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Properties of amenability of group actions

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Properties of amenability of group actions

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In this paper we will consider a generalisation of the notion of amenability of a group, which is also connected with the property A. *Amenability of an action* of a group on a topological space has been the topic of research for Brodzki, Niblo, Nowak and Wright in a series of their papers from the previous years ([2], [3], [4]), where they showed that it is equivalent to (non-)triviality of a particular class in the cohomology (homology) group with coefficients in the appropriate Banach spaces. This is a generalization of the equivalence between the amenability of a group and non-vanishing of the Block-Weinberger homology.

We will show a theorem connecting, similarly as in the case of classical amenability, the existence of a sequence of almost invariant functions (a *Reiter sequence*) with the existence of a net of almost invariant subsets (a *Følner net*). We also regard actions on non-compact spaces and, under certain (unfortunately quite strong) additional assumptions, we simplify the notion of amenability for such actions.

1 Introduction - amenable actions

Notation 1. For every function $f : G \times X \to \mathbb{R}$ we denote

$$f_g(x) = f(g, x),$$
$$\operatorname{supp}_G f = \{g \in G \,:\, f_g \neq 0\}.$$

The set $\operatorname{supp}_G f$ will be called the *G-support* of the function f. If it is finite, we define

$$\tilde{\sigma}(f) = \sum_{g \in G} f_g.$$

Definition 2. Let a group G act on a topological space X. Let $W_{00} = W_{00}(G, X)$ denote the space of bounded functions $f : G \times X \to \mathbb{R}$ with finite G-support which are continuous with respect to X and satisfy $\tilde{\sigma}(f) = c \cdot \mathbf{1}_X$ for some $c \in \mathbb{R}$.

We introduce the norm $\|\cdot\| = \|\cdot\|_{\infty,1}$ on W_{00} by the formula

$$\|f\|_{\infty,1} = \sup_{x \in X} \sum_{g \in G} |f_g(x)|$$

and an isometric action of G on both coordinates, that means

$$(g \cdot f)(g_1, x) = f(g^{-1}g_1, g^{-1} \cdot x).$$

Definition 3. Let $W_0 = W_0(G, X)$ be the closure of $W_{00}(G, X)$ in the norm $\|\cdot\|$ (taken in the space of all functions on $G \times X$ of finite norm)

Fact 4 (implicitly resulting from [4] combined with [3])**.** If X is compact and G is countable, we can equivalently define W_0 as the space of bounded continuous functions $X \to l^1(G)$.

The proof will be omitted. (It is analogous to the proof of fact 24).

Definition 5. We set $\sigma : W_{00} \to \mathbb{R}$ to be the restriction of the general summing operation $\tilde{\sigma}$:

$$\tilde{\sigma}(f) = \sigma(f) \cdot \mathbf{1}_X \qquad \text{for } f \in W_{00}.$$

This is a continuous functional on W_{00} and so it has a unique continuous extension onto W_0. We will denote this extension by σ.

Now we are ready to introduce the main definition.

Definition 6. A sequence of functions $f^n \in W_{00}(G, X)$ is called a *Reiter sequence* for the action of G on a topological space X if:

1) For all $g \in G$ and $x \in X$, the inequality $f^n(g, x) \geq 0$ holds.
2) We have $\sum_{g \in G} f^n_g = \mathbf{1}_X$.
3) For all $g \in G$, the convergence $||f^n - gf^n||_{\infty,1} \longrightarrow 0$ holds.

Definition 7. We call the action of G on X *amenable* if there exists a Reiter sequence for it.

Fact 8 ([4])**.** The amenability of a group G is equivalent to the amenability of its action on the one-point space. Also, G has property A iff it acts amenably on its Stone-Čech compactification.

Remark 9. If G is amenable then every its action is amenable. It is sufficient to take f^n to be a constant function on the n-th Følner set.

Definition 10. An *invariant measure* for an action of a group G on a topological space X is a functional $\mu \in W_0(G, X)^{**}$ which is invariant with respect to the induced action of G on W_0^{**} and satisfies the condition $\mu(\sigma) = 1$.

Theorem 1 ([3, Theorem A])**.** *Let a countable group G act on a compact topological space X. The following conditions are equivalent:*

1) G acts on X amenably.

2) There exists an invariant measure for the action of G on X.

Theorem 2 ([4, Theorem 9])**.** *An action of a group G on a space X is amenable iff the homology class of the functional σ in $H(G, W_0(X, l^1(G))^*)$ is non-trivial.*

Notably, it is sufficient for amenability that there exists a sequence satisfying somehow weaker conditions than those in Definition 6.

Fact 11. If $f^n \in W_{00}$ is a sequence of functions satisfying the conditions:

1) For all $g \in G$ and $x \in X$, the inequality $f^n(g, x) \geq 0$ holds.
2') We have $\sum_{g \in G} f^n(g, \cdot) \geq \mathbf{1}_X$.
3) For all $g \in G$, the convergence $||f^n - gf^n||_{\infty,1} \longrightarrow 0$ holds,

then the sequence $\tilde{f}^n(g, x) = \frac{f^n(g,x)}{\sum_{g \in G} f^n(g,x)}$ satisfies the conditions from Definition 6.

Proof. Conditions 1) and 2) are obvious.

