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Gromov boundaries of torsion free hyperbolic groups are Markov compacta

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We provide a construction which allows presenting boundaries of torsion-free hyperbolic groups as Markov compacta. The latter notion was introduced in [1] by Dranishnikov, who claims the concept inspired by the work of Gromov, and refers to inverse limits of polyhedra with certain additional regularity conditions. The existence of such presentation is known (see M. Kapovich, http://aimath.org/pggt/Boundaries), but it seems that there has been no proof provided yet.

Definition 0.1. A topological space V is a *Markov compactum* if it is an inverse limit of some sequence of spaces K_i and maps $f_i : K_{i+1} \to K_i$ which satisfy the following conditions:

- K_i are simplicial complexes of bounded dimension;
- For every i, there exists a subdivision $\overline{K}_i$ of K_i such that $f_i : K_{i+1} \to \overline{K}_i$ is a simplicial map;
- Each simplex in $\amalg K_i$ can be assigned one of finitely many *types* so that if $s \in K_i$ and $s' \in K_j$ are simplices of the same type, then there exist simplicial isomorphisms $i_k : f^{-k}(s) \to f^{-k}(s')$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} s & \xleftarrow{f_i} & f^{-1}(s) & \longleftarrow & \cdots & \longleftarrow & f^{-k}(s) & \xleftarrow{f_{i+k}} & f^{-(k+1)}(s) & \longleftarrow & \cdots \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & & & \downarrow{\scriptstyle i_k} & & \downarrow{\scriptstyle i_{k+1}} & & \\ s' & \xleftarrow{f_j} & f^{-1}(s') & \longleftarrow & \cdots & \longleftarrow & f^{-k}(s') & \xleftarrow{f_{j+k}} & f^{-(k+1)}(s') & \longleftarrow & \cdots \end{array}$$

  where f^{-k} denotes the inverse image under the appropriate composition of f_i's.

Definition 0.2. V is a Markov compactum with the *mesh property* if the diameters of simplices in K_i converge to 0 as $i \to \infty$.

The aim of this paper is to prove the following result.

Theorem 1. The boundary of every torsion-free hyperbolic group G is a Markov compactum with mesh property.

In Section 1 we introduce the result of M. Coornaert and A. Papadopoulos which allows to present the boundary of a hyperbolic group as the topological quotient of a Cantor-like space of words by some convenient relation defined letterwise. In Section 2 we introduce a convenient quasimetric on this space which is connected with the above presentation, and bi-Lipschitz equivalent to the canonical metric on ∂G. In Section 3 we build a presentation of ∂G as the inverse limit of a sequence of its subsequently finer covers (which will consist of *all* balls of a given radius $r_n \to 0$ in our quasimetric). In the last two sections we check that this presentation satisfies the Markov condition.

Remark 0.3. The result of our construction will turn out to satisfy a strengthened definition of Markov compactum, in which every f_i is required to take the vertices of K_{i+1} to the vertices of the barycentric subdivision of K_i.

1 A description of ∂G by infinite words

Let G be a torsion-free δ-hyperbolic group.

Notation 1.1. We denote by S_i the sphere in G with center in e and radius i.

Definition 1.2. Let V be a subset of S_i. Its *projection* is the set

$$P(V) = \{x \in S_{i-1} \mid \operatorname{dist}(x, V) = 1\}.$$

Definition 1.3. Let T_{part} be the tree whose vertices are the subsets of G which satisfy the condition

$$V \subset S_n \text{ for some } n \qquad \text{and} \qquad \operatorname{diam} P^i(V) \leqslant 20\delta \text{ for all } 0 \leqslant i \leqslant n,$$

and an edge conects the set with its projection. The root of T_{part} is the vertex $\{e\}$.

In [3] the authors introduce a finite alphabet Σ and a *type function* $T : T_{part} \to \Sigma$ such that

- No vertex has two children of the same type.
- For every two vertices of the same type, their children have the same types.
 (thus we can speak about type a being a *child* of type b)

In addition, we will use another property of the type function in Fact 1.4: if $T(u) = T(v)$, then there exists an isometry $\tau : G \to G$ such that $\tau(u) = v$.

By a *word* we will mean a sequence of types (acting as letters) from Σ in which every type is a child of the previous one. Every vertex v in T_{part} determines the path joining the root with v; this in turn determines the word $w(v)$ which we can identify with v. Similarly, an element of ∂T_{part} (i.e. an infinite path) can be identified with an infinite word.

Unless explicitly stated otherwise, every sequence of types appearing in this paper is meant to form a valid word in the above sense.

We introduce a relation in Σ (its detailed definition will be needed only in the proof of Lemma 2.2):

$$a \sim b \qquad \Leftrightarrow \qquad \exists_{v,v' \subset T_{part}} \Big(T(v) = a, \quad T(v') = b, \quad \exists_j \ v, v' \subset S_j, \quad \operatorname{dist}(v, v') \leqslant 4\delta\Big),$$

where $\operatorname{dist}(v, v')$ is the minimal distance in G between the elements of the sets v and v'. This induces a relation in the set of words:

$$\alpha \sim \beta \qquad \Leftrightarrow \qquad \forall_i \ \alpha_i \sim \beta_i.$$

These two relations are reflexive and symmetric but transitivity holds only in ∂T_{part}, that is, for infinite words starting from $T(\{e\})$. The above definition of $\sim$ is somehow different from [3] but their equivalence is proven by the following fact.

