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Cylindrical transformations with full-dimensional set of discrete orbits

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CYLINDRICAL TRANSFORMATIONS WITH FULL-DIMENSIONAL SET OF DISCRETE ORBITS

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Abstract. A cylindrical transformation (cylinder) is a mapping of the form $T_f\colon X\times\mathbb{R}\ni(x,t)\mapsto(Tx,t+f(x))$ where $T\colon X\to X$ is a homeomorphism of a topological space, and $f\colon X\to\mathbb{R}$ is a continuous function. It may display varied dynamical behaviour. In particular, Frączek and Lemańczyk found for every minimal rotation T of a torus $\mathbb{T}^d$ a function f for which T_f has both dense and discrete orbits (a Besicovitch cylinder). Also, for $d=1$ and almost every irrational rotation they constructed f such that the Hausdorff dimension of the set of points with discrete orbit under T_f is at least $3/2$. By enhancing their construction we show that for these rotations this set can actually have full dimension (i.e. 2). We recall also an application to differential equations.

Introduction

Chaotic behaviour in dynamical systems has been of particular interest of dynamicists since about the second half of the 20th century.[1] Few examples had been studied earlier, thus it must have been surprising to Abram Besicovitch to discover a homeomorphism of the cylinder $\mathbb{T}\times\mathbb{R}$ with both dense and discrete orbits ([Be1], [Be2]). His example was what would later be called a cylindrical transformation, or, more generally, a skew product. *Cylindrical transformations* or *cylinders* are mappings of the form

$$T_f\colon X\times\mathbb{R}\ni(x,t)\mapsto(Tx,t+f(x))$$

where $T\colon X\to X$ is, in the most general setting, a homeomorphism of a topological space, and $f\colon X\to\mathbb{R}$ is a continuous function. They arise naturally in ergodic theory, as their iterates are the products of

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[1] Let us remind that one of the most popular definitions of chaos, the Devaney chaos, comprises dense orbits (transitivity), dense set of periodic orbits and sensitivity to initial conditions (the last condition may be often omitted).

respective iterates of T and ergodic sums of f over T. For this reason they were introduced independently, and even earned own chapter in a textbook on topological dynamics ([GoHe], Chapter 14). Apart from ergodic applications, cylinders are sections of the flows derived from some differential equations (see Section 4). They also give rise to transformations of other spaces – e.g. Besicovitch viewed his construction as a homeomorphism of punctured plane and expanded it to a homeomorphism of plane.

Most recently, cylinders have been studied by, among others, a group of researchers from Nicolaus Copernicus University in Toruń; some of their findings are summarised in Subsection 1.2. In particular, Krzysztof Frączek and Mariusz Lemańczyk sought for cylinders with both dense orbits and discrete orbits (hereafter called Besicovitch cylinders), especially such that either are somehow regular or have a large amount of discrete orbits.[2] One of their results in [FrLe] were Besicovitch cylinders for every minimal rotation of a torus $\mathbb{T}^d$, along with such that the points with discrete orbits have altogether Hausdorff dimension at least $3/2$ – these for almost every minimal rotation of the one-dimensional torus. However, it was unknown whether higher Hausdorff dimensions can be achieved and it was left as an open question. The (positive) answer to this question is the main topic of this paper: by enhancing the techniques from [FrLe] we have constructed similar Besicovitch cylinders with full Hausdorff dimension of discrete orbits for almost every minimal rotation of $\mathbb{T}$. Thus our article can be regarded as a continuation of the paper [FrLe], as well as of the first term paper [Dy], in which we have outlined some properties of cylinders from a more topological point of view.

The present paper consists of four sections. Section 1 contains a few facts about continued fraction expansions and details on Besicovitch cylinders. Our construction depends heavily on the properties of these expansions, and some of them do not seem to be commonly known. In Section 2 we present our construction, define some subsets of $\mathbb{T}$ and prove that points in these sets have discrete orbits. The Hausdorff dimension of these sets is computed in Section 3, where we also discuss how it depends on the choice of the rotation. The last section explains shortly how our result applies to special differential equations.

[2]It is known that such cylinders cannot be too regular nor can have too much discrete points – see Subsection 1.2.

1. Preliminaries

We begin with some preliminary facts in order to keep the paper self-contained. This includes properties of the continued fraction expansion and of the (Besicovitch) cylindrical transformations.

1.1. Continued fraction expansions (based on [Kh]).

Every irrational number α has a unique infinite continued fraction expansion of the form

$$\begin{aligned}\alpha = [a_0(\alpha); a_1(\alpha), a_2(\alpha), a_3(\alpha), \dots] &= [a_0; a_1, a_2, a_3, \dots] \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}\end{aligned}$$

where the *partial quotients* a_n are integers and $a_1, a_2, \dots > 0$ ([Kh], Section II.5, Theorem 14). This gives a sequence of *convergents*:

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$$

which can be equivalently defined as ([Kh], Section I.1, Theorem 1)

$$\begin{aligned} q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \\ p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2}. \end{aligned}$$

Therefore, for irrational numbers in $(0, 1)$ (i.e. with $a_0 = 0$) there is a one-to-one correspondence between irrational numbers and the sets of sequences $(a_n)_{n\geq 1}$, $(p_n)_{n\geq 1}$ or $(q_n)_{n\geq 1}$ fulfilling the above conditions.

The fractions $\frac{p_n}{q_n}$ are the best approximants of α among all rational numbers with denominators at most q_n; they are less or greater than α, alternately. The error of the approximation varies, but remains within the following range ([Kh], Sections I.3 and I.4, Theorems 8, 9, 13):

$$\frac{1}{2q_n q_{n+1}} < (-1)^n \left(\alpha - \frac{p_n}{q_n}\right) = \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}. \tag{1}$$

In our paper we will be particularly interested in irrational numbers with somewhat bounded sequence of elements. It is thus worth knowing how much such numbers are there. Observe first that the sequence of denominators (q_n) grows at least exponentially, as does the smallest of such sequences, the Fibonacci sequence, which consists of denominators of the golden ratio.

All numbers with bounded (a_n) form a zero-measure set ([Kh], Section III.13, Theorem 29); however, if we require only a bounded subsequence (a_{k_n}), then we can find a full-measure set of such numbers;[3] this persists even if we require a constant subsequence. On the other hand, the numbers for which the denominators grow only slightly faster:

$$q_{n+1} \leq C q_n^{\gamma} \quad \text{for some } \gamma > 1 \text{ and all } n \geq 0$$

form a set of full measure (it follows from the second assertion of Theorem 32, [Kh], Section III.14).

