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A factorization theorem for maps on Bing spaces and transfinite dimensions $tr\,Ind_m$

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A FACTORIZATION THEOREM FOR MAPS ON BING SPACES AND TRANSFINITE DIMENSIONS $trInd_m$

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Abstract. We prove a factorization theorem for continuous maps on Bing spaces with $trInd_m = \alpha$, where $trInd_m$ are transfinite dimensions, recently investigated by V. V. Fedorchuk ($trInd_2$ is the classical large transfinite dimension). Our approach is based on Wallman representations of lattices of closed sets in compact spaces, which also simplifies proofs of some factorization theorems in the literature.

1. Introduction

The transfinite dimension trInd is a transfinite extension of the classical large inductive dimension, cf.[4]

Recently, V.V. Fedorchuk, [6], [7], refined this notion, introducing transfinite dimensions $trInd_m$ for $m = 2, 3, ...$ ($trInd = trInd_2$).

In particular, Fedorchuk, [6] section 3, showed that the classical factorization theorem for $trInd$ can be extended to the dimensions $trInd_m$.

A Bing space is a compact Hausdorff space whose every component is a hereditarily indecomposable continuum, cf. [12] section 3.8 and [2].

The main result of this paper is the following theorem, where $w(S)$ denotes the weight of the space S, cf. [5].

Theorem 1.1. *Let $f : X \to Z$ be a continuous map of a Bing space X onto a compact space Z, let $F \subset X$ be a compact set and $m \geq 2$ a natural number. Then there is a Bing space Y and continuous surjections $g : X \to Y$, $h : Y \to Z$ such that $f = h \circ g$, $w(Y) \leq w(Z)$ and $trInd_m f(F) \leq trInd_m(F)$.*

The proofs in the literature of the factorization theorems for $trInd$ or $trInd_m$ are rather involved and they are based on inverse system methods, cf. [1], [6]

Our approach, motivated by [8], [9], [10] and [14], exploits Wallman representations of lattices of closed sets in compact spaces, and the way we use this technique was strongly influenced by a lecture of E.Pol on her joint work with K.P.Hart [10].

It is our feeling that this approach provides simpler proofs of factorization theorems in the literature concerning large transfinite dimensions and continuous maps of compact spaces. Its flexibility is also very convenient to deal simultaneously with Bing spaces.

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The Wallman representation and its links with factorization of continuous maps on compact spaces is discussed in section 2.

The transfinite dimensions $trInd_m$ for compact spaces and also their counterparts for lattices of closed sets in compact spaces, are discussed in section 3, and in section 4 we recall an important characterization of Bing spaces given by Krasinkiewicz and Minc [11], which is vital for our reasonings.

The proof of Theorem 1.1, given in section 6, is based on some results on extension of sublattices of closed sets in compact spaces, presented in section 5.

The results of this paper for transfinite dimension $trInd = trInd_2$ were obtained in the author's Master Thesis presented to the University of Warsaw [13]. In the last section, we shall outline also some other results from this Thesis.

2. Wallman representations of lattices of closed sets in compact spaces

Let X be a compact space. A family 2^X of all closed subsets of X, with operations $\cap$ and $\cup$, is a distributive lattice, and the inclusion relation defines a partial order $\leq$:

$$A \leq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B \Leftrightarrow A \subset B.$$

The empty set $\emptyset$ is the smallest element in the lattice 2^X and X is its biggest element.

Any family $\mathcal{C} \subset 2^X$ closed under $\cap$ and $\cup$ and containing $\emptyset$ and X will be called a *sublattice*.

For a given sublattice $\mathcal{C} \subset 2^X$, a subfamily $\mathcal{F} \subset \mathcal{C}$ is called a *filter* in $\mathcal{C}$ whenever for any $A, B \in \mathcal{F}$ and $C \in \mathcal{C}$ the following conditions are met:

(1) $\emptyset \notin \mathcal{F}$
(2) $A \subset C \Rightarrow C \in \mathcal{F}$
(3) $A \cap B \in \mathcal{F}$

By an *ultrafilter (a maximal filter)* in $\mathcal{C}$ we mean a filter $\mathcal{F}$ which has the following property:

(1) for any filter $\mathcal{F}'$ in $\mathcal{C}$ such that $\mathcal{F} \subset \mathcal{F}'$ we have $\mathcal{F} = \mathcal{F}'$.

For any filter $\mathcal{F} \subset \mathcal{C}$ there exists an ultrafilter $\mathcal{F}' \subset \mathcal{C}$ such that $\mathcal{F} \subset \mathcal{F}'$.

Let us recall the following useful fact.