Fact 1.4. Let $a_1 a_2$ and $b_1 b_2$ be two words in relation $\sim$. Let $u_1, v_1 \subset S_i$ be two vertices in T_{part} resp. of type a_1 and b_1 which satisfy $\operatorname{dist}(u_1, v_1) \leqslant 4\delta$. Then there exist $u_2, v_2 \in T_{part}$ resp. of type a_2 and b_2 which are resp. children of u_1 and v_1 such that $\operatorname{dist}(u_2, v_2) \leqslant 4\delta$.

Proof. Since $a_1 a_2$ is a word, every vertex of type a_1 has a unique child of type a_2. This uniquely determines u_2; analogically we choose v_2. From the conditions

$$\operatorname{dist}(u_1, u_2) = \operatorname{dist}(v_1, v_2) = 1, \qquad \operatorname{diam}(u_1), \operatorname{diam}(v_1) \leqslant 20\delta, \qquad \operatorname{dist}(u_1, v_1) \leqslant 4\delta, \tag{1}$$

we obtain $\operatorname{dist}(u_2, v_2) \leqslant 44\delta + 2$. We should prove that this distance is not greater than 4δ.

Since $a_2 \sim b_2$, there exists $\tilde{u}, \tilde{v} \in T_{part}$ resp. of type a_2 and b_2 satisfying $\mathrm{dist}(\tilde{u}, \tilde{v}) \leqslant 4\delta$ and both contained in some S_k. From the definition of type we have an isometry $\tau : G \to G$ which takes $\tilde{u}$ to u_2. Then a reasoning analogous to the proof of [3, Proposition 7.3.5] shows that $\tau(\tilde{v}) = v_2$ ([1]). □

Theorem 2 ([3])**.** ∂G is (naturally) homeomorphic to the quotient space $\partial T_{part} / \sim$. Moreover, there exists a constant W such that the equivalence classes of $\sim$ contain at most W elements.

Remark 1.5. If G is a δ_0-hiperbolic group, it is also δ-hiperbolic for $\delta \geqslant \delta_0$. We observe that the definition of T_{part} and the type function in [3] depend on the choice of δ; Theorem 2 holds independently for each appropriate value of δ. In this paper, we assume that $\delta \geqslant 6\delta_0$. (This assumption will be used only in a few places).

The equivalence class of a word α with respect to $\sim$ will be denoted by $[\alpha]$. In particular, $\alpha \in [\beta]$ means $\alpha \sim \beta$.

Notation 1.6. We will write $\alpha \sim_n \beta$ if $\forall_{i \leqslant n}\ \alpha_i \sim \beta_i$. We will then say that α is a *n-prefix companion* of β. If $\alpha \sim \beta$, we will call α a *companion* of β.

We will write simply $\alpha \sim \beta$ instead of $\alpha \sim_n \beta$ when n is the length of the shorter of two words.

Fact 1.7. We may shrink the relation $\sim$ among the letters, without changing the induced relation among infinite words, so that every pair of related letters would occur in some related words, that is: $a \sim b \Leftrightarrow \exists_{\alpha,\beta}\ a\alpha \sim b\beta$. □

2 A quasimetric on ∂G

Definition 2.1. Let $a > 1$ be the parameter used in the definition of the visual metric on ∂G. Define a function (as we will prove, a quasimetric) $d : \partial G \times \partial G \to \mathbb{R}$ by the formula

$$d([\alpha], [\beta]) = \min\{a^{-n}\ :\ \alpha \sim \gamma \sim_n \delta \sim \beta \text{ dla pewnych } \gamma, \delta\}.$$

This definition is correct because $\sim$ is an equivalence relation for infinite words. As an immediate consequence of the definition, we have

Lemma 2.2. The distance function d is Lipschitz with respect to the visual metric d_v on ∂G:

$$d([\alpha], [\beta]) \leqslant C d_v([\alpha], [\beta])$$

Remark 2.3. In fact, these two quasimetrics are bi-Lipschitz equivalent, but we will not need this. Nevertheless, note that this allows an alternative way of proving Lemma 2.4.

Proof of Lemma 2.2. Let $d([\alpha], [\beta]) = a^{-n}$. Without loss of generality we can assume that $\alpha \sim_n \beta$ and $\alpha \not\sim_{n+1} \beta$. Choose some geodesic γ conecting $[\alpha]$ with $[\beta]$ which has the maximal possible distance from e. Denote $\mathrm{dist}(e, \gamma) = d$. By the properties of the visual metric, we have

$$C_1^{-1} d_v([\alpha], [\beta]) \leqslant a^{-d} \leqslant C_1 d_v([\alpha], [\beta])$$

[1]For completness, we will present this reasoning, using the notions defined in [3]. We assume that N (the parameter of type) is sufficiently large, and that $\delta > 1$. For any $g \in \tilde{u}$, $h \in \tilde{v}$ we have $d(g, h) \leqslant 44\delta$; therefore, as g and $\tau(g)$ have the same N-type and $|h| = |g|$, we have $|\tau(h)| = |\tau(g)| = i + 1$ and moreover h has the same $(N - 44\delta)$-type as $\tau(h)$. Then $\tau(\tilde{v})$ is contained in S_{i+1} and has the same $(N - 44\delta)$-type as $\tilde{v}$, and thus also as v_2. However, $\mathrm{dist}(\tau(\tilde{v}), v_2) \leqslant \mathrm{dist}(\tilde{v}, \tilde{u}) + \mathrm{dist}(u_2, v_2) \leqslant 50\delta$. Then the equality $\tau(\tilde{v}) = v_2$ is a result of [3, Proposition 7.5.1] applied to $\delta' = 13\delta$.

for some constant C_1. Now, it is enough to show that $\alpha \sim_{d-C_2} \beta$ for some constant C_2.