To show 3), for a given g_0 and ε, take n such that

$$\|f^n - g_0 f^n\|_{\infty,1} < \varepsilon. \tag{1}$$

Then we can express the value of $\|\tilde{f}^n - g\tilde{f}^n\|_{\infty,1}$ as follows:

$$\begin{aligned}\|\tilde{f}^n - g_0\tilde{f}^n\|_{\infty,1} &= \sup_{x\in X}\sum_{g_1\in G}\left|\frac{f^n(g_1,x)}{\sum_{g\in G} f^n(g,x)} - \frac{f^n(g_0^{-1}g_1,\ g_0^{-1}\cdot x)}{\sum_{g\in G} f^n(g_0^{-1}g,\ g_0^{-1}\cdot x)}\right| = \\ &= \sup_{x\in X}\sum_{g_1\in G}\left|\frac{f^n(g_1,x)}{\sum_{g\in G} f^n(g,x)} - \frac{(g_0f^n)(g_1,x)}{\sum_{g\in G}(g_0f^n)(g,x)}\right|.\end{aligned}$$

For a fixed x it remains to estimate a sum of the form

$$\sum_{g_1\in G}\left|\frac{a(g_1)}{\sum_{g\in G}a(g)} - \frac{b(g_1)}{\sum_{g\in G}b(g)}\right| = \left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\|$$

where $a = f^n(\cdot,x)$, $b = (g_0f^n)(\cdot,x)$ are understood as elements of $l^1(G)$. The condition 2') implies that $\|a\|,\|b\| \geq 1$, and from (1) we have $\|a-b\| < \varepsilon$. So:

$$\left\|\tfrac{a}{\|a\|} - \tfrac{b}{\|b\|}\right\| \leq \left\|a\left(\tfrac{1}{\|a\|} - \tfrac{1}{\|b\|}\right)\right\| + \left\|\tfrac{1}{\|b\|}(a-b)\right\| = \|a\|\cdot\tfrac{\big|\|a\|-\|b\|\big|}{\|a\|\cdot\|b\|} + \tfrac{\|a-b\|}{\|b\|} \leq 2\tfrac{\|a-b\|}{\|b\|} < 2\varepsilon.$$

Thus $\|\tilde{f}^n - g_0\tilde{f}^n\|_{\infty,1} < 2\varepsilon$, which finishes the proof. □

2 Følner sets

With the normal definition of amenability, the existence of a Reiter sequence is equivalent to the existence of a Følner net.

In this section we suppose that a group G acts homeomorphically on a compact topological space X. We equip $G\times X$ with the product topology where G is treated as a discrete space.

Definition 12. A subset $U \subseteq G\times X$ will be called *normal* when it is Borel and its image under the projection to G is finite.

Notation 13. For a normal subset $U\subseteq G\times X$ and an element $x\in X$, we denote

$$\begin{aligned} U^x &= \{g\in G\ :\ (g,x)\in U\},\\ \operatorname{supp}_G U &= \bigcup_{x\in X} U^x,\\ |U| &= \max_{x\in X}\#\{U^x\},\\ m(U) &= \min_{x\in X}\#\{U^x\}.\end{aligned}$$

Remark 14. Note that $|U| = \|\mathbf{1}_U\|$.

Definition 15. A *Følner net* ("a net of almost invariant sets") for the action of G on X is a net $\{U_\alpha\}$ of normal subsets of $G\times X$ which satisfies the conditions:

1) For all $g\in G$, we have $\frac{|U_\alpha \,\triangle\, gU_\alpha|}{|U_\alpha|} \longrightarrow 0$.

2) We have $\frac{|U_\alpha|}{m(U_\alpha)} \longrightarrow 1$.

Now we can formulate the main theorem of this section.

Theorem 3. *Let an infinite countable group G act on a compact topological space X. Then the following conditions are equivalent:*

1) G acts amenably on X.

2) There exist a Følner net for the action of G on X.

The proofs of both implications (and the necessary lemmata) appear in subsections 2.1 and 2.3. The assumption of G being infinite is used only to prove implication 1) $\Rightarrow$ 2), the remaining assumptions are needed only for the opposite direction.

2.1 Construction of a Følner net

Fact 16. Let A_1, A_2, B_1, B_2 be sets such that $A_1 \cup B_1$ is disjoint from $A_2 \cup B_2$. Then

$$(A_1 \triangle B_1) \amalg (A_2 \triangle B_2) = (A_1 \cup A_2) \triangle (B_1 \cup B_2).$$

Corollary 17. If $A_i \cup B_i$ is disjoint from $A_j \cup B_j$ for all $i \neq j$, then

$$\coprod_{i=1}^{n} (A_i \triangle B_i) = \bigcup_{i=1}^{n} A_i \;\triangle\; \bigcup_{i=1}^{n} B_i.$$

Notation 18. If A, B are subsets of a group G, we denote $AB = \{ab : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$.

Fact 19. Let A be a finite subset of an infinite group G. Then there exists $k \in G$ such that $Ak \cap A = \emptyset$.

Proof. We can take k to be any element not in $A^{-1}A$. □

Corollary 20. Let A be a finite subset of an infinite group G. Then for any m there exist elements $k_1, \ldots, k_m \in G$ such that:

$$Ak_i \cap Ak_j = \emptyset \qquad \text{for } i \neq j.$$

Proof. We can define the elements k_i inductively, for instance take k_1 to be e and choose k_{i+1} by applying Fact 19 to the set $\bigcup_{j \leq i} Ak_j$. □

Lemma 21. *Let $g_1, \ldots, g_n \in G$ and f be a non-negative element of $W_{00}(G, X)$ such that $\sum_{g \in G} f_g = \mathbf{1}_X$, and the inequality $\|f - g_i f\| < \varepsilon$ holds for all i. Then, for sufficiently large $J \in \mathbb{N}$, there exists a function $\bar{f} : G \times X \to \mathbb{R}$ satisfying the following conditions:*

1) $\bar{f}$ takes its values in the set $\{\frac{n}{J} : 0 \leq n \leq J\}$.

2) $\mathbf{1}_X \leq \sum_{g \in G} \bar{f}_g \leq (1 + \varepsilon) \cdot \mathbf{1}_X$.

3) $\forall_i \left\|\bar{f} - g_i \bar{f}\right\| < 2\varepsilon$.

Proof. Let $J > \frac{M}{2\varepsilon}$ where $M = \#\{\operatorname{supp}_G f\}$. Define $\bar{f}$ as $\pi \circ f$, where $\pi : [0,1] \to [0,1]$ is the function assigning to a number x the least multiple of $\frac{1}{J}$ which is greater or equal than x. We claim that the function $\bar{f}$ defined as above satisfies conditions 1)–3).