Lemma 2.1. *Let $\mathcal{F}$ be an ultrafilter in a lattice $\mathcal{C} \subset 2^X$ for some compact space X. Then for any set $A \in \mathcal{C}$ we have:*

$$A \in \mathcal{F} \Leftrightarrow \forall_{B \in \mathcal{F}} A \cap B \neq \emptyset. \tag{2.1}$$

Proof. Implication from the left to the right is an obvious consequence of the definition of filters.

Suppose that there is a set $A \notin \mathcal{C} \setminus \mathcal{F}$ such that $\forall_{B \in \mathcal{F}} A \cap B \neq \emptyset$. Then

$$\mathcal{F}' = \mathcal{F} \cup \{A \cap B : B \in \mathcal{F}\}$$

is a filter in $\mathcal{C}$, such that $\mathcal{F} \subset \mathcal{F}'$ and $A \in \mathcal{F}' \setminus \mathcal{F}$, which contradicts with maximality of $\mathcal{F}$. $\square$

For any sublattice $\mathcal{C} \subset 2^X$ we shall consider a space

$$w\mathcal{C} = \{\mathcal{F} \subset \mathcal{C} : \mathcal{F} \text{ is an ultrafilter}\}$$

of all ultrafilters on $\mathcal{C}$, with base of closed sets generated by sets

$$A^* = \{\mathcal{F} \in w\mathcal{C} : A \in \mathcal{F}\}, \tag{2.2}$$

for all $A \in \mathcal{C}$. The space $w\mathcal{C}$ is the *Wallman representation* of the lattice $\mathcal{C}$.

For any pair of lattices $\mathcal{C} \subset \mathcal{F} \subset 2^X$ we define conditions $D(\mathcal{C}, \mathcal{F})$ and $(*)(\mathcal{C}, \mathcal{F})$ as follows:

(1) We say that condition $D(\mathcal{C}, \mathcal{F})$ is met, if for every pair of different sets $A, B \in \mathcal{C}$ there is a set $C \in \mathcal{F}$ such that exactly one of the intersections $A \cap C$, $B \cap C$ is empty.
(2) We say that condition $(\star)(\mathcal{C}, \mathcal{F})$ is met, if for every pair of disjoint sets $A, B \in \mathcal{C}$ there are sets $C, D \in \mathcal{F}$ such that $C \cup D = X$, $A \cap C = \emptyset = B \cap D$.

If a lattice $\mathcal{C}$ meets condition $D(\mathcal{C}, \mathcal{C})$, it is called *disjunctive*, and if it satisfies condition $(*)(\mathcal{C}, \mathcal{C})$ it is called *normal.* It is easily seen, that for any compact space X lattice 2^X is disjunctive and normal.

Wallman showed in [15], that for any sublattice $\mathcal{C} \subset 2^X$ with compact X, if $\mathcal{C}$ is disjunctive and normal, then its Wallman representation $w\mathcal{C}$ is a compact space, and the function $f : \mathcal{C} \to w\mathcal{C}$ defined by

$$f(A) = A^*,$$

see (2.2), is an isomorphism onto a lattice which forms a base of closed sets in $w\mathcal{C}$.

If $f : X \to Z$ is a map and $\mathcal{A} \subset 2^Z$, we write

$$f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}.$$

The following fact concerning factorization of maps between compact spaces will be useful in the sequel.

Lemma 2.2. *Let $f : X \to Z$ be a continuous map between compact spaces. Let $\mathcal{A}$ be a base of closed subsets of Z and let $\mathcal{C}$ be a normal sublattice of 2^X such that $f^{-1}(\mathcal{A}) \subset \mathcal{C}$. Then there are continuous maps $g : X \to w\mathcal{C}$ and $h : w\mathcal{C} \to Z$ such that $f = h \circ g$ and $g(X) = w\mathcal{C}$.*

Proof. For every $x \in X$ we let

$$\mathcal{C}_x = \{A \in \mathcal{C} : x \in A\}. \tag{2.3}$$

Then $\mathcal{C}_x$ is a filter in $\mathcal{C}$, and hence $\mathcal{C}_x$ can be extended to an ultrafilter $\mathcal{F}$ on $\mathcal{C}$, which is element of $w\mathcal{C}$ and we define $g(x) = \mathcal{F}$.

In effect, we have

$$\mathcal{C}_x \subset g(x),\ g(x) \in w\mathcal{C} \text{ for every } x \in X. \tag{2.4}$$

Let us check, that g is a surjection. Indeed, if $\mathcal{F} \in w\mathcal{C}$, then there exists $x \in \cap\mathcal{F}$, cf. [13]. Because $\mathcal{F}$ is an ultrafilter, we have $\mathcal{F} = \mathcal{C}_x$, and hence $g(x) = \mathcal{F}$.

