ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Dominika Pawlik

Uniwersytet Warszawski

A (co)homological criterion for Property A of metric spaces

Praca semestralna nr 2

(semestr zimowy 2011/12)

Opiekun pracy:

A (co)homological criterion for Property A of metric spaces

Dominika Pawlik

1 Introduction

In this paper we present an extension of the criteria for amenability of group actions (in particular, for Property A of groups) given in [5], [3] and [4] to detect Property A for general metric spaces. In Section 4 we will show how this theory works on a concrete example of a space without Property A which was described in [1].

Throughout the whole paper we assume that X is a metric space with the following properties:

- X is *uniformly discrete*: there exists $C > 0$ such that $d(x, y) > C$ if $x \neq y$; (without loss of generality we assume that $C = 1$)
- X has *bounded geometry*: for every R there exists K_R such that for all x, the ball $B(x, R)$ contains at most K_R elements;
- X is infinite. (every finite metric space clearly has Property A)

Among many equivalent definitions of Property A, we choose as our starting point the one taken from [7, Lemma 3.5]:

Definition 1. A space X has *Property A* if and only if there exists a sequence (a_n) of functions from X to $l^1(X)$ satisfying the following conditions:

(a) for all n and x, we have $a_n(x) \geq 0$ and $||a_n(x)||_1 = 1$;

(b) for all n, there exists R such that for all x, the support of $a_n(x)$ is contained in the ball of radius R around x;

(c) for every K and $\varepsilon > 0$ there exists N such that for every $n > N$ and arbitrary pair $x, y : d(x, y) < K$ we have

$$||a_n(x) - a_n(y)||_1 < \varepsilon.$$

We will reformulate the above definition using the space $W_0(X)$ analogous to $W_0(G, X)$ appearing in [3].

Definition 2. Let $W_{00}(X)$ denote the space of functions $w : X \times X \to \mathbb{R}$ with the following properties:

(a) there exists $c \in \mathbb{R}$ such that $\sum_{y \in X} w(x, y) = c$ for all $x \in X$; (the functional which, given w, returns the value of c will be denoted by π)

(b) there exists $r \in \mathbb{R}$ such that $w(x,y) > 0$ implies $d(x,y) \leq r$;
(the lowest such r will be called the *support size* of w and denoted by $|\operatorname{supp} w|$)

(c) the expression $\sup_{x\in X} \sum_{y\in X} |w(x,y)|$ is a finite number.
(by this formula we define a norm $W_{00}(X)$, denoted by $||\cdot||_{\infty,1}$)

Definition 3. The space $W_0(X)$ is defined as the closure of $W_{00}(X)$ in the space of all functions from $X \times X$ to $\mathbb{R}$ which have finite $||\cdot||_{\infty,1}$-norm.

According to the above definitions, we may modify Definition 1: a function $a_n : X \to l^1(X)$ satisfying conditions (a) and (b) can be equivalently replaced by an element $w_n \in W_{00}(X)$ meeting the condition

(a') $w_n \geq 0$ and $\pi(w_n) = 1$.

Note that condition (b) in Definition 1 translates exactly to the condition $w_n \in W_{00}(X)$. We have still not simplified (c), which will take place in the next section.

2 Criteria for Property A

2.1 The action of bijection group

Definition 4. Let $G_R(X)$ denote the undirected graph whose vertices are the points of X, and an edge joins x with y if and only if $d(x,y) \leq R$.

Definition 5. We say that a bijection $f : X \to X$ is *close to the identity* if $\sup_{x\in X} d(f(x),x) < \infty$. The group of all such bijections will be denoted by $B_{id}(X)$.

Lemma 6 ([10, Lemma 1.13])**.** For a given R, the edges of $G_R(X)$ can be labeled with finitely many colours so that two edges of the same color never meet in a common adjacent vertex.

Lemma 7. For a given R, there exists a finite subset $B \subseteq B_{id}(X)$ such that for every pair (x,y) with $d(x,y) \leq R$ there exists $b \in B$ such that $b(x) = y$.

Proof. By Lemma 6, $G_R(X)$ can be labeled by a M colors for some M. For each i, define a bijection b_i close to the identity as follows:

- if there is an edge of color i joining x with y, set $f(x) = y$;
  (then also $f(y) = x$)
- if there is no edge of color i adjacent to x, set $f(x) = x$.

