Lets denote this basis of $l^2(\mathbb{Z})$ as $B_{[M,M]}$. We will also use notation $\varphi_{[m,k]}$ and $\psi_{[m,k]}$ respectively for sequences $\Phi(\varphi_{m,k})$ and $\Phi(\psi_{m,k})$.

For $T < M$ we can choose orthonormal basis of S_M in another way taking $B_{M,T} := \{\varphi_{T,k}\}_{k\in\mathbb{Z}} \cup \{\psi_{m,k}\}_{T\le m<M,k\in\mathbb{Z}}$ and therefore get the orthonormal basis of $l^2(\mathbb{Z})$:

$$B_{[M,T]} := \{\varphi_{[T,k]}\}_{k\in\mathbb{Z}} \cup \{\psi_{[m,k]}\}_{T\le m<M,k\in\mathbb{Z}}. \tag{3.16}$$

These bases are consisting of the so-called scale and wavelet sequences. Using relations (3.10) and (3.12) we can simply specify the recursive algorithm for receiving elements from consecutive bases:

$$\begin{cases} \varphi_{[M,k]}[n] = \delta_{k,n} \\ \varphi_{[m,k]}[n] = \sum_j h_{j-2k}\varphi_{[m+1,j]}[n], \ \textit{for } m < M \\ \psi_{[m,k]}[n] = \sum_j g_{j-2k}\varphi_{[m+1,j]}[n], \ \textit{for } m < M. \end{cases} \tag{3.17}$$

Each $\{s[n]\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ can be express as a series of elements from $B_{[M,T]}$ for $T \le M$, i.e. we can find the coefficients $c_{[T,k]}$ and $d_{[T,k]}$ such that:

$$\begin{aligned} s[n] &= \sum_{k\in\mathbb{Z}} c_{[T,k]}\varphi_{[T,k]}[n] \ \textit{for } T = M, \\ s[n] &= \sum_{k\in\mathbb{Z}} c_{[T,k]}\varphi_{[T,k]}[n] + \sum_{T\le m<M,k\in\mathbb{Z}} d_{[m,k]}\psi_{[m,k]}[n] \ \textit{for } T < M, \end{aligned} \tag{3.18}$$

where

$$\begin{cases} c_{[M,k]} = s[k] \\ c_{[m,k]} = \sum\limits_{n\in\mathbb{Z}} s[n]\varphi_{[m,k]}[n], \ m < M \\ d_{[m,k]} = \sum\limits_{n\in\mathbb{Z}} s[n]\psi_{[m,k]}[n], \ m < M. \end{cases} \tag{3.19}$$

Coefficients $d_{[m,k]}$ for $\{s[n]\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, given by (3.19), are called Discrete Wavelet Transform (DWT) of s. The use of (3.17) and (3.19) quickly gives us the so-called pyramidal algorithm (cascade algorithm) established by Mallat [6] which allows for very efficient calculation of $c_{[m,k]}$ and $d_{[m,k]}$ coefficients:

$$\begin{cases} c_{[M,k]} = s[k], \\ c_{[m,k]} = \sum\limits_{j\in\mathbb{Z}} h_{j-2k} c_{[m+1,j]}, \; m < M \\ d_{[m,k]} = \sum\limits_{j\in\mathbb{Z}} g_{j-2k} c_{[m+1,j]}, \; m < M. \end{cases} \tag{3.20}$$

Comparing above algorithm for discrete wavelet decomposition with relation (3.11) and (3.13) obtained for wavelet series it is obvious that receiving WST and DWT coefficients distinguishes the initialization step only. This factor may cause significant differences between $c_{m,k}$ and $c_{[m,k]}$ or between $d_{m,k}$ and $d_{[m,k]}$. Knowing the exact values of CWT or WST are important in fields such as for example the signal analysis, however in practice these coefficients have to be approximated by the DWT. This raises the question of the accuracy of such approximation and the problem of optimal choice of the initial coefficients basing on discrete values of s. There are several papers dedicated to this issue, such as: [1], [13], [8], [9], [4], [5] and [14]. Let us consider first the situation that for some $M \in \mathbb{Z}$ coefficients $c_{M,k}$ are equal to $c_{[M,k]}$. This take place when: $s(t) = \sum_{k\in\mathbb{Z}} s[k]\varphi_{M,k}(t)$. In general case taking $c_{M,k} = s[k]$ for initialization of pyramidal algorithm (what is commonly applied) is in effect making multi-resolution decomposition not for s but for function $\widetilde{s}$ defined as:

