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Strongly generic sets in real Lie groups

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Abstract

We give a description of strongly generic subsets in definably compact o-minimal torus $S^1 \times S^1$ as well as in the definably simple $SL(2,\mathbb{R})$. Then we discuss the saturated case and give some results related to endomoprhisms induced by the minimal subflows of the group $SL(2,-)$.

1 Introduction

In [2, 3, 4], Newelski described the applications of topological dynamics in model theory. In this section, we recall the basic definitions and results, leading up to the notion of strongly generic sets and putting them in the context of o-minimal setting. This exposition will be brief and succint: we encourage the reader to refer to [2], [3] as well as [4] for a more in-depth explanation, and motivation for the work.

First we recall some classical notions from the topological dynamics. By a (point-transitive) *G-flow* we mean a left action of the group G on a compact topological space X by homeomorphisms that contains a dense orbit. Each $g \in G$ defines a homeomorphism $\pi_g : X \to X$, an element of the topological space X^X (with the usual product topology). We let $E(X)$ to be the closure of the set $\{\pi_g : g \in G\}$. The structure $(E(X), *)$ with $*$ denoting the operation of function composition is a semigroup and we call it the *Ellis semigroup* of the G-flow. The $E(X)$ itself is also a G-flow.

Important objects associated with an Ellis semigroup $E(X)$ are its minimal ideals, denoted by $I \lhd_m E(X)$. These ideals are sub-semigroups and they turn out to be exactly its *minimal G-subflows*: minimal nonempty closed subsets which are closed under the G-action. Every minimal ideal I is the closure of the G-orbit of p for any $p \in I$. Each $p \in E(X)$ such that $\mathrm{cl}(Gp)$ is a minimal subflow is called *almost periodic*. A point $p \in E(X)$ is called an *idempotent* if $p * p = p$.

For $I \lhd_m E(X)$, let $J(I)$ be the set of idempotents of I. We have that $I = \mathrm{cl}(Gu)$ for any $u \in J(I)$. Moreover,

$$I = \dot{\bigcup}_{u \in J(I)} uI,$$

where every $(uI, *)$ is a group (with the group identity being u), and all those groups are isomorphic to each other, even for different I. We call these groups the *ideal subgroups*.

We now put these notions in (for now, general) model theoretic context. Fix an arbitrary structure M, a group G definable in it, and an $|M|^+$-saturated $\mathfrak{C} \succ M$. We say that a set $U \subset M$ is *externally definable* if it is of the form $U' \cap M$ for some $\mathfrak{C}$-definable $U' \subset \mathfrak{C}$. We denote the set of all externally definable subsets of $G(M)$ by $Def_{ext,G}(M)$. We let $S_{ext,G}(M) = S(Def_{ext,G}(M))$, the set of all ultrafilters on $Def_{ext,G}(M)$. The elements of $S_{ext,G}(M)$ are called *external types*.

A family $\mathcal{A} \subset P(G)$ is said to be a *G-algebra* if it is closed under the boolean operations and under taking of G-translates. We note that $Def_{ext,G}(M)$ is a $G(M)$-algebra. The set $S_{ext,G}(M)$ with the natural $G(M)$-action on it forms a $G(M)$-flow. It has been shown in [3] that it is isomorphic to its Ellis semigroup. The semigroup operation $*$ can be given explicitly as follows: for $\mathcal{U}_1, \mathcal{U}_2 \in S_{ext,G}(M)$,

$$U \in \mathcal{U}_1 * \mathcal{U}_2 \Leftrightarrow \{g \in G(M) : g^{-1}U \in \mathcal{U}_2\} \in \mathcal{U}_1.$$

Let $\mathcal{E}_G(M) = \mathrm{End}_G(Def_{ext,G}(M))$, the semigroup of G-endomorphisms of the G-algebra $Def_{ext,G}(M)$, with the function composition as the semigroup operation. Any $p \in S_{ext,G}(M)$ determines a $d_p \in \mathcal{E}_G(M)$ given by

$$d_p(U) = \{g \in G(M) : g^{-1}U \in p\}.$$

It has been shown that $d : S_{ext,G}(M) \to \mathcal{E}_G(M)$ is in fact a semigroup isomorphism.

Recall the definitions of some canonical objects from the model theory:

Definition. *Let $U \subset G(M)$.*

(i) U is called (left) generic if finitely many of its (left) $G(M)$-translates cover $G(M)$.

(ii) U is said to be weakly generic if there exists a non-generic $V \subset G(M)$ such that $U \cup V$ is generic.

(iii) A formula $\phi(x) \in L(M)$ is said to be (weakly) generic if the set $\phi(M) \cap G(M)$ is (weakly) generic.

