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Introduction to Besicovitch transformations

Praca semestralna nr 1

(semestr letni 2010/11)

Opiekun pracy: Mariusz Lemańczyk

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INTRODUCTION TO BESICOVITCH TRANSFORMATIONS

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1. Introduction

Henri Poincaré and Stanisław Ulam – these famous mathematicians are the sources of the problems that are considered beneath. Poincaré was studying systems of differential equations, especially the dynamical behaviour of the flows generated by them ([Po]). Ulam asked whether there exists a homeomorphism of the Euclidean space which has a dense orbit (noted in *The Scottish Book*, a unique set of problems posed by the mathematicians of Lwów School of Mathematics – see [Sc, Problem 115]). This question was answered positively for $\mathbb{R}^2$ by Besicovitch in [Be1]; moreover, in [Be2] he constructed a homeomorphism of the plane of particular kind exhibiting amazing behaviour of intertwining chaos and regularity: it has both a dense orbit and a discrete one, with one limit point. This kind of examples was recently studied by Frączek and Lemańczyk in [FrLe], and used to show that some of the systems considered by Poincaré display similar irregularity.

The example of Besicovitch can be considered today as a *skew product* of a rotation of the circle and the line, i.e. in their Cartesian product the first variable is transformed by the rotation, and the second one is transformed by homeomorphisms varying with the first variable. Specifically, it is a *cylindrical transformation*, i.e. the homeomorphisms of lines are adding a number which depends on the first variable. Formally:
A homeomorphism T of a topological space X is given, as well as a topological group $(G, +)$ and a continuous mapping $f\colon X \longrightarrow G$. Then the cylindrical transformation T_f is defined:

$$T_f\colon X \times G \ni (x, t) \mapsto (Tx, t + f(x)) \in X \times G.$$

Frączek and Lemańczyk consider cylindrical transformations with T as a minimal rotation of the torus $X = \mathbb{T}^d$ and $G = \mathbb{R}$,[1] above all

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[1]For $d = 1$, this yields a cylinder, which is homeomorphic to punctured plane, and the example of Besicovitch on $\mathbb{R}^2$ induces a homeomorphism of $\mathbb{T} \times \mathbb{R}$.

Besicovitch transformations/cylinders, i.e. with both dense orbit and discrete orbit. They prove that these can be generated by every minimal rotation of a torus. They also find Besicovitch cylinders with special properties, which includes Hölder continuity, small Fourier coefficients or high regularity of the function f, and having a relatively large number of discrete orbits (see Theorems 4.1, 4.2). Finally, they apply these results to some differential equations, including these studied by Poincaré (Section 5).

This term paper is based on the article [FrLe] and collective work within a seminar on Nicolaus Copernicus University in academic year 2010/2011, led by Prof. Lemańczyk, and attended mainly by Joanna Kułaga and the author. It is meant to be an introduction to the topic of Besicovitch transformations for readers interested in topological dynamics. It discusses the features of 'having a dense orbit' (transitivity and, stronger, minimality), the properties of (mainly real) cylindrical transformations, and summarises the results from [FrLe].

In particular, Section 2 includes relations among various definitions of transitivity and positive transitivity (which are present for topological spaces with Baire property, with a countable base or without isolated points; if all the three conditions are met, all these definitions are equivalent (Oxtoby)), equivalence of minimality and positive minimality for compact spaces, and a proof that minimal sets exist in compact setting without referring to the axiom of choice (Glasner).

Section 3 presents the dynamics of cylindrical transformations of minimal homeomorphisms on compact metric spaces. It starts with the "trivial" cases of real cylindrical transformations, in which the dynamics can be readily described – when $\int_X f \neq 0$ (all orbits are discrete) and when f is a *coboundary*, i.e. $f(x) = g(x) - g(Tx)$ for some continuous $g\colon X \longrightarrow \mathbb{R}$ (then the whole space decomposes into invariant compact sets – copies of the graph of g) and a characterisation of coboundaries by Gotschalk-Hedlund (e.g. f is a coboundary if and only if there are bounded orbits). Next, a minimality issues are discussed. Cylindrical transformations are never minimal (Besicovitch; this applies also to groups including $\mathbb{R}$ as a direct summand), but, unless being trivial, they are always transitive (Lemańczyk and Mentzen). If f is of bounded variation, there are no minimal sets at all (Matsumoto and Shishikuro, Mentzen and Siemaszko). Real cylindrical transformations are never positively minimal either, which follows from a general theorem of Gottschalk: there are no positively minimal homeomorphisms on locally compact, noncompact space without isolated points.

Sections 4 and 5 summarise the results of [FrLe], as stated above.

