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Cylindrical transformations with full-dimensional set of discrete orbits. II

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CYLINDERICAL TRANSFORMATIONS WITH FULL-DIMENSIONAL SET OF DISCRETE ORBITS. II

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Introduction

This paper is a continuation of the term paper [Dy2] – the reader is encouraged to read the introduction and Section 1.2 of [Dy2] before reading this article. However, we summarize below the necessary facts. We consider *cylindrical transformations* (*cylinder*), usually over minimal rotations of the circle, that is mappings of the form

$$T_f \colon \mathbb{T} \times \mathbb{R} \ni (x,t) \mapsto (Tx, t + f(x)) \in \mathbb{T} \times \mathbb{R}$$

where $Tx = x + \alpha$ for some $\alpha \in [0,1] \setminus \mathbb{Q}$ and the *cocycle* $f \colon \mathbb{T} \to \mathbb{R}$ is continuous. It is easy to establish the dynamical behaviour of the cylinder when $\int_{\mathbb{T}} f \, \mathrm{d}\lambda \neq 0$ or when the average is zero and T_f has only bounded orbits (equivalently, it has some bounded orbits, or f is of the form $g - g \circ T$ for some continuous g; see [GoHe]). In the remaining cases the cylinders are always transitive ([LeMe]),[1] but never minimal ([Be2]). In [Dy2], by developing the ideas from [FrLe], we have constructed for almost every irrational α a transitive cylinder over T with closed discrete orbits[2] (a *Besicovitch cylinder*),such that *discrete points*, i.e. points with discrete orbit, form a set of full Hausdorff dimension. (Note that the set of discrete points is here of measure zero – [FrLe].) In the present article we modify this construction, so that it works for every irrational $\alpha$and the exposition is less sophisticated. We also observe that these cylinders satisfy a definition of chaos, which

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[1]Recall that in this setting both usual definitions of transitivity (existence of dense orbit and dense orbits of all open sets) are equivalent. The proof was reminded in [Dy1].

[2]For the sake of brevity, from now on we will write "discrete" instead of "closed discrete".

is a possible generalization of the Devaney chaos to the noncompact dynamical systems.

1. Construction of a Besicovitch cylinder

Let α be an irrational number in $[0,1)$ and $(p_n/q_n)_{n\geq 0}$ its sequence of convergents. Recall that then

$$\frac{1}{2q_nq_{n+1}} < (-1)^n\left(\alpha - \frac{p_n}{q_n}\right) = \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_nq_{n+1}}. \tag{1}$$

Because $q_n \to \infty$, one can choose a subsequence $(q_{k_n})_{n\geq 1}$ that grows quickly enough:

$$q_{1+k_1} \geq 9, \tag{2}$$

$$q_{k_{n+1}} \geq 5q_{k_n},\ q_{1+k_{n+1}} \geq 5q_{1+k_n}, \tag{3}$$

$$\frac{1}{n}\log q_{k_n} \to \infty. \tag{4}$$

We may also assume that

$$\text{all } k_n \text{ are even or all are odd.} \tag{5}$$

For example, we can set $k_n := 4n^2+1$, because always $q_6 \geq \mathrm{Fib}_6 = 13$, $q_{n+4} \geq \mathrm{Fib}_4\, q_n = 5q_n$ and the sequences $q_{n^2} \geq \mathrm{Fib}_{n^2}$ grow superexponentially. Put additionally

$$A_n := \lfloor (3/4)^n q_{1+k_n} \rfloor > (3/4)^n q_{1+k_n} - 1 \quad \text{for } n \geq 1. \tag{6}$$

