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On the price of stability in network formation games, a case study

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Abstract

Exact bounds on the price of stability in global congestion game constitute an open problem, the best known upper bound is $O(\log n)$ and the highest lower bound equals 2.245. The case of planar graphs is also open, and in an attempt to close this gap I analyze instances where the underling graph is a cycle. The main contribution of this article is the proof that in the considered setting the price of stability is exactly 3/2. This is a part of joint work with Gianpiero Monaco and Piotr Sankowski.

1 Introduction

If there is a single example that embodies the essence of most important strategic decisions, then resource usage and allocation would be it. Which type of transport to use, what sort of software and hardware to order, what kind of customer range to aim at—these are all decisions that companies struggle with every day. Even in our private, daily lives we seek the shortest traffic routes or pick entertainments enjoyable to ourselves and at the same time easily shared within friends. The class of game-theoretical models that can be used to analyze such situations are called *congestion games*.

This family is especially noteworthy for it coincides with *potential games* and thus it is guaranteed for a pure equilibrium to exist. However, as it is hard to investigate a game with an abstract cost function, we turn to a subclass called network formation games with fair cost allocation.

Consider a number of companies that want to transfer goods between their branches in cities across the country. As not all the pairs of towns can be connected and each mile of a road is very expensive, they want to share the burden of new highway system and agree that the cost of section will be split equally between companies that are going to use it. In this setting a strategy of a player (company) is a set of paths the cargo will be routed through. The network to be build is formed by their decisions.

Naturally, there is an optimal solution that maximizes social welfare (minimizes the total cost of highway system), however, selfish players may choose to deviate from it with regard for their own profit only. The problem arises, how bad the outcome may be? The stable states will be precisely the Nash equilibria, and in game-theoretic wording the last question asks for *the price of anarchy* and *the price of stability* of corresponding game, that is for the ratio of, respectively, the worst and the best Nash equilibrium with respect to the optimal solution.

This article attempts to address the open question of the price of stability of undirected graphs, specifically the case of planar graphs. The paper's contribution is the exact bound for rings, that hopefully may be used for proving the general case. The rest of the document is organized as follows. The next section contains the definitions of the already informally introduced concepts. Related results are described in chapter 3. The core work is presented in the section 4 and the last part discusses possible avenues for future research.

2 Definitions and basic concepts

2.1 Congestion games

Let E be a set of resources that are to be used by n players [1]. Each player $i \in [n]$ has a finite set of possible strategies $S_i \subseteq 2^E$, mixed strategies are not allowed unless explicitly stated otherwise. For each state $s = \langle s_i \rangle_{i \in [n]}$ where $s_i \in S_i$, we define a load function $d_s : E \to \mathbb{N}$ that denotes the number of players using resource $e \in E$ in state s, that is $d_s(e) = \#\{i \mid e \in s_i\}$.

The game is called a congestion game if for each $e \in E$ there is a cost function $\hat{c}_e : \mathbb{N} \to \mathbb{R}^+$ such that the payment of a player i in state s can be given by

$$u_i(s) = \sum_{e \in s_i} \hat{c}_e \left(d_s(e) \right).$$

To avoid confusion, it is worth emphasizing that functions u_i denote *the cost*, not the utility. For this reason players try to minimize u instead of maximizing it, as it is frequently the case in papers in the domain of game theory. Also, functions $\hat{c}_e$ define individual prices (the same for all players), not the total payments of whole group using the resource e, which will be denoted later as c_e, that is, without the hat. Finally, as in all games considered in this text, the set of players is finite, and so is the set of all the states.

A game is called potential game if there exists potential function compatible with it. Let $\mathscr{S}$ be the set of all the states, $\mathscr{S} = \{\langle s_i \rangle_{i \in [n]} \mid s_i \in S_i\}$. A function $\Phi : \mathscr{S} \to \mathbb{R}^+$ is called

- *ordinal* potential function if $u_i(s) - u_i(r) > 0 \iff \Phi(s) - \Phi(r) > 0$,
- *weighted* potential function if $u_i(s) - u_i(r) = w_i(\Phi(s) - \Phi(r))$,
- *exact* potential function if $u_i(s) - u_i(r) = \Phi(s) - \Phi(r)$,

for any player $i \in [n]$ and any two neighbouring states $s, r \in \mathscr{S}$, that means states that differ only by strategy of the i-th player, $s_j = r_j \iff j \neq i$ for all $j \in [n]$.