1.2. **Cylindrical transformations (based on** [FrLe]**).** The cylindrical transformations are a special case of the concept of skew product (see [FoKS], Subsection 10.1.3) in ergodic theory, transferred in a natural way to the topological setting. In general, they can be defined for a minimal homeomorphism T of a compact metric space X ("the base") with a T-invariant measure μ defined on the Borel σ-algebra, and a real continuous function $f\colon X \longrightarrow \mathbb{R}$ (which we will customarily call a *cocycle*). In the next sections we will confine ourselves to minimal rotations on tori with Lebesgue measure.[4] The functions T and f generate a *cylindrical transformation* (or a *cylinder*):

$$T_f\colon X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$$
$$T_f(x,t) := (Tx, t + f(x))$$

The iterations of T_f are of the form $T_f^n(x,t) = (T^n x, t + f^{(n)}(x))$, where $f^{(n)}$ is given by the formula:

$$f^{(n)}(x) := \begin{cases} f(x) + f(Tx) + \cdots + f(T^{n-1}x) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -f(T^{-1}x) - f(T^{-2}x) - \cdots - f(T^n x) & \text{for } n < 0. \end{cases}$$

[3]This can be proved in a few ways; e.g. it follows from the fact that there exists $B > 0$ such that for almost every α and n, $q_n < e^{Bn}$ ([Kh], Section III.14, Theorem 31) or that $\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n}$ is the same for almost every α ([FoKS], Section 7.4, Theorem 4). Another proof, that requires less sophisticated facts, uses the Gauss mapping $Tx := (1/x) - \lfloor 1/x \rfloor$, which acts like shift on continued fraction expansions: $T[0; a_1, a_2, \dots] = [0; a_2, a_3, \dots]$. It is a mixing with respect to some invariant measure, equivalent to Lebesgue measure ([FoKS], Section 7.4, Theorem 3). Since the set of α with $a_1(\alpha) = c$ (for any c) has positive measure, almost every α returns to it infinitely many times – there exists a subsequence k_n such that $a_{k_n}(\alpha) = a_1(T^{k_n}\alpha) = c$.

[4]Minimal rotations on compact groups do not always exist – groups possessing them are called *monothetic*. All tori $\mathbb{T}^n$ are monothetic, and a rotation on $\mathbb{T}^n$ is minimal if its coordinates are irrational and $\mathbb{Q}$-linearly independent; moreover, these rotations are ergodic with respect to Lebesgue measure.

Observe that the dynamics of a point (x, t) does not depend on t, because the mappings

$$\tau_{t_0}\colon X \times \mathbb{R} \ni (x, t) \mapsto (x, t + t_0) \in X \times \mathbb{R}$$

(for arbitrary $t_0 \in \mathbb{R}$) are in the topological centralisator of T_f:

$$T_f(\tau_{t_0}(x, t)) = T_f(x, t + t_0) = (Tx, t + t_0 + f(x)) = \tau_{t_0}(Tx, t + f(x)) = \tau_{t_0}(T_f(x, t)).$$

Unlike the compact case, a homeomorphism of a locally compact space (as here $X \times \mathbb{R}$) need not have a minimal subset. The cylinder on a compact space is never minimal (see Theorem 1.1), thus it is meaningful to study minimal subsets. As pointed out in the introduction to [FrLe], there are two cases in which the minimal subsets can be easily described:

(a) when $\int_X f \mathrm{d}\mu \neq 0$, all points have discrete orbits, or equivalently: for all $x \in X$: $|f^{(n)}(x)| \xrightarrow{n\to\infty} \infty$.
(b) when the cocycle is a *coboundary*, i.e. of the form $f = g - g \circ T$ for a (continuous) *transfer function* $g\colon \mathbb{T} \longrightarrow \mathbb{R}$, the minimal sets are vertically translated copies of the graph of g in $X \times \mathbb{R}$. Conversely, if some orbit under T_f is bounded, then so are all of them, and the cocycle is a coboundary (Gottschalk-Hedlund Theorem, [GoHe], Theorem 14.11).

Notice that if f is a coboundary and T is measure-preserving, then $\int_X f \mathrm{d}\mu = 0$. Also, in both cases the phase space decomposes into minimal sets. In what follows we will call cocycles that fulfil (a) or (b) *trivial* and summarise some results about minimal sets in the nontrivial cases.

Theorem 1.1. (Besicovitch) *A cylinder originating from a minimal homeomorphism T of a compact phase space X is never minimal (see* [Be2]*).*
(Lemańczyk, Mentzen) *If, additionally, X is a topological group, T is a rotation and a cocycle f is not trivial, then the cylinder T_f is transitive (see* [LeMe]*: this fact follows from Lemmas 5.2 – in contraposition – and 5.3, and from the minimality of T).*
Therefore, if f is of average zero, but T_f has a discrete orbit, then it is automatically transitive (by (b) coboundaries have only bounded orbits).

As in [FrLe] we consider cylindrical transformations that display both transitive and discrete behaviour, called *Besicovitch transformations* or *Besicovitch cylinders*, because their first example was given in [Be2]. Given a homeomorphism of the base, we will also call a cocycle which generates a Besicovitch cylinder a *Besicovitch cocycle*. By virtue of the condition (a) and Theorem 1.1, *a cocycle is Besicovitch if and only if it has average zero and the resulting cylinder has a discrete orbit.* This characterisation will be used in our paper.

Unfortunately, Besicovitch cylinders are not easy to find, because too regular cocycles yield no minimal sets at all, as has been proved by Matsumoto and Shishikuro, and, independently, Mentzen and Siemaszko:

Theorem 1.2 ([MaSh, Theorem 1], [MeSi, Theorem 2.4])**.** *If the cocycle on $\mathbb{T}$ is nontrivial and of bounded variation, then the cylinders which it generates have no minimal sets. In particular, they have no discrete orbits.*

For brevity, given a cylinder T_f, we will denote

$$\begin{aligned}\mathcal{D} &:= \{x \in X\colon o_{T_f}(x,t) \text{ is discrete for every } t \in \mathbb{R}\}\\ &= \{x \in X\colon o_{T_f}(x,0) \text{ is discrete}\}.\end{aligned}$$

Then the set of points in $X \times \mathbb{R}$ with discrete orbits equals $\mathcal{D} \times \mathbb{R}$. Clearly, if T is minimal and $\mathcal{D} \neq \emptyset$, then both $\mathcal{D}$ and $\mathcal{D} \times \mathbb{R}$ are dense in ambient spaces, as $\mathcal{D}$ is T-invariant.

In [FrLe] the authors construct a Besicovitch cocycle for any minimal rotation of a torus. They also find ones with some special properties, in particular with "large" $\mathcal{D}$.[5]

Theorem 1.3 ([FrLe])**.** *For every irrational rotation of $\mathbb{T}$ there exist Besicovitch cocycles* ([FrLe, Section 2])*. The cocycles can be chosen in such a way that $\mathcal{D}$ is uncountable* ([FrLe, Proposition 6])*. Moreover, for almost every irrational rotation one can find Besicovitch cocycles such that the Hausdorff dimension of $\mathcal{D}$ is at least* $1/2$ ([FrLe, Theorem 9])*.*

As for the higher dimensional tori, Besicovitch cocycles again exist for every minimal rotation of every $\mathbb{T}^d$ ([FrLe, remark on p. 2])*. Moreover, given $\varepsilon > 0$, for every $d \geq 3$ and for uncountably many minimal rotations of $\mathbb{T}^d$ there exist Besicovitch cocycles of class C^1 and with the Hausdorff dimension of $\mathcal{D}$ greater than* $(d/2) - \varepsilon$ ([FrLe, Theorem 11, Proposition 12])*.*

It was unknown and left as an open problem whether the coefficient $1/2$ could be improved or not. We answer it partially by developing the techniques from [FrLe]: for almost every irrational rotation of $\mathbb{T}$ we have obtained Besicovitch cylinders with $\mathcal{D}$ of full Hausdorff dimension (Conclusion 3.1). This construction is presented in the next section.