Condition 1) and the first inequality in 2) are obvious. The remaining inequalities are corollaries from $|\bar{f} - f| < \frac{1}{J}$. Namely, for all x we can bound the sum $\sum_g \bar{f}_g(x)$ from above by $1 + \frac{M}{J}$. This proves 2). Also, the sum $\sum_g \left|\bar{f}_g(x) - (g_i \bar{f})_g(x)\right|$ is bounded by $\varepsilon + \frac{2M}{J}$ and this proves 3). □

The proof of the implication 1) ⇒ 2) in Theorem 3 reduces to the following lemma:

Lemma 22. *Let an infinite group G act amenably on a compact space X. Then, for any ε and for any elements $h_1, \dots, h_k \in G$, there exists a normal subset $V = V^{\varepsilon}_{h_1,\dots,h_k} \subseteq G \times X$ such that*

$$\forall_i \ \frac{|V \triangle h_i V|}{|V|} < \varepsilon, \qquad \frac{|V|}{m(V)} < 1 + \varepsilon. \tag{2}$$

Indeed, from the lemma we deduce that $\{V_\alpha\}_{\alpha \in A}$ is a correct Følner net, where A is the family of all finite subsets of G ordered by inclusion, and for $\alpha = \{h_1, \dots, h_k\}$ we set

$$V_\alpha = V^{1/k}_{h_1,\dots,h_k}.$$

Proof (of the lemma). We know that the action is amenable, so there exists a Reiter sequence (f^n) for it. Choose n so large that $\|f^n - h_i f^n\| < \frac{\varepsilon}{2}$ for all $i \le k$. Applying Lemma 21 to the function f^n and the elements h_i, we obtain a number $J \in \mathbb{N}$ and a function $\bar{f}$ with the properties listed in the lemma. In particular,

$$\left\|\bar{f} - h_i \bar{f}\right\| < \varepsilon \qquad \text{for every } i.$$

Define

$$V_j = \left\{(g,x) \in G \times X \ : \ \bar{f}(g,x) \ge \frac{j}{J}\right\}.$$

Then $\bar{f} = \frac{1}{J}\sum_j \mathbf{1}_{V_j}$. The following inequality holds:

$$\epsilon > \left\|\bar{f} - h_i \bar{f}\right\| = \frac{1}{J}\left\|\sum_j \left(\mathbf{1}_{V_j} - h_i \mathbf{1}_{V_j}\right)\right\| = \frac{1}{J}\left\|\sum_j \left(\mathbf{1}_{V_j} - \mathbf{1}_{h_i V_j}\right)\right\| = \frac{1}{J}\sum_j \left\|\mathbf{1}_{V_j} - \mathbf{1}_{h_i V_j}\right\|. \tag{3}$$

The last equality results from that the sequences of sets $(V_j)_j$ and $(h_i V_j)_j$ are descending, so, for a fixed element (g,x), all functions $\mathbf{1}_{V_j} - \mathbf{1}_{h_i V_j}$ take only non-negative or only non-positive values.

Let $H = \{e, h_1, \dots, h_n\}$ and $A = H \cdot \operatorname{supp}_G V$. From Corollary 20 we get elements $k_1, \dots, k_J$ such that the sets Ak_i are pairwise disjoint. Now define the "corrected" version of V_j by the formula

$$\tilde{V}_j = \left\{(x, g \cdot k_j) \ : \ (x,g) \in V_j\right\}.$$

Moreover we set

$$V = \bigcup_j \tilde{V}_j.$$

Note that, for all i, $\operatorname{supp}_G(\tilde{V}_j \cup h_i \tilde{V}_j)$ is contained in Ak_j, so the sets $\tilde{V}_j \cup h_i \tilde{V}_j$ are pairwise disjoint for different values of j. Moreover, $\|\mathbf{1}_{\tilde{V}_j} - \mathbf{1}_{h_i \tilde{V}_j}\| = \|\mathbf{1}_{V_j} - \mathbf{1}_{h_i V_j}\|$, because these functions differ only by a right translation on the G-coordinate. So, from (3) we deduce that

$$J \cdot \varepsilon > \sum_j \|\mathbf{1}_{V_j} - \mathbf{1}_{h_i V_j}\| = \sum_j \|\mathbf{1}_{\tilde{V}_j} - \mathbf{1}_{h_i \tilde{V}_j}\| = \sum_j \|\mathbf{1}_{\tilde{V}_j \,\triangle\, h_i \tilde{V}_j}\| \geq \Big\| \sum_j \mathbf{1}_{\tilde{V}_j \,\triangle\, h_i \tilde{V}_j} \Big\| =$$
$$= \|\mathbf{1}_{\coprod_j (\tilde{V}_j \,\triangle\, h_i \tilde{V}_j)}\| = \Big| \coprod_j (\tilde{V}_j \triangle h_i \tilde{V}_j) \Big| \stackrel{\text{Cor. 17}}{=} \Big| (\coprod_j \tilde{V}_j) \triangle h_i (\coprod_j \tilde{V}_j) \Big| = |V \triangle h_i V| .$$

This proves the first condition in (2). Now, in fact it remains to check that V satisfies the conditions

$$|V| \geq J, \qquad \frac{|V|}{m(V)} \leq 1 + \varepsilon. \tag{4}$$

Choose any $x \in X$. We will prove that

$$J \leq \#\{V^x\} \leq J \cdot (1 + \varepsilon), \tag{5}$$

which is enough to show (4). Once again we use the fact that $\tilde{V}_j$ differs V_j only by a right translation along G, so

$$\#\{V^x\} = \#\Big\{ \big(\coprod_j \tilde{V}_j \big)^x \Big\} = \sum_j \#\Big\{ \big(\tilde{V}_j\big)^x \Big\} = \sum_j \#\{(V_j)^x\} .$$

The number $\#\{(V_j)^x\}$ is equal to the number of elements $g \in G$ such that $\bar{f}(x,g) \geq \frac{j}{J}$. We know that the values of $\bar{f}$ are of the form $\frac{n}{J}$, $0 \leq n \leq J$. Therefore, whenever $\bar{f}(x, g_1) = \frac{n}{J}$, it follows that (x, g_1) belongs to exactly n sets of all the sets V_j. This means that

$$\sum_j \#\{(V_j)^x\} = J \cdot \sum_{g \in G} \bar{f}(x,g) \in [J, J(1+\varepsilon)],$$

which proves (5) and thus also (4). □

2.2 Around the Riesz theorem

In the classical case of amenability of a group, the existence of a Følner net for G is known to imply the existence of a countably additive G-invariant measure on G. The idea of the proof is to consider the image in $l_1^{**}(G)$ of a net of constant functions on the Følner sets. By the Banach-Alaoglu theorem they have a limit point, which turns out to be the desired measure.