(iv) A collection p of sets (formulas) which is closed under intersections (conjunctions) is called (weakly) generic if each $U \in p$ is (weakly) generic.

Note that the property of being generic is elementary, while that of being nongeneric (or weakly generic) is not.

Paper [4] introduced the notion of *strongly generic* subsets, the main object of interest in this paper:

Definition. *A set $U \subset G(M)$ is called strongly generic if the G-algebra generated by U consists only of generic sets, and the empty set.*

The motivation for introducing this notion comes from the interpretation of topological dynamics concepts in the model theoretical setting. The G-flow $S_{ext,G}(M)$ contains minimal ideals, split into ideal subgroups. Each such an ideal is generated by an almost periodic $p \in S_{ext,G}(M)$. It has been shown in general that p is weakly generic. Crucial to understanding p, the minimal ideal generated by p, and its ideal subgroups are the image and the kernel of the induced endomorphism d_p. As it turns out, the image of any $d_p \in \mathcal{E}_G(M)$ for an almost periodic p is a $G(M)$-algebra of externally definable strongly generic sets, and every externally definable strongly generic set is in the image of some such d_p.

The results in [3] indicate strong connections between the ideal subgroups and G/G^{00} (in particular, in the stable case these two are isomorphic). They also relate externally definable strongly generic subsets to G^{00}. We have, for instance, that if there are boundedly many externally definable strongly generic sets in G, then G^{00} exists and equals G^{∞}.

In this paper we aim to explore the notion of strongly generic sets in definable o-minimal groups. So our ambient theory is $\mathrm{Th}(\mathbb{R}, +, \cdot, 0, 1)$. By the classic result of Pillay [5, Proposition 2.5], a definable o-minimal group is a group manifold. In particular, such a group interpreted over $\mathbb{R}$ is a real Lie group. Moreover, any $p \in S_G(\mathbb{R})$ is definable. We recall the following results:

Fact 1. *Let G be an o-minimal group. Then $Def_{ext,G}(\mathbb{R}) = Def_G(\mathbb{R})$ and every $p \in S_G(\mathbb{R})$ extends uniquely to a $p' \in S_{ext,G}(\mathbb{R})$. The sets $S_G(\mathbb{R})$ and $S_{ext,G}(\mathbb{R})$ are isomorphic as $G(\mathbb{R})$-flows.*

Fact 2. *Let G be a definably compact o-minimal group and $\mathbb{R} \prec M$.*

(i) Every weakly generic $p \in S_{ext,G}(M)$ is generic.

(ii) Every generic $p \in S_{ext,G}(\mathbb{R})$ extends to a unique generic $p' \in S_{ext,G}(M)$. Moreover, for every every $I \lhd_m S_{ext,G}(\mathbb{R})$, the set $I' = \{p' \in S_{ext,G}(M) : p'|_{Def_{ext,G}(\mathbb{R})} \in I\}$ is a minimal subflow of $S_{ext,G}(M)$.

(iii) Every externally definable strongly generic $U \subset G(M)$ is of the form $V \cdot G^{00}(M)$ where V is a definable strongly generic subset of $G(\mathbb{R})$.

The rest of this paper is organized as follows. In the second section, we examine the definably compact case of $S^1 \times S^1$, where externally definable strongly generic sets admit a particularly nice description. We also indicate about how the result generalizes to general definably compact case. The third section extends the work done in [1] to describe the case of $SL(2, \mathbb{R})$.

2 The $S^1 \times S^1$

In this section we describe the externally definable strongly generic subsets of $G = S^1 \times S^1$ (with the canonical group operation). Since this is a definably compact o-minimal group, by Fact 2 we can work in $\mathbb{R}$ to obtain the complete description of strongly generic sets, and by Fact 1 we can work with the $G(\mathbb{R})$-flow $S_G(\mathbb{R})$ rather than $S_{ext,G}(\mathbb{R})$. We can freely identify $p \in S_G(\mathbb{R})$ with the corresponding ultrafilter on $Def_G(\mathbb{R})$. We are interested in images of endomorphisms d_p induced by weakly generic $p \in S_G(\mathbb{R})$. We know that these types are in fact generic. Observe that for any $h \in G(\mathbb{R}), U \in Def_G(\mathbb{R})$,

$$\begin{aligned} d_{hp}(U) = \{g \in G(M) : g^{-1}U \in hp\} &= \{g \in G(M) : (gh)^{-1}U \in p\} = \\ \{g \in G(M) : g^{-1}U \in p\}h^{-1} &= d_p(U)h^{-1}. \end{aligned}$$

Since $\mathrm{Im}(d_p(U))$ is a $G(\mathbb{R})$-algebra, this shows that $\mathrm{Im}(d_p(U))$ and $\mathrm{Im}(d_{hp}(U))$ are $G(\mathbb{R})$-conjugates, so they are equal, as G is abelian. Note that every $p \in S_G(\mathbb{R})$ is a $G(\mathbb{R})$-translate of some "infinitesimal" p' contained in G^{00}. So we can restrict our attention to the generic types there in order to describe the algebras $\mathrm{Im}(d_p)$.