Let (u_i), (v_i) be the paths in T_{part} corresponding to the words α, β. We choose inductively a point $g_i \in u_i$ such that $d(g_i, g_{i+1}) = 1$; then (g_i) is a geodesic because $g_i \in S_i$. From the definition of the function $\partial T_{part} \to \partial G$ we deduce that (g_i) converges to $[\alpha]$. Analogously, we define a geodesic (h_i) inside (v_i) converging to $[\beta]$. Consider the geodesic triangle Δ with sides γ, (g_i), (h_i). It is $12\delta_0$-slim. In particular, every g_i is $(12\delta_0)$-close to the union of sides (h_i) and γ. However, if $i < d - 12\delta_0$, then for any $\overline{g} \in \gamma$ we have

$$d(g_i, \overline{g}) \geqslant d(e, \overline{g}) - d(e, g_i) \geqslant \operatorname{dist}(e, \gamma) - d(e, g_i) = d - i > 12\delta_0,$$

so there exists h_j such that $d(g_i, h_j) \leqslant 12\delta_0$. Then

$$|i - j| = |d(e, g_i) - d(e, h_j)| \leqslant d(g_i, h_j) = 12\delta_0,$$

so $d(g_i, h_i) \leqslant 24\delta_0 \leqslant 4\delta$. Then we have $\operatorname{dist}(u_i, v_i) \leqslant 4\delta$ for $i < d - 12\delta_0$. From the definition we obtain that $\alpha \sim_{d-12\delta_0-1} \beta$, which finishes the proof. □

Lemma 2.4. There exists a constant L such that if $\alpha \sim_n \beta \sim_n \gamma$, then $\alpha \sim_{n-L} \gamma$.

Proof. If this does not hold, we will obtain a contradiction by constructing in ∂T_{part} three sequences

$$\tilde{\alpha} a \overline{\alpha} \sim \tilde{\beta} b \overline{\beta} \sim \tilde{\gamma} c \overline{\gamma} \qquad \text{such that} \qquad a \not\sim c, \tag{2}$$

which is impossible by the transitivity of $\sim$ in ∂T_{part}.

Assume the negation of the claim of the Lemma. Then it follows easily that for any i there exist $\alpha_i \sim_{n_i} \beta_i \sim_{n_i} \gamma_i$ in ∂T_{part} such that $\alpha_i \not\sim_{n_i - i} \gamma_i$. Since $\alpha_i \not\sim \gamma_i$, we can determine the lowest "witness index" k_i such that $\alpha_i \not\sim_{k_i} \gamma_i$. Then,

$$\alpha_i \sim_{k_i + i} \beta_i \sim_{k_i + i} \gamma_i. \tag{3}$$

By the finiteness of the alfabet, the sequence of triples $\big(\alpha_i(k_i), \beta_i(k_i), \gamma_i(k_i)\big)$ must contain some value (a_1, b_1, c_1) infinitely many times on certain positions $(i_{1,j})_{j=0}^{\infty}$; we can assume that $i_{1,1} \geqslant 1$. Analogically, in the subsequence

$$\big(\alpha_{i_{1,j}}(k_{i_{1,j}} + 1), \beta_{i_{1,j}}(k_{i_{1,j}} + 1), \gamma_{i_{1,j}}(k_{i_{1,j}} + 1)\big),$$

(which is formed by the letters following (a_1, b_1, c_1) in those $(\alpha_i, \beta_i, \gamma_i)$ in which they appeared), some value (a_2, b_2, c_2) occurs on the positions $(i_{2,j})_{j=0}^{\infty}$; we can assume that $i_{2,1} \geqslant 2$. In the same way we obtain the subsequent values (a_l, b_l, c_l). We define

$$a = a_1, \qquad \overline{\alpha} = a_2 a_3 \ldots, \qquad \tilde{a} = \alpha_{i_{1,1}}(1) \ldots \alpha_{i_{1,1}}(k_{i_{1,1}} - 1)$$

and similarly $\tilde{\beta} b \overline{\beta}$ and $\tilde{\gamma} c \overline{\gamma}$. Since $i_{l,1} \geqslant l$ for any l, we obtain from (3) that $a_l \sim b_l \sim c_l$. Therefore (2) holds. □

Corollary 2.5. The distance function d is a *quasimetric*, i.e. it satisfies the triangle inequality with some constant a^{2L}. Moreover, we have

$$d([\alpha], [\gamma]) \leqslant a^{2L} \cdot \max\big(d([\alpha], [\beta]), d([\beta], [\gamma])\big).$$

Fact 2.6. For every n and every word α, there exists at most $W \cdot |\Sigma|$ pairwise distinct words x of length n such that $\alpha \sim_n x$, where W is the constant from Theorem 2.