The case of amenability of an action is similar. However, here constant functions on Følner sets belong to the space $l^\infty(X, l^1(G))$, while we want to obtain an invariant measure in W_0^{**} (using Theorem 1). Thus we want a way to pass from $l^\infty(X, l^1(G))$ to W_0^{**}. In this subsection we construct for this purpose an inclusion $C(X, l^1(G))^*$ into a space which is dual to some subspace of $l^\infty(X, l^1(G))$ (which will turn out to be sufficiently large).

In the case when $G = \{e\}$, this inclusion can be easily obtained from the Riesz theorem [7, Theorem 6.19]: indeed, every continuous functional on $C(X)$ is a regular measure with respect to which one can also integrate any bounded function (provided that it is Borel). However, in the general case, the existence of this inclusion is not such an easy corollary and it requires using the techniques from the proof of the Riesz theorem.

Notation 23. We will consider the following Banach spaces:

- $l^\infty_B(X, l^1(G))$ — the space of Borel functions $f : X \to l^1(G)$ such that $\|f\|_{\infty,1} < \infty$, equipped with the norm $\|\cdot\|_{\infty,1}$.
- $l^\infty_{cB}(X, l^1(G))$ — the closure of the subspace in $l^\infty_B(X, l^1(G))$ consisting of functions with finite G-support.
- $C(X, l^1G)$ — the subspace of continuous functions in $l^\infty_B(X, l^1(G))$.
- $\mathrm{rca}(X)$ — the space of regular Borel countably additive measure on X.

Fact 24. If G is countable, the functions with finite G-support form a dense subspace in $C(X, l^1(G))$.

Proof. Let $G = \{g_n\}_{n\in\mathbb{N}}$ and $f \in C(X, l^1(G))$. We call by f^n the restriction of f to the first n elements in G:

$$f_n = f \cdot \sum_{i=1}^{n} \mathbf{1}_{\{g_i\}\times X}.$$

Then for all x the convergence $\|f_n(\cdot, x) - f(\cdot, x)\|_1 \longrightarrow 0$ holds. However, the left-hand side depends continuously on x, so by the compactness of X we are given that the convergence is uniform. □

Corollary 25. $C(X, l^1(G))$ is contained in $l^\infty_{cB}(X, l^1(G))$.

Notation 26. We denote by $\mathrm{rca}^G(X)$ the space of functions from G to $\mathrm{rca}(X)$. These functions will be also called *measures*. For $\mu \in \mathrm{rca}^G(X)$ we denote $\mu_g = \mu(g)$.

Fact 27. Every continuous functional $\varphi \in C(X, l^1(G))^*$ has the form $\varphi(f) = \sum_{g\in G} \int f_g \, d\mu_g$ for some $\mu \in \mathrm{rca}^G(X)$.

Proof. For $g \in G$, let V_g denote the subspace of functions $f \in C(X, l^1(G))$ whose G-support is contained in $\{g\}$. From Fact 24 we deduce that φ is uniquely determined by its restrictions to all V_g. However, every V_g is isomorphic to $C(X)$, so the Riesz representation theorem implies that applying $\varphi\big|_{V_g}$ means integrating with respect to some measure $\mu_g \in \mathrm{rca}(X)$. Passing to linear combinations and to limits, we get the result. □

Notation 28. Let $\mu \in \mathrm{rca}^G(X)$. For every family of disjoint, measurable subsets $\mathcal{A} = \{A_i\}_{i\in I}$ we define the value

$$v_{\mathcal{A}}(\mu) = \sum_{i\in I} \sup_{g\in G} |\mu_g(A_i)|. \tag{6}$$

Definition 29. With this notation, we call by the *variation* of μ the number $\sup_{\mathcal{A}} v_{\mathcal{A}}(\mu)$. We will denote it by $|\mu|$.

Notation 30. By $\mathrm{brca}^G(X)$ we denote the space of measures $\mu \in \mathrm{rca}^G(X)$ with finite variation, which is set as the norm.

Remark 31. Instead of the space $\mathrm{brca}^G(X)$ we could consider the space $\mathrm{rca}(X, l^\infty(G))$ of regular, countably additive *vector measures* [6] on X with values in $l^\infty(G)$. Note that every such measure naturally determines a family of measures $\{\nu_g\}_{g\in G} \subseteq \mathrm{rca}(X)$, and then its variation on a family $\mathcal{A}$, understood in the classical sens, coincides with the formula (6).

However, we cannot easily judge whether $\mathrm{brca}^G(X) = \mathrm{rca}(X, l^\infty(G))$ because the elements of the second space must satisfy additional conditions of regularity. Intuitively, $\mathrm{rca}(X, l^\infty(G))$ looks like a better candidate for the dual of $C(X, l^1(G))$. However, in the following text we will show that $C(X, l^1(G))^*$ is isomorphic to $\mathrm{brca}^G(X)$.

Definition 32. Let $\mu \in \mathrm{rca}(X)^G$ and let $f \in l^\infty_{cB}(X, l^1(G))$ be a function with finite G-support. We define

$$I(\mu)(f) = \int f \, d\mu := \sum_{g \in G} \int f_g \, d\mu_g,$$

We can easily check that if $|\mu| < \infty$ then $\left|\int f \, d\mu\right| \leq \|f\| \cdot |\mu|$. So, $I(\mu)$ extends continuously to $l^\infty_{cB}(X, l^1(G))$. This gives us a continuous injection

$$I : \mathrm{brca}^G(X) \to l^\infty_{cB}(X, l^1(G))^*.$$

Fact 33. Let $\mu \in \mathrm{rca}^G(X)$ and let φ be the restriction of $I(\mu)$ to the subspace of continuous functions with finite G-support. Then $|\mu| \leq \|\varphi\|$. (In particular, $\|\varphi\| < \infty$ implies $|\mu| < \infty$).