The included proofs were usually based on the original ones; sources are always stated. The remaining ones were based on the mentioned classes.

2. Basic notions

In the first section we deal with some basic notions of topological dynamics. To begin with, we will recall some fundamental definitions and notation.

We will assume henceforth that a metric space (X, d) is given and a homeomorphism T is defined thereon (thus T generates a *discrete dynamical system*, for which X is called the *phase space*). We will denote the *orbit* (or *trajectory*) of a point $x \in X$ by $o_T(x)$ or simply $o(x)$; sometimes this will also denote the orbit of a set. For the sake of brevity, we will usually refer to the *positive semi-orbit* as the *semi-orbit.*We will also consider T-*invariant* / *positively invariant* subsets, i.e. $Y \subset X$ such that $T(Y) = Y$ / $T(Y) \subset Y$.

Transitivity. A dynamical system is called (*topologically*) *transitive* if it has a dense orbit, and *positively transitive* if it has a dense semi-orbit.[2] A point that has a dense (semi-)orbit is also called (*positively*) *transitive.*

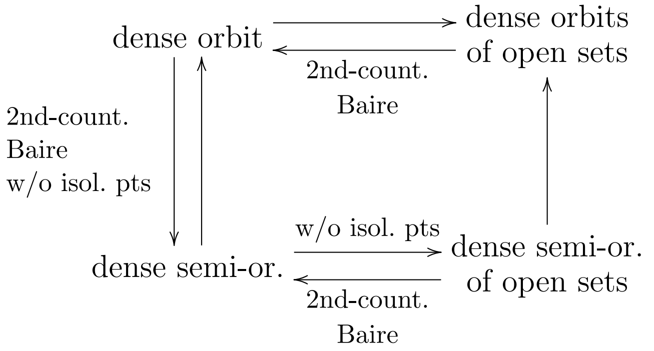
(Positive) transitivity is sometimes defined in terms of open subsets: every open set passes through every other one along its trajectory, i.e. for every open U, V there is $n \in \mathbb{Z}$ ($n \in \mathbb{N}$) such that $T^{-n}U \cap V \neq \emptyset$; in other words, every open set has a dense orbit. These definitions are not equivalent in general, but turn out to be under relatively mild conditions.

Proposition 2.1 (Definitions of transitivity)**.** *If there exists a dense orbit, then every nonempty open set has a dense orbit. If X is a second-countable Baire space, then the converse implication is also true.*[3]
The same holds with 'orbit' replaced by 'semi-orbit', except that, for the former implication to be true, X cannot contain isolated points.
Finally, if X is a second-countable Baire space without isolated points, then all the mentioned notions of transitivity and positive transitivity coincide.

This proposition is summarised by the diagram below.

[2]Since some point "transits" the whole space.

[3]Recall that for metrisable spaces second countability is equivalent to separability. A *Baire space* is a topological space satisfying the conclusion of the Baire category theorem, i.e. such that the intersection of a countable family of dense open sets is always dense. In particular, both locally compact and completely metrisable spaces are Baire.



Proof. First, observe that a dense orbit passes through every nonempty open set, thus the orbit of such set contains this dense orbit, and consequently is itself dense. In the positively transitive case, again, the open set intersects the dense semi-orbit, yet now its semi-orbit contains the dense one possibly apart from several initial points. However, if these points are not isolated, then the rest of the (point) semi-orbit is still dense, and so is the semi-orbit of the set.

To establish the converse implication for semi-orbits, fix a countable base of X, say, $\{U_i\}_{i\geq 0}$, and consider the set $\bigcap_{i\geq 0}\bigcup_{n\in\mathbb{N}} T^{-n}U_i$. The sets $\bigcup_{n\in\mathbb{N}} T^{-n}U_i$ are open and dense (note that this is the only moment that we need open sets to have dense semi-orbits), and therefore their intersection is also dense. It remains now to notice that this intersection consists of all dense semi-orbits. Indeed, $x \in \bigcap_{i\geq 0}\bigcup_{n\in\mathbb{N}} T^{-n}U_i$ means that every U_i contains some $T^n x$, i.e. $\{T^n x\}_{n\in\mathbb{N}}$ is dense. The proof for orbits is analogous – one considers the set $\bigcap_{i\geq 0}\bigcup_{n\in\mathbb{Z}} T^{-n}U_i$.