It follows that for $n \geq 2$ on the one hand

$$\begin{aligned}\frac{A_n q_{1+k_{n-1}}}{A_{n-1}q_{1+k_n}} &\overset{(6)}{>} \frac{((3/4)^n q_{1+k_n} - 1)\, q_{1+k_{n-1}}}{(3/4)^{n-1}q_{1+k_{n-1}}q_{1+k_n}} \\ &= \frac{3}{4} - \frac{(4/3)^{n-1}}{q_{1+k_n}} \overset{(3)}{\geq} \frac{3}{4} - \frac{4}{3q_{1+k_2}} \overset{(2),(3)}{>} \frac{18}{25},\end{aligned} \tag{7}$$

and on the other hand

$$\begin{aligned}\frac{A_{n-1} q_{1+k_n}}{A_n q_{1+k_{n-1}}} &\overset{(6)}{>} \frac{((3/4)^{n-1} q_{1+k_{n-1}} - 1)\, q_{1+k_n}}{(3/4)^n q_{1+k_n}q_{1+k_{n-1}}} \\ &= \frac{4}{3} - \frac{(4/3)^n}{q_{1+k_{n-1}}} \overset{(3)}{\geq} \frac{4}{3} - \frac{16}{9q_{1+k_1}} \overset{(2)}{>} 1.1,\end{aligned} \tag{8}$$

hence altogether

$$1.1 < \frac{q_{1+k_n}}{A_n} : \frac{q_{1+k_{n-1}}}{A_{n-1}} < 25/18. \tag{9}$$

This also proves that

$$q_{1+k_n}/A_n \text{ rises exponentially.} \tag{10}$$

For the sake of brevity, we will also denote

$$L_n := q_{k_n} q_{1+k_n}/n^2, \quad \text{for } n \geq 1.$$

We consider a modification of the example from [FrLe, Section 2]: we define f_n to be L_n-Lipschitz, $1/(A_n q_{k_n})$-periodic and even continuous function (hence also $f_n(\frac{1}{A_n q_{k_n}} - x) = f_n(x)$) by the formulas:

$$f_n(x) := \begin{cases} 0, & \text{for } 0 \leq x \leq \dfrac{1}{12 A_n q_{k_n}}, \\ L_n\left(x - \dfrac{1}{12 A_n q_{k_n}}\right), & \text{for } \dfrac{1}{12 A_n q_{k_n}} \leq x \leq \dfrac{5}{12 A_n q_{k_n}}, \\ \dfrac{q_{1+k_n}}{3 A_n n^2}, & \text{for } \dfrac{5}{12 A_n q_{k_n}} \leq x \leq \dfrac{1}{2 A_n q_{k_n}}. \end{cases}$$

By periodicity:

$$\begin{aligned} |f_n(x+\alpha) - f_n(x)| &= \left| f_n\left(x + \alpha - \frac{A_n p_{k_n}}{A_n q_{k_n}}\right) - f_n(x) \right| \\ &\leq L_n \left| \alpha - \frac{p_{k_n}}{q_{k_n}} \right| < \frac{q_{k_n} q_{1+k_n}}{n^2} \frac{1}{q_{k_n} q_{1+k_n}} = 1/n^2, \end{aligned}$$

so the series

$$\varphi(x) := \sum_{l=1}^{\infty} (f_l(x+\alpha) - f_l(x))$$

converges uniformly and yields a continuous cocycle of average zero. Moreover, it is easy to verify that for every $m \in \mathbb{Z}$:

$$\varphi^{(m)}(x) = \sum_{l=1}^{\infty} (f_l(x + m\alpha) - f_l(x)). \tag{11}$$

where $\varphi^{(m)}(x)$ is the second coordinate of $T_\varphi^m(x, 0)$. One can also check that $T_\varphi^m(x,t) = (T^m(x), t + \varphi^{(m)}(x))$ for every x and t.