To see that g is continuous let us take $x \in X$ and let for some $A \in \mathcal{C}$,

$$\mathcal{V} = \{\mathcal{F} \in w\mathcal{C} : A \notin \mathcal{F}\}$$

be a base neighborhood of $g(x)$. It means, that $A \notin g(x)$, so there is $B \in g(x)$ such that $A \cap B = \emptyset$, see Lemma 2.1. The lattice $\mathcal{C}$ is normal, and we can find sets $C, D \in \mathcal{C}$, such that $X = C \cup D, A \cap C = \emptyset = B \cap D$. From (2.4), $\mathcal{C}_x \subset g(x)$, hence $x \notin D$, and $V = X \setminus D$ is a neighborhood of x. For every $y \in V$ we have $y \in C$, so $C \in \mathcal{C}_y \subset g(y)$, and we infer that $A \notin g(y)$. This means that $g(V) \subset \mathcal{V}$.

Similarily, for any open set $\mathcal{V}$ in $w\mathcal{C}$ containing $g(x)$, one can find in X an open set V containing x such that $g(V) \subset \mathcal{V}$. This proves the continuity of g.

To define h let us notice that $f^{-1}(\mathcal{A}) \subset \mathcal{C}$, so for every $\mathcal{F} \in w\mathcal{C}$, $\bigcap\{f(A) : A \in \mathcal{F}\}$ is a singleton. We set

$$\{h(\mathcal{F})\} = \bigcap\{f(A) : A \in \mathcal{F}\}, \text{ for } \mathcal{F} \in w\mathcal{C}. \tag{2.5}$$

For every $x \in X$, we have $f^{-1}(f(x)) = \bigcap\{f^{-1}(A) : A \in \mathcal{A}, f(x) \in A\}$, hence

$$\{f(x)\} = \bigcap\{f(A) : A \in \mathcal{C}_x\} \tag{2.6}$$

and from (2.3), (2.4), (2.5) and (2.6) we get

$$\{h(g(x)))\} = \bigcap\{f(A) : A \in g(x)\} = \{f(x)\},$$

i.e., $h(g(x)) = f(x)$.

Now, for any closed set $F \subset Z$, $h^{-1}(F) = g(f^{-1}(F))$ and from continuity of f and g we infer that $h^{-1}(F)$ is compact, hence closed in $w\mathcal{C}$

We checked that h is continuous, which ends the proof of the lemma. □

Remark 2.3. *Notice, that the map g defined in Lemma 2.2 has the following property: for every $A \in \mathcal{C}$ we have $g(A) = A^*$, cf. (2.2).*

3. Transfinite dimensions $trInd_m$ and lattices of closed sets in compact spaces

Let X be a compact space, and let $A_1, ..., A_m$ be pairwise disjoint closed subsets of X. A *partition* between sets $A_1, ..., A_n$ is any set $X \setminus \bigcup_{i=1}^{n} U_i$, where U_i are parwise disjoint open sets such that $A_i \subset U_i$ fo every $i = 1, 2, ..., n$.

To every compact space X we can assign the following transfinite inductive dimension of X with parameter $m \in \mathbb{N}, m \geq 2$, as follows:

(1) $trInd_m X = -1$ iff $X = \emptyset$
(2) $trInd_m X \leq \alpha$ iff for every pairwise disjoint closed sets $A_1, A_2, ..., A_m$ in X there is a partition L in X between these sets, such that $trInd_m L < \alpha$.
(3) $trInd_m X = \alpha$ iff $trInd_m X \leq \alpha$ and for every $\beta < \alpha$ $trInd_m X > \beta$

It is easily seen that for every compact space X and $n > m \geq 2$ we have $trInd_n X \leq \alpha \Rightarrow trInd_m X \leq \alpha$ i.e., $trInd_m X \leq trInd_n X$. For $m = 2$, $trInd_2$ coincides with the classical large transfinite dimension $trInd$.

For any sublattice $\mathcal{C} \subset 2^X$ and $A \in 2^X$

$$\mathcal{C}|A = \{A \cap B : B \in \mathcal{C}\} \tag{3.1}$$

is a sublattice of 2^A - the *trace* of $\mathcal{C}$ on A.

We shall consider a counterpart of $trInd_m$ for sublattices of the lattices 2^X.