Theorem 2.1 (Rosenthal [8]). *Every congestion game admits an exact potential function, namely*

$$\Phi(s) = \sum_{e \in E} \sum_{k=0}^{d_s(e)} \hat{c}_e(k) \tag{1}$$

Proof. Let s and r be two states that differ only by strategy of i-th player. Set

$$\begin{aligned} E_\varnothing &= \overline{s_i \cup r_i} = \{e \in E \mid e \notin s_i, e \notin r_i\} \\ E_s &= s_i \setminus r_i = \{e \in E \mid e \in s_i, e \notin r_i\} \\ E_r &= r_i \setminus s_i = \{e \in E \mid e \notin s_i, e \in r_i\} \\ E_{sr} &= s_i \cap r_i = \{e \in E \mid e \in s_i, e \in r_i\} \end{aligned}$$

[1]The set of players $\{1, 2, \ldots, n\}$ will be denoted by $[n]$.

then

$$\Phi(s) = \sum_{e \in E_\varnothing} \sum_{k=0}^{d_s(e)} \hat{c}_e(k) + \sum_{e \in E_s} \sum_{k=0}^{d_s(e)} \hat{c}_e(k) + \sum_{e \in E_r} \sum_{k=0}^{d_s(e)} \hat{c}_e(k) + \sum_{e \in E_{sr}} \sum_{k=0}^{d_s(e)} \hat{c}_e(k)$$

and

$$\begin{aligned} \forall e \in E_\varnothing.\ d_s(e) &= d_r(e) \\ \forall e \in E_s.\ d_s(e) &= d_r(e) + 1 \\ \forall e \in E_r.\ d_s(e) &= d_r(e) - 1 \\ \forall e \in E_{sr}.\ d_s(e) &= d_r(e) \end{aligned}$$

so

$$\begin{aligned} \Phi(s) - \Phi(r) &= \sum_{e \in E_s} \sum_{k=0}^{d_s(e)} \hat{c}_e(k) - \sum_{e \in E_s} \sum_{k=0}^{d_r(e)} \hat{c}_e(k) + \sum_{e \in E_r} \sum_{k=0}^{d_s(e)} \hat{c}_e(k) - \sum_{e \in E_r} \sum_{k=0}^{d_s(e)} \hat{c}_e(k) \\ &= \sum_{e \in E_s} \hat{c}_e(d_s(e)) - \sum_{e \in E_r} \hat{c}_e(d_r(e)) \\ &= \sum_{e \in E_{sr}} \hat{c}_e(d_s(e)) + \sum_{e \in E_s} \hat{c}_e(d_s(e)) - \sum_{e \in E_{sr}} \hat{c}_e(d_r(e)) + \sum_{e \in E_r} \hat{c}_e(d_r(e)) \\ &= u_i(s) - u_i(r) \end{aligned}$$

□

The following theorem is not of major significance for this article and was included, without a proof, for the sake of completeness.

Theorem 2.2 (Monderer and Shapley [6]). *Every finite exact potential game is isomorphic to a congestion game.*

On the other hand, the next result constitutes one of the reasons why the potential games have received so much attention.

Theorem 2.3. *In a potential game, if state $s \in \mathcal{S}$ is a local minimum with respect to the potential function, then it is a pure Nash equilibrium.*

Proof. Being a local minimum means that for every $r \in \mathcal{S}$ that differs from s by a strategy of a single player we have $\Phi(s) \leq \Phi(r)$. However, by the definition of potential function, we have $u_i(s) > u_i(r) \iff \Phi(s) > \Phi(r)$, so no player has an incentive to deviate and this completes the proof. □

Corollary 2.4. *In a potential game, the state $s \in \mathcal{S}$ that minimizes the potential is a pure Nash equilibrium.*

Corollary 2.5. *Every congestion game has a pure Nash equilibrium.*

Proof. The corollary follows directly from the fact that the space of states is finite, thus there exists a state that minimizes the potential. □

Corollary 2.6. *In congestion games the best-response dynamic always converges to a pure Nash equilibirum.*

Proof. The best-response dynamic is a process in which the potential decreases over time. In a game with finite space of states it has to converge to some local minimum. □

