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NOTE ON REFLEXIVITY OF SOME SPACES OF CONTINUOUS INTEGER-VALUED FUNCTIONS

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Abstract. Given a topological group G we denote by $G^\wedge$ the group of characters on G and reflexivity of G means that the natural map from G to $G^{\wedge\wedge}$ is a topological isomorphism.

We show that for any zero-dimensional realcompact k-space X and a discrete finitely generated abelian group $\mathbb{A}$, the group $\mathbb{A}^X$ of continuous maps from X to $\mathbb{A}$ with pointwise addition and compact-open topology is reflexive, and we construct a countable non-reflexive closed subgroup of $\mathbb{Z}^X$, where X is a countable subspace of the plane (this group embeds as a closed subgroup in the product $(\sum \mathbb{Z})^{\mathfrak{c}}$ of continuum many copies of the discrete Specker group). We show also that, for metrizable separable X, analyticity of $(\mathbb{Z}^X)^\wedge$ is equivalent to complete metrizability of X.

1. Introduction

Given topological spaces X, Y, we denote by Y^X the space of continuous functions from X to Y, equipped with the compact-open topology. If G is a topological group, so is G^X under the pointwise group operation.

For a topological group G, $G^\wedge$ is the subgroup of $\mathbb{T}^G$, $\mathbb{T}$ being the complex unit circle, consisting of continuous homomorphisms, i.e., characters of G. We call G reflexive, if the evaluation map

$$\Gamma : G \to G^{\wedge\wedge},\ \Gamma(g)(\phi) = \phi(g) \text{ for } \phi \in G^\wedge \tag{1}$$

is a topological isomorphism.

We shall prove the following theorem.

Theorem 1.1. *Let X be a zero-dimensional realcompact k-space and let $\mathbb{A}$ be a finitely generated abelian group. Then the topological group $\mathbb{A}^X$ is reflexive.*

This result is closely related to important papers by V. V. Pestov [Pe1], [Pe2], and subsequent work by J. Galindo and S. Hernandez [GH] and L. Aussenhofer [Au1], providing deep insight into reflexivity of topological groups $\mathbb{T}^X$ (in particular, $\mathbb{T}^X$ is reflexive for X as in Theorem 1.1). Also, S.Hernandez and V.Uspenskij [HU], Theorem 2.6, established reflexivity of some additive groups $\mathbb{R}^X$, in particular, for realcompact k-spaces X. We did not see, however, a direct way of deriving Theorem 1.1 from these results.

The groups $\mathbb{A}^X$ considered in Theorem 1.1 are prodiscrete, cf. sec. 2, which links our subject with a recent paper by J. Galindo, L. Recoder-Núñez and M. Tkachenko [GRT]. In

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particular, there is an example in [GRT] of a prodiscrete group of countable character, which is not reflexive. The following example shows that such a group may be even countable (our group is also torsion-free, which answers a question in [GRT] preceding Problem 11).

Example 1.2. *There is a countable subspace X of the euclidean plane and a countable closed subgroup G of $\mathbb{Z}^X$ which is not reflexive.*

The group G embeds as a closed subgroup in the power $(\sum \mathbb{Z})^{2^{\aleph_0}}$ of the discrete Specker group $\sum \mathbb{Z}$ of finitely supported functions from natural numbers $\mathbb{N}$ to integers $\mathbb{Z}$, cf. Comment 6.3. The group G can be also represented as $\mathbb{Z}^Y$ for some $Y \subset \mathbb{R}^3$ containing no non-trivial continuum; for such Y, both groups $\mathbb{T}^Y$ and $\mathbb{R}^Y$ are reflexive, by [GH] and [HU], cf. Example 4.2.

Let us recall that a regular topological space X is analytic if X is a continuous image of the irrationals.

The following theorem is closely related to results of Christensen [Chr1] concerning analyticity in function spaces on separable metrizable spaces.

Theorem 1.3. *For any zero-dimensional separable metrizable space X, the following conditions are equivalent:*

(i) $(\mathbb{Z}^X)^\wedge$ *is analytic,*
(ii) $(\mathbb{Z}^X)^\wedge$ *equipped with the pointwise topology is analytic,*
(iii) X *is completely metrizable.*

In particular, for the rationals $\mathbb{Q}$, the group $(\mathbb{Z}^\mathbb{Q})^\wedge$ is not analytic. In contrast, by Pestov's results [Pe2], the group $(\mathbb{T}^\mathbb{Q})^\wedge$ is countable, being the free topological abelian group generated by $\mathbb{Q}$.

2. Some background

For basic notions concerning topological groups we refer to S. Morris [Mo] and for topological notions to R. Engelking [En]. A reach source of information concerning the subject is a dissertation of L. Aussenhofer [Au1].

Let G be a prodiscrete group, i.e., a complete topological group whose open subgroups form a basis of neighbourhoods of the identity (equivalently - G embeds as a closed subgroup in a product of discrete groups). Then the evaluation map $\Gamma : G \to G^{\wedge\wedge}$ in (1) is an isomorphism taking open sets to open sets, cf. [Au1], Corollary 21.5 and [GRT], Theorem 3.1.