[5] Recall that the set of discrete orbits is of first category and of measure zero for minimal rotations of tori.

2. Construction of a Besicovitch cylinder

Let α be an irrational number in $[0,1)$ and $(p_n/q_n)_{n\geq 0}$ its sequence of convergents. Because $q_n \to \infty$, one can choose a subsequence $(q_{k_n})_{n\geq 0}$ that satisfies the inequalities

$$q_{k_0} \geq 36, \tag{2}$$

$$q_{k_{n+1}}^2 \geq 450\, q_{k_n} q_{1+k_n} \text{ for } n \geq 0 \tag{3}$$

and altogether grows increasingly rapidly:

$$\lim_{n\to\infty} (\log q_{k_{n+1}})/(\log q_{1+k_n}) = \infty. \tag{4}$$

We may also assume that

$$\text{all } k_n \text{ are even or all are odd.} \tag{5}$$

Combining conditions (2) and (3), we obtain a uniform estimation:

$$q_{k_n} \geq 6^{n+2}. \tag{6}$$

and by (4) the sequence (q_{k_n}) grows superexponentially:

$$\frac{1}{n} \log q_{k_n} \to \infty. \tag{7}$$

We will need some more parameters to proceed with the construction. Put additionally

$$\begin{aligned} A_n &:= \lceil q_{k_n}/6^n \rceil < 1 + q_{k_n}/6^n < 2q_{k_n}/6^n && \text{for } n \geq 0, \\ M_n &:= \lceil q_{k_{n-1}}/(n+1)^2 \rceil < 1 + q_{k_{n-1}}/(n+1)^2 && \text{for } n \geq 1. \end{aligned}$$

It follows from these definitions that for $n \geq 1$:

$$\frac{q_{k_n}}{A_n} \to \infty; \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{M_n}{q_{k_{n-1}}} < \infty; \tag{9}$$

$$\frac{M_n}{A_{n-1}} = \frac{M_n}{q_{k_{n-1}}} \cdot \frac{q_{k_{n-1}}}{A_{n-1}} > \frac{6^n}{2(n+1)^2} \to \infty; \tag{10}$$

$$\frac{A_n}{q_{k_n}} \cdot \frac{q_{k_{n-1}}}{A_{n-1}} < \frac{1 + q_{k_n}/6^n}{q_{k_n}} \cdot 6^{n-1} = \left(\frac{6^{n-1}}{q_{k_n}} + \frac{1}{6} \right) \overset{(6)}{\leq} \frac{1}{216} + \frac{1}{6} < \frac{1}{5}; \tag{11}$$

$$\begin{aligned} \frac{A_n}{M_{n+1}} \cdot \frac{M_n}{A_{n-1}} &= \frac{A_n}{q_{k_n}} \cdot \frac{q_{k_{n-1}}}{A_{n-1}} \cdot \frac{q_{k_n}}{M_{n+1}} \cdot \frac{M_n}{q_{k_{n-1}}} \\ &\overset{(11)}{<} \frac{(n+2)^2}{5} \left(\frac{1}{(n+1)^2} + \frac{1}{q_{k_{n-1}}} \right) \overset{(6)}{\leq} \frac{1}{5} \left(\frac{(n+2)^2}{(n+1)^2} + \frac{(n+2)^2}{6^{n+1}} \right) \leq \frac{1}{2} \end{aligned} \tag{12}$$

$$\frac{A_{n-1}q_{k_n}q_{1+k_n}}{q_{k_{n-1}}A_{n+1}q_{k_{n+1}}} \overset{(3)}{<} \left(\frac{1}{q_{k_{n-1}}}+\frac{1}{6^{n-1}}\right)\cdot\frac{6^{n+1}q_{k_n}q_{1+k_n}}{450\,q_{k_n}q_{1+k_n}} \overset{(6)}{<} \left(\frac{1}{6^{n+1}}+\frac{1}{6^{n-1}}\right)\cdot\frac{6^{n+1}}{12(6^2+1)}=\frac{1}{12} \tag{13}$$

$$0 \le \frac{\log(q_{k_{n-1}}/A_{n-1})}{\log q_{k_n}} < \frac{(n-1)\log 6}{\log q_{k_n}} \xrightarrow{(7)} 0. \tag{14}$$

$$1 \ge \frac{\log A_n}{\log q_{k_n}} > \frac{\log q_{k_n}-n\log 6}{\log q_{k_n}} \xrightarrow{(7)} 1; \tag{15}$$

To complete the proof we will usually need the properties (8)-(14) rather than the definitions of A_n and M_n themselves. Also, out of the properties of (q_{k_n}) we will not need conditions (2) nor (3) anymore, but only their result, (6). For the sake of brevity, we will also denote

$$L_n := \frac{M_n q_{k_n} q_{1+k_n}}{q_{k_{n-1}}}, \quad \delta_n := \frac{M_n}{L_n} = \frac{q_{k_{n-1}}}{q_{k_n}q_{1+k_n}}, \quad \text{for } n \ge 1.$$

Note that

$$\frac{\delta_n}{A_{n-1}} = \frac{q_{k_{n-1}}}{q_{k_n}q_{1+k_n}A_{n-1}} \overset{(11)}{\le} \frac{q_{k_n}}{5q_{k_n}q_{1+k_n}A_n} < \frac{1}{5A_nq_{k_n}}. \tag{16}$$

Similarly to [FrLe] we define f_n to be L_n-Lipschitz, $1/(A_nq_{k_n})$-periodic continuous function:[6]

$$f_n(x) := \begin{cases} 0, & \text{for } 0 \le x \le \dfrac{\delta_n}{8A_{n-1}}, \\ L_n\left(x-\dfrac{\delta_n}{8A_{n-1}}\right), & \text{for } \dfrac{\delta_n}{8A_{n-1}} \le x \le \dfrac{5\delta_n}{8A_{n-1}}, \\ \dfrac{M_n}{2A_{n-1}}, & \text{for } \dfrac{5\delta_n}{8A_{n-1}} \le x \le \dfrac{1}{2A_nq_{k_n}}-\dfrac{5\delta_n}{8A_{n-1}}, \\ L_n\left(x-\dfrac{1}{2A_nq_{k_n}}+\dfrac{5\delta_n}{8A_{n-1}}\right)+\dfrac{M_n}{2A_{n-1}}, & \\ \qquad \text{for } \dfrac{1}{2A_nq_{k_n}}-\dfrac{5\delta_n}{8A_{n-1}} \le x \le \dfrac{1}{2A_nq_{k_n}}-\dfrac{\delta_n}{8A_{n-1}}, & \\ \dfrac{M_n}{A_{n-1}}, & \text{for } \dfrac{1}{2A_nq_{k_n}}-\dfrac{\delta_n}{8A_{n-1}} \le x \le \dfrac{1}{2A_nq_{k_n}}. \end{cases}$$

[6]All the following intervals are well-defined – cf. (16).