Before we proceed, we recall the notion of *Hardy field* in the o-minimal setting. Consider the set $\mathfrak{F}$ of all $\mathbb{R}$-definable functions mapping $(0, \infty)$ to itself. Given $f, g \in \mathfrak{F}$, there is a small $\varepsilon \in \mathbb{R}^+$ such that $f|_{(0,\varepsilon)}$ and $g|_{(0,\varepsilon)}$ are either equal, or one majorizes the other. We write $f \sim g$ in the former case, or one of $f > g$, $f < g$ in the latter. Clearly $\sim$ is an equivalence relation, and the relation $<$ linearly orders the set $\mathfrak{F}_0 = \mathfrak{F}/\sim$. We call $\mathfrak{F}_0$ the Hardy field of germs at 0. As the name suggests, it comes equipped with a field structure. However, here we are only interested in its order properties. Let $\mathfrak{F}_0^0 = \{[f]_\sim \in \mathfrak{F}_0 : \lim_{x\to 0^+} f(x) = 0\}$. Note that $[f]_\sim \in \mathfrak{F}_0^0$ also implies $\lim_{x\to 0^+} f(x) = 0$. Let $\mathfrak{G}$ be the set of *Dedekind cuts* of $\mathfrak{F}_0^0$. We identify each cut with a pair (C, D) where C and D partition $\mathfrak{F}_0^0$ and $C < D$. We note that $\mathfrak{F}_0^0$ itself is not Dedekind complete: it is easy to see that in the cut (C, D) where $C = \{[f]_\sim \in \mathfrak{F}_0^0 : \lim_{x\to 0^+} f'(x) \le 0\}$, the sets C and D have no maximum and minimum respectively.

The real torus is isomorphic to the quotient $\mathbb{R}^2/\mathbb{Z}^2$ where $\mathbb{R}^2$ with addition is its universal cover. For the remainder of the section, we identify the real torus with the set $[0, 1) \times [0, 1)$. Let $Y = [0, 1/2) \times [0, 1/2)$. Adding elements of the torus represented by points in D is faithfully represented in the cover. Consider a generic $p \in S_G(\mathbb{R})$ contained in G^{00}. p has a realization (α, β) in a sufficiently saturated $R^* \succ \mathbb{R}$, with α, β infinitesimally close to one of the corners of $[0, 1) \times [0, 1)$. It is enough to restrict our attention to types where α, β are both infinitesimally close to $(0, 0)$, that is generics in $Y \cap G^{00}$, because for any other type r, an analogous analysis can be done by considering different Y that contains a G-translate of r that is contained in G^00. As p is generic, $\alpha \notin \operatorname{dcl}(\beta)$, meaning that β is not a value of any $\mathbb{R}$-definable function at α.

For $\mathbb{R}$-definable $f, g : (0, \infty) \to (0, \infty)$ such that $f|_{(0,\delta)} < g|_{(0,\delta)}$ for some $\delta \in (0, 1/2)$ and for $\varepsilon \in (0, \delta)$ we define a nonempty, definable cell

$$U_{f,g,\varepsilon} = \{(x, y) \in \mathbb{R}^2 : x \in (0, \varepsilon), f(x) < y < g(x)\}.$$

Now consider a cut $(C, D) \in \mathfrak{G}$. Let

$$\Phi_{C,D} = \{U_{f,g,\varepsilon} : [f]_\sim \in C, [g]_\sim \in D, f \text{ is majorized by } g \text{ on } (0, \varepsilon) \text{ for a small } \varepsilon \in \mathbb{R}^+\}.$$

Lemma 3. *For any $(C, D) \in \mathfrak{G}$, the set $\Phi_{C,D}$ generates a complete, generic $\mathbb{R}$-type in G^{00}.*

Proof. Clearly $\Phi_{C,D}$ is closed under finite intersections and therefore consistent, with realizations contained in $G^{00}(\mathbb{R})$. First we show that $\Phi_{C,D}$ does not extend to a nongeneric type over $\mathbb{R}$. For if $(\alpha, \beta) \in Y(R^*)$ satisfies a nongeneric type over $\mathbb{R}$, there is an $\mathbb{R}$-definable function f such that $f(\alpha) = \beta$ and $[f]_\sim \in C$ or $[f]_\sim \in D$. In either case we use f to find a $U \in \Phi_{C,D}$ disjoint with the graph of f.