An important issue in this topic is equicontinuity of compact families of continuous functions, cf. [Au1] and [GH]. In particular, for any compact $\mathcal{K} \subset Y^X$, where Y is discrete and X is a k-space, any point $a \in X$ has a neighbourhood V such that all functions from $\mathcal{K}$ are constant on V.

Since any discrete finitely generated abelian group $\mathbb{A}$ is a free union of finitely many copies of $\mathbb{Z}$ or $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, and the product of reflexive topological groups is reflexive, cf. [Ka], to check reflexivity of $\mathbb{A}^X$ it is enough to verify reflexivity for $\mathbb{Z}^X$ and $\mathbb{Z}_m^X$.

Since $\mathbb{Z}_m$ can be identified with the subgroup of $\mathbb{T}$ consisting of m'th roots of the identity, it is convenient to identify $(\mathbb{Z}_m^X)^\wedge$ and $(\mathbb{Z}_m^X)^{\wedge\wedge}$ with groups of continuous homomorphisms into $\mathbb{Z}_m$ from $\mathbb{Z}_m^X$ or $(\mathbb{Z}_m^X)^\wedge$, respectively, keeping the additive notation.

Given $A \subset X$, we denote by

$$c_A \text{ the characteristic function of } A, \tag{2}$$

and the support of a function f into $\mathbb{Z}$ or $\mathbb{Z}_m$ is

$$supp f = \{x : f(x) \neq 0\}. \tag{3}$$

For the real line $\mathbb{R}$, $\mathbb{R}^{(\mathbb{N})}$ is the additive group of finitely supported functions from natural numbers $\mathbb{N}$ to $\mathbb{R}$, carrying the box-product topology.

A Hausdorff space X is zero-dimensional, if X has a base consisting of closed and open sets (in short, clopen sets), i.e., $ind X = 0$; $dim X = 0$ means strong zero-dimensionality of X, cf. [En].

A completely regular space X is realcompact if X can be embedded as a closed subspace of some power of the real line, cf. [En], 3.11. Let us recall that in a realcompact space X, if $C \subset X$ is not compact, there is a continuous $f : X \to \mathbb{R}$ unbounded on C, and hence a discrete sequence $U_1, U_2, ...$ of open sets in X intersecting C.

Given a separable metrizable space X, we denote by $\mathcal{K}(X)$ the space of compact subsets of X equipped with the Vietoris topology, cf. [En], 2.7.20. Let us recall that $\mathcal{K}(X)$ is separable metrizable and for each open $U \subset X$, the set $\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$ is open in $\mathcal{K}(X)$, cf. [En], 4.5.23.

3. Reflexivity of $\mathbb{Z}^X$ and $\mathbb{Z}_m^X$

The following observation provides easily continuity of the evaluation maps (1) for the topological groups $\mathbb{Z}^X$ and $\mathbb{Z}_m^X$, cf. [Au], 5.10. This is a counterpart of a reasoning of V. V. Pestov [Pe2] (cf. Claim 1 on page 308 in [Pe2] and [Au1], 13.14, 13.16 and 14.7). A distinction is that the characters of the group $\mathbb{T}^X$ investigated in [Pe2] and [Au1] have finite support, while in the case we deal with, the characters are supported by compact sets, cf. also Comment 6.4.

Lemma 3.1. *Let $\Phi \subset (\mathbb{A}^X)^\wedge$ be compact, where X is a zero-dimensional realcompact space, $\mathbb{A} = \mathbb{Z}$ ($\mathbb{A} = \mathbb{Z}_m$) and let C be the set of points in X whose each neighbourhood contains a clopen set V with $\phi(c_V) \neq 1$ ($\phi(c_V) \neq 0$), for some $\phi \in \Phi$, cf. (2). Then C is compact and $\phi(f) = 1$ ($\phi(f) = 0$), whenever $\phi \in \Phi$ and $f \in \mathbb{A}^X$ vanishes on C.*

Proof. Let us concentrate first on $\mathbb{A} = \mathbb{Z}$. Assume, striving for a contradiction, that C is not compact. Then, since C is closed, by realcompactness of X, one can find a discrete sequence of open sets $U_1, U_2, ...$ intersecting C and then, by the definition of C, a discrete collection of clopen sets $V_i \subset U_i$ such that, for some $\phi_i \in \Phi$, $\phi_i(c_{V_i}) \neq 1$, cf. (2). Pick $m_i \in \mathbb{Z}$ such that $|\phi_i(c_{V_i})^{m_i} - 1| > 1$. Since $m_i c_{V_i} \to 0$ in $\mathbb{Z}^X$, $\mathcal{K} = \{m_i c_{V_i} : i = 1, 2, ...\} \cup \{0\}$ is compact in $\mathbb{Z}^X$. But $|\phi_i(m_i c_{V_i}) - 1| > 1$ and $\phi_i(0) = 1$, which contradicts equicontinuity of Φ on $\mathcal{K}$.