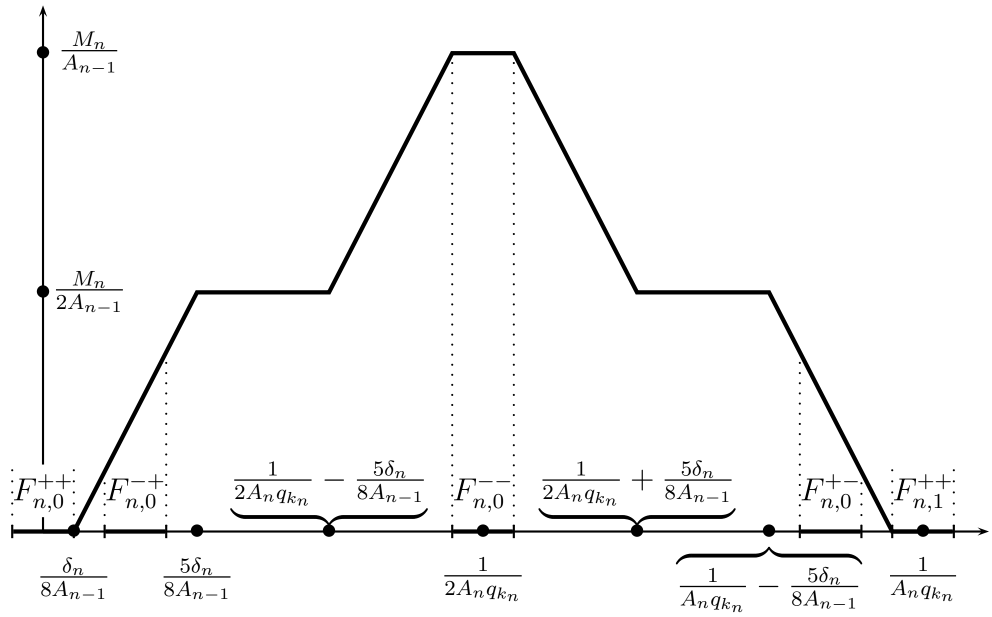

FIGURE 1. One period of f_n and the sets $F_{n,\cdot}^{s_-s_+}$.

and $f_n(\frac{1}{A_n q_{k_n}} - x) = f_n(x)$ (hence also $f_n(-x) = f_n(x)$). By periodicity:

$$|f_n(x+\alpha) - f_n(x)| = \left| f_n\left(x + \alpha - \frac{A_n p_{k_n}}{A_n q_{k_n}}\right) - f_n(x)\right|$$
$$\leq L_n \left|\alpha - \frac{p_{k_n}}{q_{k_n}}\right| < \frac{M_n q_{k_n} q_{1+k_n}}{q_{k_{n-1}}} \frac{1}{q_{k_n} q_{1+k_n}} = \frac{M_n}{q_{k_{n-1}}},$$

so the series

$$\varphi(x) := \sum_{n=1}^{\infty} (f_n(x+\alpha) - f_n(x))$$

converges uniformly by (9) and yields a continuous cocycle of average zero. Moreover, for every $m \in \mathbb{Z}$:

$$\varphi^{(m)}(x) := \sum_{n=1}^{\infty} (f_n(x+m\alpha) - f_n(x)). \tag{17}$$

Consider, for $n \geq 1$ and $j = 0, \ldots, A_n q_{k_n} - 1$:

$$F_{n,j}^{++} := \left[-\frac{\delta_n}{8A_{n-1}}, \frac{\delta_n}{8A_{n-1}}\right] + \frac{j}{A_n q_{k_n}},$$
$$F_{n,j}^{-+} := \left[0, \frac{\delta_n}{4A_{n-1}}\right] + \frac{\delta_n}{4A_{n-1}} + \frac{j}{A_n q_{k_n}},$$
$$F_{n,j}^{--} := \left[-\frac{\delta_n}{8A_{n-1}}, \frac{\delta_n}{8A_{n-1}}\right] + \frac{1}{2A_n q_{k_n}} + \frac{j}{A_n q_{k_n}},$$

$$F^{+-}_{n,j} := \left[0, \frac{\delta_n}{4A_{n-1}}\right] + \left(\frac{1}{A_n q_{k_n}} - \frac{\delta_n}{2A_{n-1}}\right) + \frac{j}{A_n q_{k_n}},$$

and for arbitrary $s_-, s_+ \in \{+, -\}$

$$F^{s_-s_+} := \bigcap_{n=1}^{\infty} \bigcup_{j=0}^{A_n q_{k_n}-1} F^{s_-s_+}_{n,j}.$$

The sets $F^{s_-s_+}$ are nonempty and even uncountable: every interval $F^{s_-s_+}_{n,j}$ contains at least

$$\left\lfloor \frac{|F^{s_-s_+}_{n,j}|}{1/(A_{n+1}q_{k_{n+1}})} \right\rfloor - 1 = \left\lfloor \frac{q_{k_{n-1}} A_{n+1} q_{k_{n+1}}}{4A_{n-1} q_{k_n} q_{1+k_n}} \right\rfloor - 1 \overset{(13)}{\geq} \lfloor 12/4 \rfloor - 1 = 2 \tag{18}$$

of the intervals $F^{s_-s_+}_{n+1,j}$, since the intervals from the $(n+1)$-st union are uniformly distributed with period $1/(A_{n+1}q_{k_{n+1}})$; therefore the intersections $F^{s_-s_+}$ are topological Cantor sets.

We will show in the subsections below that the sets $F^{s_-s_+} \times \mathbb{R}$ consist of points with discrete orbits. This, according to the equivalent condition from Subsection 1.2, will prove that φ is a Besicovitch cocycle. More precisely, we will show that for every $x \in F^{s_-s_+}$:

- if $s_+ = s_-$, then

$$\varphi^{(m)}(x) \xrightarrow{m\to\pm\infty} s_+ \infty,$$

- if $s_+ \neq s_-$, then

$$\varphi^{(m)}(x) \xrightarrow{m\to\pm\infty} (+1)/(-1)\, s_\pm \infty,$$

  where the coefficient $(+1)/(-1)$ depends on the parity of k_n: $+1$ if even, -1 if odd (cf. (5)).

The Hausdorff dimension of $F^{s_-s_+}$ is calculated in the following section. We also find rotations for which $\dim_H F^{s_-s_+}$ is positive or even 1.

To compare our construction with [FrLe], let us remark that the main differences are scaling by A_n and relatively longer "featured" intervals (of order δ_n/A_{n-1} rather than δ_n/A_n, as one might expect). This, however, allows greater flexibility, and, having made some effort to choose suitable A_n, we obtained higher Hausdorff dimension of $F^{s_-s_+}$. It may be worth noting that when using the length δ_n/A_n we were able to get the dimension 2/3 instead of 1.