As the whole $G_R(X)$ is coloured, every b_i is fully specified in this way, and the claim holds. □

Definition 8. We define an action of $B_{id}(X)$ on $W_0(X)$ by the formula

$$(gw)(x, y) = w(g^{-1}(x), y).$$

Definition 9. For a function $w \in W_0(X)$ and an element $g \in B_{id}(X)$, we will understand as the *g-variance* of w either the element $w - gw$ itself or its norm. (The meaning will be clear from the context. Note that, in both senses, g-invariance is having trivial g-variance).

Using the above definitions, we can eventually restate the definition of Property A:

Fact 10. A space X has Property A if and only if there exists a sequence $w_n \in W_{00}(X)$ satisfying the following conditions:

(a)' for all n, $w_n \geq 0$ and $\pi(w_n) = 1$ (i.e. $\sum_{y \in X} w_n(x, y) = 1$ for all x);

(c') for all $g \in B_{id}$, we have $\lim_{n \to \infty} ||w_n - gw_n||_{\infty,1} = 0$.

Proof. This is an immediate consequence of the preceding definitions and Lemma 7, which allows for any R to include all pairs (x, y) of mutual distance bounded by R into finitely many bijections close to the identity. □

Definition 11. A sequence (w_n) satisfying the above conditions will be called a *Reiter sequence* for X.

2.2 Cohomological criterion

Lemma 12. A space X has Property A if and only if there exists an element $\mu \in W_0(X)^{**}$ satisfying conditions:

$$\mu(\phi) = \mu(g\phi) \qquad \text{and} \qquad \mu(\pi) = 1.$$

The proof proceeds analogously as in [4, Theorem A].

Definition 13. By $N_0(X)$ we denote the space of these $w \in W_0(X)$ for which $\pi(w) = 0$.

Definition 14 (cf. [5])**.** We define the *Johnson cochain* by the formula

$$J(g_0, g_1)(a)(x) = [x = g_1^{-1}(a)] - [x = g_0^{-1}(a)],$$

where $[\varphi]$ is the Iverson bracket standing for 1 when φ holds and 0 otherwise.

Theorem 1. A space X has Property A if and only if the Johnson class $[J] \in H_b^1(B_{id}(X), N_0(X)^{**})$ vanishes.

The proof proceeds analogously as in [4, Theorem B]. It is easy to check that, just as in [4], $[J]$ is the image of $[\mathbf{1}_X]$ under the connecting map arising from the Snake Lemma.

2.3 Reduction of the group and homological criterion

The group B_{id} seems too large; in particular, it is usually not finitely generated. Note, however, that Fact 10 and Theorem 1 will still hold if we replace B_{id} by any its subgroup H satisfying the claim of Lemma 7. For this it is sufficient that H be *exhausting* in the following sense.

Definition 15. A subgroup H of $B_{id}(X)$ is *exhausting* if for every $g \in B_{id}(X)$ there exist finitely many elements $h_1, \ldots, h_n \in H$ such that for all $x \in X$, we have $g(x) = h_i(x)$ for some i.

Remark 16. Recall that a space X is called *coarsely connected* if the graph $G_{R_0}(X)$ is connected for some R_0, and *quasi-geodesic* if it is coarsely connected and the metric in X is quasi-isometric to the graph-path metric in $G_{R_0}(X)$ (cf. [9, Definition 1.4.6]). For such spaces, there exists a finitely generated exhausting group; an example is the group H generated by the set B of elements obtained in Lemma 7 for $R = R_0$. Let us verify that H is exhausting. Let $C > 1$ be the constant setting up the quasi-isometry between X and $G_{R_0}(X)$ (we may use a single constant since we assume that all distances in X are separated from zero). Let $\overline{B}$ the set of the compositions of all sequences of length at most $C \cdot k$ of elements of B. Now, let g be any element of $B_{id}(X)$; then $\sup_{x\in X} d(g(x), x) < kR$ for some k. Then for any $x \in X$ there is a path $x = x_0, x_1, \ldots, x_M = g(x)$ in $G_{R_0}(X)$ with $M \le C \cdot k$. It follows from the definition of B that for every i there exists $b_i \in B$ such that $b_i(x_i) = x_{i+1}$. Therefore an appropriate composition of these b_i belongs to $\overline{B}$ and takes x to $g(x)$, which finishes the proof.