To get the second part of the assertion for $\mathbb{A} = \mathbb{Z}$, let us fix $\phi \in \Phi$ and $f \in \mathbb{Z}^X$ vanishing on C. For an arbitrary $\epsilon > 0$, continuity of ϕ at zero provides a compact set $L \subset X$ such that

$$|\phi(g) - 1| < \epsilon, \text{ whenever } g \in \mathbb{Z}^X \text{ vanishes on } L. \tag{4}$$

Since $supp f \cap C = \emptyset$, cf. (3), $supp f \cap L$ can be covered by pairwise disjoint clopen sets $V_1, ..., V_m$ such that $\phi(c_{V_i}) = 1$, $V_i \cap (L \setminus supp f) = \emptyset$ and f is constant on each V_i. Then $f - \sum_i k_i c_{V_i}$, where $f|V_i = k_i$, vanishes on L, $\phi(k_i c_{V_i}) = 1$, and by (4) we get $|\phi(f) - 1| < \epsilon$. Since this is true for arbitrary ϵ, $\phi(f) = 1$, as required.

In case $\mathbb{A} = \mathbb{Z}_m$ the reasoning is parallel, but there is no need to use ϵ and (4) is replaced by the condition that $\phi(g) = 0$, whenever g vanishes on L. □

Proof of Theorem 1.1. As was explained in section 2, dealing with prodiscrete groups we have only to verify continuity of the evaluation map Γ defined in (1).

First, let $\mathbb{A} = \mathbb{Z}$ and let us consider a neighbourhood of the identity in $(\mathbb{Z}^X)^{\wedge\wedge}$,

$$\mathcal{V} = \{\gamma \in (\mathbb{Z}^X)^{\wedge\wedge} : |\gamma(\phi) - 1| < \epsilon \text{ for } \phi \in \Phi\}, \tag{5}$$

where $\Phi \subset (\mathbb{Z}^X)^{\wedge}$ is compact. The compact set C associated with Φ in Lemma 3.1 determines a neighbourhood of zero in $\mathbb{Z}^X$,

$$\mathcal{U} = \{f \in \mathbb{Z}^X : f|C = 0\}. \tag{6}$$

If $f \in \mathcal{U}$ and $\phi \in \Phi$, the second part of Lemma 3.1 implies that $\Gamma(f)(\phi) = \phi(f) = 1$, i.e., $\Gamma(\mathcal{U}) \subset \mathcal{V}$, cf. (1), (5), (6).

The same arguments work for $\mathbb{A} = \mathbb{Z}_m$, if we replace $\mathcal{V}$ by a neighbourhood of zero $\{\gamma \in (\mathbb{Z}_m^X)^{\wedge\wedge} : \gamma|\Phi = 0\}$ in $(\mathbb{Z}_m^X)^{\wedge\wedge}$. □

We shall close this section with some additional observations related to Lemma 3.1.

Remark 3.2. *Let X and $\mathbb{A}$ be as in Lemma 3.1 and let, for $\phi \in (\mathbb{A}^X)^{\wedge}$, $C(\phi)$ be the set C associated in this lemma with the singleton $\Phi = \{\phi\}$. Then the map $\phi \to C(\phi)$ is lower-semicontinuous with respect to the topology τ_p of pointwise convergence in $(\mathbb{A}^X)^{\wedge}$.*

Indeed, let $C(\phi) \cap U \neq \emptyset$, where $\phi \in (\mathbb{A}^X)^{\wedge}$ and $U \subset X$ is open. By the definition of $C(\phi)$, there is a clopen $V \subset U$ such that $\phi(c_V)$ is not the unit in $\mathbb{A}$ and let $\mathcal{W} = \{\psi \in (\mathbb{A}^X)^{\wedge} : \psi(c_V)$ is not the unit in $\mathbb{A}\}$. Then $\mathcal{W}$ is a neighbourhood of ϕ in $((\mathbb{A}^X)^{\wedge}, \tau_p)$ and for any $\psi \in \mathcal{W}$, c_V can not vanish on $C(\psi)$ by Lemma 3.1, i.e., $C(\psi)$ intersects V and hence $C(\psi) \cap U \neq \emptyset$.

Proposition 3.3. *Let X be a strongly zero-dimensional paracompact space and let $\mathbb{A}$ be either $\mathbb{Z}$ or $\mathbb{Z}_m$. Then each homomorphism $\phi : \mathbb{A}^X \to \mathbb{T}$ continuous on every compact set in $\mathbb{A}^X$ is continuous, i.e., the topological group $(\mathbb{A}^X)^{\wedge}$ is complete.*

Proof. Let $\phi : \mathbb{A}^X \to \mathbb{T}$ be a homomorphism continuous on compact sets in $\mathbb{A}^X$ and let C be the set associated in Lemma 3.1 with the singleton $\Phi = \{\phi\}$. The reasoning in the proof of this lemma shows that sequential continuity of ϕ yields compactness of C.

It is enough to check that ϕ takes the neighbourhood $\{f \in \mathbb{A}^X : f|C = 0\}$ of zero in $\mathbb{A}^X$ to the unit in $\mathbb{T}$.