2.1. **The case of F^{++}.** Choose an $x \in F^{++}$. There exists $(j_l)_{l=1}^{\infty} \subset \mathbb{N}$ such that $x \in F^{++}_{l,j_l}$ for every $l \in \mathbb{N}$. Let also $x = x_l + \frac{j_l}{A_l q_{k_l}}$, where $|x_l| \leq \delta_l/(8A_{l-1})$. Additionally, fix an integer $|m| \geq q_{k_0}$. It follows from the periodicity and parity of f_l that

$$
\begin{aligned}
(19)\quad f_l(x+m\alpha)-f_l(x) &= f_l(x_l+m\alpha)-f_l(x_l)=f_l(x_l+m\alpha)\\
&= f_l\left(x_l+m\alpha-\frac{mA_lp_{k_l}}{A_lq_{k_l}}\right)=f_l\left(\left|x_l+m\alpha-\frac{mp_{k_l}}{q_{k_l}}\right|\right),
\end{aligned}
$$

whence in particular:

$$
(20)\qquad f_l(x+m\alpha)-f_l(x)\geq 0.
$$

Take now $n=n(m)\geq 1$ for which $\frac{2q_{k_{n-1}}}{3A_{n-1}}\leq |m|<\frac{2q_{k_n}}{3A_n}$ (it exists by (8), it is unique by (11), and when $|m|$ tends to infinity, so does $n(m)$). These constraints along with the inequalities from Subsection 1.1 imply that

$$
\begin{aligned}
\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right| &\geq |m|\left|\alpha-\frac{p_{k_n}}{q_{k_n}}\right|-|x_n|\\
&\overset{(1)}{\geq} \frac{q_{k_{n-1}}}{3q_{k_n}q_{1+k_n}A_{n-1}}-\frac{\delta_n}{8A_{n-1}}=\frac{\delta_n}{3A_{n-1}}-\frac{\delta_n}{8A_{n-1}},
\end{aligned}
$$

and on the other hand:

$$
\begin{aligned}
\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right| &\leq |x_n|+|m|\left|\alpha-\frac{p_{k_n}}{q_{k_n}}\right|\\
&\overset{(1)}{\leq} |x_n|+\frac{|m|}{q_{k_n}q_{1+k_n}}\leq\frac{\delta_n}{8A_{n-1}}+\frac{2}{3q_{1+k_n}A_n}.
\end{aligned}
$$

We wish to get some useful lower bound on $f_n\left(\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right|\right)$. Since

$$
\begin{aligned}
&\frac{\delta_n}{8A_{n-1}}+\frac{2}{3A_nq_{1+k_n}}\leq\frac{\delta_n}{8A_{n-1}}+\frac{1}{A_nq_{k_n}}-\frac{1}{3A_nq_{1+k_n}}\\
&\overset{(11)}{\leq}\frac{\delta_n}{8A_{n-1}}+\frac{1}{A_nq_{k_n}}-\frac{1}{3q_{1+k_n}}\frac{q_{k_{n-1}}}{A_{n-1}q_{k_n}}=\frac{1}{A_nq_{k_n}}-\left(\frac{\delta_n}{3A_{n-1}}-\frac{\delta_n}{8A_{n-1}}\right)
\end{aligned}
$$

and owing to its shape, the function f_n takes the minimum at the left endpoint of the interval in question:

$$
\begin{aligned}
f_n\left(\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right|\right) &\geq f_n\left(\frac{\delta_n}{3A_{n-1}}-\frac{\delta_n}{8A_{n-1}}\right)\\
&= L_n\cdot\left(\frac{1}{3}-\frac{2}{8}\right)\frac{\delta_n}{A_{n-1}}=\frac{M_n}{12A_{n-1}}\xrightarrow[(10)]{m\to\infty}\infty,
\end{aligned}
$$

which finally yields the required divergence:

$$
\begin{aligned}
\varphi^{(m)}(x) &\overset{(17)}{=}\sum_{l=1}^{\infty}(f_l(x+m\alpha)-f_l(x))\overset{(20)}{\geq} f_{n(m)}(x+m\alpha)-f_{n(m)}(x)\\
&\overset{(19)}{=} f_n\left(\left|x_n+m\alpha-\frac{mp_{k_n}}{q_{k_n}}\right|\right)\xrightarrow{m\to\infty}\infty.
\end{aligned}
$$

2.2. **The case of F^{-+}.** Choose an $x \in F^{-+}$. As in the previous case, we find a sequence $(j_l)_{l=1}^{\infty} \subset \mathbb{N}$ and (x_l) such that for every l: $x \in F_{l,j_l}^{-+}$, $x = x_l + \frac{\delta_l}{4A_{l-1}} + \frac{j_l}{A_l q_{k_l}}$ and $0 \leq x_l \leq \frac{\delta_l}{4A_{l-1}}$. We fix also an integer $|m| \geq q_{k_0}$ and $n = n(m)$ for which $\frac{q_{k_{n-1}}}{5A_{n-1}} \leq |m| < \frac{q_{k_n}}{5A_n}$. We will assume that m has the same sign as $\alpha - \frac{p_{k_l}}{q_{k_l}}$, that is, is positive if k_l are even and negative if they are odd (cf. (5)), so that always $m\alpha - \frac{mp_{k_l}}{q_{k_l}} = \left|m\alpha - \frac{mp_{k_l}}{q_{k_l}}\right|$. Then

$$f_l(x+m\alpha)-f_l(x) = f_l\left(x_l + \frac{\delta_l}{4A_{l-1}} + \left|m\alpha - \frac{mp_{k_l}}{q_{k_l}}\right|\right) - f_l\left(x_l + \frac{\delta_l}{4A_{l-1}}\right)$$

Consider now $l > n(m)$. On the one hand

$$x_l + \frac{\delta_l}{4A_{l-1}} + |m|\left|\alpha - \frac{p_{k_l}}{q_{k_l}}\right| \overset{(1)}{<} \frac{\delta_l}{2A_{l-1}} + \frac{q_{k_n}}{5A_n} \cdot \frac{1}{q_{k_l} q_{1+k_l}} \overset{(16),(11)}{<} \frac{1}{10A_l q_{k_l}} + \frac{q_{k_l}}{25A_l} \cdot \frac{1}{q_{k_l} q_{1+k_l}} < \frac{1}{2A_l q_{k_l}}$$

and on the other hand

$$x_l + \frac{\delta_l}{4A_{l-1}} + |m|\left|\alpha - \frac{p_{k_l}}{q_{k_l}}\right| \overset{(1)}{>} x_l + \frac{\delta_l}{4A_{l-1}} + \frac{q_{k_{n-1}}}{5A_{n-1}} \cdot \frac{1}{2q_{k_l} q_{1+k_l}} > 0$$