2.2 Network formation games

Let $G = \langle V, E \rangle$ be an undirected graph and $c : E \to \mathbb{R}^+$ be the cost function. Further, let n denote the number of players, each of whom has two vertices p_i, q_i to connect, while the set of strategies of the i-th player $S_i \subseteq 2^E$ is the set of simple paths from p_i to q_i. This setting, along with the cost functions

$$u_i(s) = \sum_{e \in s_i} \frac{c(e)}{d_s(e)}$$

will be called the *global connection game* with fair cost allocation. In this context the cost $c(e)$ of the edge e is paid only once for all the players, i.e. it is shared among them. The cost $\mathrm{cost}(s)$ of a state s is given by the total price paid by all the players in the considered state:

$$\mathrm{cost}(s) = \sum_i \sum_{e \in s_i} \frac{c(e)}{d_s(e)}$$

which is equal to the sum of costs of all used edges:

$$\mathrm{cost}(s) = \sum_{e \in E,\ d_s(e) > 0} c(e)$$

Theorem 2.7. *In global connection game*

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{d_s(e)} \frac{c(e)}{k} = \sum_{e \in E} c(e) \mathcal{H}_{d_s(e)} \tag{2}$$

is an exact potential function[2].

Proof. Global connection game is a congestion game with individual price functions given by

$$\hat{c}_e(k) = \begin{cases} \frac{c(e)}{k}, & \text{for } k > 0 \\ 0, & \text{otherwise} \end{cases}$$

and in this setting the function Φ is equal to its counterpart in equation (1), thus is an exact potential function. □

Let OPT $\in \mathcal{S}$ be the state that minimizes the global cost function. Obviously, the edges with positive load will form a Steiner forest of minimum-cost. However, it might not be a Nash equilibrium. This gives a rise to the concepts of price of anarchy (PoA) and price of stability (PoS). If we denote the set of states that are Nash equilibria by $\mathrm{NE} \subseteq \mathcal{S}$, then

$$PoA = \frac{\max_{s \in \mathrm{NE}} \mathrm{cost}(s)}{\mathrm{cost}(\mathrm{OPT})}$$

$$PoS = \frac{\min_{s \in \mathrm{NE}} \mathrm{cost}(s)}{\mathrm{cost}(\mathrm{OPT})}$$

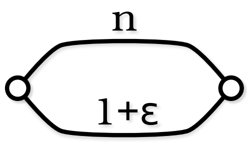
The above notions have nice practical interpretation that justifies their names. In fact the price of anarchy measures how much we could lose if there would be no control whatsoever in the system and all the players would be motivated only by their costs and utilities. Analogously, the price of stability compares the optimal solution to the best possible stable state—how much we have to lose to be in a state in which no player deviates from his strategy.

[2] $\mathcal{H}_k$ denotes the k-th harmonic number with $\mathcal{H}_0$ set to 0.

3 Related work

The two aforementioned prices concepts have received significant amount of attention in recent years. However, the following result gives some idea why the price of anarchy is not as useful as the price of stability. More detailed information on the subject can be found in chapter 19 of [7].

Theorem 3.1. *The price of anarchy for network formation game is $\Omega(n)$, where n denotes the number of players involved.*



Proof. Consider the graph presented above. Let the cost of the upper edge be n and set the cost of the lower edge to $1+\varepsilon$ for some small $\varepsilon > 0$. Then, a strategy in which all the n players use the upper edge is a Nash equilibrium, while the cost of optimum strategy equals to $1+\varepsilon$ instead of n. Thus, $PoA = \Omega(n)$. □

The price of stability has much more promising bounds, e.g. the one implied by potential function method. The original proof comes from [1].

Theorem 3.2. *In global connection game with fair cost allocation with n players, the price of stability is at most $O(\log n)$.*

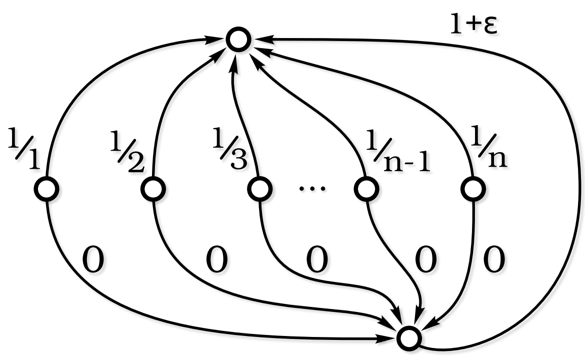
Proof. First, observe that for any state $\text{cost}(s) \le \Phi(s) \le \mathcal{H}_n \text{cost}(s)$. It follows, that

$$\frac{\min_{s\in NE}\text{cost}(s)}{\text{cost}(\text{OPT})} \le \frac{\text{cost}(\text{MIN})}{\text{cost}(\text{OPT})} \le \frac{\mathcal{H}_n\Phi(\text{MIN})}{\Phi(\text{OPT})} \le \frac{\mathcal{H}_n\Phi(\text{MIN})}{\Phi(\text{MIN})} \le \mathcal{H}_n = O(\log n).$$

□

It happens, that for the directed graph, the bound is tight, that is there exists a family of instances with $PoS = \mathcal{H}_n$ for any number n of players.