For a finitely generated exhausting subgroup $H \le B_{id}$, we may use the duality between homology and cohomology as it is done in [5, Main Theorem] and [3, Theorem 8]. This leads to the following theorem.

Theorem 2. Let X be a space and $H \le B_{id}(X)$ a finitely generated exhausting subgroup. Then X has Property A if and only if the class $[\pi]$ does not vanish in $H_0(H, W_0^*(X))$.

Remark 17. If X is the Cayley graph of a group, the group of all left shifts by elements of X is an exhausting subgroup of $B_{id}(X)$. In this way, we may obtain as special cases from Theorems 1 and 2 the original criteria for Property A of groups which were given in [5], [3] and [4].

Definition 18. If H is an exhausting subgroup of $B_{id}(X)$ with a finite set of generators $h_1, \ldots, h_n$ specified in the context, we will mean by the *H-variance* of w the maximum of its h_i-variances. If H is also specified, we will say that w is *ε-invariant* if its H-variance is below ε. The motivation is that it is sufficient for being a Reiter sequence that w_n is ε_n-invariant, where ε_n may be any sequence converging to zero.

3 Reductions of coefficients

3.1 Symmetrisation

Lemma 19. If a non-amenable space X with a f. g. subgroup of $B_{id}(X)$ has Property A, then there exists a Reiter sequence for X consisting of *symmetric* elements, i.e. satisfying $w_n(x_1, x_2) = w_n(x_2, x_1)$.

Proof. By [6], if X is not amenable, there exist two coarse inclusions $i_1, i_2 : X \to X$ with disjoint images. For any $w \in W_{00}$ denote

$$p(w) = \sup_{x \in X} \sum_{y \in X} w(x, y).$$

This is a finite number bounded by $||w||$ times the support size of w. Now let

$$w'(x,y) = \begin{cases} \frac{1}{2}w(x,y') & \text{if } y = i_1(y'), \\ \frac{1}{2}w(x,y'') & \text{if } y = i_2(y''), \\ 0 & \text{otherwise} \end{cases}$$

Then $w' \in W_{00}$ and we have

$$||w'|| = ||w||, \qquad \pi(w') = \pi(w), \qquad p(w') = \tfrac{1}{2}p(w).$$

Moreover, this operation is linear and preserves the action of B_{id}; thus $||w' - gw'|| = ||w - gw||$. Therefore by iterating it we may force $p(w)$ to become arbitrarily small, at the same time keeping the g-variance of w unchanged for all g. This means that if X is a non-amenable space with Property A, we may choose a Reiter sequence (w_n) with the following property:

$$\text{if } w_n \text{ is } \varepsilon\text{-invariant, then } p(w) < \varepsilon$$

Now, for such a sequence, we define its symmetrisation $\bar{w}_n$ by the formula

$$\bar{w}_n(x_1, x_2) = w_n(x_1, x_2) + w_n(x_2, x_1).$$

If w_n is ε-invariant, then $\bar{w}_n$ is (2ε)-invariant since the norm of the second summand does not exceed $p(w_n)$ and this is less than ε. In order to obtain a Reiter sequence, we must correct every $\bar{w}_n$ so that it has equal sums of values on all vertical fibers of the form $\{x\} \times X$. Observe that the sum of $\bar{w}_n$ on any such fiber belongs to $[1, 1+\varepsilon]$. Moreover, by the symmetry, the sums on the horizontal and vertical fibers passing through (x, x) are equal. Thus, we can construct $\tilde{w}_n$ by increasing the diagonal values of $\bar{w}_n$ so that the sum on all horizontal and vertical fibers becomes exactly $1 + \varepsilon$. Since what we added has norm at most ε, the new function $\tilde{w}_n$ is (3ε)-invariant. The last step is to multiply $\tilde{w}_n$ by $\frac{1}{1+\varepsilon}$ which makes $\pi(\tilde{w}_n)$ equal one. □

Remark 20. The above lemma shows that, in our definition of Property A, the space $W_{00}(X)$ can be (in the case of a non-amenable space) replaced by its subspace

$$\widetilde{W}_{00}(X) = \left\{ w \in W_{00}(X) \ : \ \forall_{y \in X} \sum_{x \in X} w(x, y) = \pi(w) \right\}$$

considered with the same norm $|| \cdot ||_{\infty,1}$. Note that the lemma implies the existence of a Reiter sequence even in a smaller subspace $\overline{W}_{00}(X)$ of symmetric elements. However, unlike $\widetilde{W}_{00}(X)$, this space is not closed under the action of $B_{id}(X)$, which means that investingating functionals on it will not lead to a description of Property A.