Let $f \in \mathbb{A}^X$ and $suppf \cap C = \emptyset$. Then each $x \in suppf$ has a neighbourhood U_x such that $\phi(c_V) = 1$, whenever $V \subset U_x$ is clopen. Strong zero-dimensionality and paracompactness of X provide a collection $\mathcal{V}$ of pairwise disjoint clopen sets in X such that $suppf = \cup\mathcal{V}$ and $\mathcal{V}$ refines $\{U_x : x \in X\}$.

The subspace of $\mathbb{A}^X$ consisting of functions constant on each member of $\mathcal{V}$ and vanishing outside of $suppf$ can be identified with the product $\mathbb{Z}^{\mathcal{V}}$ equipped with the pointwise topology. Since the restriction $\phi|\mathbb{Z}^{\mathcal{V}} \to \mathbb{T}$ is continuous on each compact set in $\mathbb{Z}^{\mathcal{V}}$, it is continuous on $\mathbb{Z}^{\mathcal{V}}$, by a theorem of Noble [No2], Theorem 5.6.

But ϕ takes value 1 on a a dense subset of $\mathbb{Z}^{\mathcal{V}}$ consisting of functions with infinite supports, hence ϕ is constant on $\mathbb{Z}^{\mathcal{V}}$, and in particular $\phi(f) = 1$.

□

4. A countable prodiscrete non-reflexive group

We shall define the group G described in Example 1.2. To that end, for natural numbers $p, q \geq 1$, we let

$$a_{pq} = \left(\frac{1}{p}, \frac{1}{p+q}\right), b_{pq} = \left(\frac{1}{p}, \frac{1}{p+q} - 1\right), A_{pq} = \{a_{pq}, b_{pq}\}, \tag{7}$$

and let

$$X = \bigcup_{p,q} A_{pq} \cup \{b_p : p = 1, 2, ...\} \cup \{a\}, b_p = \left(\frac{1}{p}, -1\right), a = (0, 0), \tag{8}$$

be a subspace of the euclidean plane.

Our G is the subgroup of the topological group $\mathbb{Z}^X$, defined as follows:

$$G = \{g \in \mathbb{Z}^X : g \text{ is constant on each } A_{pq} \text{ and } g(a) = g(b_p) = 0 \text{ for } p = 1, 2, ...\}. \tag{9}$$

Since $\mathbb{Z}^X$ is prodiscrete, so is its (pointwise) closed subgroup G. Let us check that

$$\text{if } g \in G \text{ then } suppg \text{ is finite.} \tag{10}$$

Indeed, since $g(a) = 0$, by continuity there is p_0 with $g(a_{pq}) = 0$ for $p \geq p_0$ and by (9) and (7), also $g(b_{pq}) = 0$ for $p \geq p_0$. Since $g(b_p) = 0$, again by continuity and the fact that g is constant on each A_{pq}, for any $p \leq p_0$ there is q_p such that g vanishes on $\bigcup_{q \geq q_p} A_{pq}$. It follows that g vanishes at all but finitely many points.

We shall verify next that

$$\text{all compact sets in } G \text{ are finite.} \tag{11}$$

Let $K \subset G$ be compact. To begin with, let us make sure that

$$\{p : g(a_{pq}) \neq 0 \text{ for some } q \text{ and } g \in K\} \text{ is finite.} \tag{12}$$

For otherwise, we could pick $p_1 < p_2 < \dots$ such that, for some q_i and $g_i \in K$, $g_i(a_{p_i q_i}) \neq 0$. But, since $a_{p_i q_i} \to a$ and $g_i(a) = 0$, this violates equicontinuity of K at a.

Now, aiming at a contradiction, assume that the compact set K is infinite. Then, by (10) and (12), one can find p and sequences $q_1 < q_2 < \dots$, $g_i \in K$ such that $g_i(a_{pq_i}) \neq 0$. But then, by (9) and (7), also $g_i(b_{pq_i}) \neq 0$. However, $b_{pq_i} \to b_p$ and $g_i(b_p) = 0$, which contradicts equicontinuity of K at b_p.

With (11) in hand, we are ready to demonstrate discontinuity of the evaluation $\Gamma : G \to G^{\wedge\wedge}$, cf. (1).

For each p, q, let $\phi_{pq} : G \to \mathbb{T}$ be defined by

$$\phi_{pq}(g) = (-1)^{g(a_{pq})},\ \phi_{pq} \in G^{\wedge}. \tag{13}$$

If $g \in G$, then $\phi_{pq}(g) = 1$, whenever $a_{pq} \notin suppg$ and therefore, by (10), each neighbourhood of the identity of $G^{\wedge}$, in the pointwise topology, contains all but finitely many ϕ_{pq}. By (11), the pointwise topology in $G^{\wedge}$ coincides with the compact-open topology, and therefore

$$\mathcal{K} = \{\phi_{pq} : p, q = 1, 2, \dots\} \cup \{\mathbf{1}\} \text{ is compact in } G^{\wedge}, \tag{14}$$

where $\mathbf{1}$ stands for the indentity in $G^{\wedge}$. Let

$$\mathcal{V} = \{\gamma \in G^{\wedge\wedge} : |\gamma(\phi) - 1| < 1 \text{ for } \phi \in \mathcal{K}\} \tag{15}$$

be a basic neighbourhood of the identity in $G^{\wedge\wedge}$ determined by $\mathcal{K}$.