As for the last statement,[4] it will follow from the previous ones if we show that transitivity implies positive transitivity in the given setting. Dense semi-orbits can be found in the same way as above – they form the set $\bigcap_{i\geq 0}\bigcup_{n\in\mathbb{N}} T^{-n}U_i$. The only difference is that now the semi-orbits $\bigcup_{n\in\mathbb{N}} T^{-n}U_i$ are not readily dense; to complete the proof, it suffices to demonstrate that, with $i, j \geq 0$ arbitrarily fixed, $U_j \cap T^{-n}U_i \neq \emptyset$ for some $n \in \mathbb{N}$. Due to transitivity, this certainly holds for some $n_0 \in \mathbb{Z}$. However, if $n_0 < 0$, then the set $U_j \cap T^{-n_0}U_i$, being open and infinite (because there are no isolated points), contains an infinite part of some dense orbit, so there is an x such that x, $T^m x \in U_j \cap T^{-n_0}U_i$ for some $m \geq -n_0$, and this eventually yields $U_j \cap T^{-(m+n_0)}U_i \neq \emptyset$, because $x = T^{-m}(T^m x) \in T^{-(m+n_0)}U_i$. □

Remark 2.2. Note that the set of points with dense (semi-)orbit (described in the proof), if nonempty, is a countable intersection of dense open sets, that is, a G_δ-dense set. Also, if a measurable setting is

[4] This proof is based on [Ox]; see p. 70 thereof.

given, namely, X is equipped with Borel measure of full support and T is *ergodic*, this set is of full measure.

Minimality. This notion corresponds to systems that are irreducible in some way. Namely, a *minimal set* is a closed invariant set that contains no other nonempty closed invariant set,[5] and a system is *minimal* if the phase space itself is a minimal set or all orbits are dense. *Positively minimal* set and dynamical system are defined by means of positive invariance and semi-orbit.

As to the definition of minimal system, *the conditions "X is (positively) minimal" and "all (semi-)orbits are dense" are equivalent*, because every orbit lying in a nontrivial closed invariant set is not dense, and the closure of a nondense orbit is nontrivial, closed and invariant. In particular, minimal systems are always transitive.

It is also worth noting that *in the compact setting minimality and positive minimality are equivalent.* Obviously, the former always follows from the latter. On the other hand, in a minimal system any semi-orbit contains in its closure the so-called positive limit set, which is closed, nonempty and bilaterally invariant, thus it is the whole space.

Also *in compact spaces minimal sets always exist (more general: in compact invariant sets minimal subsets always exist)*, and the classical proof uses the axiom of choice in disguise of the Zorn lemma. Indeed, if we consider the family of all nonempty closed (hence compact) invariant subsets, partially ordered by reverse inclusion, then every chain has an upper bound (e.g. its intersection – nonempty because of compactness), so there exists a maximal element, which is a minimal set. However, the axiom of choice is not necessary, as suggested by Glasner (Exercise 1.1.4 in [Gl] with appended hints):

Proof. Pick a countable base $\{U_i\}_{i\in\mathbb{N}}$ (it always exists for compact metric spaces). Either X itself is minimal – then we are done – or not. If not, then the trajectory of some nonempty open set does not cover X (e.g. the complement of a nontrivial closed invariant set – existing by nonminimality – which is invariant); in particular, trajectory of some U_i does not cover X. Let then n_1 be the smallest of such i and $X_1 := X \setminus o(U_{n_1})$. Again, if X_1 is not minimal, then the trajectory of some U_i with $i > n_1$ does not cover X_1 (remember that each U_i for $i < n_1$ contains the whole X in its trajectory), and we define n_2 to be the smallest of such i and $X_2 := X_1 \setminus o(U_{n_2})$.

Proceeding inductively, we either obtain a minimal subset or a decreasing sequence of compacts. Note that in every step $U_{n_{k'}} \cap X_k = \emptyset$ for $k' \le k$, while $o(U_i) \supset X_k$ for $i \notin \{n_{k'} \colon k' \le k\}$, $i < n_k$. We claim

[5] Accordingly, all closed invariant subsets of a minimal set are the empty set and the set itself – which are called *trivial* when the set is fixed.

that if all X_k are not minimal, then their intersection $M := \bigcap_{k\geq 1} X_k$ is minimal. To this end, pick a closed invariant $C \subsetneq M$. There exists some U_{i_0} such that $U_{i_0} \subset X \setminus C$ and $U_{i_0} \cap M \setminus C \neq \emptyset$, and this cannot be any of U_{n_k}, since they are disjoint with M. Now each other U_i contains M in its trajectory, so $M \subset o(U_{i_0}) \subset X \setminus C$, whence eventually $C = \emptyset$. □

Both proofs are valid for positive minimality, if one replaces 'orbit' ('trajectory'), 'minimal' and 'invariant' with their counterparts.

3. Cylindrical transformations

In the following section we introduce the cylindrical transformations (a special case of the Anzai skew products) and study their dynamical properties, especially minimal sets. In particular, they are transitive in all "nontrivial" cases, but they are never minimal.