3.2 Invariant chains with respect to a torsion generator

We will now consider a special case of a group G generated by two elements a, b such that a is a torsion element while b is not. Then, for any $w \in W_{00}(G)$ we define its "partially averaged" version $\bar{w}$ by the formula

$$\bar{w}(x_1, x_2) = \frac{1}{n}\sum_{i=1}^{n} w(x_1 \cdot a^i, x_2),$$

where n is the rank of a. Now, $\bar{w}$ is a-invariant, and its b-variance is at most the b-variance of w times a constant independent of w. Thus, if G has Property A, it must have a Reiter sequence consisting of a-invariant elements.

Lemma 21. Let $v \in N_{00}(G)$ be the b-variance of some $w \in W_{00}(G)$. Then w must be equal to $\bar{w}$ which is defined by the formula

$$\bar{w}(x_1, x_2) = \sum_{i=0}^{\infty} v(b^{-i} \cdot x_1, x_2). \tag{1}$$

Proof. First, note that $\bar{w} - b\bar{w} = v$. Then, the element $\tilde{w} = \bar{w} - w$ is b-invariant. If we had $\tilde{w}(x, y) \neq 0$ for some (x, y), we would also have $\tilde{w}(b^i \cdot x, y) = \tilde{w}(x, y) \neq 0$ for all i, so the support size of $\tilde{w}$ would be infinite, which is a contradiction. Thus $\tilde{w} = 0$, so $w = \bar{w}$. □

Definition 22. Define $\bar{N}_{00}(X)$ as the space of possible b-variances of a-invariant elements of $W_{00}(X)$.

Corollary 23. Under the assumptions imposed on G, G has Property A if and only if the operator on $\bar{N}_{00}(X)$ defined by the formula (1) is not continuous.

4 Constructing a witness for not having Property A

In this section we show how one can try — in the case when X does not have Property A — to construct a chain $c \in C_1(H, W_0^*(X))$ such that $\pi = \partial c$. In a particular case, we will perform an explicit construction.

4.1 General form of the witness functional

Lemma 24. Let $h_1, \ldots, h_n$ be generators of an exhausting group H for a space X. Then the map

$$\delta : \qquad W_0 \ni w \longmapsto (w - h_1 w,\ \ldots,\ w - h_n w) \in W_0^n$$

is injective.

Proof. Suppose that $\delta w = 0$ for some $w \in W_0$. For any x, y, z there exists $h \in H$ taking x to y; on the other hand, $w = hw$, so $w(x, z) = w(y, z)$. Therefore either w is zero or it has infinite support size; the latter is impossible. □

Observe that the existence of a Reiter sequence (that is, Property A) is equivalent to the inverse operator δ^{-1} (considered on the image of δ) being not continuous. If X does not have Property A, we can, by Hahn-Banach Theorem, extend $\pi \circ \delta^{-1}$ from $\operatorname{im} \delta$ onto whole W_0^n, without increasing its norm. Let ϕ_i be the restriction of this extended functional to the i-th summand W_0 in the product W_0^n. Then we have

$$\sum_{i=1}^{n} \phi_i(w) - (h_i^{-1}\phi_i)(w) = \sum_{i=1}^{n} \phi_i(w - h_i w) = \phi(\delta(w)) = \pi(w). \tag{2}$$

In order to define a chain c, it suffices to set its values on all 1-simplices of the form (ε, h) for $h \in H$, with the restriction that $c(\varepsilon, h)$ be zero for almost all h. In our case, we take

$$c(\varepsilon, h_i) = -\phi_i \tag{3}$$

and $c(\varepsilon, h) = 0$ for all other elements h. Note that then ∂c is a chain which in $\varepsilon \in G$ takes the value

$$\sum_{i=1}^{n} \phi_i - h_i^{-1}\phi_i \overset{(2)}{=} \pi,$$

so $\partial c = \pi$.

4.2 Description of a space without Property A

In [1], there is an example of a space X without property A which is interesting as it admits an inclusion into a Hilbert space. By Theorem 2, the fact that X does not have Property A must be demonstrated by some chain $c \in W_0(X)^*$ such that $\pi = \partial c$. Below we will construct an example of such chain. We will start from a description of the space X.

