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Note on analyticity of function spaces

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NOTE ON ANALYTICITY OF FUNCTION SPACES

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Abstract. A regular topological space X is analytic if X is a continuous image of $\mathbb{N}^{\mathbb{N}}$, and X is K-analytic if X is an image under an upper semicontinuous (usco) compact set-valued mapping of $\mathbb{N}^{\mathbb{N}}$. Given a topological group G we denote by $G^\wedge$ the group of characters on G. We recall some recent results considering analyticity and K-analyticity of $G^\wedge$ for some specific topological groups G, and give some closer looks on the proofs of theses results.

1. Introduction

We shall consider only completely regular spaces. For abelian topological groups X and Y by $Hom_p(X,Y)$ and $Hom_c(X,Y)$ we denote the set $Hom(X,Y)$ of all continuous homomorphisms from X into Y endowed with the pointwise and compact-open topology, respectively. We set $X_p^\wedge = Hom_p(X,\mathbb{T})$ and $X_c^\wedge = Hom_c(X,\mathbb{T}) = X^\wedge$, where $\mathbb{T}$ denotes the unit circle of the complex plane.

Given topological spaces X, Y, we denote by Y^X the space of continuous functions from X to Y, equipped with the compact-open topology.

Let us recall that a regular topological space X is *analytic* if X is a continuous image of the irrationals (or equivalently $\mathbb{N}^{\mathbb{N}}$), and X is *K-analytic* if X is an image under an upper semicontinuous compact set-valued mapping of irrationals (or $\mathbb{N}^{\mathbb{N}}$).

For a topological abelian group X the coarsest group topology on X for which all elements of $X^\wedge$ are continuous is called the *Bohr topology*; we denote this topology by $\sigma(X, X^\wedge)$.

A topological space X is said to have *countable tightness* if for each set $A \subset X$ and any $x \in \overline{A}$ (the closure of A) there exists a countable subset $B \subset A$ whose closure contains x.

A topological space X is *Lindelof* if every open cover in it, has a countable subcover.

Given a separable metrizable space X, we denote by $\mathcal{K}(X)$ the space of compact subsets of X equipped with the Vietoris topology. Let us recall that $\mathcal{K}(X)$ is separable metrizable and for each open $U \subset X$, the set $\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$ is open in $\mathcal{K}(X)$, cf. [En], 4.5.23.

In Section 2 we take a look on the following theorem (Kakol, Martin-Peinador, Moll [KMM]):

Theorem 1.1. *Let X be a locally compact abelian group. The following conditions are equivalent:*

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(i) *X is metrizable,*
(ii) *$X_p^\wedge$ is σ-compact,*
(iii) *$X_p^\wedge$ is K-analytic,*
(iv) *$(X, \sigma(X, X^\wedge))$ has countable tightness.*

Moreover, if X is Lindelof, then any condition above is equivalent to

(v) *$X_c^\wedge$ is metric complete and separable.*

In Section 3 we extend the proof of the following theorem by Pol and Smentek [PS]:

Theorem 1.2. *For any zero-dimensional separable metrizable space X, the following conditions are equivalent:*

(i) *$(\mathbb{Z}^X)^\wedge$ is analytic,*
(ii) *$(\mathbb{Z}^X)^\wedge$ equipped with the pointwise topology is analytic,*
(iii) *X is completely metrizable.*

In Section 4 we take a closer look at the theorem by Kakol and Lopez Pellicer [KL], concerning approach introduced in [CO]:

Theorem 1.3. *If F is a locally convex Baire space covered by an ordered family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of relatively countably compact sets, F is K-analytic. In fact then F is a separable locally convex Frechet space.*

2. K-analyticity of $X_p^\wedge$

The proof of Theorem 1.1 is derived from the following lemmas:

Lemma 2.1. *A locally compact Lindelof topological group X is hemicompact.*

Indeed, take an open neighbourhood of the unit U whose closure $\overline{U}$ is compact. Since $X = \cup_{x\in X} xU$ and X is Lindelof, there exists a sequence $(x_n)_n$ such that $X = \cup_n x_n\overline{U}$. Let us set $K_n = \cup_{i=1}^n x_i\overline{U}$. Then $(K_n)_n$ is a fundamental sequence of compact sets in X, so X is hemicompact.