Proof. If this were false, there would exist $W+1$ pairwise distinct words $x_1, ..., x_{W+1} \sim_n \alpha$ all having the same n-th letter. Note that by Fact 1.7 we can construct $\overline{\alpha}$ and $\overline{\beta}_1$ such that

$$\alpha =_n \overline{\alpha} \sim \overline{\beta}_1 =_n x_1.$$

As all x_i have the same n-th letter, we can copy the suffix from $\overline{\beta}_1$ to obtain

$$\overline{\alpha} \sim \overline{\beta}_i =_n x_i.$$

This means that $\overline{\alpha}$ has at least $W+1$ companions, which contradicts Theorem 2. □

3 A presentation of ∂G as a limit of polyhedra

In this section we discuss a method to present ∂G as the inverse limit of a sequence of polyhedra. (In section 5 we will show that this sequence satisfies the Markov condition). We use the construction described in the proof of Theorem 1.13.2 in [2], though under slightly different assumtions (which we will show to be sufficient for its corectness). We will start from studying covers of ∂G consisting of *all* balls of a fixed radius, with respect to the quasimetrics d discussed in Section 2.

The letters $\alpha, \beta, \ldots$ will denote infinite words, while the letters $x, y, \ldots$ will stand for finite words. Let L be the constant from Lemma 2.4 and let $M = 9L$. Recall that d is a quasimetric with constant a^{2L}.

Definition 3.1. Let $n \geqslant 2$ be an integer. By a ball *of order* n in ∂G we mean any "closed" ball with respect to d of radius a^{-nM}. (Here "closed" relates to the inequality $d([\alpha],[\beta]) \leqslant a^{-nM}$ rather than to closedness in any topology). The notation $B(n)$ will mean that B is (some) ball of order n.

Remark 3.2. It will result from the following facts that the center of a ball is often not unique. For this reason, we do not include it in the notation.

Remark 3.3. Unless stated otherwise, when speaking about a *ball* we will mean a ball of some integer order n together with the information about the value of n. Even when $B(n)$ and $B'(n)$ contain the same points, we will treat them as distinct. (Formally, one can think about pairs: $(B,n) \neq (B',n')$).

Remark 3.4. We introduce the condition $n \geqslant 2$ to make Definition 4.5 always applicable.

Fact 3.5. For any balls $B_1(n) \cap B_2(n) \neq \emptyset$, if $[\alpha],[\beta] \in B_1 \cup B_2$, then $\alpha \sim_{nM-8L} \beta$. In particular, this holds when $B_1 = B_2$.

Proof. It follows from a multiple application of Corollary 2.5. □

Fact 3.6. The number of all n-th order balls is finite. Moreover, there exists some constant T (independent of n) such that any ball of order n intersects only at most T other balls of the same order.

Proof. Let $S(B)$ be the set of all representatives of all centers of B. For any set $A \subset \partial T_{part}$, let $P_k(A)$ denote the set of k-prefixes of the words from A.

The first part of the claim follows from that the set $P_{nM}(S(B))$ uniquely determines B. Indeed:

$$[\gamma] \in B \quad \Leftrightarrow \quad \gamma \sim \delta \sim_{nM} \eta \in S(B) \quad \Leftrightarrow \quad \gamma \sim \delta \sim_{nM} x \in P_{nM}(S(B)).$$

Now, fix some center $[\alpha]$ of $B(n)$. Let S be the union of the sets $S(B')$ for all $B'(n)$ intersecting B. From Fact 3.5 we have $\gamma \sim_{nM-8L} \alpha$ for all $\gamma \in S$ (where M is independent of n). Therefore, by Fact 2.6, $|P_{nM-8L}(S)| \leqslant W \cdot |\Sigma|$, so $|P_{nM}(S)| \leqslant W \cdot |\Sigma| \cdot |\Sigma|^{8L}$. However, as the set $P_{nM}(S(B')) \subset P_{nM}(S)$ uniquely determines B', we can bound the number of distinct balls B' not disjoint with B by $T = 2^{W \cdot |\Sigma|^{8L+1}}$. □

Lemma 3.7 (Star Lemma)**.** If a family of balls $B_1, B_2, \dots, B_k$ of order n has non-empty intersection, then there exists a ball $B'(n-1)$ containing every B_i.

Proof. Let $[\alpha] \in \bigcap B_i$. For any $[\beta] \in B_i$ it follows from Fact 3.5 that $\alpha \sim_{nM-8L} \beta$, which implies $\alpha \sim_{(n-1)M} \beta$. Therefore all B_i are contained in the $(n-1)$-th order ball centered in $[\alpha]$. □

Definition 3.8. Let $\mathcal{U}_n$ be the cover of ∂G consisting of all balls of order n. (The balls do not have to be open in the topology of ∂G; however, it follows from Lemma 2.2 that every ball contains an open neighbourhood of its center).

Construction 3.9. Let K_n be the nerve of $\mathcal{U}_n$. For a vertex $x \in K_{i+1}$ corresponding to a ball B, we define $f_i(x)$ as the barycenter of the simplex $s \in K_i$ spanned by all the vertices corresponding to the balls containing B. We then define $f_i : K_{i+1} \to K_i$ by the linear extension on each simplex. These complexes and maps form an inverse system. We will denote its limit by K.

Finally, for every $x \in \partial G$ denote by $K_i(x)$ the simplex in K_i spanned by all vertices corresponding to the balls containing x. Clearly, the simplices $K_i(x)$ with the restrictions of f_i form an inverse system whose limit $\varprojlim K_i(x)$ is naturally contained in K.

Theorem 3. For every $x \in \partial G$, the inverse limit $\varprojlim K_i(x)$ consists of exactly one point $f(x) \in K$. Moreover, the map $f : \partial G \to \varprojlim K_i$ defined in this manner is a homeomorphism.