In general, we may be concerned with a minimal homeomorphism T of a compact metric space X with a T-invariant measure μ defined on the Borel σ-algebra, and a real continuous function $f\colon X \longrightarrow \mathbb{R}$ (which we will customarily call a *cocycle*). Usually, without further notice, we will confine ourselves to minimal rotations on compact topological groups with the *Haar measure*.[6] The functions T and f generate a *cylindrical transformation* (or a *cylinder*):

$$T_f\colon X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$$
$$T_f(x,t) := (Tx, t + f(x))$$

The iterations of T_f are of the form $T_f^n(x,t) = (T^n x, t + f^{(n)}(x))$, where $f^{(n)}$ is given by the formula:

$$f^{(n)}(x) := \begin{cases} f(x) + f(Tx) + \cdots + f(T^{n-1}x) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -f(T^{-1}x) - f(T^{-2}x) - \cdots - f(T^{n}x) & \text{for } n < 0. \end{cases}$$

One can also consider cylindrical transformations for any topological group instead of $(\mathbb{R}, +)$. At the end of this section we will present a simple corollary about cylinders on more general groups.

Observe that the dynamics of a point (x,t) does not depend on t, because the mappings

$$\tau_{t_0}\colon X \times \mathbb{R} \ni (x,t) \mapsto (x, t + t_0) \in X \times \mathbb{R}$$

[6]Minimal rotations do not always exist – groups possessing them are called *monothetic*. All tori $\mathbb{T}^n$ are monothetic, and a rotation on $\mathbb{T}^n$ is minimal if its coordinates are irrational and $\mathbb{Q}$-linearly independent; moreover, these rotations are ergodic with regard to the Lebesgue measure.

(for arbitrary $t_0 \in \mathbb{R}$) are topological conjugations:

$$T_f(\tau_{t_0}(x,t)) = T_f(x, t+t_0) = (Tx, t+t_0+f(x)) \\ = \tau_{t_0}(Tx, t+f(x)) = \tau_{t_0}(T_f(x,t)).$$

As pointed in the introduction in [FrLe], there are two cases, when the minimal subsets can be easily described:

(1) when $\int_X f \, \mathrm{d}\mu \neq 0$, all points have discrete orbits, because for all $x \in X$: $|f^{(n)}(x)| \overset{n\to\infty}{\longrightarrow} \infty$. For the details, see Theorems 6.19 and 6.20 in [Wa].
(2) when the cocycle is a *coboundary*, i.e. of the form $f = g - g \circ T$ for a (continuous) *transfer function* $g\colon X \longrightarrow \mathbb{R}$, the minimal sets are vertically translated copies of the graph of g in $X \times \mathbb{R}$.

Note that if f is a coboundary and T is measure-preserving, then $\int_X f \, \mathrm{d}\mu = 0$.
To start a more detailed analysis, we will give an explicit characterisation of coboundaries:

Gottschalk-Hedlund Theorem (3.1).
For a minimal homeomorphism T on a compact phase space X and a cocycle f the following conditions are equivalent:

(i) *f is a coboundary,*
(ii) *$\{f^{(n)}(x)\colon n \in \mathbb{Z}\}$ is bounded for some $x \in X$,*
(iii) *the orbit $o_{T_f}(x, 0)$ is bounded for some $x \in X$,*
(iv) *$\{f^{(n)}(x)\colon n \in \mathbb{Z}\}$ is bounded for* each *$x \in X$,*
(v) *the orbit $o_{T_f}(x, 0)$ is bounded for* each *$x \in X$.*

Proof ([GoHe]*).* Both pairs (ii), (iii) and (iv), (v) are equivalent by the very definition of $f^{(n)}$. If (ii) holds for some $x \in X$, then all $f^{(n)}(T^k x)$ are uniformly bounded. But since $\{T^k x\}_{k\in\mathbb{Z}}$ is dense in X, all $f^{(n)}$ are uniformly bounded on the whole X, i.e. (iv) holds; thus (ii) and (iv) are also equivalent. Moreover, if f is a coboundary, say, $f = g - g \circ T$, then one easily calculates that $f^{(n)} = g - g \circ T^n$, therefore (i) implies (ii) and (iv).

It remains to establish (i) when assuming (ii). We know that if f was a coboundary generated by g, then $g(T^n x) = g(x) - f^{(n)}(x)$ for all x. So we will define g on a dense set $\{T^k x_0\}_{k\in\mathbb{Z}}$ by this formula (with x_0 and $g(x_0)$ arbitrarily chosen), and show that it extends to a continuous function on X, which then satisfies $f = g - g \circ T$ on the whole X.