Lemma 2.2. *If X is a metrizable locally compact abelian group, $X_c^\wedge$ is a hemicompact k-space.*

[AUSSENHOFER - DO UZUPEŁNIENIA]

Lemma 2.3. *If a topological abelian group X is hemicompact, then $X_c^\wedge$ is metrizable.*

[AUSSENHOFER - DO UZUPEŁNIENIA]

Lemma 2.4. *Let X be a Tychonoff space and assume that $C_p(X, \mathbb{R})$ has countable tightness. Then $C_p(X, Y)$ has also countable tightness for any metric space (Y, d).*

Indeed, let $A \subset C_p(X, Y)$ and assume that $f \in \overline{A}$ (the closure in $C_p(X, Y)$). Definea continuous map $T : C_p(X, Y) \to C_p(X, \mathbb{R})$ by $T(g)(x) := d(g(x), f(x))$, where $g \in C_p(X, Y)$ and $x \in X$. Note that $0 = T(f) \in T(\overline{A}) \subset \overline{T(A)}$. By assumption there exists in A a countable subset B such that $T(f) \in \overline{T(B)}$; hence $f \in \overline{B}$.

Proof of Theorem 1.1. (i)$\Rightarrow$ (ii): By Lemma 2.2 the group $X_c^\wedge$ is hemicompact. So $X_p^\wedge$ is σ-compact.

(ii)$\Rightarrow$ (iii): If $(B_n)_n$ is an increasing sequence of compact sets covering $X_p^\wedge$, let us set $T(\alpha) := B_{n_1}$ for $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$. It is clear that T is upper semi-continuous, with compact values which cover $X_p^\wedge$.

(iii)$\Rightarrow$ (iv): Since $X_p^\wedge$ is K-analytic, then any finite product $(X_p^\wedge)^n$ is Lindelof. By [A, Theorem II.1.1] the space $C_p(X_p^\wedge, \mathbb{R})$ has countable tightness. Now Lemma 2.4 applies to deduce that the space $C_p(X_p^\wedge, \mathbb{C})$ has also countable tightness. Therefore $(X, \sigma(X, X^\wedge))$ (as topologically included in $C_p(X_p^\wedge, \mathbb{C})$) has countable tightness.

(iv)$\Rightarrow$ (i): Since X is a locally compact group, there exist a compact subgroup G of X, $n \in \mathbb{N} \cup \{0\}$, and a discrete subset $D \subset X$ such that X is homeomorphic to the product $\mathbb{R}^n \times D \times G$, see [CM, Theorem 1, Remark(ii)]. Therefore the induced topology $\sigma(X, X^\wedge)|G$ coincides with the original one of G. Hence G has countable tightness. Since a compact group with countable tightness is metrizable, so X is metrizable as well.

For the last statement observe that under the assumption that X is Lindelof, Lemmas 2.2 and 2.3 apply to deduce that $X_c^\wedge$ is a separable metric space. It is also complete since the dual group of a locally compact abelian group is also locally compact. The result follows now from the fact that $X_p^\wedge$ is the continuous image of $X_c^\wedge$ by the identity mapping. □

3. Analyticity of $(\mathbb{Z}^X)^\wedge$

We shall now prove Theorem 1.2.

Let us recall two lemmas from [PS]:

Lemma 3.1. *Let $\Phi \subset (\mathbb{A}^X)^\wedge$ be compact, where X is a zero-dimensional realcompact space, $\mathbb{A} = \mathbb{Z}$ ($\mathbb{A} = \mathbb{Z}_m$) and let C be the set of points in X whose each neighbourhood contains a clopen set V with $\phi(c_V) \neq 1$ ($\phi(c_V) \neq 0$), for some $\phi \in \Phi$, cf. (2). Then C is compact and $\phi(f) = 1$ ($\phi(f) = 0$), whenever $\phi \in \Phi$ and $f \in \mathbb{A}^X$ vanishes on C.*

Lemma 3.2. *Let X and $\mathbb{A}$ be as in Lemma 3.1 and let, for $\phi \in (\mathbb{A}^X)^\wedge$, $C(\phi)$ be the set C associated in this lemma with the singleton $\Phi = \{\phi\}$. Then the map $\phi \to C(\phi)$ is lower-semicontinuous with respect to the topology τ_p of pointwise convergence in $(\mathbb{A}^X)^\wedge$.*

[BEZ DOWODU?]