Proof. The proof is strongly based on the construction described in [2] and the proof given there, even though they start with different assumptions. It is easy to check that both the construction and the proof in [2] — except for the proof of the continuity of functions f_j — remain valid under the following assumptions about $\mathcal{U}_i$:

(a) The covers $\mathcal{U}_i$ are finite;

(b) There exists n such that all $\mathcal{U}_i$ are of order n, that is, at most n different elements of $\mathcal{U}_i$ may have non-empty intersection.

(c) For any family $\mathcal{U} \subset \mathcal{U}_{i+1}$ with non-empty intersection, there exists $U \in \mathcal{U}_i$ which contains all elements of $\mathcal{U}$ (condition (3) in [2]);

(d) For any $x \neq x' \in \partial G$ there exists i such that no element $\mathcal{U}_i$ can contain simultaneously x and x' (this condition appears in the last part of the proof in [2] and accounts for the injectivity of function f)

(The key observation is that the condition (2) in [2], requiring (in particular) that the diameters of the elements of $\mathcal{U}_{i+1}$ be bounded by the Lebesgue's number of the cover $\mathcal{U}_i$ — which does not make sense in our case — is needed there only to prove (c), (d) and continuity of f_j).

In our case, conditions (a) and (b) follow immediately from Fact 3.6, (c) — from Lemma 3.7 and (d) — from the fact that the elements of $\mathcal{U}_i$ are balls (in a quasimetric) with radii $a^{-iM} \xrightarrow[i\to\infty]{} 0$. It remains to prove continuity of the maps f_j. For this, we will modify a part of the proof from [2], using the notation introduced there.

Fix $x_0 \in X = \partial G$, $j \in \mathbb{N}$ and $\varepsilon > 0$. There exists $i \geqslant j$ such that $\left(\frac{n}{n+1}\right)^{i-j} < \frac{\varepsilon}{4}$; then from condition (6) in [2] we have that π^i_j is $\frac{\varepsilon}{4}$-lipschitz. Let $B = U_{i,k}$ be the ball of order i (with respect to d) centered in x_0. From Lemma 2.2 we know that B contains some ball U with respect to the visual metric d_v centered in in x_0. In particular, U is open in the visual metric topology on ∂G. For any $x \in U$ we have

$$U_{i,k} = B \in \mathcal{U}_i(x_0) \cap \mathcal{U}_i(x) \quad \Rightarrow \quad p_{i,k} \in K_i(x_0) \cap K_i(x) \quad \Rightarrow \quad \delta\Big(\pi^i_j\big(K_i(x_0) \cup K_i(x)\big)\Big) < \varepsilon.$$

From the fact that $f_j(y) \in \pi^i_j(K_i(y))$ for any $y \in \partial G$ we conclude that

$$\sigma\big(f_j(x_0), f_j(x)\big) < \varepsilon \qquad \text{dla dowolnego } x \in U.$$

Thus f_j is continous at x_0, which ends the proof. □

4 Definition of the ball type

Unless stated otherwise, the letters B, C and D (with various decorations) will denote balls of some integer order. Recall that $x, y, z, \dots$ stand for finite words, while α, β, γ denote infinite words. For convenience, we will use notation $x^{(n)}$ to indicate that x is a word of length nM (analogously to $B(n)$). A superscript $x^{(\geqslant n)}$ will mean that x has a finite length not shorter than nM (and not necessarily divisible by M).

Definition 4.1. For a fixed ball $B(n)$, the set of its *accepted prefixes* is the set of all words $x^{(\geqslant n)}$ such that for any β beginning with x we have $[\beta] \in B$.

Fact 4.2. If $[\alpha]$ is some center of a ball $B(n)$ and $x^{(n)} \sim \eta \sim \alpha$, then $x \in \mathrm{AP}(B)$. □

Corollary 4.3. For a ball $B(n)$ and any $k \geqslant nM$, every point of B can be represented by a word beginning with an accepted prefix of B of length k.

Proof. Let $[\alpha]$ be some center of $B(n)$ and $[\beta] \in B$. Then, from the definition,

$$\beta \sim xy\gamma \sim_{nM} \delta \sim \alpha \qquad \text{for some } x^{(n)}, |y| = k - nM, \gamma, \delta,$$

so $[\beta]$ has a presentation starting with xy. Since $x \in \mathrm{AP}(B)$, we have also $xy \in \mathrm{AP}(B)$. □

Fact 4.4. If $B \subset B'$, then $\mathrm{AP}(B) \subset \mathrm{AP}(B')$.

Definition 4.5. For a ball $B(n)$, we define the the set of *truncated accepted prefixes* for B, denoted $\mathrm{TAP}(B)$, as the set of accepted prefixes with their $(n-2)M$ first characters removed:

$$\mathrm{TAP}(B) = \big\{ s \;\big|\; \exists_{x^{(n-2)}}\; xs \in \mathrm{AP}(B) \big\}.$$

Remark 4.6. For the most of the paper, it would suffice to define the TAP-sets by truncating the $(n-1)M$-prefix (which would make them contanining less information). The choice of $(n-2)M$ will be needed in the proof of Fact 5.4.

Lemma 4.7. The balls of all orders have together only finitely many different TAP-sets.