Definition 3.1. *Let $\mathcal{C} \subset \mathcal{F}$ be sublattices of 2^X for some compact space X.*

(A) *Property $trInd_m(\mathcal{C}, \mathcal{F}) = 0$ means that for every pairwise disjoint closed sets $A_1, ..., A_m \in \mathcal{C}$ there are sets $C_1, ..., C_m \in \mathcal{F}$, such that $A_i \subset X \setminus C_i$, $C_i \cup C_j = X$ for any pair of distinct indices i, j, and $\bigcap_{i=1}^{m} C_i = \emptyset$.*

(B) *Property $trInd_m(\mathcal{C}, \mathcal{F}) \leq \alpha$, with ordinal number $\alpha > 0$ means, that for every pairwise disjoint sets $A_1, ..., A_m \in \mathcal{C}$ there are sets $C_1, ..., C_m \in \mathcal{F}$, such that $A_i \subset X \setminus C_i$, $C_i \cup C_j = X$ for any pair of distinct indices i, j, and $trInd(\mathcal{C}|\bigcap_{i=1}^{m} C_i, \mathcal{F}|\bigcap_{i=1}^{m} C_i) < \alpha$, see* (3.1)*.*

It is easily seen that $trInd_m X \leq \alpha \Leftrightarrow trInd_m(2^X, 2^X) \leq \alpha$.

Lemma 3.2. *For any compact space X and a sublattice $\mathcal{C} \subset 2^X$ we have:*

$$trInd_m(\mathcal{C}, \mathcal{C}) \leq \alpha \Rightarrow trInd_m w\mathcal{C} \leq \alpha. \tag{3.2}$$

Proof. We shall proceed by induction on α.

Let $trInd_m(\mathcal{C}, \mathcal{C}) = 0$. Let us take any pairwise disjoint closed sets $A_1, ..., A_m \subset w\mathcal{C}$. Family $\{A^* : A \in \mathcal{C}\}$ is a base of closed sets in $w\mathcal{C}$, so we can find pairwise disjoint sets $B_1, ..., B_m \in \mathcal{C}$, such that $A_i \subset B_i^*$ for all $i = 1, ..., m$, and by $trInd_m(\mathcal{C}, \mathcal{C}) = 0$ we can assume, extending B_i if needed, that $\bigcup_{i=1}^{m} B_i = X$. Then the sets B_i^* are pairwise disjoint, and $\bigcup_{i=1}^{m} B_i^* = w\mathcal{C}$, so $trInd_m w\mathcal{C} = 0$.

Now assume that for some α, and all sublattices $\mathcal{C}'$ of 2^X, we have

$$trInd_m(\mathcal{C}', \mathcal{C}') < \alpha \Rightarrow trInd_m w\mathcal{C}' < \alpha,$$

and let $trInd_m(\mathcal{C}, \mathcal{C}) = \alpha$, for some sublattice $\mathcal{C}$ of 2^X.

Let us take any pairwise disjoint closed sets $A_1, ..., A_m \subset w\mathcal{C}$. The family $\{A^* : A \in \mathcal{C}\}$ is a base of closed sets in $w\mathcal{C}$, so we can find pairwise disjoint sets $B_1, ..., B_m \in \mathcal{C}$, such that $A_i \subset B_i^*$ for all $i = 1, ..., m$.

Since $trInd_m(\mathcal{C}, \mathcal{C}) = \alpha$, there are sets $C_1, ..., C_m \in \mathcal{C}$ such that $B_i \subset X \setminus C_i$, $C_i \cup C_j = X$ for any pair of distinct indices i, j, and $trInd\left(\mathcal{C}|\bigcap_{i=1}^{m} C_i, \mathcal{C}|\bigcap_{i=1}^{m} C_i\right) < \alpha$. Then $C_i^* \cup C_j^* = w\mathcal{C}$ for any pair of distinct indices i, j, and $A_i \subset w\mathcal{C} \setminus C_i^*$, so $L = \bigcap_{i=1}^{m} C_i^*$ is a partition between $A_1, ..., A_m$.

But $L = \left(\bigcap_{i=1}^{m} C_i\right)^*$, cf. (2.2), so L is the Wallman representation of the lattice $\mathcal{C}|\bigcap_{i=1}^{m} C_i$, and by the induction assumption we get $trInd_m L < \alpha$. Therefore $trInd_m w\mathcal{C} \leq \alpha$. □

4. Krasinkiewicz-Minc characterization of Bing spaces

We will say that a continuum is *decomposable* if it is a union of two its proper subcontinua; otherwise a continuum is *indecomposable*. We will say that a continuum is *hereditarily indecomposable* if its every subcontinuum is indecomposable, or equivalently, whenever two of its subcontinuua meet, one is contained in the other. A space, whose every component is an indecomposable continuum, is called a *Bing continuum* or a *Bing space*.