Denotation 25. For a group G, we denote by G^2 the group generated by the squares of all elements of G.

Fact 26. G^2 is a normal subgroup in G with an abelian quotient.

Proof. Normality follows from the equality

$$g^{-1}\, g_1^2 g_2^2 \ldots g_n^2\, g = (g^{-1}g_1 g)^2 (g^{-1}g_2 g)^2 \ldots (g^{-1}g_n g)^2.$$

Observe that for any $g \in G/G^2$ we have $g = g^{-1}$. Thus

$$gh = g^{-1}h^{-1} = (hg)^{-1} = hg.$$

□

Definition 27. Let $\Gamma_0 = F_2$ and $\Gamma_n = \Gamma_{n-1}^2$. Denote by X_n the quotient F_2/Γ_n.

Lemma 28. X_n is finite.

Proof. We will perform an inductive proof. For $X_0 = F_2/\Gamma_0$, the claim is clear. Now suppose that X_{n-1} is finite. We have

$$X_{n-1} = F_2/\Gamma_{n-1} = (F_2/\Gamma_n)/(\Gamma_{n-1}/\Gamma_n) = X_n/(\Gamma_{n-1}/\Gamma_n). \tag{4}$$

By Propositions 3.8 and 3.9 in [8], Γ_{n-1} is a finitely generated free group because the quotient F_2/Γ_{n-1} is finite. Therefore Γ_{n-1}/Γ_n is finitely generated, and abelian by Fact 26, so finite (as every its element has rank 2). Thus from (4) we get that X_n is also finite. $\square$

Define a metric space X as $\bigcup_{n=0}^{\infty} X_n$, where $\bigcup$ means adjoining the spaces X_n to a rope, each appropriately separated from the other. More formally, start with endowing each quotient group X_n with the metric of its Cayley graph, and setting the distance between the neutral elements $\varepsilon_n \in X_n$ and $\varepsilon_{n+1} \in X_{n+1}$ to be triple diameter of X_{n+1}. If we chose any metric on X compatible with these settings, X would turn out not to be coarsely connected. To fix that, we introduce a "$\mathbb{Z}$-rope", i.e. a sequence of additional points $\{x_i\}_{i\in\mathbb{Z}}$, where ε_n is identified with x_{s_n} for $s_n = 3\sum_{i=2}^{n} \operatorname{diam} X_i$. We will now formally define — using the Cayley graph metrics in the X_n — the distance in X:

- $d(x_i, x_j) = |i - j|$ for $i, j \in \mathbb{Z}$;
  (in particular, $d(\varepsilon_n, \varepsilon_{n+1}) = 3 \operatorname{diam} X_{n+1}$, just as claimed before)
- $d(a, b) = d(a, \varepsilon_n) + d(\varepsilon_n, \varepsilon_m) + d(\varepsilon_m, b)$ for $a \in X_n$, $b \in X_m$, $n \neq m$;
- $d(a, x_i) = d(a, \varepsilon_n) + d(\varepsilon_n, x_i)$ for $a \in X_n$, $i \in \mathbb{Z}$.

We define also an action of the free group F_3 on X as follows:

- The first two generators a, b act on every X_m according to the natural action of F_2 on X_m,
- $c \cdot x_i = x_{i+1}$ for $i \in \mathbb{Z}$,
- in all other cases, the generators a, b, c act trivially.

This action determines an identification of F_3 with some subgroup in $B_{id}(X)$. It is easy to check that this subgroup is exhausting. (This follows, for instance, from the fact that Lemma 7 holds when taking $R = 1$ and $B = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$). An illustration of the space X together with the action of F_3 on it is given in Figure 1.

Lemma 29 (cf. [8, Proposition 3.3]). For every R there exists n_0 such that for all $n > n_0$, the natural projection $p : F_2 \to X_n$ restricted to the ball $B(\varepsilon, R)$ is an isometry and preserves the action of a i b. More precisely: if g and ag belong to $B(\varepsilon, R)$, then $p(ag) = [a]p(g)$, and analogously for b.