Proof. Let $B(n)$ have some center in $[\alpha]$. Denote

$$\mathrm{AP}^{(n)}(B) = \{x^{(n)} \mid x \in \mathrm{AP}(B)\}.$$

If $x \in \mathrm{AP}^{(n)}(B)$, then $[x\beta] \in B$ for some β and thus

$$x\beta \sim \gamma \sim_{nM} \delta \sim \alpha \qquad \Rightarrow \qquad x \sim_{nM-2L} \alpha.$$

Since $[\alpha]$ is fixed, by Fact 2.6 we obtain $|\,\mathrm{AP}^{(n)}(B)| \leqslant W \cdot |\Sigma|^{2L+1}$.

Let $G(B)$ denote the (undirected) graph with $\mathrm{AP}^{(n)}(B)$ as the set of vertices, with an edge joining x with y if and only if $x \sim y$. We introduce a labeling of vertices of $G(B)$: to every vertex $x \in \mathrm{AP}^{(n)}(B)$ we assign its suffix of length $2M$. Since the numbers $|\,\mathrm{AP}^{(n)}(B)|$, $|\Sigma|$ and M are all bounded independently of B, it follows that all graphs of the form $G(B)$ fall into finitely many classes of labeled isomorphism. We will show that the set $\mathrm{TAP}(B)$ depends only on the choice of such class.

Suppose that there exists a labeled isomorphism between $G(B(n))$ and $G(\overline{B}(m))$. Then, by choosing some numberings of elements of $\mathrm{AP}^{(n)}(B)$ and $\mathrm{AP}^{(n)}(\overline{B})$ compatible with this isomorphism, we obtain presentations with the following properties:

$$\mathrm{AP}^{(n)}(B) = \{x_i^{(n-2)} s_i^{(2)}\}, \qquad \mathrm{AP}^{(n)}(\overline{B}) = \{\overline{x}_i^{(m-2)} s_i\}, \qquad x_i s_i \sim x_j s_j \;\Leftrightarrow\; \overline{x}_i s_i \sim \overline{x}_j s_j. \tag{4}$$

We want to show that $\mathrm{TAP}(B) = \mathrm{TAP}(\overline{B})$. Let $y \in \mathrm{TAP}(B)$, then $xy \in \mathrm{AP}(B)$ for some $x^{(n-2)}$. Let $y = s^{(2)}u$; we have $xs \in \mathrm{AP}^{(n)}(B)$, so $x = x_i$ and $s = s_i$ for some i.

Choose any β such that $xy\beta$ exists. From $[xy\beta] \in B$ we obtain that

$$xy\beta \sim z\gamma \sim_{nM} \eta \sim \alpha \qquad \text{for some } z^{(n)}, \gamma, \eta,$$

where $[\alpha]$ is some center of B. By Fact 4.2, $z \in \mathrm{AP}(B)$; thus $z = x_j s_j$ for some j. Now, we use (4):

$$x_i y\beta = x_i s_i u\beta \sim x_j s_j \gamma \qquad \Rightarrow \qquad \overline{x}_i y\beta = \overline{x}_i s_i u\beta \sim \overline{x}_j s_j \gamma.$$

Since $\overline{x}_j s_j \in \mathrm{AP}^{(n)}(\overline{B}) \subset \mathrm{AP}(\overline{B})$, we obtain that $[\overline{x}_i y\beta] \in \overline{B}$. As β was arbitrary, it follows that $y \in \mathrm{TAP}(\overline{B})$, which proves that $\mathrm{TAP}(B) \subset \mathrm{TAP}(\overline{B})$.

By a symmetric reasoning one obtains $\mathrm{TAP}(B) = \mathrm{TAP}(\overline{B})$, which finishes the proof. □

Definition 4.8. Two balls $B_1(n)$, $B_2(n)$ of the same order will be called *neighbours* if they have non-empty intersection. We will consider B being its own neighbour. The set of neighbours of B will be denoted by $N(B)$.

Definition 4.9. We define the *type* of a ball $B(n)$ by the formula

$$T(B) = \{\,\mathrm{TAP}(B') \mid B' \in N(B)\}$$

Remark 4.10. It follows immediately from Fact 3.6 Lemma 4.7 that, for a given group G, the balls of all orders have together only finitely many types.

Note also that it will follow from Lemma 3.7 and Fact 5.1 that the sets $\mathrm{TAP}(B')$ are pairwise distinct among all the neighbours B' of a fixed ball B.

5 Proof of Markovian property

We continue to use the notations introduced in Section 4.

Fact 5.1. Let $B_1(n) \cap B_2(n) \neq \emptyset$ and $C_1(n+1), C_2(n+1) \subset B_1 \cup B_2$. If $\mathrm{TAP}(C_1) = \mathrm{TAP}(C_2)$, then $C_1 = C_2$.

Proof. Let $[\alpha] \in C_1$. By Corollary 4.3 we have $\alpha \sim x^{(n-2)}s^{(2)}\beta$, where $s \in \mathrm{TAP}(C_1)$. Then $[\overline{x}s\beta] \in C_2$ for some $\overline{x}^{(n-2)}$. From Fact 3.5 it is apparent that $x \sim \overline{x}$, so $[\alpha] = [xs\beta] = [\overline{x}s\beta] \in C_2$. Then $C_1 \subset C_2$; analogically, we prove the second inclusion. □

Lemma 5.2. If $\mathrm{TAP}(B(n)) = \mathrm{TAP}(B'(m))$, then for any ball $C(n+1) \subset B$ there exists a ball $C'(m+1) \subset B'$ such that $\mathrm{TAP}(C) = \mathrm{TAP}(C')$.