2. Discrete orbits

Consider, for $n \geq 1$ and $j = 0, \dots, A_n q_{k_n} - 1$:

$$F_{n,j}^{++} := \left[-\frac{1}{12A_n q_{k_n}}, \frac{1}{12A_n q_{k_n}}\right] + \frac{j}{A_n q_{k_n}},$$
$$F_{n,j}^{-+} := \left[\frac{1}{6A_n q_{k_n}}, \frac{1}{3A_n q_{k_n}}\right] + \frac{j}{A_n q_{k_n}},$$
$$F_{n,j}^{--} := \left[\frac{5}{12A_n q_{k_n}}, \frac{7}{12A_n q_{k_n}}\right] + \frac{j}{A_n q_{k_n}} = F_{n,j}^{++} + \frac{1}{2A_n q_{k_n}},$$
$$F_{n,j}^{+-} := \left[\frac{2}{3A_n q_{k_n}}, \frac{5}{6A_n q_{k_n}}\right] + \frac{j}{A_n q_{k_n}} = F_{n,j}^{-+} + \frac{1}{2A_n q_{k_n}}$$

and for arbitrary $s_-, s_+ \in \{+, -\}$

$$F^{s_- s_+} := \bigcap_{n=1}^{\infty} \bigcup_{j=0}^{A_n q_{k_n} - 1} F_{n,j}^{s_- s_+}.$$

The sets $F^{s_- s_+}$ are nonempty and even uncountable: every interval $F_{n-1,j}^{s_- s_+}$ contains at least

$$\left\lfloor \frac{|F_{n-1,j}^{s_- s_+}|}{1/(A_n q_{k_n})} \right\rfloor - 1 = \left\lfloor \frac{A_n q_{k_n}}{6A_{n-1} q_{k_{n-1}}} \right\rfloor - 1 \tag{12}$$
$$= \left\lfloor \frac{1}{6} \frac{A_n q_{1+k_{n-1}}}{A_{n-1} q_{1+k_n}} \cdot \frac{q_{1+k_n}}{q_{1+k_{n-1}}} \cdot \frac{q_{k_n}}{q_{k_{n-1}}} \right\rfloor - 1 \overset{(7),(3)}{>} \lfloor \frac{1}{6} \cdot \frac{18}{25} \cdot 25 \rfloor - 1 = 2$$

of the intervals $F_{n,j}^{s_- s_+}$, since the intervals from the n-th union are uniformly distributed with period $1/q_{k_n}$; therefore the intersections $F^{s_- s_+}$ are topological Cantor sets.

We will now show that the products $F^{s_- s_+} \times \mathbb{R}$ consist of discrete points, which proves that T_φ is a Besicovitch cylinder.[3] More precisely, we will show that for every $x \in F^{s_- s_+}$:

- if $s_+ = s_-$, then
$$\varphi^{(m)}(x) \xrightarrow{m \to \pm\infty} s_+ \infty,$$
- if $s_+ \neq s_-$, then
$$\varphi^{(m)}(x) \xrightarrow{m \to \pm\infty} (-1)^{k_n} s_{\pm} \infty,$$
where the coefficient $(-1)^{k_n}$ is constant (cf. (5)).

Later, in the next section, we will verify that these sets are of full Hausdorff dimension.

[3] It is transitive, because the average $\int_{\mathbb{T}} \varphi \, d\lambda$ is zero and there is an unbounded orbit – see Introduction.

2.1. **The case of F^{++}.** Fix an element $x \in F^{++}$ and an integer $|m| > q_{1+k_1}/(3A_1)$. We wish to bound the summands $f_l(x+m\alpha) - f_l(x)$ from below. To this end, recall that x determines a sequence $(j_l)_{l=1}^{\infty}$ such that $x \in F^{++}_{l,j_l}$ for every $l \in \mathbb{N}$ and let x_l be given by $x_l := x - j_l/(A_l q_{k_l})$; then $|x_l| \leq 1/(12A_l q_{k_l})$. Now, by the properties of f_l:

$$\begin{aligned} f_l(x+m\alpha) - f_l(x) &= f_l(x_l + m\alpha) - f_l(x_l) = f_l(x_l + m\alpha) \\ &= f_l\left(x_l + m\alpha - \frac{mA_l p_{k_l}}{A_l q_{k_l}}\right) = f_l\left(x_l + m\left(\alpha - \frac{p_{k_l}}{q_{k_l}}\right)\right), \end{aligned} \tag{13}$$

which implies that:

$$f_l(x+m\alpha) - f_l(x) \geq 0. \tag{14}$$

Because of (10) there exists a unique $n = n(m)$ which satisfies

$$q_{1+k_{n-1}}/(2A_{n-1}) \leq |m| < q_{1+k_n}/(2A_n),$$

and when $|m|$ tends to infinity, so does $n(m)$. This helps us to estimate the n-th summand of $\varphi^{(m)}$:

- $m\left(\alpha - \frac{p_{k_n}}{q_{k_n}}\right) \overset{(1)}{<} \frac{q_{1+k_n}}{2A_n} \cdot \frac{1}{q_{k_n} q_{1+k_n}} = \frac{1}{2A_n q_{k_n}},$
- $m\left(\alpha - \frac{p_{k_n}}{q_{k_n}}\right) \overset{(1)}{>} \frac{q_{1+k_{n-1}}}{2A_{n-1}} \cdot \frac{1}{2q_{k_n} q_{1+k_n}} = \frac{1}{4A_n q_{k_n}} \cdot \frac{q_{1+k_{n-1}}}{A_{n-1}} \cdot \frac{A_n}{q_{1+k_n}} \overset{(7)}{>} \frac{1}{4A_n q_{k_n}} \cdot \frac{18}{25} = \frac{9}{50A_n q_{k_n}}.$

Therefore, owing to the bound for x_n,

$$\frac{1}{12} + \frac{1}{75} = \frac{1}{12} + \frac{9}{50} < \left(x_n + m\alpha - \frac{mp_{k_n}}{q_{k_n}}\right) A_n q_{k_n} < \frac{1}{12} + \frac{1}{2} < 1 - \frac{1}{12} - \frac{1}{75}.$$

This leads to the bound we seek, since f_n is symmetrical and unimodal on $[0, 1/(A_n q_{k_n})]$:

$$\begin{aligned} f_n\left(x_n + m\left(\alpha - \frac{p_{k_l}}{q_{k_l}}\right)\right) &> f_n\left(\frac{1}{12A_n q_{k_n}} + \frac{1}{75A_n q_{k_n}}\right) \\ &= L_n \cdot \frac{1}{75A_n q_{k_n}} = \frac{q_{1+k_n}}{75A_n n^2} \to \infty, \end{aligned}$$

and finally proves the required divergence:

$$\begin{aligned} \varphi^{(m)}(x) &\overset{(11)}{=} \sum_{l=1}^{\infty} (f_l(x+m\alpha) - f_l(x)) \overset{(14)}{\geq} f_{n(m)}(x+m\alpha) - f_{n(m)}(x) \\ &\overset{(13)}{=} f_n\left(x_n + m\left(\alpha - \frac{p_{k_n}}{q_{k_n}}\right)\right) \xrightarrow{|m|\to\infty} \infty. \end{aligned}$$

2.2. **The case of F^{--}.** The behaviour of functions f_n on the set $F^{--}_{n,j}$ is symmetrical to the situation on $F^{++}_{n,j}$, and the calculations are analogous.

2.3. **The case of F^{-+} and F^{+-}.** Choose an $x \in F^{-+} \cup F^{+-}$. Again, there is $(j_l)_{l=1}^{\infty}$ such that $x \in F^{-+}_{l,j_l} \cup F^{+-}_{l,j_l}$ for every $l \in \mathbb{N}$, and we denote by x_l the respective "reductions" $x - j_l/(A_l q_{k_l})$; then

$$x_l \in \left[\frac{1}{6A_l q_{k_l}}, \frac{1}{3A_l q_{k_l}}\right] \ (s_+ = +) \text{ or } x_l \in \left[\frac{2}{3A_l q_{k_l}}, \frac{5}{6A_l q_{k_l}}\right] \ (s_+ = -).$$