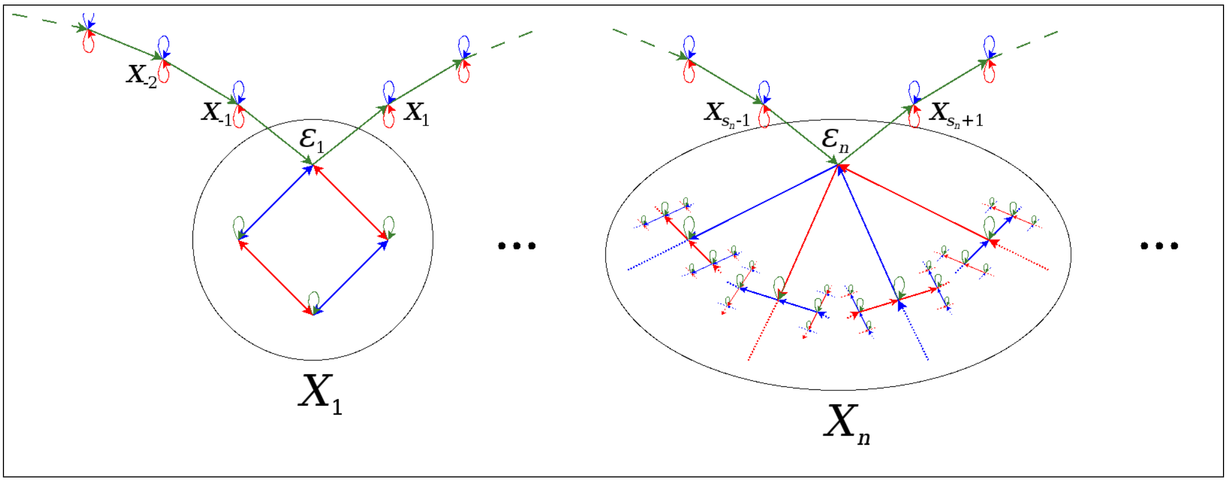

Figure 1: An illustration of the space X together with the action of F_3 on it. The blue, red and green arrows show respectively the actions of the generators a, b i c.

4.3 Description of the witness chain

Let $l_1, l_2 : F_2 \to \mathbb{R}$ be bounded functions such that

$$(l_1 - al_1) + (l_2 - bl_2) = \mathbf{1}_{F_2},$$

where a, b are the canonical generators of F_2. The existence of such functions follows from [2] because F_2 is not amenable. Choose $n > 0$. Let m be the radius for which the ball $B(\varepsilon_m, n)$ in X_m is isometric with a corresponding ball in F_2 (in the sense of Lemma 29). Note that

$$(5) \qquad (l_1 - al_1) + (l_2 - bl_2) = \mathbf{1} \qquad \text{in the ball } B(\varepsilon, n-1).$$

Let $f_i : X \to \mathbb{R}$ assign to a point $x \in B(\varepsilon_m, n)$ the value $l_i(x)$ obtained by identifying the balls in X_m and F_2, and the value 0 to all other points in X. Define functionals ϕ_i^n on $W_0(X)$ by the formula

$$\phi_i^n(w) = \frac{1}{\#(X_m)} \sum_{x_1, x_2 \in X_m} w(x_1, x_2) \cdot f_i(x_1 \cdot x_2^{-1}).$$

It is easy to check that ϕ_i^n is continuous with norm bounded by $||l_i||_\infty$. Then we can define $\phi_i \in W_0(X)^*$ as the Banach limit of ϕ_i^n; more precisely, as the composition

$$W_0(X) \xrightarrow{(\phi_i^n)_{n=1}^\infty} l^\infty(\mathbb{N}) \xrightarrow{\lim \mathrm{B}} \mathbb{R}.$$

Theorem 3. For all $w \in W_0(X)$, we have

$$(6) \qquad \phi_1(w - aw) + \phi_2(w - bw) = \pi(w).$$

With this Theorem, we see that, after setting $\phi_3 = 0$, the chain c defined as in (3) will satisfy $\partial c = \pi$, which will demonstrate that X does not have Property A.