Proof. Let $\mathrm{TAP}^-(B)$ be the set of elements $\mathrm{TAP}(B)$ with their M-prefixes removed.

Fix some center $[\alpha]$ of ball C and choose any $x^{(n-1)}s^{(2)}\beta \sim \alpha$. By Fact 4.2, $s \in \mathrm{TAP}(C)$. Since

$$(*) \qquad \mathrm{TAP}(C) \subseteq \mathrm{TAP}^-(B) = \mathrm{TAP}^-(B'),$$

we have $s \in \mathrm{TAP}^-(B')$, so there exists $y^{(m-1)}s \in \mathrm{AP}(B')$. We define $C'(m+1)$ as the ball centered in $[\alpha'] = [ys\beta]$. We will now check that $[\alpha']$ is independent from the choice of $xs\beta$. Indeed, if $x'^{(n)}s'^{(2)}\beta' \sim xs\beta$, then analogically there exists $y'^{(m-1)}s' \in \mathrm{AP}(B')$. By Corollary 3.5 we have $y' \sim y$; hence $y's'\beta' \sim ys\beta \sim \alpha'$.

Let $s_1^{(\geqslant 2)} \in \mathrm{TAP}(C)$. Then there exist sequences $x_1^{(n-1)}s_1 \in \mathrm{AP}(C)$ and, by virtue of (*), $x_1'^{(m-1)}s_1 \in \mathrm{AP}(B')$. We will show that $x_1's_1 \in \mathrm{AP}(C')$.

Fix any γ. Then $[x_1s_1\gamma] \in C$, which means that

$$(**) \qquad x_1s_1\gamma \sim \overline{x}_1\overline{s}_1\overline{\gamma} \sim_{(n+1)M} xs\beta \sim \alpha \qquad \text{for some } \overline{x}_1^{(n-1)}, \overline{s}_1^{(2)}, \overline{\gamma}, x^{(n-1)}, s^{(2)}, \beta.$$

From Fact 4.2 we obtain $\overline{s}_1 \in \mathrm{TAP}(C)$, so by (*) there exists a sequence $\overline{x}_1'^{(m-1)}\overline{s}_1 \in \mathrm{AP}(B')$. Then Fact 3.5 applied in B' implies that $x_1' \sim \overline{x}_1' \sim y$ (where y is such that $ys\beta \sim \alpha'$). This and (**) give

$$x_1's_1\gamma \sim \overline{x}_1'\overline{s}_1\overline{\gamma} \sim_{(m+1)M} ys\beta \sim \alpha' \qquad \Rightarrow \qquad [x_1's_1\gamma] \in C'.$$

As γ was arbitrary, we have $s_1 \in \mathrm{TAP}(C')$, so $\mathrm{TAP}(C) \subseteq \mathrm{TAP}(C')$. Analogically, $\mathrm{TAP}(C') \subseteq \mathrm{TAP}(C)$. □

Fact 5.3. Let

$$C(n+1) \subset \bigcap_{i=1}^{k} B_i(n), \qquad \bigcap_{i=1}^{k} B_i'(m) \neq \emptyset, \qquad \mathrm{TAP}(B_i) = \mathrm{TAP}(B_i').$$

Then there exists $C'(m+1) \subset \bigcap B_i'$ such that

$$(5) \qquad \mathrm{TAP}(C') = \mathrm{TAP}(C).$$

Moreover, for any i, C' is the only ball contained in B_i' which satisfies (5).

Proof. For any i, by Fact 5.2 there exists $C_i' \subset B_i'$ satisfying (5). Then from Fact 5.1 we know that every C_i' is unique, and as well that all of them must be pairwise equal. □

Fact 5.4. Let $B_1(n+1), B_2(n+1) \subset B_0(n)$ and $B'_1(m+1), B'_2(m+1) \subset B'_0(m)$, where

$$\mathrm{TAP}(B_i) = \mathrm{TAP}(B'_i), \qquad i = 0, 1, 2.$$

Then $B_1 \cap B_2 \neq \emptyset$ implies $B'_1 \cap B'_2 \neq \emptyset$.

Proof. Let $[\alpha] \in B_1 \cap B_2$. By Corollary 4.3, for every $i = 1, 2$ we have

(*) $$\alpha \sim x_i s_i \beta_i \qquad \text{for some } x_i^{(n-1)} s_i^{(2)} \in \mathrm{AP}(B_i).$$

In particular, $s_i \in \mathrm{TAP}(B_i) = \mathrm{TAP}(B'_i)$, so there exist $x_i'^{(n-1)}$ such that $\alpha'_i = x'_i s_i \beta \in B'_i$. Then, by Corollary 3.5 applied in B'_0, we know that $x'_1 \sim x'_2$. On the other hand, (*) implies $s_1\beta_1 \sim s_2\beta_2$. Thus $\alpha'_1 \sim \alpha'_2$, so $[\alpha'_1] = [\alpha'_2] \in B'_1 \cap B'_2$. □

Lemma 5.5. If $T(B(n)) = T(B'(m))$, then for every ball $C(n+1) \subset B$ there exists $C'(m+1) \subset B'$ such that $T(C) = T(C')$.

Remark 5.6. By previous facts, there is a unique $C'(m+1) \subset B'$ such that $\mathrm{TAP}(C) = \mathrm{TAP}(C')$. Thus, proving the above lemma reduces to checking that $T(C) = T(C')$.