For an arbitrary basic neighbourhood of zero in G, $\mathcal{U} = \{g \in G : g|C = 0\}$, where $C \subset X$ is compact, one can find p, q such that $A_{pq} \cap C = \emptyset$, cf. (7) and (8). Let g be the characteristic function of A_{pq}. Then $g \in \mathcal{U}$, but $\phi_{pq}(g) = -1$, cf. (13), i.e., $\Gamma(g)(\phi_{pq}) = -1$ and hence $\Gamma(g) \notin \mathcal{V}$, cf. (15).

Remark 4.1. *The group $G^{\wedge\wedge}$ is discrete.*

Indeed, let for $z \in \mathbb{T}$, $p, q = 1, 2, \dots.$, $\phi_{z,pq} \in G^{\wedge}$ be defined by $\phi_{z,pq}(g) = z^{g(a_{pq})}$, cf. (13).

One can check that $\mathcal{K}' = \{\phi_{z,pq} : z \in \mathbb{T}, p, q = 1, 2, \dots.\}$ is compact in $G^{\wedge}$, cf. (14), and since $G^{\wedge\wedge} = \Gamma(G)$, $\mathcal{V}' = \{\gamma \in G^{\wedge\wedge} : |\gamma(\phi) - 1| < 1 \text{ for } \phi \in \mathcal{K}'\}$, cf.(15), contains only the identity of $G^{\wedge\wedge}$.

Non reflexivity of G can be also deduced from (11), non-discreteness of G (verified at the end of our proof) and results of Aussenhofer [Au1], Corollary 4.7 and J. M. Chasco [Ch], Theorem 1. Indeed, since G is countable, (11) implies that $G^{\wedge}$ is metrizable, and by these results, $G^{\wedge\wedge}$ is a k-space.

The construction in this section can be also used to the following effect.

Example 4.2. *There exists $Y \subset \mathbb{R}^3$ such that $\mathbb{Z}^Y$ is not reflexive, but both groups $\mathbb{T}^Y$ and $\mathbb{R}^Y$ are reflexive.*

To define Y, let us fix for each natural p a connected subset E_p of the square $[0, 1] \times [\frac{1}{p+1}, \frac{1}{p}]$ without non-trivial continua such that E_p hits the boundary of the square exactly at the points $(0, \frac{1}{p})$, $(1, \frac{1}{p})$, and let E be a connected subset of $[0, 1] \times [0, 1]$ without non-trivial continua such that the only points from E on the boundary of the square are $(0,0)$ and $(\frac{1}{p},1)$, $p=1,2,...$, cf. [En], 6.3.23. We let

$$Y = E\times\{0\}\cup\bigcup_{p=1}^{\infty}\bigcup_{q=1}^{\infty}\left\{\frac{1}{p}\right\}\times E_{3pq}.$$

Let us notice that $L = Y\cap([0,1]\times\{0,1\}\times[0,1])$ is homeomorphic to the space X, cf. (8). Now, the restriction map $f\to f|L$ takes $H=\{f\in\mathbb{Z}^Y : f(0,0,0)=0\}$ homeomorphically onto a subgroup of $\mathbb{Z}^L$ which can be identified with the topological group G, cf. (9), as each $f\in\mathbb{Z}^Y$ is constant on every component $E\times\{0\}$ or $\{\frac{1}{p}\}\times E_{3pq}$ of the space Y. Therefore, $\mathbb{Z}^Y = H\oplus\mathbb{Z}$ is not reflexive.

On the other hand, Y contains no non-trivial continuum and, by a theorem of Galindo and Hernandez [GH], $\mathbb{T}^Y$ is reflexive. Also, by theorem of Hernandez and Uspenskij [HU], $\mathbb{R}^Y$ is reflexive.

5. Analyticity of $(\mathbb{Z}^X)^\wedge$

We shall prove Theorem 1.3. Throughout this section, X is a zero-dimensional separable metrizable space.

For each $\phi\in(\mathbb{Z}^X)^\wedge$, $C(\phi)$ is the set of points in X whose each neighbourhood contains a clopen set V with $\phi(c_V)\neq 1$, cf. Lemma 3.1.

Let τ_p be the pointwise topology in $(\mathbb{Z}^X)^\wedge$. Since the compact-open topology is stronger than τ_p, implication $(i)\to(ii)$ in Theorem 1.3 is evident.

To explain why $(ii)\to(iii)$, let us fix a continuous surjection $p:\mathbb{P}\to((\mathbb{Z}^X)^\wedge,\tau_p)$ on the irrationals $\mathbb{P}$ and let $\sigma:\mathbb{P}\to\mathcal{K}(X)$, cf. section 2, be defined by $\sigma(t)=C(p(t))$. By Remark 3.2, the map is Borel.