Proof. Suppose, to the contrary, that $\|\varphi\| = M < \infty$ but $|\mu| > M$. We will construct a continuous function f with finite G-support and satisfying the condition $\left|\int f \, d\mu\right| > M \cdot \|f\|$.

From the definition of variation we have that for some $\varepsilon > 0$ there exists a family $\mathcal{A} = \{A_i\}$ such that the inequality $v_{\mathcal{A}}(\mu) > M + \varepsilon$ holds. So, we can choose a finite subfamily $\mathcal{A}' \subset \mathcal{A}$ such that $v_{\mathcal{A}'}(\mu) > M + \varepsilon$. Then for every set $A_i \in \mathcal{A}'$ take $g_i \in G$ such that $\mu_{g_i}(A_i) > \sup_{g \in G} |\mu_g(A_i)| - \frac{\varepsilon}{\#\{\mathcal{A}'\}}$.

Now, we are almost ready to define a function f which we are looking for. If f had to be continuous, we would define it by the formula

$$f = \sum_i \mathrm{sgn}(\mu_{g_i}(A_i)) \cdot \mathbf{1}_{\{g_i\} \times A_i}.$$

So, we should find a continuous approximation for this f. By the regularity of measure, for every δ we can approximate (with respect to μ) every set A_i from inside using a compact set B_i^δ. Moreover, each of the sets B_i^δ can be approximated from outside by an open set U_i^δ. For every i, define a continuous function $f_i^\delta : G \times X \to [0, 1]$ equal to 1 on the set $\{g_i\} \times B_i^\delta$ and to 0 outside $\{g_i\} \times U_i^\delta$. Let $f^\delta = \max_i f_i^\delta$. For sufficiently small δ, the function f^δ satisfies the requested property. □

Corollary 34. I induces an isomorphism $\Theta : \mathrm{brca}^G(X) \to C(X, l^1(G))^*$.

Proof. The isomorphism is defined by the formula $\Theta(\mu) = I(\mu)\big|_{C(X,l^1(G))}$. From the continuity of I we deduce continuity of Θ. Then from Facts 27 and 33 we get that Θ is surjective and, moreover, an isometry. □

Corollary 35. The operator $j = I \circ \Theta^{-1}$ is an inclusion of $C(X, l^1(G))^*$ into $l^\infty_{cB}(X, l^1(G))^*$. Moreover, we have $i^* \circ j = \mathrm{id}$, where i denotes the inclusion of $C(X, l^1(G))$ into $l^\infty_{cB}(X, l^1(G))$.

2.3 Construction of an invariant measure

Let $\{V_\alpha\}$ be a Følner net for the action of G on X. Define a net of functions $\chi_\alpha = \frac{\mathbf{1}_{V_\alpha}}{|V_\alpha|} \in l^\infty_{cB}(X, l^1(G))$. We will show that after passing to the double dual we get a bounded net. Then from Banach-Alaoglu theorem it follows that $\{\chi_\alpha^{**}\}$ contains a convergent subnet. We will prove that its limit is an invariant measure (Definition 10). In this way we will prove the implication 2) $\Rightarrow$ 1) in Theorem 3.

Notation 36. For an element v in Banach space V we call by v^{**} the image of v under the canonical inclusion $i : V \to V^{**}$.

Fact 37. Let $L : V \to W$ be a continuous operator of Banach spaces. Then $L^{**} : V^{**} \to W^{**}$ is continuous with respect to the weak-* topologies on V^{**}, W^{**}.

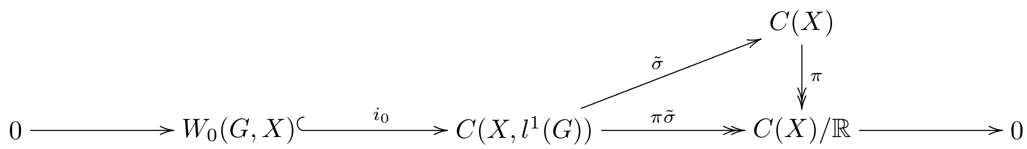
Notation 38. In the following diagrams, arrows of the form $\hookrightarrow$ will denote isometric inclusions and arrows of the form $\twoheadrightarrow$ will mean *projections* (i.e. continuous surjections setting the quotient norm in their image).

Definition 39. A *short exact sequence* of Banach spaces is a sequence which is exact as a sequence of groups, and it has the form:

$$0 \longrightarrow V \hookrightarrow W \twoheadrightarrow Z \longrightarrow 0\,.$$

Fact 40 ([5, Lemma 2.2.c]). The duality functor of Banach spaces preserves short exact sequences, in particular, it converts isometric inclusions to projections and vice versa.

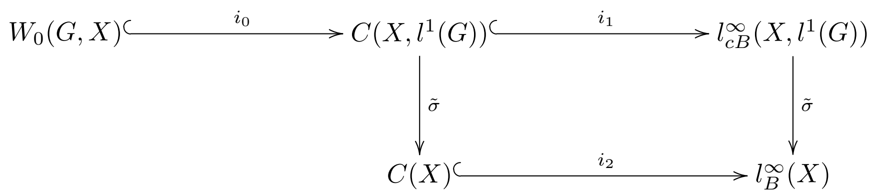
Fact 41. The following sequence is exact.



Proof of the implication 2) ⇒ 1) in Theorem 3.

1. *The main picture.* Note that the following diagram commutes:

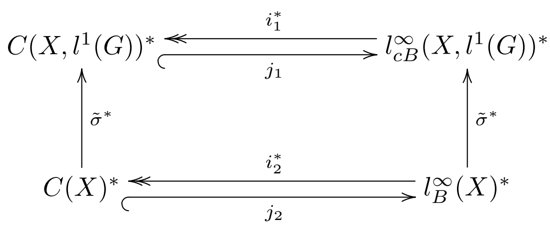
(7)



The existence of the inclusion i_1 is deduced from Corollary 25.