The following useful characterization of Bing spaces was introduced by Krasinkiewicz and Minc in [11]

Theorem 4.1. *A compact space X is a Bing space iff it has the following (KM) property:*

> *For any disjoint closed sets $A, B \subset X$ and open sets $U, V \subset X$, such that $A \subset U$ and $B \subset V$ there are closed sets X_0, X_1, X_2 in X, such that $X = X_0 \cup X_1 \cup X_2$, $A \subset X_0$, $B \subset X_2$, $X_0 \cap X_1 \subset V$, $X_1 \cap X_2 \subset U$ and $X_0 \cap X_2 = \emptyset$.*

Krasinkiewicz and Minc considered only metrizable spaces, but their arguments can be extended directly to all compact spaces, see [10].

Property (KM) can be expressed in a language of lattices of closed sets in a space X:

Definition 4.2. *Let $\mathcal{C} \subset \mathcal{F} \subset 2^X$ be sublattices of 2^X. Then condition $(KM)(\mathcal{C}, \mathcal{F})$ is met, if for any disjoint closed sets $A, B \in \mathcal{C}$ and closed sets $C, D \in \mathcal{C}$ such that $C \cup D = X, A \subset C \setminus D, B \subset D \setminus C$, there are sets $X_1, X_2, X_3 \in \mathcal{F}$, such that $X = X_0 \cup X_1 \cup X_2$, $A \subset X_0$, $B \subset X_2$, $X_0 \cap X_1 \subset D \setminus C$, $X_1 \cap X_2 \subset C \setminus D$ and $X_0 \cap X_2 = \emptyset$. We will call such triple (X_1, X_2, X_3) a fold in $\mathcal{F}$ for the quadruple (A, B, C, D).*

Let us notice that by Theorem 4.1, a space X has property (KM) iff $(KM)(2^X, 2^X)$ holds true.

5. Some extension results for lattices of closed sets in compact spaces

To prove the main theorem in section 6 we will need two lemmas concerning lattices of closed sets, cf. Section 2.

Lemma 5.1. *Let X be a Bing space and $\mathcal{D} = 2^X$. Let $\mathcal{B} \subset \mathcal{D}$ be a sublattice with cardinality $|\mathcal{B}| = \kappa$. Then one can extend $\mathcal{B}$ to a lattice $\mathcal{C}$ such that $\mathcal{B} \subset \mathcal{C} \subset \mathcal{D}$, $|\mathcal{C}| = \kappa$ and conditions $(\star)(\mathcal{B}, \mathcal{C})$, $D(\mathcal{B}, \mathcal{C})$ and $(KM)(\mathcal{B}, \mathcal{C})$ are met.*

Proof. Since X is a Bing space, conditions $(\star)(\mathcal{D}, \mathcal{D})$, $D(\mathcal{D}, \mathcal{D})$ and $(KM)(\mathcal{D}, \mathcal{D})$ are satisfied, and hence, also conditions $(\star)(\mathcal{B}, \mathcal{D})$, $D(\mathcal{B}, \mathcal{D})$ and $(KM)(\mathcal{B}, \mathcal{D})$ are satisfied.

For every pair of disjoint closed sets $A, B \in \mathcal{B}$ we can find sets $C, D \in \mathcal{D}$, such that $C \cup D = X$ and $A \cap C = \emptyset = B \cap D$. We add these sets to $\mathcal{B}$, getting in this way a family

$\mathcal{B}' \supset \mathcal{B}$. Then property $(\star)(\mathcal{B}, \mathcal{B}')$ holds true and since there are at most κ different pairs of disjoint sets from $\mathcal{B}$, we added at most κ new sets and hence $|\mathcal{B}'| = \kappa$.

Next, to get property (KM), we consider all quadruples A, B, C, D such that $A, B, C, D \in \mathcal{B}$, $A \subset C \setminus D$, $B \subset D \setminus C$ and $C \cup D = X$. There is at most κ such quadruples. Because X is a Bing space, it follows that for each of these quadruples (A, B, C, D) we can find sets $X_0, X_1, X_2 \in \mathcal{D}$, such that (X_0, X_1, X_2) is a fold for (A, B, C, D). We add all these sets to $\mathcal{B}'$, in effect we get a family $\mathcal{B}''$ such that $KM(\mathcal{B}, \mathcal{B}'')$ holds true. Also in this step we added at most κ sets, so $|\mathcal{B}''| = \kappa$.

Next, for every pair of different sets $A, B \in \mathcal{B}$ we find a set $C \in \mathcal{D}$ such that exactly one of the intersections $A \cap C, B \cap C$ is empty. We add these sets to $\mathcal{B}''$, obtaining a new family $\mathcal{B}'''$. Again, we added at most κ sets, because there is at most κ different pairs of sets from $\mathcal{B}$. So we have $|\mathcal{B}'''| = \kappa$ and transparently, condition $D(\mathcal{B}, \mathcal{B}''')$ is met.