Additionally, fix an integer $|m| > q_{1+k_1}/(12A_1)$. It follows from the periodicity of f_l that

$$\begin{aligned} f_l(x+m\alpha) - f_l(x) &= f_l(x_l + m\alpha) - f_l(x_l) \\ &= f_l(x_l + m(\alpha - p_{k_l}/q_{k_l})) - f_l(x_l), \end{aligned} \tag{15}$$

We remind that $\operatorname{sign}(\alpha - p_{k_l}/q_{k_l}) \overset{(1)}{=} (-1)^{k_l} \overset{(5)}{=} (-1)^{k_1}$. Take now $n = n(m) \geq 1$ for which

$$q_{1+k_n}/(12A_n) \leq |m| < q_{1+k_{n+1}}/(12A_{n+1}).$$

These constraints along with the inequalities (1) imply that for $l > n$

$$\left| m\left(\alpha - \frac{p_{k_l}}{q_{k_l}}\right)\right| \overset{(1)}{<} \frac{q_{1+k_{n+1}}}{12A_{n+1}} \cdot \frac{1}{q_{k_l} q_{1+k_l}} \overset{(9)}{\leq} \frac{q_{1+k_l}}{12A_l} \cdot \frac{1}{q_{k_l} q_{1+k_l}} = \frac{1}{12A_l q_{k_l}}.$$

Therefore both arguments $x_l + m(\alpha - p_{k_l}/q_{k_l})$ and x_l lie in the same interval of linearity (and monotonicity) of f_l, so the sign of the difference (15) equals $(-1)^{k_1} s_+ \operatorname{sign} m$ and the expression (15) can be estimated:

$$\begin{aligned} \left| f_l\left(x_l + m\left(\alpha - \frac{p_{k_l}}{q_{k_l}}\right)\right) - f_l(x_l)\right| &= L_l \left| m\left(\alpha - \frac{p_{k_l}}{q_{k_l}}\right)\right| \\ &\overset{(1)}{>} \frac{q_{k_l} q_{1+k_l}}{l^2} \cdot \frac{q_{1+k_n}}{12A_n} \cdot \frac{1}{2q_{k_l} q_{1+k_l}} = \frac{q_{1+k_n}}{24A_n l^2}. \end{aligned}$$

Since all these differences are of the same sign, this yields an estimate for the part of the sum (11) with $l > n$:

$$\left| \sum_{l>n} \left(f_l\left(x_l + m\left(\alpha - \frac{p_{k_l}}{q_{k_l}}\right)\right) - f_l(x_l)\right)\right| > \frac{q_{1+k_n}}{24A_n} \sum_{l>n} \frac{1}{l^2} \overset{(\star)}{>} \frac{q_{1+k_n}}{25A_n n}, \tag{16}$$

where the inequality $(\star)$ holds for n large enough, which results form the fact that the remainder $\sum_{l>n} 1/l^2$ is asymptotically equivalent to $1/n$ (thus greater than $24/(25n)$ for large n).[4]

[4]This follows from the termwise equivalence to a telescoping series of $1/n$:

$$\textstyle\sum_{l \geq n+1} \frac{1}{(l+1)^2} - \sum_{l \geq n+2} \frac{1}{(l+1)^2} = 1/(n+1)^2 \approx (1/n) - 1/(n+1),$$

and from an analogue of the Stolz-Cesàro Theorem.

As it occurs, we do not have to work hard to take the remaining summand into account – it suffices to subtract the upper bounds of the functions f_l:

$$\left|\sum_{l\le n}(f_l(x_l+m\alpha)-f_l(x_l))\right| \le \sum_{l\le n} 2\max_{x\in\mathbb{T}} f_l = \frac{2}{3}\sum_{l\le n}\frac{q_{1+k_l}}{A_l l^2} \tag{17}$$

Note that this sum behaves roughly like the sum of a finite geometric series: since q_{1+k_l}/A_l grows exponentially and, asymptotically, l^2 grows slower, the quotient for large l also grows exponentially, say:

$$\frac{q_{1+k_l}}{A_l l^2} \ge C\frac{q_{1+k_{l-1}}}{A_{l-1}(l-1)^2} \quad \text{for some } C>1 \text{ and } l \text{ large enough}$$