The lesser number is close enough to $x_l + \frac{\delta_l}{4A_{l-1}}$, so that f_l is linear between them (because it is linear on the $\frac{\delta_l}{8A_{l-1}}$-neighbourhood of $F_{l,0}^{-+}$ and $\frac{\delta_l}{8A_{l-1}} > \frac{q_{k_{n-1}}}{10A_{n-1} q_{k_l} q_{1+k_l}}$, again by (11)). Thus, as f_l is nondecreasing on $[0, 1/(2A_l q_{k_l})]$:

$$\begin{aligned} f_l(x+m\alpha) - f_l(x) &= f_l\left(x_l + \frac{\delta_l}{4A_{l-1}} + |m|\left|\alpha - \frac{p_{k_l}}{q_{k_l}}\right|\right) - f_l\left(x_l + \frac{\delta_l}{4A_{l-1}}\right) \\ &\geq f_l\left(x_l + \frac{\delta_l}{4A_{l-1}} + \frac{q_{k_{n-1}}}{10A_{n-1} q_{k_l} q_{1+k_l}}\right) - f_l\left(x_l + \frac{\delta_l}{4A_{l-1}}\right) \\ &= \frac{L_l q_{k_{n-1}}}{10A_{n-1} q_{k_l} q_{1+k_l}} = \frac{1}{10} \frac{q_{k_{n-1}}}{A_{n-1}} \frac{M_l}{q_{k_{l-1}}}. \end{aligned}$$

The expressions $f_l(x - m\alpha) - f_l(x)$ for $l > n$ are likewise bounded from above: since

$$x_l + \frac{\delta_l}{4A_{l-1}} - |m|\left|\alpha - \frac{p_{k_l}}{q_{k_l}}\right| \overset{(1)}{>} \frac{\delta_l}{4A_{l-1}} - \frac{q_{k_n}}{5A_n} \cdot \frac{1}{q_{k_l} q_{1+k_l}} \overset{(11)}{\geq} \frac{\delta_l}{4A_{l-1}} - \frac{\delta_l}{5A_{l-1}} > 0,$$

$$x_l + \frac{\delta_l}{4A_{l-1}} - |m|\left|\alpha - \frac{p_{k_l}}{q_{k_l}}\right| \overset{(1)}{<} x_l + \frac{\delta_l}{4A_{l-1}} - \frac{q_{k_{n-1}}}{5A_{n-1}} \cdot \frac{1}{2q_{k_l} q_{1+k_l}} \overset{(16)}{<} \frac{1}{2A_l q_{k_l}},$$

a similar reasoning yields

$$\begin{aligned}
f_l(x-m\alpha)-f_l(x) &= f_l\left(x_l+\frac{\delta_l}{4A_{l-1}}-|m|\left|\alpha-\frac{p_{k_l}}{q_{k_l}}\right|\right)-f_l\left(x_l+\frac{\delta_l}{4A_{l-1}}\right)\\
&\le f_l\left(x_l+\frac{\delta_l}{4A_{l-1}}-\frac{q_{k_{n-1}}}{10A_{n-1}q_{k_l}q_{1+k_l}}\right)-f_l\left(x_l+\frac{\delta_l}{4A_{l-1}}\right)\\
&= -\frac{L_l q_{k_{n-1}}}{10A_{n-1}q_{k_l}q_{1+k_l}}=-\frac{1}{10}\frac{q_{k_{n-1}}}{A_{n-1}}\frac{M_l}{q_{k_{l-1}}}.
\end{aligned}$$

Summarising, we have just established that for almost every integer r and for every $l>n(r)$

$$|f_l(x+r\alpha)-f_l(x)|\ge\frac{1}{10}\frac{q_{k_{n(r)-1}}}{A_{n(r)-1}}\frac{M_l}{q_{k_{l-1}}}$$

and the sign of $f_l(x+r\alpha)-f_l(x)$ is equal to $(-1)^{k_l}\operatorname{sign} r$. To prove the desired property of F^{-+} we want now to prove

$$\varphi^{(r)}(x)=\sum_{l=1}^{\infty}(f_l(x+r\alpha)-f_l(x))\xrightarrow{r\to\pm\infty}\pm(-1)^{k_l}\infty.$$

We have already estimated almost every term of this series and, as we will see in a moment, the sums of these terms rise to infinity. Then to finish the proof it will remain to check that the first terms can be neglected in the limit.

Fix $|r|>q_{k_0}$. Using the definition of M_n we get

$$\sum_{l=n(r)+1}^{\infty}\frac{q_{k_{n(r)-1}}}{A_{n(r)-1}}\frac{M_l}{q_{k_{l-1}}}=\frac{q_{k_{n-1}}}{A_{n-1}}\sum_{l=n+1}^{\infty}\frac{M_l}{q_{k_{l-1}}}\ge\frac{q_{k_{n-1}}}{A_{n-1}}\sum_{l=n+1}^{\infty}\frac{1}{(l+1)^2}.$$

One can verify that the remainder is asymptotically equivalent to $\frac{1}{n}$, i.e.[7]

$$\lim_{n\to\infty}n\left(\sum_{l=n+1}^{\infty}\frac{1}{(l+1)^2}\right)=1.$$

Consequently, for n large enough the remainder is greater than $1/(2n)$, which yields

$$\left|\sum_{l=n+1}^{\infty}(f_l(x+r\alpha)-f_l(x))\right|\ge\sum_{l=n+1}^{\infty}\frac{1}{10}\frac{q_{k_{n-1}}}{A_{n-1}}\frac{M_l}{q_{k_{l-1}}}\ge\frac{1}{20n}\frac{q_{k_{n-1}}}{A_{n-1}}.$$

[7] This follows from the termwise equivalence to a telescoping series of $1/n$:
$\sum_{l\ge n+1}\frac{1}{(l+1)^2}-\sum_{l\ge n+2}\frac{1}{(l+1)^2}=1/(n+1)^2\approx(1/n)-1/(n+1)$,
and from an analogue of the Stolz-Cesàro Theorem.

As for the remaining terms, the maximum of $|f_l(x+r\alpha)-f_l(x)|$ is obviously at most $2M_l/A_{l-1}$, hence

$$\left|\sum_{l=1}^{n}(f_l(x+r\alpha)-f_l(x))\right| \leq 2\sum_{l=1}^{n}\frac{M_l}{A_{l-1}} \overset{(12)}{\leq} 2\sum_{l=1}^{n}\frac{1}{2^{n-l}}\frac{M_n}{A_{n-1}} < \frac{4M_n}{A_{n-1}}.$$

We are ready to calculate the required limit. To this end, we will again apply the definition of M_n. Note that we may without loss of generality examine the absolute value, because the summand $\sum_{l=n+1}^{\infty}(f_l(x+r\alpha)-f_l(x))$ prevails, and we know that it is of appropriate sign.

$$\begin{aligned}|\varphi^{(r)}(x)| &\geq \left|\sum_{l=n+1}^{\infty}(f_l(x+r\alpha)-f_l(x))\right| - \left|\sum_{l=1}^{n}(f_l(x+r\alpha)-f_l(x))\right| \\ &> \frac{1}{20n}\frac{q_{k_{n-1}}}{A_{n-1}} - \frac{4M_n}{A_{n-1}} = \frac{q_{k_{n-1}}}{A_{n-1}}\left(\frac{1}{20n}-\frac{4M_n}{q_{k_{n-1}}}\right) \\ &\overset{(*)}{\geq} \frac{q_{k_{n-1}}}{A_{n-1}}\cdot\frac{M_n}{q_{k_{n-1}}} \xrightarrow[(10)]{r\to\pm\infty} \infty\end{aligned}$$

The inequality $(*)$ is valid for large n, which follows from the definition of M_n: since the sequence $M_n/q_{k_{n-1}} \leq 2/(n+1)^2$ is asymptotically dominated by $1/n$, eventually $1/(20n) > 5M_n/q_{k_{n-1}}$.