At the end, we take as $\mathcal{C}$ the minimal sublattice of $\mathcal{D}$ containing $\mathcal{B}'''$. Then $|\mathcal{C}| = \kappa$ and conditions $(\star)(\mathcal{B}, \mathcal{C})$, $(KM)(\mathcal{B}, \mathcal{C})$, $D(\mathcal{B}, \mathcal{C})$ are met. ☐

Lemma 5.2. *Let $\mathcal{B}$ be a sublattice of $\mathcal{D} = 2^X$ in a compact space X and let F be a closed subset of X. If $|\mathcal{B}| = \kappa$ and $trInd_m F = \alpha$, then $\mathcal{B}$ can be extended to a lattice $\mathcal{C} \subset \mathcal{D}$ such that $|C| = \kappa$ and $trInd_m(\mathcal{B}|F, \mathcal{C}|F) \leq \alpha$.*

Proof. Without loss of generality (by extending, if necessary, the lattice $\mathcal{B}$ to the smallest lattice containing $\mathcal{B}$ and the set F) we can assume, that $F \in \mathcal{B}$.

By induction we can define a tree T of height ω, in which every level T_n, $n = 0, 1, ...,$ has at most κ elements.

To begin with, we distinguish some $(m+1)$-tuples of closed sets in X as follows:

$$\mathfrak{S} = \{(A_1, ..., A_m, E) \in \mathcal{B}^m \times \mathcal{D} : A_i \cap E \text{ are nonempty and pairwise disjoint}\}, \tag{5.1}$$

and we let

$$\mathfrak{S}_G = \{(A_1, ..., A_m, E) \in \mathfrak{S} : E = G\}. \tag{5.2}$$

To every $(A_1, ..., A_m, E) \in \mathfrak{S}$ we assign some m-tuple

$$P(A_1, ..., A_m, E) = (C_1, ..., C_m) \in \mathcal{D}^m$$

satisfying the following conditions:

(1) $C_i \cup C_j = E$ for every $1 \leq i < j \leq m$,
(2) $A_i \cap C_i = \emptyset$ for every $i = 1, 2, ..., m$,
(3) $trInd_m \left(\bigcap_{i=1}^{m} C_i \right) < trInd_m E$,

and we set

$$\bigwedge P(A_1, ..., A_m, E) = \bigcap_{i=1}^{m} C_i. \tag{5.3}$$

In other words, $\bigwedge P(A_1, ..., A_m, E)$ is a partition between $A_1, ..., A_m$ in E, of dimension lower than the dimension of E.

The elements of our tree T will be certain sequences $(A_1, ..., A_m, E) \in \mathfrak{S}$, where $E \subset F$, and for every $n \in \mathbb{N}$, n-th level T_n of the tree T will consist of sequences of lenght $n+1$, i.e., $T_n \subset \mathfrak{S}^{n+1}$. We define a partial order $\leq$ in T, declaring

$$(t_0, ..., t_k) \leq (r_0, ..., r_l) \Leftrightarrow k \leq l \text{ and } \forall_{i \leq k}(t_i = r_i).$$

Then for every element $(t_0, ..., t_k) \in T$ its immediate successor is of the form $(t_0, ..., t_k, t_{k+1})$, where $t_{k+1} \in \mathfrak{S}$.

We will focus mainly on last elements of sequences from T, and it is handy to introduce a function $\Phi : T \to \mathfrak{S}$ defined by the formula

$$\Phi((t_0, ..., t_n)) = t_n$$

The level T_0 of the tree T consists of all $(m+1)$-tuples $(A_1, ..., A_m, F) \in \mathfrak{S}_F$.

Assume that n-th level $T_n \subset \mathfrak{S}^{n+1}$ is already defined, $|T_n| \leq \kappa$ and let $(A_1, ..., A_m, E) = \Phi(\sigma)$ for some sequence $\sigma \in T_n$.

If $\mathfrak{S}_{\bigwedge P(A_1,...,A_m,E)} = \emptyset$ (where $\mathfrak{S}_\emptyset = \emptyset$), see (5.2) and (5.3), then σ has no successors - σ is a maximal element of T. Otherwise, the immediate successors of σ in T are all sequences $\tau \in \mathfrak{S}^{n+2}$ such that

$$\sigma \leq \tau \text{ and } \Phi(\tau) \in \mathfrak{S}_{\bigwedge P(A_1,...,A_m,E)}.$$

Since the elements of $\mathfrak{S}_{\bigwedge P(A_1,...,A_m,E)}$ are $(m+1)$-tuples $(H_1, ..., H_m, \bigwedge P(A_1, ..., A_m, E))$ and because m-tuples $(H_1, ..., H_m)$ are elements of $\mathcal{B}^m$, every $\sigma \in T_n$ has no more than κ successors, and therefore $(n+1)$-th level of T, T_{n+1} has cardinality $\leq \kappa$.