Throughout this section, X is a zero-dimensional separable metrizable space.

For each $\phi \in (\mathbb{Z}^X)^\wedge$, $C(\phi)$ is the set of points in X whose each neighbourhood contains a clopen set V with $\phi(c_V) \neq 1$, cf. Lemma 3.1.

Let τ_p be the pointwise topology in $(\mathbb{Z}^X)^\wedge$. Since the compact-open topology is stronger than τ_p, implication $(i) \to (ii)$ in Theorem 1.2 is evident.

To explain why $(ii) \to (iii)$, let us fix a continuous surjection $p : \mathbb{P} \to ((\mathbb{Z}^X)^\wedge, \tau_p)$ on the irrationals $\mathbb{P}$ and let $\sigma : \mathbb{P} \to \mathcal{K}(X)$, cf. section 1, be defined by $\sigma(t) = C(p(t))$. By Lemma 3.2, the map is Borel.

Let us check that

$$\mathcal{K}(X) = \sigma(\mathbb{P}). \tag{1}$$

For any nonempty $C \in \mathcal{K}(X)$ (notice that $C(1) = \emptyset$ for for the unit 1 in $(\mathbb{Z}^X)^\wedge$), let $a_1, a_2, ...$ be dense in C, and let $\phi \in (\mathbb{Z}^X)^\wedge$ be defined by $\phi(f) = exp\left(i \sum_n 2^{-n} f(a_n)\right)$. Then $C(\phi) = C$ and since $\phi = p(t)$ for some $t \in \mathbb{P}$, we get (1).

We infer from (1) that the space $\mathcal{K}(X)$ of compact sets in X is analytic and by a theorem of Christensen [DO UZUPEŁNIENIA] this yields complete metrizability of X.

It remains to check that $(iii) \to (i)$, i.e., we have to derive analyticity of $(\mathbb{Z}^X)^\wedge$ from complete metrizability of X. To that end, let us associate to each compact set $C \subset X$ the set

$$\Phi(C) = \{\phi \in (\mathbb{Z}^X)^\wedge : C(\phi) \subset C\}, \tag{2}$$

and let us show that for the restriction map $r_C : \mathbb{Z}^X \to \mathbb{Z}^C$, $r_C(f) = f|C$,

$$\Phi(C) = r_C^\wedge((\mathbb{Z}^C)^\wedge),\ r_C^\wedge(\alpha) = \alpha \circ r_C. \tag{3}$$

Indeed, if $\phi = \alpha \circ r_C$, $\alpha \in (\mathbb{Z}^C)^\wedge$, then $\phi(f) = 1$ for all $f \in \mathbb{Z}^X$ that vanish on C, hence by Lemma 3.1, $C(\phi) \subset C$. On the other hand, if $\phi \in (\mathbb{Z}^X)^\wedge$ and $C(\phi) \subset C$, again by Lemma 3.1, the formula $\alpha(r_C(f)) = \phi(f)$ defines $\alpha \in \mathbb{Z}^C$, as $r_C(f) = r_C(g)$ implies $\phi(f - g) = 1$.

Now, since $\mathbb{Z}^C$ is a discrete group, the dual group $(\mathbb{Z}^C)^\wedge$ is compact and the compact-open topology in $(\mathbb{Z}^C)^\wedge$ coincides with the pointwise topology. The map $r_C^\wedge : (\mathbb{Z}^C)^\wedge \to (\mathbb{Z}^X)^\wedge$ is continuous and by (3), we get compactness of $\Phi(C)$.

Now, the correspondence $C \to \Phi(C)$ defined in (2) preserves inclusions, and hence, in terminology of [COT], the space $(\mathbb{Z}^X)^\wedge$ is determined by X [DO UZUPEŁNIENIA]. Since $(\mathbb{Z}^X)^\wedge$ has a countable net, a theorem of B. Cascales and J. Orihuela [CO] guarantees analyticity of $(\mathbb{Z}^X)^\wedge$. [DO UZUPELNIENIA I POWIAZANIA Z SECTION 4]

4. K-analyticity of spaces covered by ordered families of sets

[DO ZROBIENIA]

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