(eg. when $l^2/(l-1)^2 < 1.1/C$). Then, indeed, the sum (17) is of order of its largest term, and therefore we arrive at a satisfactory bound:

$$\frac{2}{3}\sum_{l\le n}\frac{q_{1+k_l}}{A_l l^2} \le \frac{2}{3}\sum_{l\le n}\frac{1}{C^{n-l}}\cdot\frac{q_{1+k_n}}{A_n n^2} < \frac{2}{3}\cdot\frac{C}{C-1}\cdot\frac{q_{1+k_n}}{A_n n^2} \overset{(\star\star)}{<} \frac{q_{1+k_n}}{50A_n n}, \tag{18}$$

where the inequality $(\star\star)$ also holds for large n. Combining the estimations (16), (17) and (18), we eventually obtain the required divergence:

$$|\varphi^{(m)}(x)| \overset{(11),(15)}{=\!=} \left|\sum_{l\ge 1}\left(f_l\left(x_l+m\left(\alpha-\frac{p_{k_l}}{q_{k_l}}\right)\right)-f_l(x_l)\right)\right|$$

$$\ge \left|\sum_{l>n(m)}\cdots\right| - \left|\sum_{l\le n(m)}\cdots\right| > \frac{q_{1+k_n}}{25A_n n}-\frac{q_{1+k_n}}{50A_n n} = \frac{q_{1+k_n}}{50A_n n}\xrightarrow[(10)]{m\to\pm\infty}\infty.$$

Also, the sign of $\varphi^{(m)}$ is correct, because the prevailing part has correct sign.

Remark 2.1. Observe that the calculations for F^{+-} and F^{-+} (in this and previous section) do not require all the assumptions on k_n and A_n that we made initially. Actually, we only need that k_n are all of the same parity, q_{1+k_n}/A_n grows at least geometrically, and $A_n q_{k_n} \ge 18A_{n-1}q_{k_{n-1}}$. In particular, the restriction for the growth of q_{1+k_n}/A_n (as in (7)) is redundant – for example, we may put $A_n := 1$ for every n (then we have to ensure the inequality $q_{k_n} \ge 18q_{k_{n-1}}$). Moreover, we do not use the pieces of constant value of the functions f_n. Summarizing, the sets F^{+-} and F^{-+} also consist of discrete points in the following example from [FrLe, Section 2]:

$$\varphi(x) := \sum_{n=1}^{\infty}(g_n(x+\alpha)-g_n(x))$$

where g_n are L_n-Lipschitz, $1/q_{k_n}$-periodic continuous functions:

$$g_n(x) := \begin{cases} L_n x, & \text{for } 0 \leq x \leq \frac{1}{2q_{k_n}}, \\ L_n\left(\frac{1}{q_{k_n}} - x\right), & \text{for } \frac{1}{2q_{k_n}} \leq x \leq \frac{1}{q_{k_n}}. \end{cases} \tag{19}$$

and $q_{k_n} \geq 18 q_{k_{n-1}}$ (this coefficient can be decreased by widening $F^{s_- s_+}_{n,j}$ appropriately).

3. Hausdorff dimension of $F^{s_- s_+}$

To calculate the Hausdorff dimension of $F^{s_- s_+}$, we will use methods from [Fa] (Example 4.6 and Proposition 4.1):
Consider a sequence of unions of a finite number of disjoint closed intervals in $[0, 1)$ (here: the sequence $(\bigcup_{j=0}^{A_n q_{k_n}-1} F^{s_- s_+}_{n,j})_{n \geq 1}$). *Suppose that the intervals of the n-th union ($n \geq 1$)*

- *are of length at most δ_n and $\delta_n \to 0$,*
- *are separated by gaps of length at least ε_n (with $\varepsilon_n > \varepsilon_{n+1} > 0$),*
- *contain at least $m_{n+1} \geq 2$ and at most $\bar{m}_{n+1}$ intervals of the $(n+1)$-st union.*