Now consider a generic type $p \in S_G(\mathbb{R})$ consistent with $\Phi_{C,D}$. Every $U \in p$ has a subset of the form $U_{f,g,\varepsilon}$ also belonging to p. But then $U_{f,g,\varepsilon}$ belongs to $\Phi_{C,D}$. □

So a cut $(C,D) \in \mathfrak{G}$ determines a generic $\mathbb{R}$-type in $Y \cap G^{00}$. At the same time, every generic $\mathbb{R}$-type in $Y \cap G^{00}$ realized in R^* by (α, β) is generated by the set $\Phi_{C,D}$ where $C = \{[f]_\sim : f(\alpha) > \beta\}$. So we have a description of generics in $Y \cap G^{00}$ by means of $\mathfrak{G}$. We now describe the images of the endomorphisms d_p for p generic in $Y \cap G^{00}$. As each d_p is a $G(\mathbb{R})$-endomorphism and the set $Def_G(\mathbb{R})$ is generated (as a $G(\mathbb{R})$-algebra) by cells contained in Y, it is enough to describe the images of these.

So fix a generic $p \in S_G(\mathbb{R})$ generated by $\Phi_{C,D}$ for $(C,D) \in \mathfrak{G}$ and a cell $U \subset Y$. If U is not generic, then no $G(\mathbb{R})$-translate of it belongs to p, so $d_p(U) = \emptyset$. So assume that U is generic. Without loss of generality we assume that it is open. Then U is of the form

$$U = \{(x,y) : f(x) < y < g(x), x \in I\}$$

for a suitable interval $I = (c', d') \subset (0, 1/2)$ and definable $f, g : I \to [0, 1/2)$. We want to describe the set

$$d_p(U) = \{g \in G(\mathbb{R}) : g^{-1}U \in p\}.$$

First consider $h \in U$. As $h^{-1}U$ contains a real neighbourhood of $(0,0)$, it belongs to p. By similar argument, any $h \notin \mathrm{cl}(U)$ does not belong to $d_p(U)$. So we have that $U \subset d_p(U) \subset \mathrm{cl}(U)$. We are left with points $h \in \partial U$. Consider a point $h = (a,b) \in Y$ contained in the closure of the graph of f and consider $h^{-1} \cap Y$. If $a = d'$ it is the empty set and $h \notin d_p(U)$, otherwise it is the set

$$U_{\hat{f},\hat{g},\varepsilon} = \{(x,y) \in \mathbb{R}^2 : x \in (0,\varepsilon), \hat{f}(x) < y < \hat{g}(x)\},$$

where $\hat{f}(x) = \max(f(x+a) - b, 0)$ and $\hat{g}(x) = g(x+a) - b$. We have that $\lim_{x \to 0^+} \hat{f}(x) = 0$. Now if also $\lim_{x \to 0^+} \hat{g}(x) = 0$ then $U_{\hat{f},\hat{g},\varepsilon} \in d_p(U)$ iff $U_{\hat{f},\hat{g},\varepsilon} \in \Phi_{C,D}$, i.e. $[\hat{f}]_\sim \in C$ and $[\hat{g}]_\sim \in D$. Otherwise $U_{\hat{f},\hat{g},\varepsilon}$ contains some $U_{f',g',\delta} \in \Phi_{C,D}$ iff $[\hat{f}]_\sim \in C$.

A similar analysis holds for the points contained in the closure of the graph of g and for the remaining points of ∂U. So we have shown

Proposition 4. *Let $p \in G^{00} \cap Y$ be generic $\mathbb{R}$-type. Such a type is generated by the family $\Phi_{C,D}$ determined by a cut $(C,D) \in \mathfrak{G}$. Then the $G(\mathbb{R})$-algebra*

$\mathrm{Im}(d_p)$ *is generated by all sets of the following form:*

$$\begin{aligned}\{(x,y)\in Y: 0\le x<\varepsilon, f(x)<y<g(y)\} &\cup\\ \{(x,y)\in Y: 0\le x<\varepsilon, f(x)=y=g(y), f_{x,y}\in C, g_{x,y}\in D\} &\cup\\ \{(x,y)\in Y: 0\le x<\varepsilon, f(x)<y=g(y), g_{x,y}\in D\} &\cup\\ \{(x,y)\in Y: 0\le x<\varepsilon, f(x)=y<g(y), f_{x,y}\in C\}&\end{aligned}$$

where $\varepsilon\in[0,1/2)$, $f,g:[0,\varepsilon]\to[0,1/2)$ *are* $\mathbb{R}$*-definable, and*

$$\begin{aligned}f_{x,y}(x',y') &= \max(f(x+x')-y',0),\\ g_{x,y}(x',y') &= g(x+x')-y'.\end{aligned}$$

Note that if p is determined by a cut (C,D) when D has the minimum h, the condition that $[\hat{f}]_\sim\in C$ is just saying that h majorizes $\hat{f}$ on some small interval $(0,\varepsilon)$, and similarly when C has the maximum. However by an earlier remark not all cuts enjoy this property.