Theorem 3.3. *The network presented below with n players connecting middle vertices to the top sink, has the price of stability equal to $\mathcal{H}_n$.*



Proof. The optimal solution is the one when all the players route through the bottom vertex and the edge with cost $1+\varepsilon$. However, it is not a Nash equilibrium, because $\frac{1+\varepsilon}{n} > \frac{1}{n}$, thus the n-th player has an incentive to use the direct edge. For similar reasons all players will deviate, arriving altogether at the opposite case, the only Nash equilibrium in this instance. It follows that the price of stability tends to $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}+\frac{1}{n}=\mathscr{H}_n$ as ε tends to 0. □

Finally, the most relevant result is the lower bound on the price of stability for undirected graphs. The proof presented in [2] is very technical and will not be reproduced here.

Theorem 3.4. *The price of stability in global connection game is at least* 2.245.

There are two special cases in which there are known better bounds than for general game, namely the broadcast game with $PoS = O(\log\log n)$ [4] and the multicast setting where $PoS = O(\log n/\log\log n)$ [5]. Finally, an overview of network formation games can be found in [7].

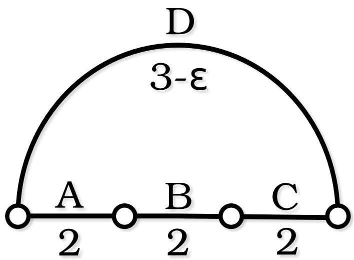
4 Cycle game

In this section we will consider a cycle game, that is a global connection game with fair allocation in which the underlying graph is a single cycle. This is a major simplification of the general setting and it is the first non-trivial one, since in a tree there is only one simple path between any two vertices. However, the problem is not easy—the proof of upper bound is long and tedious, and would be even worse without the help of computer.

Theorem 4.1. *The price of stability in the cycle game is at least* $\frac{3}{2}$.

Proof. Fix an epsilon, $0<\varepsilon<1$. We will construct an instance with only one Nash equilibrium with cost $\frac{3}{2}-\frac{\varepsilon}{6}$ times the cost of optimal solution.

Consider a cycle consisting of 4 edges $E=\{A,B,C,D\}$ as shown on a picture below. The vertices from left to right are 1, 2, 3 and 4. The numbers near the edges describe the cost function, i.e. $c(D)=3-\varepsilon$ and $c(e)$ is equal two for $e\in\{A,B,C\}$.



There are four players $\{1,2,3,4\}$ each of them trying to connect the vertices of corresponding edge:

player	edge	source	target
1	A	1	2
2	B	2	3
3	C	3	4
4	D	4	1

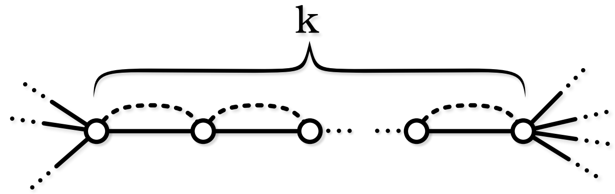
Observe, that in any state, the edges with positive load form a connected graph. It follows, that the optimal state OPT induces a minimal spanning tree, specifically $\{A, B, C\}$. Let s be the state in which each player uses its corresponding edge. It remains to show, that s is the only Nash equilibrium, the key fact being that in this instance of cycle game the potential function has only one local minimum, i.e. the global minimum MIN.

However, by Lemma 4.2 presented below, we can see that players 1, 2 and 3 have to use edges A, B and C respectively. Moreover, OPT is a neighbour of s and $\Phi(\text{OPT}) > \Phi(s)$ because $u_4(\text{OPT}) > u_4(s)$

$$\frac{2}{2} + \frac{2}{2} + \frac{2}{2} > \frac{3-\varepsilon}{1}.$$

This implies that $s = \text{MIN}$ and completes the proof. □

Consider a setting in which there are $k \geq 3$ players distributed on a part of a graph as marked by the dashed lines in the picture below, that means for every two consecutive vertices of some path of length k there is exactly one player that wants to connect them.