Then the Hausdorff dimension of the intersection of this sequence lies between the following two numbers:

$$\liminf_{n\to\infty} \frac{\log(m_2 \cdots m_n)}{-\log(m_{n+1}\varepsilon_{n+1})} \leq \liminf_{n\to\infty} \frac{\log(\bar{m}_2 \cdots \bar{m}_n)}{-\log \delta_{n+1}}.$$

First, note that $\delta_n = |F^{s_- s_+}_{n,j}| = 1/(6A_n q_{k_n}) \to 0$. Next, observe that

$$\varepsilon_n = \frac{1}{A_n q_{k_n}} - |F^{s_- s_+}_{n,j}| > \frac{1}{A_n q_{k_n}} - \frac{1}{6A_n q_{k_n}} > \frac{1}{2A_n q_{k_n}}.$$

As for m_n and $\bar{m}_n$, we have already checked that $m_n \geq 2$ (see (12)), but we need a more precise estimate. Using the inequality $\lfloor t \rfloor - 1 > t/2$ for $t \geq 3$ we conclude that:

$$m_n \geq \left\lfloor \frac{|F^{s_- s_+}_{n-1,j}|}{1/A_n q_{k_n}} \right\rfloor - 1 = \left\lfloor \frac{A_n q_{k_n}}{6A_{n-1} q_{k_{n-1}}} \right\rfloor - 1 \geq \frac{A_n q_{k_n}}{12A_{n-1} q_{k_{n-1}}}.$$

On the other hand, only one more interval can fit into:

$$\bar{m}_n \leq \left\lfloor \frac{|F^{s_- s_+}_{n-1,j}|}{1/A_n q_{k_n}} \right\rfloor \leq \frac{A_n q_{k_n}}{6A_{n-1} q_{k_{n-1}}}.$$

Consequently:

$$\begin{aligned}
m_2 \cdots m_n &\geq \frac{A_2 q_{k_2}}{12 A_1 q_{k_1}} \cdots \frac{A_n q_{k_n}}{12 A_{n-1} q_{k_{n-1}}} = \frac{A_n q_{k_n}}{12^{n-1} A_1 q_{k_1}}, \\
m_{n+1} \varepsilon_{n+1} &\geq \frac{1}{12} \frac{A_{n+1} q_{k_{n+1}}}{A_n q_{k_n}} \cdot \frac{1}{2 A_{n+1} q_{k_{n+1}}} = \frac{1}{24 A_n q_{k_n}}, \\
\bar{m}_2 \cdots \bar{m}_n &\leq \frac{A_n q_{k_n}}{6^{n-1} A_1 q_{k_1}},
\end{aligned}$$

hence eventually

$$\begin{aligned}
\dim_H F^{s_- s_+} &\geq \liminf_{n\to\infty} \frac{\log A_n q_{k_n} - (n-1)\log 12 - \log A_1 q_{k_1}}{\log A_n q_{k_n} + \log 24} \\
&= 1 - \liminf_{n\to\infty} \frac{n}{\log A_n q_{k_n}} \cdot \log 12, \\
\dim_H F^{s_- s_+} &\leq \liminf_{n\to\infty} \frac{\log A_n q_{k_n} - (n-1)\log 6 - \log A_1 q_{k_1}}{\log A_n q_{k_n} + \log 6} \\
&= 1 - \liminf_{n\to\infty} \frac{n}{\log A_n q_{k_n}} \cdot \log 6.
\end{aligned}$$

Let us remark that the coefficient 12 can be lowered nearly to 6, if m_n are larger. Nevertheless, under the assumption (4) the dimension equals 1.