Let us check that

$$\mathcal{K}(X)=\sigma(\mathbb{P}). \tag{16}$$

For any nonempty $C\in\mathcal{K}(X)$ (notice that $C(1)=\emptyset$ for for the unit 1 in $(\mathbb{Z}^X)^\wedge$), let $a_1,a_2,...$ be dense in C, and let $\phi\in(\mathbb{Z}^X)^\wedge$ be defined by $\phi(f)=exp\left(i\sum_n 2^{-n}f(a_n)\right)$. Then $C(\phi)=C$ and since $\phi=p(t)$ for some $t\in\mathbb{P}$, we get (16).

We infer from (16) that the space $\mathcal{K}(X)$ of compact sets in X is analytic and by a theorem of Christensen [Chr1] this yields complete metrizability of X.

It remains to check that $(iii)\to(i)$, i.e., we have to derive analyticity of $(\mathbb{Z}^X)^\wedge$ from complete metrizability of X. To that end, let us associate to each compact set $C\subset X$ the set

$$\Phi(C)=\{\phi\in(\mathbb{Z}^X)^\wedge : C(\phi)\subset C\}, \tag{17}$$

and let us show that for the restriction map $r_C:\mathbb{Z}^X\to\mathbb{Z}^C$, $r_C(f)=f|C$,

$$\Phi(C)=r_C^\wedge((\mathbb{Z}^C)^\wedge),\ r_C^\wedge(\alpha)=\alpha\circ r_C. \tag{18}$$

Indeed, if $\phi = \alpha \circ r_C$, $\alpha \in (\mathbb{Z}^C)^\wedge$, then $\phi(f) = 1$ for all $f \in \mathbb{Z}^X$ that vanish on C, hence by Lemma 3.1, $C(\phi) \subset C$. On the other hand, if $\phi \in (\mathbb{Z}^X)^\wedge$ and $C(\phi) \subset C$, again by Lemma 3.1, the formula $\alpha(r_C(f)) = \phi(f)$ defines $\alpha \in \mathbb{Z}^C$, as $r_C(f) = r_C(g)$ implies $\phi(f - g) = 1$.

Now, since $\mathbb{Z}^C$ is a discrete group, the dual group $(\mathbb{Z}^C)^\wedge$ is compact and the compact-open topology in $(\mathbb{Z}^C)^\wedge$ coincides with the pointwise topology. The map $r_C^\wedge : (\mathbb{Z}^C)^\wedge \to (\mathbb{Z}^X)^\wedge$ is continuous and by (18), we get compactness of $\Phi(C)$.

Now, the correspondence $C \to \Phi(C)$ defined in (17) preserves inclusions, and hence, in terminology of [COT], the space $(\mathbb{Z}^X)^\wedge$ is determined by X. Since $(\mathbb{Z}^X)^\wedge$ has a countable net, cf. [Mi], a theorem of B. Cascales and J. Orihuela [CO] guarantees analyticity of $(\mathbb{Z}^X)^\wedge$.

Remark 5.1. *Let X be a zero-dimensional, completely metrizable, separable space. Then $(\mathbb{Z}^X)^\wedge$ is a compact - covering image of the irrationals, i.e., there exists a continuous surjection $p : \mathbb{P} \to (\mathbb{Z}^X)^\wedge$ such that each compact set in $(\mathbb{Z}^X)^\wedge$ is an image under p of a compact set in $\mathbb{P}$.*

Indeed, by Lemma 3.1, each compact set in $(\mathbb{Z}^X)^\wedge$ is contained in some compact set $\Phi(C)$, cf. (17) (i.e., in terminology of [COT], $(\mathbb{Z}^X)^\wedge$ is strongly determined by X). Therefore, the results in [OC] provide a completely metrizable subspace E of $\mathcal{K}(\mathcal{K}(X)) \times (\mathbb{Z}^X)^\wedge$ whose projection onto the second axis is compact-covering. Since E is a compact covering image of $\mathbb{P}$, so is $(\mathbb{Z}^X)^\wedge$.

In particular, the map $\phi \to \phi \circ p$ embeds $(\mathbb{Z}^X)^\wedge$ isomorphically and topologically onto a closed subgroup of $\mathbb{T}^{\mathbb{P}}$, cf. Proposition 3.3.

6. Comments

6.1. **Evaluation at points of X.** Let X be a zero-dimensional k-space, and let $e : X \times \mathbb{T} \to (\mathbb{Z}^X)^\wedge$ be defined by $e(x, z)(f) = z^{f(x)}$.

One can check that e is continuous and the group generated by $e(X \times \mathbb{T})$ is dense in $(\mathbb{Z}^X)^\wedge$.

Since $\mathbb{Z}^X$ is prodiscrete, the map $\Gamma : \mathbb{Z}^X \to (\mathbb{Z}^X)^{\wedge\wedge}$ is an isomorphism, cf. section 2, and therefore, for each $\gamma \in (\mathbb{Z}^X)^{\wedge\wedge}$ there is unique $f \in \mathbb{Z}^X$ such that $\Gamma(f) = \gamma$. In particular, $\gamma(e(x, z)) = e(x, z)(f) = z^{f(x)}$.