Let us mention that the results of this subsection, despite being derived from the very definition of M_n, actually depend only on more general relations between the sequences (A_n), (M_n) and (q_{k_n}).

2.3. **The cases of F^{+-} and F^{--}.** The behaviour of functions f_n on the sets $F^{+-}_{n,j}$ and $F^{--}_{n,j}$ is symmetrical to the situation on $F^{-+}_{n,j}$ and $F^{++}_{n,j}$, respectively. The examined limits are calculated analogously, and thus we will omit these proofs.

3. Hausdorff dimension of $F^{s_-s_+}$

To estimate the Hausdorff dimension of $F^{s_-s_+}$ from below, we will use a method from [Fa] (Example 4.6):

Consider a sequence of unions of a finite number of disjoint closed intervals in $[0,1)$ (here: the sequence $(\bigcup_{j=0}^{A_nq_{k_n}-1}F^{s_-s_+}_{n,j})_{n\geq 1}$). *Suppose that the intervals of the n-th union ($n\geq 1$) are separated by gaps of length at least ε_n (with $\varepsilon_n > \varepsilon_{n+1} > 0$) and contain at least $m_{n+1} \geq 2$ intervals of the $(n+1)$-st union, and also the maximum length of intervals in the n-th union tends to 0 as $n\to\infty$. Then the Hausdorff dimension of the intersection of this sequence can be estimated from below by:*

$$\liminf_{n\to\infty}\frac{\log(m_2\cdots m_n)}{-\log(m_{n+1}\varepsilon_{n+1})}.$$

First, note that $|F_{n,j}^{s_-s_+}| = \delta_n/(4A_{n-1}) \overset{(16)}{<} 1/(20A_n q_{k_n}) \to 0$. Next, observe that

$$\varepsilon_n = \frac{1}{A_n q_{k_n}} - |F_{n,j}^{s_-s_+}| > \frac{1}{A_n q_{k_n}} - \frac{1}{20A_n q_{k_n}} > \frac{1}{2A_n q_{k_n}}.$$

As for m_{n+1},[8] it has already been roughly estimated in (18):

$$m_{n+1} \geq \left\lfloor \frac{q_{k_{n-1}} A_{n+1} q_{k_{n+1}}}{4A_{n-1} q_{k_n} q_{1+k_n}} \right\rfloor - 1 \geq \lfloor 3 \rfloor - 1 = 2.$$

Here we need a more subtle bound. To facilitate the following calculations, observe that $\lfloor t \rfloor - 1 > t/2$ if $t \geq 3$, and therefore

$$m_{n+1} > \frac{A_{n+1} q_{k_{n-1}} q_{k_{n+1}}}{8A_{n-1} q_{k_n} q_{1+k_n}};$$

We can finally estimate the very expressions from the desired limit:

$$\begin{aligned} m_2 \cdots m_n &> \frac{8}{8^n} \left(\frac{A_2 q_{k_0}\ q_{k_2}}{A_0 q_{k_1} q_{1+k_1}} \right) \left(\frac{A_3 q_{k_1}\ q_{k_3}}{A_1 q_{k_2} q_{1+k_2}} \right) \cdots \left(\frac{A_n\ q_{k_{n-2}}\ \ q_{k_n}}{A_{n-2} q_{k_{n-1}} q_{1+k_{n-1}}} \right) \\ &= \frac{8}{8^n} \left(\frac{A_2}{A_0} \frac{A_3}{A_1} \cdots \frac{A_n}{A_{n-2}} \right) \left(\frac{q_{k_0}}{q_{k_1}} \frac{q_{k_1}}{q_{k_2}} \cdots \frac{q_{k_{n-2}}}{q_{k_{n-1}}} \right) \left(\frac{q_{k_2}}{q_{1+k_1}} \frac{q_{k_3}}{q_{1+k_2}} \cdots \frac{q_{k_n}}{q_{1+k_{n-1}}} \right) \\ &\overset{(**)}{>} \frac{8}{8^n} \frac{A_{n-1} A_n}{A_0 A_1} \frac{q_{k_0}}{q_{k_{n-1}}} \frac{q_{k_n}}{q_{1+k_{n-1}}} = \frac{8 q_{k_0}}{A_0 A_1} \frac{A_{n-1} A_n q_{k_n}}{8^n q_{k_{n-1}} q_{1+k_{n-1}}}, \end{aligned}$$

(in the inequality $(**)$ we use the fact that k_l grow and all are of the same parity, so $k_l > 1 + k_{l-1}$)

$$m_{n+1}\varepsilon_{n+1} \geq \frac{1}{8} \frac{A_{n+1}}{A_{n-1}} \frac{q_{k_{n-1}}}{q_{k_n}} \frac{q_{k_{n+1}}}{q_{1+k_n}} \frac{1}{2A_{n+1} q_{k_{n+1}}} = \frac{q_{k_{n-1}}}{16A_{n-1} q_{k_n} q_{1+k_n}},$$

hence eventually

$$\begin{aligned} \dim_H F^{s_-s_+} &\geq \liminf_{n\to\infty} \frac{\log(m_2 \cdots m_n)}{-\log(m_{n+1}\varepsilon_{n+1})} \\ &\geq \liminf_{n\to\infty} \frac{\log(A_{n-1}/q_{k_{n-1}}) + \log(A_n q_{k_n}) - n\log 8 - \log q_{1+k_{n-1}} + \log \frac{8q_{k_0}}{A_0 A_1}}{\log(A_{n-1}/q_{k_{n-1}}) + \log q_{k_n} + \log q_{1+k_n} + \log 16} \\ &\quad \cdot \frac{1/q_{k_n}}{1/q_{k_n}} = \frac{2}{1 + \limsup(\log q_{1+k_n})/(\log q_{k_n})} \quad \text{(by (14), (15), (7), (4))}. \end{aligned}$$

[8]We estimate m_{n+1} rather than m_n, because this involves less complicated expressions.