4. Discrete Devaney chaos

The sole property of transitivity is enough for some dynamicists to call a dynamical system chaotic. However, over the years multiple definitions for chaos have been proposed. Let us recall the notion of the Devaney chaos, one of the most popular ones: a dynamical system (X, T) on a metric space (X, d) is *chaotic in the sense of Devaney* if:

1. it is transitive,
2. the set of periodic points is dense,
3. the system is *sensitive*, i.e there are points around every point $x \in X$ (arbitrarily close) whose orbits at least once diverge far enough from the orbit of x: there is $\varepsilon > 0$ such that for every $x \in X$ and $\delta > 0$ there are $n > 0$ and y with $d(x, y) < \delta$ and $d(T^n(x), T^n(y)) > \varepsilon$.

We remind that the the last condition follows from the remaining ones, if X is infinite ([BaBr], main theorem, or [GlWe], Corollary 1.4).

It occurs that the dynamical systems we consider in this article satisfy a bit more general condition, namely, with "periodic orbits" replaced by "discrete orbits" (note that both notions are equivalent in compact spaces). We will call this property *discrete Devaney chaos* and check this fact in a moment. A similar generalization was proposed in [GlWe], with "almost periodic" instead of "periodic" (that is, contained in a minimal set) and it was shown that this, combined with transitivity, implies sensitivity, if X is compact. Note also, that there are no periodic points in cylinders over minimal rotations, so they cannot be Devaney chaotic.

Recall first that a space or a set is *boundedly compact* if bounded closed subsets are always compact.[5] In particular, closed subsets of Euclidean spaces are boundedly compact. All such spaces are complete and separable. Also, a system is *maximally sensitive*, if it sensitive with every $\varepsilon < \operatorname{diam}(X)/2$, and *maximally chaotic*, if it is Devaney chaotic and maximally sensitive (definitions introduced in [AlPr]).

Theorem 4.1. *Let X be an infinite, boundedly compact space without isolated points, and let T be transitive with dense set of discrete points. Then the system is sensitive. If, moreover, the set of discrete nonperiodic points is dense, then the system is maximally sensitive.*

Proof. Since X is complete, separable and without isolated points, the system is even positively transitive (it has a dense semi-orbit – see [Ox], p. 70, or [Dy1], Proposition 2.1). The set of discrete points consists of periodic points and nonperiodic discrete points, both of which are invariant. Thus, one of this sets contains a positively transitive point in its closure, and so it is dense. If periodic points are dense, then, by [BaBr] or [GlWe], the system is sensitive. The new result is when the second set is dense, what we assume henceforth.

Any infinite (= nonperiodic) discrete orbit, by bounded compactness, has no bounded subsequence, so $\operatorname{diam}(X) = \infty$. Fix then any $x \in X$, $\varepsilon > 0$ and $\delta > 0$. In the δ-neighbourhood of x there is a transitive point y_1 and a discrete nonperiodic point y_2. Then, for infinitely many n the orbit of y_1 returns to x: $d(T^n(y_1), x) < \varepsilon$, and on the other hand, for n large enough the orbit of y_2 stays far away from x: $d(T^n(y_2), x) > 3\varepsilon$. Consequently, for some n: $d(T^n(y_1), T^n(y_2)) > 2\varepsilon$, and hence $d(T^n(x), T^n(y_1)) > \varepsilon$ or $d(T^n(x), T^n(y_2)) > \varepsilon$. □

Remark 4.2. The Besicovitch cylinders that we consider are of course transitive and have a dense set of discrete points (we have found discrete points in $F^{s_- s_+} \times \mathbb{R}$, but its orbits are dense). Therefore, there are examples of maximally discretely chaotic systems with full-dimensional

[5] Such spaces are also given other names in the literature: they are called *proper*, *finitely compact*, *totally complete*, *Heine-Borel* or having the *Heine-Borel property* (not to be confused with the Heine-Borel [covering] property, or precompactness, that is, "every open cover has a finite subcover").

set of relatively "regular" (almost periodic, discrete) points. This feature seems not to be studied so far. However, there are results about full Hausdorff dimension of the set of points with nondense orbits, although they are rather concerned with bounded orbits – see e.g [Kl], [Ur].

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