3 The $SL(2,\mathbb{R})$

In this section we elaborate on the results found in [1] concerning the topological structure of the $G(\mathbb{R})$-flow $S_G(\mathbb{R})$ for $G=SL(2,-)$, the group of 2×2 matrices with determinant 1. Its authors described the minimal subflow $I\lhd_m S_{ext,G}(\mathbb{R})$ and its ideal subgroups which turn out to be isomorphic to $\mathbb{Z}_2$. Given that in this case we have $G=G^{00}$, this provides an example where the ideal subgroups are not isomorphic to G/G^{00}. We will recall the results from [1], but first we need to set up the notation.

We consider the subgroup $H(\mathbb{R})<G(\mathbb{R})$ consisting of matrices $\begin{pmatrix} b & c\\ 0 & b^{-1}\end{pmatrix}$ where $b>0$ and $c\in\mathbb{R}$ and the subgroup $T(\mathbb{R})<G(\mathbb{R})$ consisting of matrices $\begin{pmatrix} x & -y\\ y & x\end{pmatrix}$. The subgroup T is isomorphic to S^1. We have that every $g\in G$ can be uniquely written as th as well as $h't'$ with $t,t'\in T$ and $h,h'\in H$.

At this point we want to explore the subgroup $H(\mathbb{R})$ and the quotient $G(\mathbb{R})/H(\mathbb{R})$ with the induced $G(\mathbb{R})$-action in more detail. We proceed in a similar way as it was done in [1], but without passing to the projective line $\mathbb{P}^1=T(\mathbb{R})/\{-I,I\}$. $SL(2,\mathbb{R})$ acts definably on the real plane $\mathbb{R}^2$ by orientation-preserving linear transformations, i.e.

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} ax+by\\ cx+dy\end{pmatrix}.$$

Now consider the space $\bar{\mathbb{R}}^2 = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\sim$ be the equivalence relation that identifies points a and b if there is a scalar $r \in \mathbb{R}^+$ such that $ra = b$. Let $V(\mathbb{R})$ be the quotient $\bar{\mathbb{R}}^2/\sim$. We can identify $V(\mathbb{R})$ with the unit circle $S^1(\mathbb{R}) \subset \mathbb{R}^2$. The $G(\mathbb{R})$ action on $\mathbb{R}^2$ induces a well-defined action on $V(\mathbb{R})$. We have $Stab_G((1,0)) = H(\mathbb{R})$ so we can identify $G(\mathbb{R})/H(\mathbb{R})$ with the set $V(\mathbb{R})$ by sending $g \in G(\mathbb{R})$ to the $\sim$-class of $g \cdot (1,0)$. We let $\pi : G(\mathbb{R}) \to V(\mathbb{R})$ to be the projection map. Its restriction $\pi|_{T(\mathbb{R})} : T(\mathbb{R}) \to V(\mathbb{R})$ is a homeomorphism. The projection induces a mapping of $S_G(\mathbb{R})$ onto $S_V(\mathbb{R})$ which we also denote by π. $G(\mathbb{R})$ acts transitively on $V(\mathbb{R})$. It makes sense to talk about *arcs* in $V(\mathbb{R})$. Fix an orientation on $S^1(\mathbb{R})$ such that points close to $(1,0)$ with positive second coordinate are on the "positive" side of $(1,0)$. Under the identification of $V(\mathbb{R})$ and $S^1(\mathbb{R})$ we can consider arcs of the form (a,b), $[a,b], [a,b)$ or $(a,b]$ on $V(\mathbb{R})$. Given the homeomorphism between $V(\mathbb{R})$ and $T(\mathbb{R})$ and the induced orientation on $T(\mathbb{R})$, we can also use the notion of arcs inside $T(\mathbb{R})$.

Crucial for obtaining some later results are the following basic facts about the $G(\mathbb{R})$-action on $V(\mathbb{R})$:

Lemma 5. *Fix any $R^* \succ \mathbb{R}$. Then*

(i) The $G(R^*)$-action on $V(R^*)$ preserves the orientation: for any $g \in G(R^*)$ and arc $[a,b) \subset V(R^*)$, the set $g[a,b)$ is also an arc $[c,d)$ for some c,d.

(ii) For any arc L in $V(R^*)$, there is $g \in G(R^*)$ such that $L \cup gL = V(R^*)$.

(iii) For every $g \in G(R^*)$, the mapping $\pi_g : V(R^*) \to V(R^*)$ defined by $\pi_g(x) = g \cdot x$ is a homeomorphism.