2. *Using section 2.2.* After applying the duality functor to (7) we obtain the diagram:

(8)



The inclusion $j_2 :\ C(X)^* \hookrightarrow l_B^\infty(X)^*$ is defined as integrating with respect to the measure, using the isomorphism $C(X)^* \simeq \mathrm{rca}(X)$ (corollary from the Riesz theorem). Note that

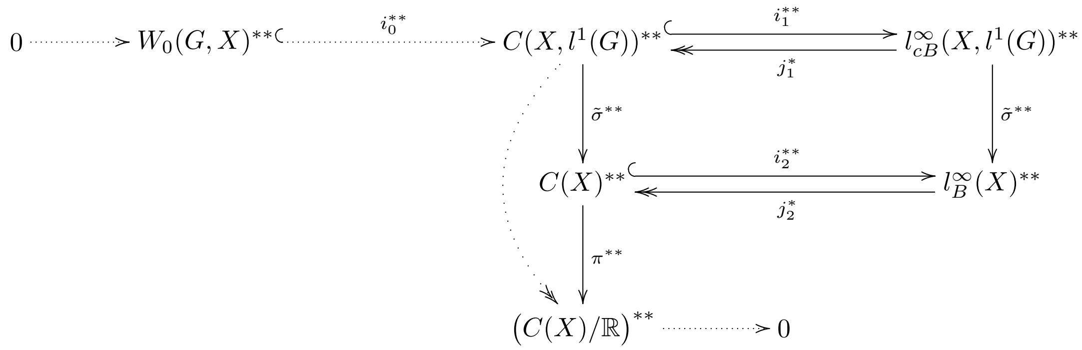
$$i_2^* \circ j_2 = \mathrm{id}\,. \tag{9}$$

The inclusion $j_1 :\ C(X, l^1G)^* \hookrightarrow l_B^\infty(X, l^1(G))^*$, defined as j in Corollary 35, is similar. (Here we again have $i_1^* \circ j_1 = \mathrm{id}$ but we will not use this).

Note that the square in (8) built from j_1, j_2 and both functions $\tilde{\sigma}^{**}$ is commutative:

$$j_1 \circ \tilde{\sigma} = \tilde{\sigma} \circ j_2. \tag{10}$$

3. *Double dual.* Once more, apply the duality functor to (8) to obtain a diagram with the most important spaces:



Note that every χ_α has norm 1 (it is corollary from the definition and Remark 14), so χ_α^{**} has norm 1 too. Then, from Banach-Alaoglu theorem we can choose a convergent subnet from $\{\chi_\alpha^{**}\}$. Let ν be its limit.

4. *Passing to W_0^{**}.* Note that it follows from condition 2) in Definition 15 that $\tilde{\sigma}(\chi_\alpha)$ uniformly converges to $\mathbf{1}_X$. From Fact 37 we deduce that $\tilde{\sigma}^{**}(\nu) = \mathbf{1}_X^{**} = i_2^{**}(\mathbf{1}_X^{**})$, so from (9) we have $j_2^*(\tilde{\sigma}^{**}(\nu)) = \mathbf{1}_X^{**}$, and finally using (10) we get

$$\tilde{\sigma}^{**}(j_1^*(\nu)) = \mathbf{1}_X^{**}. \tag{11}$$

From Facts 41 and 40 we deduce the exactness of the dotted-arrow sequence, so since the element $\mu = j_1^*(\nu)$ passes to 0 under the projection $\pi^{**}\tilde{\sigma}^{**}$, it must come from $W_0^{**}(G,X)$. Then, as we will check, μ is an invariant measure which we are looking for.

5. *Verifying the properties of μ.* It remains to show that μ satisfies the conditions from Definition 10. We check the invariance in a similar way as in the case of amenable groups: for every $g \in G$, the functions $A_g : l^\infty_{cB}(X,l^1(G)) \to l^\infty_{cB}(X,l^1(G))$ defined by $A_g(\chi) = \chi - g \cdot \chi$ are continuous and from the properties of χ_α we have that $A_g(\chi_\alpha) \to 0$; then from Fact 37 we get

$$\mu - g \cdot \mu = A_g^{**}(\mu) = 0.$$

The second condition from Definition 10 ($\mu(\sigma) = 1$) is equivalent to the condition $\mu(\tilde{\sigma}) = \mathbf{1}_X^{**}$ (due to Definition 5), which is corollary from (11). □

2.4 Interpretation of the homology of an action

Using the results of two previous subsections, we can give an elementary but complicated description of the homology of action $H_*^{uf}(G \curvearrowright X)$ (which was introduced in [4]).

Recall that the homology $H_*^{uf}(G \curvearrowright X)$ was defined as the classical homology of G with coefficients in $W_0(G,X)^*$. By Facts 40 and 41 we have

$$W_0(G,X)^* = C(X,l^1(G))^* \Big/ \big(C(X)/\mathbb{R}\big)^*.$$

From Corollary 34 we know that $C(X, l^1(G))^* \simeq \mathrm{brca}^G(X)$. To describe the divisor, we build another short exact sequence

$$0 \longrightarrow \mathbb{R} \overset{i}{\hookrightarrow} C(X) \overset{\pi}{\twoheadrightarrow} C(X)/\mathbb{R} \longrightarrow 0 ,$$

where $i(1) = \mathbf{1}_X$. From Fact 40 we get

$$\big(C(X)/\mathbb{R}\big)^* = \mathrm{rca}_0(X) := \{\mu \in \mathrm{rca}(X) \,:\, \mu(X) = 0\}.$$

So, finally

$$W_0(G, X)^* = \mathrm{brca}^G(X)/\,\mathrm{rca}_0(X).$$

The inclusion $\mathrm{rca}_0(X) \overset{I}{\hookrightarrow} \mathrm{brca}^G(X)$ is defined by the formula $I(\mu)_g = \mu$ for every $g \in G$.