The strategy depicted above with dashed lines will be called the basic strategy, and the cost of the edges used by the i-th player in it will be denoted by C_i.

Lemma 4.2. *Let us denote the smallest cost of connecting the first and the last vertex of the path without using it, i.e. through the rest of the graph, by $\varepsilon > 0$ (if no such path exists, then $\varepsilon = \infty$). If there are no other players' sources or targets on the path and $C_i \leq \frac{1}{3}\sum_i C_i$ for every i, then the basic strategy is the best one for all the k players, independently of any other players.*

Proof. Let m be the number of other players using the path and set $C = \sum_i C_i$. Further, denote the count of players who deviated by k' and the total cost of edges used by them in the basic strategy by $\hat{C}$. In the considered setting the following inequalities hold

$$\begin{aligned}
&m + 2 + k' \leq 2(m + 1 + k'),\\
&\frac{1}{m+1+k'} \leq \frac{2}{m+2+k'},\\
&\frac{C_i}{m+1+k'} \leq \frac{2C_i}{m+2+k'}\\
&\qquad\qquad\quad \leq \frac{C - C_i}{m+2+k'} = \frac{C - C_i - \hat{C}}{m+2+k'} + \frac{\hat{C}}{m+2+k'}\\
&\qquad\qquad\quad < \frac{C - C_i - \hat{C}}{m+2+k'} + \frac{\hat{C}}{m+k'} + \varepsilon.
\end{aligned}$$

The outer inequality is equivalent to $u_i(s) < u_i(r)$, where s is the state in which player i uses the basic strategy and in r a different one. It also means that $\Phi(s) < \Phi(r)$ and this completes the proof. □

Theorem 4.3. *The price of stability for the cycle game is at most $\frac{3}{2}$.*

The proof of theorem 4.3 will be split into several cases, depending on the count m of steps in which some fixed best-response dynamic reaches a Nash equilibrium starting from the optimal state OPT.

The common notation includes the number m defined above, the Nash equilibrium N obtained after m steps, as well as the names of players, meaning that i denotes the i-th different player that made the move during the dynamic run (those that stayed dormant will be referred to as "the rest"). Also, let $\mathscr{I}$ be a set of players of interest, then for any $A \subseteq \mathscr{I}$ the set D_A will mean the edges used exclusively in OPT by all the players in A, formally

$$D_A = \{e \in E \mid \forall i \in \mathscr{I}.\ e \in \mathrm{OPT}_i \iff i \in A, \neg\exists i \in \overline{\mathscr{I}}.\ e \in \mathrm{OPT}_i\},$$

while R_A will describe the edges used by players in A and at least one player from outside of $\mathscr{I}$, that is

$$R_A = \{e \in E \mid \forall i \in \mathscr{I} e \in OPT_i \iff i \in A, \exists i \in \overline{\mathscr{I}}.\ e \in \mathrm{OPT}_i\}.$$

Proof of theorem 4.3, case $m = 0$. The equality $m = 0$ trivializes the instance into an example where OPT is a Nash equilibrium, thus $PoS = 1$. □

Proof of theorem 4.3, case $m = 1$. The condition that the first player wants to move in OPT is

$$\sum_{e \in \mathrm{N}_1} \frac{c(e)}{d_{\mathrm{N}}(e)} < \sum_{e \in OPT_1} \frac{c(e)}{d_{\mathrm{OPT}}(e)}.$$

Setting $\mathscr{I} = \{1\}$ we can write

$$\sum_{e \in D_\varnothing} \frac{c(e)}{1} + \sum_{e \in R_\varnothing} \frac{c(e)}{d_{\mathrm{OPT}}(e)+1} < \sum_{e \in D_{\{1\}}} \frac{c(e)}{1} + \sum_{e \in R_{\{1\}}} \frac{c(e)}{d_{\mathrm{OPT}}(e)}$$

and with small abuse of the notation[3] simplify the above into

$$\frac{c(D_\varnothing)}{1} + \frac{c(R_\varnothing)}{\infty} < \frac{c(D_{\{1\}})}{1} + \frac{c(R_{\{1\}})}{2}.$$

The last inequality implies directly that

$$\frac{\mathrm{cost(N)}}{\mathrm{cost(OPT)}} = \frac{c(D_\varnothing) + c(R_\varnothing) + c(R_{\{1\}})}{c(D_{\{1\}}) + c(R_\varnothing) + c(R_{\{1\}})} \le \frac{c(D_{\{1\}}) + c(R_\varnothing) + \frac{3}{2}c(R_{\{1\}})}{c(D_{\{1\}}) + c(R_\varnothing) + c(R_{\{1\}})} \le \frac{3}{2}.$$

□

[3] The cost of a set of edges is defined as sum of their costs, that is $c(A) = \sum_{e \in A} c(e)$. The fraction with denominator equal to ∞ denotes 0.