This is the precise value of $\dim_H F^{s-s_+}$: since the set is covered by $A_n q_{k_n}$ intervals of length $\delta_n/(4A_{n-1})$, by Proposition 4.1 from [Fa]

$$\begin{aligned}\dim_H F^{s-s_+} &\le \liminf_{n\to\infty} \frac{\log A_n q_{k_n}}{-\log(\delta_n/(4A_{n-1}))} \\ &= \liminf_{n\to\infty} \frac{\log q_{k_n} + \log A_n}{\log q_{k_n} + \log q_{1+k_n} + \log(A_{n-1}/q_{k_{n-1}}) - \log 4} \\ &= \frac{2}{1+\limsup(\log q_{1+k_n}/\log q_{k_n})} \qquad \text{(by (14), (15))}.\end{aligned}$$

Therefore the dimension can even attain 1. This is actually possible for almost every irrational α. Namely, if there are infinitely many q_n such that

(a) $q_{1+n} \le C q_n$ for some $C > 1$,
(b) or at least $q_{1+n} \le C q_n^\gamma$ for some $C > 1$ and $\gamma > 1$,

then we can choose q_{k_n} for our construction from them and, in respective cases:

(a) $\lim(\log q_{1+k_n})/(\log q_{k_n}) = 1$ and $\dim_H F^{s-s_+} = 1$,
(b) $\limsup(\log q_{1+k_n})/(\log q_{k_n}) \le \gamma$ and $\dim_H F^{s-s_+} \ge \frac{2}{1+\gamma} > 0$.

According to remarks from Subsection 1.1, almost every α satisfies these conditions. Note also that in the case (b) we obtain twice the dimension from Theorem 8 in [FrLe].

Conclusion 3.1. *For almost every irrational rotation of $\mathbb{T}$ there exists a Besicovitch cocycle such that the Hausdorff dimension of $\mathcal{D}$ equals 1.*

4. Application to differential equations

In [Po] Poincaré was studying specific differential equations and dynamical behaviour of the flows thus generated. In the following section we will summarise the properties of his example as well as two well-behaved examples, which, after continuous perturbation, become quite chaotic; all were obtained in by Frączek and Lemańczyk in [FrLe]. In view of Conclusion 3.1, their proof leads to higher Hausdorff dimension, which is included in Property 6 and Theorem 4.1.

Basic case ([FrLe, Section 8]). The following system on $\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}$ was considered in [Po, Chapitre XIX, p. 202]:

$$(\star) \qquad \begin{cases} \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \bar{a} \\ \frac{\mathrm{d}z}{\mathrm{d}t} = 1 \\ \frac{\mathrm{d}y}{\mathrm{d}t} = f(\bar{x}, z), \end{cases}$$

where $\bar{a} \in \mathbb{T}^d$ induces a minimal rotation, denoted T, on $\mathbb{T}^d$, and $f\colon \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$ is of class C^r for some $r \ge 0$. Solving the equations, we obtain the flow

$$\Phi_t(\bar{x}, z, y) := \left(\bar{x} + t\bar{a}, z + t, \int_0^t f(\bar{x} + s\bar{a}, z + s)\,\mathrm{d}s\right).$$

For $z = 0$ we obtain a global section and the Poincaré map given by the formula

$$\mathbb{T}^d \times \{0\} \times \mathbb{R} \cong \mathbb{T}^d \times \mathbb{R} \ni (\bar{x}, y) \mapsto (T\bar{x}, y + \varphi(\bar{x})),$$

where $\varphi(\bar{x}) := \int_0^1 f(\bar{x} + s\bar{a}, s)\,\mathrm{d}s$, which is also of class C^r. This can be reversed: given smooth $\varphi\colon \mathbb{T}^d \to \mathbb{R}$, one can find equally smooth $f\colon \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$ such that φ is the Poincaré map of the corresponding flow.

It follows that the flow is topologically conjugate to the suspension flow over T_φ, which readily yields:

Properties.

1. The flow $(\Phi_t)_{t\in\mathbb{R}}$ is never minimal.

2. As $\int_{\mathbb{T}^d} \varphi(\bar{x})\,\mathrm{d}\bar{x} = \int_{\mathbb{T}^d\times\mathbb{T}} f(\bar{x}, z)\,\mathrm{d}\bar{x}\mathrm{d}z$, if the latter integral does not vanish, the flow decomposes into closed orbits, homeomorphic to $\mathbb{R}$.

3. On the other hand, if $\varphi(\bar{x}) = j(\bar{x}) - j(T\bar{x})$ for some continuous $j\colon \mathbb{T}^d \to \mathbb{R}$, then the flow also decomposes, this time into relatively compact components. Also, if some orbit is relatively compact, then so are all orbits, and φ is a coboundary.

4. If neither does the above integral vanish, nor are orbits relatively compact, then the flow is transitive.

5. In the latter case, minimal sets can exist. However, if we require f to be C^1, there are no minimal sets for $d = 1$ (in particular, there are no closed orbits) and they can exist for $d \geq 3$.

6. If $d = 1$, for almost every $\bar{a}$ the function f can be chosen so that the flow is transitive, yet there are also closed orbits, and the set of points with closed orbits, albeit of measure zero, is of full Hausdorff dimension.

Differential equations on $\mathbb{R}^3$ and $\mathbb{S}^3$ ([FrLe, Sections 8.1 & 8.2])**.** Fix an irrational α.

1) Let additionally $\alpha > 0$. Consider the following system on $\mathbb{R}^3$:

$$\begin{cases} x' = -2\pi y + 2\pi\alpha xz \\ y' = 2\pi x + 2\pi\alpha yz \\ z' = \pi\alpha(1 - x^2 - y^2 - z^2). \end{cases}$$

The phase space can be shown to decompose into a family of invariant tori and two "singular" invariant sets – a circle and a straight line:

$$A_+ := \mathbb{T} \times \{0\} \quad \text{and} \quad A_- := \{0\} \times \{0\} \times \mathbb{R}.$$

2) Consider the following system on $\mathbb{S}^3 = \{(z_1, z_2)\colon |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$:

$$\begin{cases} z_1' = 2\pi i\alpha z_1 \\ z_2' = 2\pi i z_2 \end{cases}$$

The phase space is also foliated into invariant tori, given by the formula $\{(z_1, z_2)\colon |z_1|^2 = c,\ |z_2|^2 = 1 - c\}$ for all $c \in (0, 1)$, and two circles, corresponding to $c = 0,\ 1$:

$$A_+ := \{(z_1, z_2)\colon |z_1| = 1,\ z_2 = 0\} \text{ and } A_- := \{(z_1, z_2)\colon z_1 = 0,\ |z_2| = 1\}.$$

Both systems, when singular sets A_+ and A_- are removed from the phase space, can be perturbed continuously so that the results are isomorphic to the system ($\star$) for $d = 1$ (via a real-analytic change of coordinates), and the sets A_+ and A_- correspond to the "infinities" in $\mathbb{T}^2 \times \mathbb{R}$, namely $\mathbb{T}^2 \times \{+\infty\}$ and $\mathbb{T}^2 \times \{-\infty\}$; therefore, the following holds:

Theorem 4.1 ([FrLe, Theorems 13 & 14] + Conclusion 3.1)**.**
For both systems of equations and for every considered α there exists a continuous perturbation of the system such that the flows corresponding to the perturbed systems are transitive and for an arbitrary choice of $s_1,\ s_2 \in \{+, -\}$ the set of points with the ω-limit set equal to A_{s_1} and the α-limit set equal to A_{s_2} is dense. Moreover, for almost every α the perturbation can be chosen so that those sets are of full Hausdorff dimension, i.e. 3.

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