Similarly, considering $e_m : X \to (\mathbb{Z}_m^X)^\wedge$ defined by $e_m(x)(f) = f(x)$, one can check that e_m is continuous, $e_m(X)$ generates a dense subgroup of $(\mathbb{Z}_m^X)^\wedge$ and to each character $\gamma \in (\mathbb{Z}_m^X)^{\wedge\wedge}$ corresponds unique $f \in \mathbb{Z}_m^X$ such that $\gamma(e(x)) = f(x)$.

6.2. **The group G and k_R-property for $\mathbb{Z}_2^X$ with X metrizable.** A non-reflexive prodiscrete group can not be a k-space, cf. [No1], and our definition of G was inspired by some constructions in [Po] and [GTZ] concerning function spaces which are not k-spaces.

From hindsight, some essential elements of this construction resemble also a construction of W.Banaszczyk [Ba], 17.7, 17.9 of a countable non-reflexive closed subgroup of $\mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{\mathbb{N}}$, cf. sec. 2 (this group is not prodiscrete).

In the construction of G, $\mathbb{Z}$ can be replaced by $\mathbb{Z}_m$. Let us notice also that for X described in section 4, one can consider in $\mathbb{Z}_2^X$ clopen sets

$$\mathcal{W}(p, q) = \{f \in \mathbb{Z}_2^X : f|A_{pq} = 1, f(a) = 0, f(b_q) = 0\},$$

and then the union $\mathcal{W} = \bigcup_{p,q} \mathcal{W}(p,q)$ is an open set in $\mathbb{Z}_2^X$ whose intersection with every compact set is closed, but $0 \in \overline{\mathcal{W}} \setminus \mathcal{W}$.

This shows that $\mathbb{Z}_2^X$ is not a k_R-space (the characteristic function of $\mathcal{W}$ being continuous on each compact set in $\mathbb{Z}_2^X$ but not at zero), and in effect, one can add to the conditions in Corollary 3.11 from the paper by G.Gruenhage, B. Tsaban and L. Zdomsky [GTZ] an equivalent statement about k_R-property.

6.3. **Embedding the group G into $(\sum \mathbb{Z})^{2^{\aleph_0}}$.** Let $\mathcal{C}$ be the collection of compact sets in the space X defined in section 4, and let for $C \in \mathcal{C}$, $r_C : \mathbb{Z}^X \to \mathbb{Z}^C$ be the restriction map. Then the diagonal map $\triangle_{C\in\mathcal{C}} r_C : \mathbb{Z}^X \to \prod_{C\in\mathcal{C}} \mathbb{Z}^C$ is an isomorphic and topological embedding onto a closed subgroup of the product. Since, for infinite $C \in \mathcal{C}$, C has only finitely many accumulation points, the discrete group $\mathbb{Z}^C$ is isomorphic to the Specker group $\sum \mathbb{Z}$ (recalled in sec. 2), cf. [Au2], Corollary 3.6 and [GRT], Problem 11.

6.4. **Abelian groups $C(X,\mathbb{Z})$.** In a series of penetrating papers [EKiO], [EKaO], [Oh], the authors investigated the algebraic structure of abelian groups $C(X,\mathbb{Z})$ of continuous integer-valued functions on zero-dimensional spaces.

In particular, extending classical results of Specker, they established for zero-dimensional X, that each homomorphism $h : C(X,\mathbb{Z}) \to \mathbb{Z}$ has a compact support $C(h)$ (i.e., $h(f) = 0$, whenever f vanishes on $C(h)$) if, and only if, X is $\mathbb{N}$-compact (i.e., X embeds onto a closed subspace of some power of natural numbers), cf. [EKiO], Corollary 6.7, and that, for $\mathbb{N}$-compact spaces X, Y, any homomorphism $h : C(X,\mathbb{Z}) \to C(Y,\mathbb{Z})$ is continuous with respect to the compact-open topology in the function spaces, cf. [Oh].

It was also proved in [EKaO], using subtle combinatorical reasonings, that the groups $C(\mathbb{P},\mathbb{Z})$ and $C(\mathbb{Q},\mathbb{Z})$ are not isomorphic.

Let us notice that this follows instantly from the result in [Oh] that an algebraic isomorphism between such topological groups is also a topological isomorphism and Theorem 1.3.

In fact, one can derive from [Oh] and a theorem of Christensen [Chr2] a stronger result: *if $h : C(X,\mathbb{Z}) \to C(Y,\mathbb{Z})$ is a surjective homomorphism, X and Y are zero-dimensional separable metrizable spaces, and X is completely metrizable, so is Y.*

Indeed, by [Oh], h is continuous with respect to the compact-open topology in the function spaces, which allows one to associate with each compact set $K \subset Y$ a compact set $S(K) \subset X$ - a minimal compact set such that, whenever $f \in C(X,\mathbb{Z})$ vanishes on $S(K)$, $h(f)$ vanishes on K (such set $S(K)$ is uniquely determined by K).