Proof. First we note that statements (i)-(iii) of the lemma are elementary as all arcs are uniformly definable. So we only need to prove it for $R^* = \mathbb{R}$.

(i) We may identify a set $X \subset S^1(\mathbb{R})$ with $C_X = \mathbb{R}^+ X \subset \bar{\mathbb{R}}^2$ and consider the action of $G(\mathbb{R})$ on $\bar{\mathbb{R}}^2$. For an arc $[a,b)$ the set $C_{[a,b)}$ is the area spanned between the lines $\mathbb{R}^+ a$ and $\mathbb{R}^+ b$, including the first and excluding the second. As $G(\mathbb{R})$ acts on $\mathbb{R}^2$ (and $\bar{\mathbb{R}}^2$) by linear transformations that preserve the orientation, for any $g \in G(\mathbb{R})$ the set $gC_{[a,b)}$ is of the form $C_{[c,d)}$ for some c,d and thus $g[a,b) = [c,d)$.

(ii) This is similar to the previous point. Every arc L contains an arc of the form $[a,b)$, and for such an arc there is an orientation+preserving linear transformation mapping $C_{[a,b)}$ to $C_{[b,a)}$. Therefore there is $g \in G(\mathbb{R})$ such that $[a,b) \cup g[a,b) = V(\mathbb{R})$.

(iii) This is trivial. □

Now we give the main results from [1]. Let $p_0 \in S_H(\mathbb{R})$ to be the type of the matrix $\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix}$ with b positive infinite and c positive infinite over b (meaning that c is infinite over $\mathbb{R} \cup \mathrm{dcl}(b)$). Let $q_0 \in S_T(\mathbb{R})$ be the type of the matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ with y infinitesimal and positive, and x positive. Finally let $r_0 = \mathrm{tp}(th/\mathbb{R})$ where $h \models p_0$, and t realizes the (unique) coheir of q_0 over h. Let $I = \mathrm{cl}(G(\mathbb{R})r_0)$.

Proposition 6. *[1, Corollary 3.8, Theorem 3.17] With the notation as above,*

(i) I is a minimal ideal in $S_G(\mathbb{R})$.

(ii) The restriction of the projection map π to the ideal I maps it homeomorphically to the minimal ideal $S_{V,na}(\mathbb{R})$ consisting of all nonalgebraic types in $S_V(\mathbb{R})$.

(iii) r_0 is an idempotent.

(iv) $r_0 * I$ is isomorphic to $\mathbb{Z}_2$.

The structure of $S_{V,na}(\mathbb{R})$ is well understood (see for example [4]): it is the union of two $T(\mathbb{R})$-orbits $T(\mathbb{R})p_+$ and $T(\mathbb{R})p_-$ where p_+ and p_- are respectively the types of $\pi(t/H)$ and $\pi(t^{-1}/H)$ for t realizing q_0. So by Proposition 6(ii), I is also the union of two $T(\mathbb{R})$-orbits for some $p', p'' \in S_G(\mathbb{R})$. We know that we can take p' to be r_0, with $\pi(r_0) = p_+$. As for the other type, define $r_1 = \mathrm{tp}(t^{-1}h/\mathbb{R})$ where $h \models p_0$, and t realizes the (unique) coheir of q_0 over h. The proofs from [1] all work with r_1 in place of r_0 and show that r_1 is an idempotent. We have that $\pi(r_1) = p_-$. So we take $p'' = r_1$. As the group $T(\mathbb{R})$ is sufficient to traverse the orbits of r_0 and r_1, Lemma 5 will now prove useful.

In order to understand I and the associated endomorphisms $d_p, p \in I$, we first consider their kernels. Write $t_{x,y}$ for $\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in T$ and $h_{b,c}$ for $\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} \in H$. We can also unambiguously write t_y for $t_{x,y}$ where $x > 0$.

Lemma 7. *A definable $U \subset G(\mathbb{R})$ belongs to r_0 iff for all sufficiently small $z \in \mathbb{R}^+$ we have $U \cap t_z H \in t_z p_0$.*

Proof. From right to left, let $t_y h_{b,c} \models r_0$. We have that $h_{b,c} \models p_0$ and t_y realizes a coheir of q_0 over b, c. Consider the formula $\phi(z) = h_{b,c} \in t_z^{-1}U \cap H$

definable over $\mathbb{R}, b, c$. By the assumtpion, $\phi(\mathbb{R})$ contains some small $(0,\varepsilon)^{\mathbb{R}}$. Let $R^* \succ \mathbb{R}$ contain b and c but not y and consider $\phi(R^*)$. This set must contain an interval $(\alpha, \beta) \supset (0,\varepsilon)^{\mathbb{R}}$ with $\alpha > 0$ infinitesimal and not infinite over c. But t_y is a coheir of q_0 over $\mathbb{R}, b, c$, meaning that y is infinitesimal over $\mathbb{R}$ but infinite over c, so also infinite over α. So $R^{**} \models \phi(y)$ for some $R^{**} \succ R^*$ and $t_y h_{b,c} \in U \cap t_y H$.