Proof of the Theorem. It suffices to show that (6) holds on W_{00} since it is dense in W_0 and both sides in (6) depend continuously on w. Let $\bar{w}$ be arbitrary element of W_{00} and let k be its support size. It follows from (5) that, for $n > k+1$ and m chosen as in the definition of ϕ_i^n, we have

$$(f_1 - af_1) + (f_2 - bf_2) = \mathbf{1} \qquad \text{in the ball } B(\varepsilon_m, k).$$

This means that

$$\phi_1^n\big(\bar{w} - a\bar{w}\big) = \frac{1}{\#(X_m)} \sum_{x_1,x_2 \in X_m} \bar{w}(x_1,x_2) \cdot \Big(f_1(x_1 \cdot x_2^{-1}) - (af_1)(x_1 \cdot x_2^{-1})\Big),$$
$$\phi_2^n\big(\bar{w} - b\bar{w}\big) = \frac{1}{\#(X_m)} \sum_{x_1,x_2 \in X_m} \bar{w}(x_1,x_2) \cdot \Big(f_2(x_1 \cdot x_2^{-1}) - (bf_2)(x_1 \cdot x_2^{-1})\Big),$$

so the sum of these expression is

$$\frac{1}{\#(X_m)} \sum_{x_1,x_2 \in X_m} \bar{w}(x_1,x_2) \cdot \big(f_1 - af_1 + f_2 - bf_2\big)(x_1 \cdot x_2^{-1}) = \frac{1}{\#(X_m)} \sum_{x_1,x_2 \in X_m} \bar{w}(x_1,x_2).$$

What we obtained is the sum of values of $\bar{w}$ on whole $X_m \times X_m$, divided by the number of vertical fibers of the form $\{x_1\} \times X_m$. Now define K_m as the number of these $x_m \in X_m$ for which there exists $y \notin X_m$ such that $\bar{w}(x_m, y) \neq 0$. In fact, K_m is exactly the size of the external $|\operatorname{supp} \bar{w}|$-boundary of X_m in X. Clearly, (X_m) is a Følner sequence in X, which implies that for all $\delta > 0$ we have $\frac{K_m}{\#(X_m)} < \delta$ for sufficiently large m. This means that the sum of $\bar{w}$ on $X_m \times X_m$ equals — with a relative error less than δ — the sum on $X_m \times X$, which is $\#(X_m) \cdot \pi(\bar{w})$. Therefore, after passing to the limit with respect to m, we obtain (6) for ϕ_1^n, ϕ_2^n and $\bar{w}$. Now passing to the limit with respect to n finishes the proof. □

References

[1] G. Arzhantseva, E. Guentner, J. Špakula, *Coarse non-amenability and coarse embeddings*, preprint, `http://www.unige.ch/math/folks/arjantse/Abs/CE-not-A.pdf`.

[2] J. Block, S. Weinberger, *Aperiodic tilings, positive scalar curvature, and amenability of spaces*, J. of Amer. Math. Soc. Vol. 5, No. 4 (1992), pp. 907–918, `http://math.uchicago.edu/~shmuel/Aperiodic%20tilings.pdf`.

[3] J. Brodzki, G. Niblo, P. Nowak, N. Wright, *A homological characterization of topological amenability*, preprint, `http://www.mimuw.edu.pl/~pnowak/index/research_files/homology.pdf`.

[4] J. Brodzki, G. Niblo, P. Nowak, N. Wright, *Amenable actions, invariant means and bounded cohomology*, preprint, `http://www.mimuw.edu.pl/~pnowak/index/research_files/actions_coho.pdf`.

[5] J. Brodzki, G. Niblo, N. Wright, *Pairings, duality and bounded cohomology*, preprint, `http://eprints.soton.ac.uk/79402/1/Pairings,_duality,_amenability_and_bounded_cohomology_final_refereed_version.pdf`.

[6] W. Deuber, M. Simonovits, V. T. Sós, *A note on paradoxical metric spaces*, Studia Scientiarum Math. Hung., 30 (1995), pp. 17–23, http://renyi.hu/~miki/walter07.pdf.

[7] N. Higson, J. Roe, *Amenable group actions and the Novikov conjecture*, J. reine angew. Math. 519 (2000), pp. 143–153, http://web.me.com/ndh2/math/Papers_files/Higson,%20Roe%20-%202000%20-%20Amenable%20group%20actions%20and%20the%20Novikov%20conjecture.pdf.

[8] R. C. Lyndon, P. E. Schupp, *Combinatorial group theory*, Springer, 1977.

[9] P. Nowak, G. Yu, *Large scale geometry*, EMS publishing house, to appear, http://www.math.tamu.edu/~pnowak/book_etb/book_etb.pdf.

[10] D. Pawlik, *Teoria homologii grup Blocka-Weinbergera i jej zastosowania*, MSc thesis, 2010.