To make the pattern more distinct, let us define functions $\text{left}_k, \text{right}_k : \mathscr{S} \to \mathbb{R}$ for any players $k \in \mathscr{I}$. Set $\mathscr{J}_k$ to be the collection of all the subsets of $\mathscr{I}$ that contain k, i.e. $\mathscr{J}_k = \{A \in 2^{\mathscr{I}} \mid k \in A\}$. Also, define the usage by the players' of intrest $\hat{d}_s : 2^E \to \mathbb{N}$ as $\hat{d}_s(A) = \#\{i \in \mathscr{I} \mid A \subseteq s_i\}$. Then, for

$$\text{left}_k(s) = \sum_{A \in \mathscr{J}_k} \sum_{e \in D_A} \frac{c(e)}{d_s(e)} + \sum_{A \in \mathscr{J}_k} \sum_{e \in R_A} \frac{c(e)}{\infty} = \sum_{A \in \mathscr{J}_k} \frac{c(D_A)}{\hat{d}_s(D_A)},$$
$$\text{right}_k(s) = \sum_{A \in \mathscr{J}_k} \sum_{e \in D_A} \frac{c(e)}{d_s(e)} + \sum_{A \in \mathscr{J}_k} \sum_{e \in R_A} \frac{c(e)}{\hat{d}_s(R_A)+1} = \sum_{A \in \mathscr{J}_k} \frac{c(D_A)}{\hat{d}_s(D_A)} + \frac{c(R_A)}{\hat{d}_s(R_A)+1},$$

the following inequalities hold for any $s \in \mathscr{S}$:

$$\text{left}_k(s) \leq u_k(s) \leq \text{right}_k(s). \tag{3}$$

The role of functions left_u and right_u is to weaken the inequalities between player's utilities in some neighbouring states, so that they become manageable. As we don't know the exact usage of edges in sets R_A, it would be hard to derive the precise bounds. However, we can assume the worst case. This means that on the left-hand side we introduce the maximum possible number of dormant players and on the right-hand side we make it as small as possible, e.g. for optimal state OPT we have $\hat{d}_{\text{OPT}}(R_A) = |A| + 1$. The next case will demonstrate the usage of functions left_k and right_k.

Proof of theorem 4.3, case $m = 2$. In a setting in which $m = 2$, the following must hold

$$\Phi(\text{N}) < \Phi(s) < \Phi(\text{OPT})$$

where $s \in \mathscr{S}$ is the state between OPT and N, and it implies the next set of formulae

$$\begin{array}{ccccccc} \text{left}_1(s) & \leq & u_1(s) & < & u_1(\text{OPT}) & \leq & \text{right}_1(\text{OPT}), \\ \text{left}_2(\text{N}) & \leq & u_2(\text{N}) & < & u_2(s) & \leq & \text{right}_2(s). \end{array}$$

Intuitively it means that in state OPT player 1 has an incentive to move, and in the next step it is the player 2 who wants to change his strategy. The outer set of inequalities written explicitely presents as below:

$$\frac{c(D_\varnothing)}{1} + \frac{c(R_\varnothing)}{\infty} + \frac{c(D_{\{2\}})}{2} + \frac{c(R_{\{2\}})}{\infty} < \frac{c(D_{\{1\}})}{1} + \frac{c(R_{\{1\}})}{2} + \frac{c(D_{\{1,2\}})}{2} + \frac{c(R_{\{1,2\}})}{3}$$
$$\frac{c(D_\varnothing)}{2} + \frac{c(R_\varnothing)}{\infty} + \frac{c(D_{\{1\}})}{1} + \frac{c(R_{\{1\}})}{\infty} < \frac{c(D_{\{2\}})}{2} + \frac{c(R_{\{2\}})}{2} + \frac{c(D_{\{1,2\}})}{1} + \frac{c(R_{\{1,2\}})}{2}$$