$$\widetilde{s(t)} := \sum_{k\in\mathbb{Z}} s[k]\varphi_{M,k}(t). \tag{3.21}$$

We will show however that taking $s[k]$ for the beginning in the recursion given by (3.11) and (3.13) is not pointless. Indeed we have following theorem:

Theorem 4 (Frazier, [3])**.** *Lets suppose that:*

(1) $s(t) \in L^2(\mathbb{R})$ *and* $\varphi(t) \in L(\mathbb{R}) \cap L^2(\mathbb{R})$*,*
(2) $\int_{\mathbb{R}} \varphi^2(t)\, dt = 1$*,*
(3) $s(t)$ *satisfy Lipschitz condition with constant* L*,*
(4) $C := \int_{\mathbb{R}} |t\varphi(t)|\, dt < \infty$*.*

Then denoting $s_m[k] := a_m f(k2^{-m})$ *for* $a_m = \int_{\mathbb{R}} 2^{-\frac{m}{2}}\varphi(t)\, dt$ *we have:*

$$\Big| \langle s, \varphi_{m,k}\rangle - s_m[k] \Big| \le C \cdot L \cdot 2^{-\frac{3m}{2}}.$$

Proof. For all $m \in \mathbb{R}$ we have $a_m = \int_{\mathbb{R}} 2^{-\frac{m}{2}}\varphi(t)\, dt = \int_{\mathbb{R}} 2^{\frac{m}{2}}\varphi(2^m t - k)\, dt$. It gives:

$$\begin{aligned} \Big| \langle s, \varphi_{m,k}\rangle - s_m[k] \Big| &= \Big| \int_{\mathbb{R}} s(t) 2^{\frac{m}{2}} \varphi(2^m t - k)\, dt - a_m s(k2^{-m}) \Big| \\ &= \Big| \int_{\mathbb{R}} \big(s(t) - s(k2^{-m})\big) 2^{\frac{m}{2}} \varphi(2^m t - k)\, dt \Big| \\ &\le 2^{\frac{m}{2}} \int_{\mathbb{R}} \big|s(t) - s(k2^{-m})\big| \cdot \big|\varphi(2^m t - k)\big|\, dt \\ &= 2^{-\frac{m}{2}} \int_{\mathbb{R}} \big|s(h2^{-m} + k2^{-m}) - s(k2^{-m})\big| \cdot \big|\varphi(h)\big|\, dh \\ &\le 2^{-\frac{3m}{2}} L \int_{\mathbb{R}} \big|h\big| \cdot \big|\varphi(h)\big|\, dh = C \cdot L \cdot 2^{-\frac{3m}{2}} \end{aligned}$$

□

Taking measurements in a suitably small intervals of time we have therefore the guarantee (if the assumptions of Theorem 4 are satisfied and m is big enough) that value of $s[k]$ (defined as (3.14)) is close to the value $c_{m,k}$ of rescaled function s. The analysis of such function and applying pyramidal algorithm is still valid since rescaling does not change the qualitative character of signal. It can be also concluded that assumptions of Theorem 4 are not very strong - in particular any differentiable function with derivative bounded by L meets condition(3) and any reasonable scaling function will satisfy assumption (4).

4. Conclusion

In this paper we presented various types of wavelets transforms for real valued functions. Using natural connections between Hilbert spaces $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$ we showed how DWT approach could be derived from WST, especially how pyramidal algorithm for sampled signals could be obtained from analogous one prepared for real domain functions. Coefficients for CWT and WST can not be calculated exactly and have to be approximated by DWT which is implemented for sequences (discrete signals). We showed that (under some assumptions) taking sequence formed with measurements of signal as initialization step in the Mallat pyramidal algorithm one can approximate the DWT coefficients and we established the maximum error of such approximation for functions satisfying a Lipschitz condition.

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