For the other direction, by definability of types over $\mathbb{R}$ the condition on the right is definable. Assume that it fails. Then for all sufficiently small $y \in \mathbb{R}^+$ we have $U \cap t_y H(\mathbb{R}) \notin t_y p_0$. Then an analogous argument shows that $U \notin r_0$. □

Corollary 8. *Let $t \in T(\mathbb{R})$, and let a definable $U \subset G(\mathbb{R})$ be of the form $XH(\mathbb{R})$ where $X \subset T(\mathbb{R})$. Then $U \in tr_0$ iff X contains an arc (t, s). Similarly, $U \in tr_1$ iff X contains an arc (s, t).*

Proof. The first statement is equivalent to saying that $t^{-1}U \in r_0$ iff X contains an arc $(I, t^{-1}s)$. But this what Lemma 7 asserts for the sets of the form $XH(\mathbb{R})$. □

We can now give the description of kernel of the endomorphism d_{r_0}.

Proposition 9. *The kernel $\mathrm{Ker}(d_{r_0})$ consists of all definable $U \subset G(\mathbb{R})$ such that the set*

$$\{t \in T(\mathbb{R}) : U \cap tH \in tp_0\}$$

is finite.

Proof. Fix a definable $U \subset G(\mathbb{R})$. We have $U \in \mathrm{Ker}(d_{r_0})$ iff no $G(\mathbb{R})$-translate of U belongs to r_0. So suppose $\{t \in T(\mathbb{R}) : U \cap tH(\mathbb{R}) \in tp_0\}$ is not finite. Again by definability of types over $\mathbb{R}$ it contains an open arc $(t, t') \subset T(\mathbb{R})$. Then by Lemma 7, $t^{-1}U$ belongs to r_0, so $U \notin \mathrm{Ker}(d_{r_0})$.

Now assume that $\{t \in T(\mathbb{R}) : U \cap tH(\mathbb{R}) \in tp_0\}$ is finite. Clearly, any $T(\mathbb{R})$-translate of U does not belong to r_0 so we only need to show that for $H(\mathbb{R})$-translates. Fix an $h \in H(\mathbb{R})$. Let

$$C = \{t \in T(\mathbb{R}) : hU \cap tH(\mathbb{R}) \in tp_0\} \quad = \\ \{t \in T(\mathbb{R}) : U \cap h^{-1}tH(\mathbb{R}) \in h^{-1}tp_0\}.$$

There are $t' = t'(t) \in T(\mathbb{R}), h' = h'(t) \in H(\mathbb{R})$ such that $h^{-1}t = t'h'$, so C is equal to

$$\{t \in T(\mathbb{R}) : U \cap t'h'H(\mathbb{R}) \in t'h'p_0\}.$$

Now as p_0 is $H(\mathbb{R})$-invariant and of course $h'H = H$, C is equal to

$$\{t \in T(\mathbb{R}) : U \cap t'H(\mathbb{R}) \in t'p_0\}.$$

But by Lemma 5(iii) the mapping $t \mapsto t'$ induced by h is a bijection and C is finite by the assumption. $\square$

By [3, Corollary 1.10], this described kernel is common to all endomorphisms $d_p, p \in I$.

We now turn our attention to the image algebras $\mathrm{Im}(d_p), p \in I$. Consider the $G(\mathbb{R})$-algebra $\mathcal{A}_+$ generated by all sets of the form $[t, t')H$ where $[t, t')$ is an arc in $T(\mathbb{R})$, and the $G(\mathbb{R})$-algebra $\mathcal{A}_-$ generated by all sets of the form $(t, t']H$ where $(t, t']$ is an arc in $T(\mathbb{R})$.

Lemma 10. *Every $U \in \mathcal{A}_+$ is of the form XH where X is a union of finitely many arcs in $T(\mathbb{R})$ of the form $[a, b)$. Moreover, if U is nonempty then it is 2-generic, i.e. $G(\mathbb{R})$ can be covered by at most two $G(\mathbb{R})$-translates of U.*

Proof. Clearly all finite unions of such arcs belong to $\mathcal{A}_+$. It is also clear that the intersection of two sets of such form, as well as complement of such a set, is also of that form. We are left with checking that this is also true for $G(\mathbb{R})$-translates of these sets. But it is enough to check this for a single arc, and this is proven in Lemma 5(i).