Without the loss of generality we could add the following constraints:

$$\sum_{e \in OPT} c(e) \leq 1,$$

$$\forall e \in E.\ c(e) \geq 0.$$

In this setting, maximizing $c(D_\varnothing) - \frac{1}{2}c(D_{\{1,2\}})$ with respect to the above inequalities yields

$$c(D_\varnothing) - \frac{1}{2}c(D_{\{1,2\}}) \leq \frac{5}{11} < \frac{1}{2}.$$

Further, the state N after two steps is a Nash equilibrium, where

$$\text{cost(N)} = c(D_\varnothing) + c(R_\varnothing) + c(D_{\{1\}}) + c(R_{\{1\}}) + c(D_{\{2\}}) + c(R_{\{2\}}) + c(R_{\{1,2\}}),$$
$$\text{cost(OPT)} = c(D_{\{1,2\}}) + c(R_\varnothing) + c(D_{\{1\}}) + c(R_{\{1\}}) + c(D_{\{2\}}) + c(R_{\{2\}}) + c(R_{\{1,2\}}).$$

Combining the upper bound on $c(D_\varnothing) - \frac{1}{2}c(D_{\{1,2\}})$ with formulae for cost(N) and cost(OPT) we obtain

$$\frac{\text{cost(N)}}{\text{cost(OPT)}} \leq \frac{16}{11} < \frac{3}{2}$$

□

Lemma 4.4. *In cycle game, if there are two or more neighbouring states of* OPT *with smaller potential, then* $PoS \leq \frac{3}{2}$.

Proof. Fix two neighbour states of OPT s and r, and fix two players α and β that change strategy in those state respectively. Let $\mathscr{I} = \{\alpha, \beta\}$, then

$$\frac{c(D_\varnothing)}{1} + \frac{c(R_\varnothing)}{\infty} + \frac{c(D_{\{\beta\}})}{2} + \frac{c(R_{\{\beta\}})}{\infty} < \frac{c(D_{\{\alpha\}})}{1} + \frac{c(R_{\{\alpha\}})}{2} + \frac{c(D_{\{\alpha,\beta\}})}{2} + \frac{c(R_{\{\alpha,\beta\}})}{3}$$
$$\frac{c(D_\varnothing)}{1} + \frac{c(R_\varnothing)}{\infty} + \frac{c(D_{\{\alpha\}})}{2} + \frac{c(R_{\{\alpha\}})}{\infty} < \frac{c(D_{\{\beta\}})}{1} + \frac{c(R_{\{\beta\}})}{2} + \frac{c(D_{\{\alpha,\beta\}})}{2} + \frac{c(R_{\{\alpha,\beta\}})}{3}$$

Summing the above up we get

$$\frac{2c(D_\varnothing)}{1} < \frac{c(D_{\{\alpha,\beta\}})}{1} + \frac{c(D_{\{\alpha\}}) + c(D_{\{\beta\}}) + c(R_{\{\alpha\}}) + c(R_{\{\beta\}})}{2} + \frac{2c(R_{\{\alpha,\beta\}})}{3} \leq \text{cost(OPT)}$$

and that completes the proof. □

Proof of theorem 4.3, case $m = 3$. Analogously to previous case, we will construct a suitable linear program. Let p, q, r, s be the consecutive steps in the best-response dynamic run, specifically $p = \text{OPT}$ and $s = \text{N}$. Let α, β, γ denote the players moving.

If $\alpha = 1 = \gamma$ then s is a neighbour of p with $\Phi(s) < \Phi(p)$, therefore there are two players, namely α and β that have an incentive to change their strategy in OPT and by Lemma 4.4 $PoS \leq \frac{3}{2}$.

Let us assume that $1 = \alpha \neq \gamma = 3$, then for $\mathscr{I} = \{1,2,3\}$ the following inequalities:

$$\Phi(p) < \Phi(q) < \Phi(r) < \Phi(s)$$

can be simplified to

$$\text{left}_1(q) < \text{right}_1(p)$$
$$\text{left}_2(r) < \text{right}_2(q)$$
$$\text{left}_3(s) < \text{right}_3(r)$$

Along with cost(OPT) ≤ 1 and maximization target $c(D_\varnothing) - \frac{1}{2}c(D_{\{1,2,3\}})$ this constitutes a linear programming with a solution

$$c(D_\varnothing) - \frac{1}{2}c(D_{\{1,2,3\}}) \leq \frac{2}{5} < \frac{1}{2}.$$