Pick then any bounded orbit. It is relatively compact, so there is a (compact) minimal set in its closure – call it M. We claim that M is a graph of a continuous function. First, its projection onto X is dense, because M (as an invariant set) contains a T_f-orbit, which projects to a dense T-orbit. On the other hand, its projection is compact, so it must be X. Now, if there were more then one point in M over any x, say, (x,s), $(x,t) \in M$, then $\tau_{s-t}(M)$ would also be minimal, but clearly $M \cap \tau_{s-t}(M) \supset \{(x,s)\} \neq \emptyset$, which cannot hold for two distinct minimal sets. Thus M is a compact graph of some function, which is hence continuous.[7] If we fix now any $x_0 \in X$, choose anyhow $g(x_0)$ and define g on $\{T^k x_0\}_{k\in\mathbb{Z}}$ as above, then its graph is contained in M (M is T_f-invariant, and this graph is a T_f-orbit), so it extends uniquely to X with M as the graph of the extension. □

In what follows we will call the above-mentioned cases (when f is of zero average or a coboundary) *trivial* and restrict the study to nontrivial cylinders. These have been proved to be always transitive in some important cases. However, they cannot be minimal. The latter fact was proved by Besicovitch in [Be2] for $X = \mathbb{T}$ from a slightly different point of view, but the proof can be adapted for general case.

Theorem 3.2. (Besicovitch) *A cylinder originating from a minimal homeomorphism T of a compact phase space X is never minimal.*
(Lemańczyk, Mentzen) *If, additionally, X is a topological group, T is a rotation and a cocycle gives a nontrivial cylinder, then the cylinder is transitive.*

Proof. To prove the first part, fix a real cocycle f. We may assume that T_f is transitive, and by Proposition 2.1 it is also positively transitive. Pick then a point $(x_0, 0)$ with dense semi-orbit. For every $j \in \mathbb{N}$ there exist integers $m_j < n_j < m'_j$ such that

$$f^{(m_j)}(x_0),\ f^{(m'_j)}(x_0) < -j, \quad f^{(n_j)}(x_0) > j,$$

since the set $\{T^n(x_0)\}_{n>m_j}$ is also dense. We may choose n_j so that $f^{(n_j)}(x_0) = \max_{n=m_j,\dots,m'_j} f^{(n)}(x_0)$. It turns out that

$$\lim_{j\to\infty} (m'_j - n_j) = \lim_{j\to\infty} (n_j - m_j) = +\infty,$$

because f is bounded, so $|f^{(n+1)}(x) - f^{(n)}(x)| = |f(Tx)| \leq \sup_X |f|$ for all n and x.

After passing to a subsequence, if necessary, we may assume that $T^{n_j} x_0 \longrightarrow z_0$ for some $z_0 \in \mathbb{T}$. Consider the orbit of $(z_0, 0)$:

[7] If it were not, then some sequence $(x_n, g(x_n))_{n\in\mathbb{N}}$ with $x_n \longrightarrow x_\infty$ would be separated from $(x_\infty, g(x_\infty))$ (where g denotes the function in question). But it would have to contain a subsequence, convergent in M, and thus convergent to $(x_\infty, g(x_\infty))$.

for every $n \in \mathbb{Z}$

$$\begin{aligned} T_f^n(z_0, 0) &= \lim_{j\to\infty} T_f^n(T^{n_j}x_0, 0) = \lim_{j\to\infty}(T^{n+n_j}x_0, f^{(n)}(T^{n_j}x_0)) \\ &= \lim_{j\to\infty}(T^{n+n_j}x_0, f^{(n+n_j)}(x_0) - f^{(n)}(x_0)). \end{aligned}$$

Now for j large enough $m_j < n + n_j < m_j'$, so $f^{(n+n_j)}(x_0) < f^{(n)}(x_0)$. This proves that $T_f^n(z_0, 0) \in X \times (-\infty, 0]$ for every n, that is, $(z_0, 0)$ is not a transitive point.

For the proof of the second part, see Theorem 1, p. 236 in [LeMe]. □

For $X = \mathbb{T}$ this fact also follows from a much more general theorem of Yoccoz and Le Calvez: there are no minimal homeomorphisms on a plane with a finite number of points removed; see [LeYo] for the original proof and [Fra] for a much shorter one, using the Conley index theory. Unfortunately, nothing more is known for higher-dimensional manifolds.