After investigating the coefficients of homology, we can describe them from the point of view of the original paper by Block and Weinberger [1]. Analogically to the elementary unification of $H_*^{uf}(G)$ with $H(G, l^\infty(G))$ (see [2]), the homology of action $H_*^{uf}(G \curvearrowright X)$ can be obtained by the following method:

- Define C_* as the Block-Weinberger chain complex for G with coefficients in $\mathrm{rca}(X)$, that is, C_n consists of bounded functions $G^{n+1} \to \mathrm{rca}(X)$ vanishing on simplices with sufficiently large diameter.
- Let D_* be the subcomplex of such $f \in C_n$ that for every $x \in G^{n+1}$ the measure $\mu \in \mathrm{rca}^G(X)$ given by the formula $\mu_g = f(g \cdot x)$ is an element of $\mathrm{brca}^G(X)$.
- Define E_* as the subcomplex of such $f \in D_n$ that for every $x \in G^{n+1}$, $g \in G$ the conditions

$$f(x) \in \mathrm{rca}_0(X), \qquad f(x) = f(g \cdot x)$$

  hold.
- Let $\tilde{D}_*$ be the quotient complex D_*/E_*.
- Equip $\tilde{D}_*$ with the differential induced from C_*.

Then the homology of the action G on X is the homology of $\tilde{D}_*$.

Finally, let us describe the special chain in dimension 0 corresponding to the functional σ, which is concerned in Theorem 2. It can be expressed by the formula

$$\sigma(g) = \mu \qquad \text{for } g \in G$$

for any $\mu \in \mathrm{rca}(X)$ with finite variation (in the common sense used in $\mathrm{rca}(X)$) and such that $\mu(X) = 1$. Choosing another μ leads to the same σ because we work in the quotient by E_*.

3 The dependence of amenability on the underlying space

In this section we will be interested in non-compact G-spaces and equivariant maps between them. We will state conditions under which amenability of the action on the first space implies amenability of the action on the second one.

Lemma 42. *If $F : X \to Y$ is a continuous function preserving the action (i.e. $F(gx) = gF(x)$) and the action of G on Y is amenable then the action of G on X is also amenable.*

Proof. Let (f^n) be a Reiter sequence for the action of G on Y. Define

$$\bar{f}^n = f^n \circ F. \tag{12}$$

Then the functions $\bar{f}^n$ obviously form a Reiter sequence for the action of G on X. □

Corollary 43. If G acts on a discrete space X as well as it acts amenably on a space Y and there exists a function $F : X \to Y$ preserving the action then the action of G on X is amenable.

Remark 44. Even for a discontinuous function F, the functions $\bar{f}^n$ defined by the formula (12) satisfy conditions 1)–3) from Definition 6. However, in such case $\bar{f}^n \notin W_{00}(G, X)$.

In the following part of this section we show that under certain assumptions (about the action) one can construct an appropriate sequence $\bar{f}^n$.

3.1 Amenability and partitions of unity

Definition 45. Let G act on X. We define a *G-invariant partition of unity* as a family of continuous functions $\phi_\alpha : X \to [0, 1]$, where the indices α belong to a G-set A, satisfying the conditions:

1) Every point $x \in X$ belongs to supports of finitely many functions from ϕ_α.

2) The equality $\sum_\alpha \phi_\alpha = \mathbf{1}_X$ holds.

3) For every α and $g \in G$ we have $\phi_{g\alpha} = g\phi_\alpha$.

Theorem 4. *Let G act on X as well as on Y and let $F : X \to Y$ be a (not necessarily continuous) function preserving the action. Assume that the action of G on Y is amenable and that there exists an invariant partition of unity for the action on X. Then the action of G on X is amenable.*

Proof. Let f^n be a Reiter sequence for the action on Y and $\{\phi_\alpha\}_{\alpha \in A}$ be a G-invariant partition of unity for the action on X. Fix a basepoint $x_0 \in X$. For every G-orbit in A choose a representative of it and denote the set of chosen elements as $\{\alpha_i\}_{i \in I}$. Define $\bar{f}^n$ by the formula

$$\bar{f}^n_g(x) = \sum_{i \in I} \sum_{h \in G} \phi_{h\alpha_i}(x) \cdot f^n_g(F(h \cdot x_0)).$$

It is a locally finite sum of non-negative continuous functions and so it is well-defined non-negative continuous function. Let us check that $\bar{f}^n \in W_{00}(G, X)$. We have:

$$\sum_{g \in G} \bar{f}^n_g(x) = \sum_{g} \sum_{i,\, h} \phi_{h\alpha_i}(x) \cdot f^n_g(F(h \cdot x_0))$$

By the finiteness of the G-support of f^n, we can change the order of summation to obtain:

$$\sum_{i,\, h} \sum_{g} \phi_{h\alpha_i}(x) \cdot f^n_g(F(h \cdot x_0)) = \sum_{i,\, h} \phi_{h\alpha_i}(x) \sum_{g \in G} f^n_g(F(h \cdot x_0)) = \sum_{i,\, h} \phi_{h\alpha_i}(x) = 1.$$

So, the functions $\bar{f}^n$ belong to W_{00} and satisfy conditions 1)–2) from Definition 6. Condition 3) is corollary from the G-invariance of $\{\phi_h\}$ and from condition 3) of Definition 6 for the sequence (f^n). For a given $k \in G$ and $\varepsilon > 0$, choose such N that for $n > N$ we would have

$$\|f^n - k^{-1} \cdot f^n\| < \varepsilon.$$

Then, for any $x \in X$, we have:

$$\begin{aligned}
&\sum_g \left| \left(\bar{f}^n - k^{-1} \cdot \bar{f}^n\right)_g (x) \right| = \sum_g \left| \bar{f}^n_g(x) - \bar{f}^n_{kg}(k \cdot x) \right| = \\
&= \sum_g \left| \sum_{i,h} \phi_{h\alpha_i}(x) \cdot f^n_g(F(h \cdot x_0)) - \sum_{i,h} \phi_{h\alpha_i}(k \cdot x) \cdot f^n_{kg}(F(h \cdot x_0)) \right| = \\
&= \sum_g \left| \sum_{i,h} \phi_{h\alpha_i}(x) \cdot f^n_g(F(h \cdot x_0)) - \sum_{i,h} \phi_{k^{-1}h\alpha_i}(x) \cdot f^n_{kg}(F(h \cdot x_0)) \right| = \\
&= \sum_g \left| \sum_{i,h} \phi_{h\alpha_i}(x) \cdot f^n_g(F(h \cdot x_0)) - \sum_{i,h} \phi_{h\alpha_i}(x) \cdot f^n_{kg}(F(kh \cdot x_0)) \right| = \\
&= \sum_g \left| \sum_{i,h} \phi_{h\alpha_i}(x) \cdot \Big(f^n_g(F(h \cdot x_0)) - f^n_{kg}(F(kh \cdot x_0)) \Big) \right| \le \\
&\le \sum_g \sum_{i,h} \phi_{h\alpha_i}(x) \cdot \left| f^n_g(F(h \cdot x_0)) - f^n_{kg}(F(kh \cdot x_0)) \right| = \\
&= \sum_{i,h} \sum_g \phi_{h\alpha_i}(x) \cdot \left| f^n_g(F(h \cdot x_0)) - f^n_{kg}(F(kh \cdot x_0)) \right| = \\
&= \sum_{i,h} \phi_{h\alpha_i}(x) \sum_g \left| f^n_g(F(h \cdot x_0)) - f^n_{kg}(F(kh \cdot x_0)) \right| = \\
&= \sum_{i,h} \phi_{h\alpha_i}(x) \sum_g \left| \left(f^n - k^{-1} \cdot f^n\right)_g (F(h \cdot x_0)) \right| \le \sum_{i,h} \phi_{h\alpha_i}(x) \cdot \varepsilon = \varepsilon.
\end{aligned}$$

This finishes the proof. □

3.2 Absolutely discontinuous actions

Definition 46. An open subset $U \subseteq X$ will be called *wandering* if $gU \cap U = \emptyset$ for $g \neq e$.

Remark 47. If $U \subseteq X$ is wandering and $g \neq h$, then $gU \cap hU = \emptyset$.

Definition 48. A subset $U \subseteq X$ will be called a *fundamental domain* for the action of G if it is wandering and $\overline{\bigcup_{g \in G} gU} = X$.

Definition 49. We say that an action of G on X is *absolutely discontinuous* if every point $x \in X$ has a wandering neighbourhood.

An example is the action of a discrete group on itself by left translation.

Lemma 50. *If G acts absolutely discontinuously on space X, then there exist a fundamental domain for this action.*

Proof. Choose any point $x \in X$. Then (from Kuratowski-Zorn lemma) there exists its maximal wandering neighbourhood U. Suppose, to the contrary, that the set $V = \overline{\bigcup_{g \in G} gU}$ is not the whole X. Take any $y \in X \setminus V$. Then it must have a wandering neighbourhood U_1, and we can claim that U_1 is disjoint from V. Also, gU_1 must be disjoint from V for any $g \in G$. (If $gu_1 = hu$, where $g, h \in G$, $u_1 \in U_1$ and $u \in U$, then $u_1 = g^{-1}hu \in V$). Thus the set $U \cup V$ is wandering and this gives a contradiction with the assumption that U is maximal. □

Lemma 51. *Let $U \subseteq X$ be a fundamental domain for the action of G on a normal topological space X. Assume that there exists only finitely many $g \in G$ such that $g\overline{U} \cap \overline{U} \neq \emptyset$. Then there exists a G-invariant partition of unity on X.*

Corollary 52. Under the assumptions of the lemma, the amenability of an action of G on X is equivalent to the amenability of the action of G on itself by left translations.

In this situation, we can naturally define the homology of an action with coefficients in any group instead of the group $\mathbb{R}$.

Remark 53. The assumptions of the lemma are satisfied for the action of any finitely generated group on its Cayley's graph.

Proof of the lemma. Take a set U satisfying the assumptions of the lemma and denote $U_g = gU$. We know that the closure of U_e intersects only finitely many closures of the sets U_g. We will define the family of functions $\{\phi_g\}_{g\in G}$ leading to a G-invariant partition of unity. First, define a supplementary family of functions $\{f_g\}_{g\in G}$. Let f_e be any function satisfying the following conditions:

- f_e is continuous and $f_e(x) \in [0,1]$ for $x \in X$,
- $f_e(x) = 1$, if $x \in \overline{U_e}$,
- $f_e(x) = 0$, if $x \notin \bigcup_{i\in I} U_i$, where $I = \{i \in G : \overline{U_i} \cap \overline{U_e} \neq \emptyset\}$.

Such a function exists e.g. by the Tietze-Urysohn lemma. Using f_e define the functions f_g by the formula $f_g = g \cdot f_e$. Let us check that the sum

$$\sum_{g\in G} f_g(x) \tag{13}$$

is locally finite. We will consider two cases:

- $x \in U_g$. If the inequality $f_{g_1}(x) > 0$ holds, then $g_1 = g$ or U_{g_1} is a neighbour of U_g, so there are only finitely many such elements g_1.
- $x \in \overline{U_g}$ and x lies outside all the sets U_g. Here the argument is similar. Suppose again that $f_{g_1}(x) > 0$. Then, since f_{g_1} is continuous and positive in $x \in \partial U_g$, it must be positive somewhere inside U_g. This can hold when $g = g_1$ or U_{g_1} is a neighbour of U_g. Moreover, x has a neighbourhood V contained in a finite sum of closures of the sets U_g such that in whole this neighbourhood the sum (13) is finite.

Now, we are ready to define the functions forming the partition of unity. Let

$$\phi_g(x) = \frac{f_g(x)}{\sum_{g\in G} f_g(x)}.$$

It is a G-invariant family because the family $\{f_g\}$ is invariant and the denominator of the fraction is an invariant function. The remaining properties required from a partition of unity are satisfied too. □

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