Substituting it into the ratio of the costs we arrive at

$$\frac{\text{cost(N)}}{\text{cost(OPT)}} \leq \frac{7}{5} < \frac{3}{2}$$

□

Proof of theorem 4.3, case $m \geq 4$. This case is different, because we don't know m and thus whether $D_{\{1,2,3,4\}}$ belongs to N or not. However it suffices to consider maximization target $c\left(D_{\varnothing}\right)$ subject to the usual set of constraints:

$$\text{left}_1(q) < \text{right}_1(p) \qquad \text{left}_2(r) < \text{right}_2(q)$$
$$\text{left}_3(s) < \text{right}_3(r) \qquad \text{left}_\delta(t) < \text{right}_\delta(s)$$

and

$$\text{cost(OPT)} \leq 1,$$

where p,q,r,s,t are states of the best-response dynamic run and $\alpha = 1, \beta = 2, \gamma = 3, \delta \in \{1,2,4\}$ are corresponding moving players (if $\gamma = 1$, then $PoS \leq \frac{3}{2}$ from Lemma 4.4).
For $\delta = 1$, $\delta = 2$ and $\delta = 4$ the bounds are $\frac{3}{7}$, $\frac{63}{131}$ and $\frac{114}{253}$ respectively. The $PoS \leq \frac{3}{2}$ is implied by

$$\frac{\text{cost(N)}}{\text{cost(OPT)}} \leq \frac{\text{cost(OPT)} + c\left(D_{\varnothing}\right)}{\text{cost(OPT)}} \leq \frac{\text{cost(OPT)} + \lambda \cdot \text{cost(OPT)}}{\text{cost(OPT)}} \leq 1 + \lambda < \frac{3}{2}$$

where $\lambda \in \left\{\frac{3}{7}, \frac{63}{131}, \frac{114}{253}\right\}$.

□

Corollary 4.5. *The price of stability for the cycle game equals $\frac{3}{2}$.*

The unfortunate thing is that the presented proof does not give any intuition behind the bound. However, it happens that the solutions to the linear programs are very regular, namely (almost) any instance with m players can be transformed into another instance with at least the same bound such that $D_{\{1\}}, D_{\{1,2\}}, \ldots, D_{\{1,\ldots,m\}}$ are the only edges with positive costs. The curious thing is that optimal costs $c\left(D_{\{1,\ldots,m\}}\right), \ldots, c\left(D_{\{1,2\}}\right), c\left(D_{\{1\}}\right)$ approximate function $\frac{1}{\pi}\sqrt{\frac{1}{x} - 1}$, which is equal to $\frac{1}{2}$ times the probability density function of β-distribution with coefficients $\frac{1}{2}$ and $\frac{3}{2}$. Nevertheless, no deep connection formalizing this aspect has yet been found.

5 Future work

This is a part of joint work that attempts to close the gap between the lower and upper bounds in global connection games with fair cost allocation. The cycle case is crucial in the sense that it is the first non-trivial class of graphs with regard to the price of stability (in trees obviously $PoS = 1$).

It was shown that in this setting the price of stability is equal to $\frac{3}{2}$. Please observe, that the main result can be interpreted in the following way: if the cost of some set of edges (namely $c\left(D_{\varnothing}\right)$) is big enough, then there exists a Nash equilibrium that does not uses it.

There is a possibility that the above could be used for bounding the price of stability for games on planar graphs. Every such graph can be decomposed into faces, each of which gives a rise to a cycle, and maybe then, the edges that are not used in optimal solution might be bounded in terms of cost(OPT). This leads us to the conjecture stated below.

Conjecture 5.1. *The price of stability for global connection game on planar graph is $O(1)$.*

There is yet another area in which the study of the price of stability can help, namely the computational complexity of finding a pure Nash equilibrium. It is known, that this problem in a general network formation game is PLS-complete [3]. On the other hand, this not seem to be the case for the class of cycle games—the experimental tests show that a modification of best-response dynamic might actually converge in linear time. The performed investigation leads me to believe that the two problems are highly related, and leaves hope, that the underlying structure of Nash equilibria might be spotted in polynomial time.

Conjecture 5.2. *There is a polynomial time algorithm for calculating a pure Nash equilibrium in the cycle game.*

In present moment it is far from clear how to approach above propositions in a systematic way, however, the work presented in this paper shows some significant insights that will definitely help in future research.

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