The theorem of Besicovitch can be applied to more general cylinders:

Corollary 3.3. *Consider a group product of* $(\mathbb{R}, +)$ *with some abelian topological group* G*, a minimal homeomorphism* T *on a compact phase space* X *and a (continuous) cocycle* $f\colon X \longrightarrow \mathbb{R}\times G$*. Then the cylinder* T_f *on* $X \times \mathbb{R} \times G$ *is not minimal. In particular, cylinders for cocycles* $f\colon X \longrightarrow \mathbb{R}^d$ *are never minimal.*

Proof. Let $f = (f_\mathbb{R}, f_G)$, where $f_\mathbb{R}$ $(f_G)\colon X \longrightarrow \mathbb{R}$ (G) and $e \in G$ is the identity element; then $f^{(n)} = (f_\mathbb{R}^{(n)}, f_G^{(n)})$ for all $n \in \mathbb{Z}$. $T_{f_\mathbb{R}}$ is a nonminimal transformation of $X\times\mathbb{R}$. Pick thus a point $(x_0, 0) \in X\times\mathbb{R}$ having nondense $T_{f_\mathbb{R}}$-orbit. Then $T_f(x_0, 0, e) = (T_{f_\mathbb{R}}(x_0, 0), f_G(x))$, so

$$o_{T_f}(x_0, 0, e) \subset o_{T_{f_\mathbb{R}}}(x_0, 0) \times G \subset \overline{o_{T_{f_\mathbb{R}}}(x_0, 0)} \times G,$$

which is a closed invariant proper subset of $X \times \mathbb{R} \times G$. □

If the cocycle is regular enough, there are no minimal sets at all, as has been proved by Matsumoto and Shishikuro, and, independently, Mentzen and Siemaszko:

Theorem 3.4 ([MaSh, Theorem 1], [MeSi, Theorem 2.4])**.** *If the cocycle on* $\mathbb{T}$ *is of bounded variation, then the cylinder which it generates has no nontrivial minimal sets. In particular, it has no discrete orbits.*

As to positive minimality, its lack can be derived from more general results on non-existence of minimal homeomorphisms. We present such a result for locally compact, noncompact space without isolated points.

Theorem 3.5 (Gottschalk, compiled by Kubiś)**.** *There are no positively minimal mappings on locally compact, noncompact space without isolated points.*

Proof. (For the original proof see [Go]; the compiled version is included in [Ku]. This was also proved differently in [Ber].) Fix a space X as in the assumption and its transformation T. We will prove that the set of positively transitive points

$$D := \{x \in X \colon \{T^n x\}_{n\in\mathbb{N}} \text{ is dense}\}$$

has empty interior. Suppose to the contrary that $\operatorname{int} D \neq \emptyset$; it thus contains some nonempty open relatively compact set U with its closure: $\overline{U} \subset \operatorname{int} D$. Put, for $n \geq 0$:

$$U_n := U \cup T(U) \cup \cdots \cup T^{n-1}(U).$$

The sequence (U_n) is strictly increasing: otherwise, if $U_n = U_{n+1}$ for some n, then $\overline{U_n}$ would be a positively invariant closed proper subset of X (invariant, since $T(U_n) \subset U_{n+1}$; proper, since $\overline{U_n}$ is compact), which is impossible, because it contains a positively transitive point. Consequently, there exists $x_n \in U$ such that $T^n(x_n) \notin U_{n-1}$ for every $n \geq 1$. Pick then a sequence of such x_n[8] and its cluster point: $y = \lim_{j\to\infty} x_{n_j}$ for some $n_j \to \infty$. The limit is contained in $\overline{U}$, so it has a dense semi-orbit. Additionally, it is not isolated, in particular $U \neq \{y\}$, so there is $k > 0$ such that $T^k(y) \in U$. Hence also $T^k(x_{n_0}) \in U$ for some $n_0 > k$, which implies that

$$T^{n_0}(x_{n_0}) = T^{n_0-k}(T^k(x_{n_0})) \in T^{n_0-k}(U) \subset U_{n_0}.$$

This contradicts the choice of x_{n_0}. □

4. Besicovitch transformations

Following Frączek and Lemańczyk we consider cylindrical transformations that display both transitive and discrete behaviour, called *Besicovitch transformations* or *Besicovitch cylinders*, because their first example was given in [Be2]. We will also call a cocycle which generates a Besicovitch cylinder a *Besicovitch cocycle*.

In [FrLe] the authors construct a Besicovitch cocycle for any irrational rotation of a torus. They also find ones with some special properties, in particular with a large number of discrete orbits.[9]

We will summarise their results on existence of specific Besicovitch transformations (on minimal rotations of tori), some of which will be used in the next section for differential equations. For brevity, we will

[8]The axiom of choice, or its weaker version, the axiom of countable choice, have to be used here.