Moreover, for every $\sigma \in T$ and its immediate successor τ, if $\Phi(\sigma) = (A_1, ..., A_m, E)$ and $\Phi(\tau) = (H_1, ..., H_m, \bigwedge P(A_1, ..., A_m, E))$ then we have $trInd_m(\bigwedge P(A_1, ..., A_m, E)) < trInd_m E$.

This fact implies that on every branch of the tree T we can find an element σ for which $\Phi(\sigma) = (A_1, ..., A_m, E)$ and $\mathfrak{S}_{\bigwedge P(A_1,...,A_m,E)} = \emptyset$, so it has no successors. It means, that all branches of T are finite.

Let us notice also that if σ is in level T_n of the tree T, and $\Phi(\sigma) = (A_1, ..., A_m, E)$, then for every $(m+1)$-tuple $(B_1, ..., B_m, E) \in \mathfrak{S}_E$ there is an element $\sigma' \in T_n$ such that $\Phi(\sigma') = (B_1, ..., B_m, E)$ (σ and σ' are successors of the same element from the level T_{n-1}, provided $n > 0$).

Now, as $\mathcal{C}$ we will take the smallest sublattice of $\mathcal{D}$ that contains $\mathcal{B}$ and all sets $C_1, ..., C_m \in \mathcal{D}$ such that, for some $\sigma \in T$ we have

$$\Phi(\sigma) = (A_1, ..., A_m, E) \text{ and } (C_1, ..., C_m) = P(A_1, ..., A_m, E).$$

To complete our proof, we shall check that $trInd_m(\mathcal{B}|F, \mathcal{C}|F) \leq \alpha$. We will prove this by showing a little stronger fact, that if $E \in \mathcal{D}$ and for some $\sigma \in T$ we have $\Phi(\sigma) \in \mathfrak{S}_E$, then $trInd_m(\mathcal{B}|E, \mathcal{C}|E) \leq trInd_m E$.

To that end we shall proceed by induction on $trIndE$.

Let $E \in \mathcal{D}$, $trInd_m E = 0$ and assume that there is $\sigma \in T$ such that $\Phi(\sigma) \in \mathfrak{S}_E$. Then, as we stated earlier, for every $(m+1)$-tuple $(B_1, ..., B_m, E) \in \mathfrak{S}_E$ there is $\sigma' \in T$ such that $\Phi(\sigma') = (B_1, ..., B_m, E)$. Therefore we have $trInd_m(\mathcal{B}|E, \mathcal{C}|E) = 0$.

Assume now, that we have checked this property for all $E \in \mathcal{D}$ with $trInd_m E < \alpha$. Let $E \in \mathcal{D}$, $trInd_m E = \alpha$ and assume that there is $\sigma \in T$ such that $\Phi(\sigma) \in \mathfrak{S}_E$. Then for every $(m+1)$-tuple $(B_1, ..., B_m, E) \in \mathfrak{S}_E$ there is $\sigma' \in T$ such that $\Phi(\sigma') = (B_1, ..., B_m, E)$. Since we have $trInd_m(\bigwedge P(B_1, ..., B_m, E)) < \alpha$, from the induction assumption we infer that

$$trInd_m(\mathcal{B}| \bigwedge P(B_1, ..., B_m, E), \mathcal{C}| \bigwedge P(B_1, ..., B_m, E)) < \alpha.$$

In effect, condition (B) from Definition 3.1 is met, and we have $trInd_m(\mathcal{B}|E, \mathcal{C}|E) \leq \alpha = trInd_m E$. □

6. Proof of the factorization theorem

We are ready to give a proof of Theorem 1.1 - the main result of this paper. For $m = 2$ this was established in [13], Theorem 5.2.5. As in [13], the reasoning will be based on results from (3.1), and Lemmas 2.2, 5.1, 5.2.

Let $\mathcal{A}$ be a base of closed sets in Z, and $|\mathcal{A}| = \kappa = w(Z)$. Without loss of generality we can assume that $\mathcal{A}$ is a sublattice of 2^Z. We let $\mathcal{D} = 2^X$.

Then conditions $(\star)(\mathcal{D}, \mathcal{D})$, $D(\mathcal{D}, \mathcal{D})$, $(KM)(\mathcal{D}, \mathcal{D})$ are met, and since we have $trInd_m F = \alpha$, $trInd_m(\mathcal{D}|F, \mathcal{D}|F) \leq \alpha$.