Moreover, for each compact set $L \subset X$, since $\mathbb{Z}^L$ is countable and $\mathbb{Z}^{\mathbb{N}}$ is uncountable, one can check that the set $T(L) = \{y \in Y : S(\{y\}) \subset L\}$ is compact.

This defines a correspondence $L \to T(L)$ from $\mathcal{K}(X)$ to $\mathcal{K}(Y)$ which preserves inclusions and for each compact set $K \in \mathcal{K}(Y)$, $K \subset T(S(K))$, cf. section 5. By a theorem of Christensen [Chr2], complete metrizability of X yields complete metrizability of Y.

This scheme resembles some reasonings concerning linear continuous maps between spaces $C_p(X)$, $C_p(Y)$ of real-valued continuous functions, equipped with the pointwise topology, cf. Baars, de Groot and Pelant [BdGP], but we did not see formal connections.

It can be also noticed that if $h : C(X,\mathbb{Z}) \to C(Y,\mathbb{Z})$ is a surjective homomorphism, X and Y are zero-dimensional separable metrizable spaces and X is σ-compact, so is Y.

Indeed, Theorem 1 in [Oh] shows that h is a continuous map from $\mathbb{Z}^X$ onto $\mathbb{Z}^Y$, and by a theorem of Christensen [Chr1], σ-compactness of X yields analyticity of $\mathbb{Z}^X$, hence analyticity of $\mathbb{Z}^Y$ and in effect, again by the Christensen theorem, σ-compactness of Y.

A strengthening of Christensen's theorem due to J. Calbrix [Ca] allows one to extend this result to a wider class of spaces.

6.5. **A remark on P-groups.** Let $(\mathbb{Z}_2^\tau)_\delta$ be the product of τ copies of $\mathbb{Z}_2$, equipped with the G_δ-topology, i.e., the topology whose base are G_δ-sets in the product space $\mathbb{Z}_2^\tau$ (in [GRT], section 2, this space is denoted by $P\mathbb{Z}_2^\tau$).

Galindo, Recoder-Núñez and Tkachenko proved that the group $(\mathbb{Z}_2^\tau)_\delta$ is reflexive, cf. [GRT], Theorem 3.4 and asked, if this is also the case for subgroups G of $(\mathbb{Z}_2^\tau)_\delta$ such that $|\mathbb{Z}_2^\tau/G| \le \aleph_0$, containing all elements of $\mathbb{Z}_2^\tau$ with countable support, cf. [GRT], Problem 9.

We shall note that an observation made in [BP], combined with some results from [GRT], provides under CH a positive answer, even if $|\mathbb{Z}_2^\tau/G| \le \aleph_1$.

To see this, let us consider $E \subset (\mathbb{Z}_2^\tau)_\delta$ such that $\aleph_1$ translates of E cover $\mathbb{Z}_2^\tau$ (i.e., $E+A = \mathbb{Z}_2^\tau$ for some $A \subset \mathbb{Z}_2^\tau$ of cardinality $\le \aleph_1$).

Then the space E can not be covered by $\aleph_1$ nowhere dense sets in $(\mathbb{Z}_2^{\aleph_1})_\delta$, as the space $(\mathbb{Z}_2^{\aleph_1})_\delta$ has this property, cf. [BP], $(*)$ on page 266. Since the weight of $(\mathbb{Z}_2^{\aleph_1})_\delta$ is continuum, under CH, E has a dense set of cardinality $\le \aleph_1$. Therefore, repeating a reasoning from [BP], page 266, showing that under CH, $(\mathbb{Z}_2^{\aleph_1})_\delta$ is a Namioka space, one concludes that E is a Namioka space.

Now, let G be a subgroup of $(\mathbb{Z}_2^{\aleph_1})_\delta$ with $|\mathbb{Z}_2^{\aleph_1}/G| \le \aleph_1$ and let us assume CH. Then, as we have just noticed, G is a Namioka space.

Let $K \subset G^\wedge$ be a compact. Since compact sets in G are finite, $G^\wedge$ is equipped with the pointwise topology and the Namioka property applied to the duality map $<,>: G \times K \to \mathbb{Z}_2$, $< g, \phi >= \phi(g)$ provides a nonempty open set V in G such that the functions in K are constant on V, and the elements of K being characters, one can take as V a neighbourhood of the identity in G. Using Corollary 3.3 from [GRT] one concludes that G is reflexive.

Finally, let τ be an abitrary uncountable cardinal and let G be a subgroup of $(\mathbb{Z}_2^\tau)_\delta$ such that $|\mathbb{Z}_2^\tau/G| \le \aleph_1$, containing all elements in $\mathbb{Z}_2^\tau$ with countable support. Then, for each $J \subset \tau$ of cardinality $\le \aleph_1$, and the projection $\pi_J : \mathbb{Z}_2^\tau \to \mathbb{Z}_2^J$, $|\mathbb{Z}_2^J/\pi_J(G)| \le \aleph_1$ and hence assuming CH, we infer that $\pi_J(G)$ is reflexive. By Theorem 4.9 in [GRT], we conclude assuming CH, that G is reflexive.

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