Proof of Lemma 5.5. Let C' be as in Remark 5.6 and let $D \in N(C)$. We will find a ball $D' \in N(C')$ such that $\mathrm{TAP}(D) = \mathrm{TAP}(D')$.

From the Star Lemma 3.7 we have that C and D are both contained in a neighbour B_1 of B. Since $T(B) = T(B')$, there exists a neighbour B'_1 of B' such that $\mathrm{TAP}(B_1) = \mathrm{TAP}(B'_1)$. From Fact 5.3 we know that $C' \subset B' \cap B'_1$.

Since $D \subset B$, by Lemma 5.2 there exists $D'(m+1) \subset B'_1$ such that $\mathrm{TAP}(D') = \mathrm{TAP}(D)$. Then $D' \in N(C')$ by Fact 5.4. This proves that $T(C) \subset T(C')$; the opposite inclusion is analogous. □

Theorem 4. The sequence K from Construction 3.9 defines a Markov compactum.

Proof. Clearly, the dimension of every complex K_i is uniformly bounded by the constant T from Fact 3.6. It also follows directly from the definition that the maps f_i are simplicial (and, even more, satisfy the conditions from Remark 0.3). Hence it remains to check the last condition from Definition 0.1. In checking it, we may freely assume that $i, j \geqslant 2$, since every simplex of K_1 can be assigned its own unique type.

For $i \geqslant 2$, we define the *type* of a simplex $s \in K_i$ as the set of types of its vertices, treated as balls in ∂G. Note that adjacent vertices correspond to intersecting balls, which have different types by Lemma 3.7 and 5.1. In particular, two simplices of the same type must have equal dimension.

If A, B are subcomplexes of any of the K_i's, we will call a map $f : A \to B$ a *typed isomorphism* if it is an isomorphism of simplicial complexes which preserves the types of vertices (and thus also of simplices).

To prove the Markov property, it is sufficient to show that for every two simplices $s \in K_i$ and $s' \in K_j$ of the same type, there exist typed isomorphisms $I : s \to s'$, $J : f_i^{-1}(s) \to f_j^{-1}(s')$ such that the following diagram commutes:

(6) $$\begin{array}{ccc} s & \xleftarrow{f_i} & f_i^{-1}(s) \\ \Big\downarrow{\scriptstyle I} & & \Big\downarrow{\scriptstyle J} \\ s' & \xleftarrow{f_j} & f_j^{-1}(s') \end{array}$$

Let $s \in K_n$, $s' \in K_m$, with $n, m \geqslant 2$, be spanned correspondingly by $B_1, ..., B_l$ and $B'_1, \ldots, B'_l$, where $T(B_i) = T(B'_i)$. Assume that $f_n^{-1}(s)$ contains a vertex representing a ball C of type τ. Since $f_n(C)$ belongs to s, it follows from the construction of K that $f_n(C)$ is the barycenter of a simplex spanned by some of the B_i's; let these be B_{i_j}, $j = 1, \ldots, k$. Then B_{i_j} are exactly these balls of order n which contain C.

From Lemma 5.5 we have that, for every B'_{i_j} with $j \in 1, 2, ..., k$, there exits C'_j of type τ such that $C'_j \subset B'_{i_j}$. By Lemma 5.3, all C'_j are equal to some unique C'. We have $C' \subset \bigcap_{j=1}^{l} B'_{i_j}$; we will show that C' is not contained in any other ball of order m. Suppose, to the contrary, that $C' \subset \overline{B}'$. Then $\overline{B}' \cap B'_{i_j} \neq \emptyset$ for every j, so (by Lemma 3.7 and Fact 5.1) we have $\mathrm{TAP}(\overline{B}') \neq \mathrm{TAP}(B'_{i_j})$. Moreover, the equality $T(B'_{i_1}) = T(B_{i_1})$ implies existence of some $\overline{B}$ such that

$$(*) \qquad \overline{B} \cap B_{i_1} \neq \emptyset \quad \text{and} \quad \mathrm{TAP}(\overline{B}) = \mathrm{TAP}(\overline{B}') \neq \mathrm{TAP}(B'_{i_j}) \text{ for every } j.$$

Since $C' \subset \overline{B}' \cap B'_{i_1}$, by Lemma 5.3 we know that there exists a ball $\overline{C}$ such that

$$\overline{C} \subset \overline{B} \cap B_{i_1} \quad \text{and} \quad T(\overline{C}) = T(C) = \tau.$$

On the other hand, C is the only ball of type τ contained in B_{i_1}, so $\overline{C} = C$. Concluding, $\overline{B}$ contains C, so it must be one of the B_{i_j}'s, which contradicts the condition (*). This proves that $f_m(C') \in s'$.

We can now define $J(C)$ to be C' determined as above for every $C \in f_n^{-1}(s)$. We also define I on the vertices of s by $I(B_i) = B'_i$, and extend both I and J to their whole domains linearily on each simplex. Since f_n and f_m are also linear on simplices, the commutativity of (6) needs to be verified only on the vertices. This follows from the above considerations: if $C \in f_n^{-1}(s)$, then $f_n(C)$ is the barycenter of some vertices B_{i_j}, while $f_m(J(C)) = f_m(C')$ is the barycenter of the vertices $I(B_{i_j})$, which is exactly $I(f_n(C))$. This finishes the proof. □

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