Let us note that $\mathcal{B} = f^{-1}(\mathcal{A})$ is a sublattice of the lattice $\mathcal{D}$. First, we define by induction a sequence $(\mathcal{B}_i)_{i=0}^{\infty}$ of sublattices of $\mathcal{D}$, such that

$$\mathcal{B} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset ...,$$

for every $i = 0, 1, ...$ we have $|B_i| = |B_{i+1}| = \kappa$ and conditions $(\star)(\mathcal{B}_i, \mathcal{B}_{i+1})$, $D(\mathcal{B}_i, \mathcal{B}_{i+1})$, $(KM)(\mathcal{B}_i, \mathcal{B}_{i+1})$, $trInd_m(\mathcal{B}_i|F, \mathcal{B}_{i+1}|F) \leq \alpha$ are met.

Next, we take

$$\mathcal{C} = \bigcup_{i=0}^{\infty} \mathcal{B}_i.$$

Then we have $\mathcal{B} \subset \mathcal{C} \subset \mathcal{D}$, $|\mathcal{C}| = \kappa$ and all conditions $(\star)(\mathcal{C}, \mathcal{C})$, $D(\mathcal{C}, \mathcal{C})$, $(KM)(\mathcal{C}, \mathcal{C})$, $trInd_m(\mathcal{C}|F, \mathcal{C}|F) \leq \alpha$ are met. In effect, the Wallman representation $Y = w\mathcal{C}$ of the lattice $\mathcal{C}$ together with mappings g and h from Lemma 2.2 satisfy the assertions of the theorem.

7. Comments

7.1. **A factorization theorem for trdim.** A natural transfinite extension of the covering dimension of a compact space X is denoted by *trdim*, cf. [4]. Let us recall the definition of this dimension.

Definition 7.1.1. *For a set L we denote by $FinL$ the collection of all non-empty finite subsets of L. For a subset M of $FinL$ and an element $a \in L$ we put*

$$M^a = \{\sigma \in FinL : \sigma \cup \{a\} \in M \text{ and } a \notin \sigma\}.$$

Definition 7.1.2. *Let L and M be as in Definition 7.1.1. We define the ordinal number $OrdM$ inductively as follows:*

$OrdM = 0$ *iff* $M = \emptyset$,
$OrdM \leq \alpha$ *iff for every* $a \in L$, $OrdM^a < \alpha$,
$OrdM = \alpha$ *iff* $OrdM \leq \alpha$ *and* $OrdM < \alpha$ *is not true.*

We write $OrdM = \infty$ *iff* $OrdM > \alpha$ *for every ordinal number* α.

Definition 7.1.3. *Let* X *be a space. We put*

$$L(X) = \{(A, B) : A \text{ and } B \text{ are disjoint closed sets in } X\}.$$

A collection $\sigma = \{(A_i, B_i) : i = 1, ..., n\} \in FinL(X)$ *is called inessential if there are partitions* L_i *between* A_i *and* B_i *for each* $i = 1, ..., n$ *such that* $\bigcap_{i=1}^{n} L_i = \emptyset$. *Otherwise,* σ *is called essential. For arbitrary* $L \subset L(X)$ *we set*

$$M_L = \{\sigma \in FinL : \sigma \text{ is essential in } X\}.$$

Definition 7.1.4. *For a space* X *we put*

$$trdimX = OrdM_{L(X)}.$$

Using the approach based on Wallman representations exploited in this paper, we obtained in [13] the following result

Theorem 7.1.5. *For every continuous mapping* $f : X \to Z$ *from a Bing space* X *into a compact space* Z, *there is a Bing space* Y *and continuous mappings* $g : X \to Y$ *and* $h : Y \Rightarrow Z$, *such that* $trdimY \leq trdimX$, $w(Y) \leq w(Z)$, $g(X) = Y$ *and* $f = h \circ g$.

The Wallman representations method provides in particular a simpler proof of factorization theorems for *trdim* in the literature, cf. [16].

7.2. **Examples.** For each infinite cardinal number κ and a limit countable ordinal number α, there is hereditary indecomposable continuum $L_{\kappa,\alpha}$ of weight κ, such that $trIndL_{\kappa,\alpha} = trdimL_{\kappa,\alpha} = \alpha$.

Such continua were constructed by the author in [13]. For $\kappa \leq 2^{\aleph_0}$ examples of this kind, with some interesting additional properties, were described by Charalambous and Krzempek [3]. However, for $\kappa > 2^{\aleph_0}$, the examples obtained in [13] seem to be new.

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