[9]Recall that the set of discrete orbits of minimal rotations of a torus is of first category and of measure zero for – see Remark 2.2.

denote:

$$\begin{aligned}\mathcal{D} &:= \{x \in \mathbb{T}^d \colon o_{T_f}(x,t) \text{ is discrete for every } t \in \mathbb{R}\} \\ &= \{x \in \mathbb{T}^d \colon o_{T_f}(x,0) \text{ is discrete}\}\end{aligned}$$

where T_f is the cylinder in question.

Theorem 4.1 (Frączek, Lemańczyk)**.** *For every irrational rotation of* $\mathbb{T}$ *there exist real cocycles which generate Besicovitch transformations* ([FrLe, Section 2])*. The cocycles can be chosen in such a way that* $\mathcal{D}$ *is uncountable* ([FrLe, Proposition 6])*.*

Moreover, for almost every irrational rotation one can find Besicovitch cocycles bearing some additional properties:

- γ*-Hölder continuous for* $0 < \gamma < 1/2$*; note that for* $\gamma = 1/2$ *such cocycles have also been proved to exist, but only for a measure-zero set of rotations* ([FrLe, Theorem 3])*;*
- *such that, for the cylinder thus generated, the Hausdorff dimension of* $\mathcal{D}$ *is at least* $1/2$ ([FrLe, Theorem 9])*.*

Finally, for some irrational rotations there exist Besicovitch cocycles with Fourier coefficients $\mathrm{O}(\log|n|/|n|)$ ([FrLe, Theorem 5])*.*

In fact, the proof of the last statement can be used to obtain cocycles with even smaller Fourier coefficients, namely $\mathrm{O}(\varphi(|n|)/|n|)$ if only $\lim_{n\to\infty}\varphi(n) = +\infty$. We remark that a Besicovitch cocycle on $\mathbb{T}$ cannot have Fourier coefficients of order $o(1/|n|)$ because of Theorem 3.4 (a function of bounded variation has Fourier coefficients of order $\mathrm{O}(1/|n|)$). Whether there are Besicovitch cocycles with Fourier coefficients of order $\mathrm{O}(1/|n|)$, remains yet to be found.

Theorem 4.2 (Frączek, Lemańczyk)**.** *Continuous Besicovitch cocycles exist for every minimal rotation of* $\mathbb{T}^d$ *for every* $d \geq 1$*,* $d \in \mathbb{Z}$ ([FrLe, remark on p. 2])*.*
Let additionally $d \geq 3$ *and* $a > 1$ *such that* $\bar{a} := 2a^{d-1} - a^d - 1 > 0$ *and* $\bar{a} \notin \mathbb{Z}$*. Then for uncountably many minimal rotations of the torus* $\mathbb{T}^d$ *there exists a real Besicovitch cocycle of class* $C^{\lceil \bar{a} \rceil}$ *and such that the Hausdorff dimension of* $\mathcal{D}$ *is at least* $\frac{d}{1+a^d}$ ([FrLe, Theorem 11, Proposition 12])*.*

For a fixed d, the variable $\bar{a}$ reaches at least $(2^d/ed) - 1$ (see [FrLe, Remark 11]), so the cocycles for general tori can be of any finite degree of smoothness. Note also that the mentioned Hausdorff dimension can be arbitrarily close to $d/2$; however, the maximum of $\bar{a}$ is attained beyond 1, at $a = 2(d-1)/d$, so the closer $\frac{d}{1+a^d}$ is to $d/2$, the lesser (down to 1) the degree of smoothness is.

We highlight the coefficient $1/2$, which appears as a limit for Hölder coefficients and twice for Hausdorff dimension. It is not known whether it is a coincidence, nor whether it is an actual upper bound in any of these cases.

5. Application to differential equations

In [Po] Poincaré was studying specific differential equations and dynamical behaviour of the flows thus generated. In the following section we will summarise the properties of his example as well as two well-behaved examples which, after continuous perturbation, become quite chaotic; all were obtained in [FrLe], using the results from the two previous sections.

Basic case ([FrLe, Section 8]). The following system on $\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}$ was considered in [Po, Chapitre XIX, p. 202]:

$$(\star) \qquad \begin{cases} \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \bar{a} \\ \frac{\mathrm{d}z}{\mathrm{d}t} = 1 \\ \frac{\mathrm{d}y}{\mathrm{d}t} = f(\bar{x}, z), \end{cases}$$

where $\bar{a} \in \mathbb{T}^d$ induces a minimal rotation, denoted T, on $\mathbb{T}^d$, and $f \colon \mathbb{T}^d \times \mathbb{T} \longrightarrow \mathbb{R}$ is of class C^r for some $r \geq 0$. Solving the equations, we obtain the flow

$$\Phi_t(\bar{x}, z, y) := \left(\bar{x} + t\bar{a}, z + t, \int_0^t f(\bar{x} + s\bar{a}, z + s)\,\mathrm{d}s \right).$$