For the moreover part, let $U \in \mathcal{A}_+$ be nonempty. Then U contains the set LH for some arc $L \subset T(\mathbb{R})$. By Lemma 5(ii) there is $g \in G(\mathbb{R})$ such that $gLH = L^c H$ and so LH (and therefore U) is 2-generic. $\square$

Proposition 11. *$\mathcal{A}_+$ is the $G(\mathbb{R})$-algebra $\mathrm{Im}(d_{r_0})$.*

Proof. $\mathcal{A}_+$ can be generated using sets of the form LH where L is of the form $[I, t) \subset T(\mathbb{R})$. Let $U = LH$ for L of such form. We show that $d_{r_0}(U) = U$. Consider any $t'h' \in G(\mathbb{R})$ for $t' \in T(\mathbb{R})$, $h' \in H(\mathbb{R})$. We have

$$t'h' \in d_{r_0}(U) \Leftrightarrow h'^{-1}t'^{-1}[I, t)H \in r_0 \quad \Leftrightarrow \\ h'^{-1}[t'^{-1}, t'^{-1}t)H \in r_0.$$

By 8, the last formula is true if and only if $I \cdot H \in h^{-1}[t'^{-1}, t'^{-1}t)H$. But as as h'^{-1} fixes H, this is equivalent to $1 \cdot H \in [t'^{-1}, t'^{-1}t)H$, which means $t' \in [I, t)$. So we have shown that $t'h' \in d_{r_0}(U)$ iff $t'h' \in U$. Thus $\mathcal{A}_+ \subset \mathrm{Im}(d_{r_0})$. Now consider any $p \in I, p \neq r_0$. p is of the form tr_0 or tr_1 with $t \in T(\mathbb{R})$. Again by Corollary 8, for each such p there exists $LH \subset T(\mathbb{R})$

that omits it. Thus the $G(\mathbb{R})$-algebra $\mathcal{A}_+$ separates the types of I so by [4, Proposition 2.7], $\mathcal{A}_+$ is a full image algebra. By [3, Remark 1.11] it must be precisely $\mathrm{Im}(d_{r_0})$. □

The corollary below is proven entirely analogous.

Corollary 12. $\mathcal{A}_-$ *is the* $G(\mathbb{R})$*-algebra* $\mathrm{Im}(d_{r_1})$.

Theorem 13. *Every externally definable, strongly generic subset of* $SL(2, \mathbb{R})$ *is a conjugate (equivalently: right translation) of some* $U \in \mathcal{A}_+ \cup \mathcal{A}_-$.

Proof. Working over $\mathbb{R}$, we consider only definable sets. By [4, Lemma 1.2] it is enough to describe image algebras d_p for p contained in any single minimal ideal in $S_G(\mathbb{R})$. We have that I consists of $G(\mathbb{R})$-orbits of r_0 and r_1, and that $\mathcal{A}_+ = \mathrm{Im}(d_{r_0})$ and $\mathcal{A}_- = \mathrm{Im}(d_{r_1})$. By definition, $\mathrm{Im}(d_{gp}) = \mathrm{Im}(d_p)g^{-1}$, and we are done. □

We conclude by providing the description of some definable strongly generic subsets in arbitrary $R^* \succ \mathbb{R}$. Recall from the proof of Proposition 11 that in $SL(2, \mathbb{R})$, every $U \in \mathrm{Im}(d_{r_0})$ is 2-generic and that by Theorem 13 the same is true for all strongly generic sets in $SL(2, \mathbb{R})$. Fix any $R^* \succ \mathbb{R}$. We show that the scheme for producing strongly generic sets developed previously in this section works for R^* as well.

Let the $G(R^*)$-algebra $\mathcal{A}^*_+$ be generated by all sets of the form $[t, t')H$ where $[t, t')$ is an arc in $T(R^*)$, and the $G(R^*)$-algebra $\mathcal{A}^*_-$ be generated by all sets of the form $(t, t']H$ where $(t, t']$ is an arc in $T(R^*)$. Remembering that Lemma 5 holds in R^*, Lemma 10 can be applied word by word to $\mathcal{A}^*_+$ and $\mathcal{A}^*_-$.

Corollary 14. *(i) Every* $U \in \mathcal{A}^*_+$ *is of the form* XH *where* X *is a union of finitely many arcs in* $T(R^*)$ *of the form* $[a, b)$*. If moreover* U *is nonempty, then it is* 2*-generic.*

(ii) Every $U \in \mathcal{A}^*_-$ *is of the form* XH *where* X *is a union of finitely many arcs in* $T(R^*)$ *of the form* $(a, b]$*. If moreover* U *is nonempty, then it is* 2*-generic.*

This shows that there are unboundedly many definable strongly generic subsets of $SL(2, R^*)$, as also for an infinitisemial arc $[a, b) \subset T(R^*)$ the set XH is 2-generic.

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