For $z = 0$ we obtain a global section and the Poincaré map given by the formula

$$\mathbb{T}^d \times \{0\} \times \mathbb{R} \cong \mathbb{T}^d \times \mathbb{R} \ni (\bar{x}, y) \mapsto (T\bar{x}, y + \varphi(\bar{x})),$$

where $\varphi(\bar{x}) := \int_0^1 f(\bar{x} + s\bar{a}, s)\,\mathrm{d}s$, which is also of class C^r. This can be reversed: given smooth $\varphi \colon \mathbb{T}^d \longrightarrow \mathbb{R}$, one can find equally smooth $f \colon \mathbb{T}^d \times \mathbb{T} \longrightarrow \mathbb{R}$ such that φ is the Poincaré map of the corresponding flow.

It follows that the flow is topologically conjugate to the suspension flow over T_φ, which readily leads to the following properties:

1. The flow $(\Phi_t)_{t\in\mathbb{R}}$ is never minimal.

2. As $\int_{\mathbb{T}^d} \varphi(\bar{x})\,\mathrm{d}\bar{x} = \int_{\mathbb{T}^d \times \mathbb{T}} f(\bar{x}, z)\,\mathrm{d}\bar{x}\mathrm{d}z$, if the latter integral does not vanish, the flow decomposes into closed orbits, homeomorphic to $\mathbb{R}$.

3. On the other hand, if $\varphi(\bar{x}) = j(\bar{x}) - j(T\bar{x})$ for some continuous $j \colon \mathbb{T}^d \longrightarrow \mathbb{R}$, then the flow also decomposes, this time into relatively compact components. Also, if some orbit is relatively compact, then so are all orbits, and φ is a coboundary.

4. If neither does the above integral vanish, nor are orbits relatively compact, then the flow is transitive.

5. In the latter case, minimal sets can exist. However, if we require φ to be C^1, there are no minimal sets for $d = 1$ (in particular, there are no closed orbits) and they can exist for $d \geq 3$.

Differential equations on $\mathbb{R}^3$ and $\mathbb{S}^3$ ([FrLe, Sections 8.1 & 8.2]). Fix an irrational α.

1) Let additionally $\alpha > 0$. Consider the following system on $\mathbb{R}^3$:

$$\begin{cases} x' = -2\pi y + 2\pi\alpha xz \\ y' = 2\pi x + 2\pi\alpha yz \\ z' = \pi\alpha(1 - x^2 - y^2 - z^2). \end{cases}$$

The phase space can be shown to decompose into a family of invariant tori and two "singular" invariant sets – a circle and a straight line:

$$A_+ := \mathbb{T} \times \{0\} \quad \text{and} \quad A_- := \{0\} \times \{0\} \times \mathbb{R}.$$

2) Consider the following system on $\mathbb{S}^3 = \{(z_1, z_2)\colon |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$:

$$\begin{cases} z_1' = 2\pi i\alpha z_1 \\ z_2' = 2\pi i z_2 \end{cases}$$

The phase space is also foliated into invariant tori, given by the formula $\{(z_1, z_2)\colon |z_1|^2 = c, |z_2|^2 = 1 - c\}$ for all $c \in (0, 1)$, and two circles, corresponding to $c = 0$, 1:

$$A_+ := \{(z_1, z_2)\colon |z_1| = 1,\ z_2 = 0\}, \quad A_- := \{(z_1, z_2)\colon z_1 = 0,\ |z_2| = 1\}.$$

Both systems, when singular sets A_+ and A_- are removed from the phase space, can be perturbed continuously so that the perturbation is isomorphic to the system $(\star)$ for $d = 1$ (via a real-analytic change of coordinates), and the sets A_+ and A_- correspond to the "infinities" in $\mathbb{T}^2 \times \mathbb{R}$, namely $\mathbb{T}^2 \times \{+\infty\}$ and $\mathbb{T}^2 \times \{-\infty\}$; therefore, the following holds:

Theorem 5.1 ([FrLe, Theorems 13 & 14])**.** *For both systems of equations and for every considered α there exists a continuous perturbation of the system such that the flows corresponding to the perturbed systems are transitive and for an arbitrary choice of s_1, $s_2 \in \{+, -\}$ the set of points with the ω-limit set equal to A_{s_1} and the α-limit set equal to A_{s_2} is dense. Moreover, for almost every α the Hausdorff dimension of all these sets is at least $5/2$.*
Also, for the systems on $\mathbb{S}^3$, for every $0 < \gamma < 1/2$ and almost every α the perturbation can be chosen γ-Hölder continuous. The same holds for $\gamma = 1/2$, but only for a measure-zero set of α.

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