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Notes on valuation theory

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Abstract

The paper is a survery on various notions and results for valued fields from both purely algebraic and model theoretical points of view. The main object of investigations presented here is that of $AKE^{\exists}$ Principle, a model theoretical notion considered for (elementary) classes of valued fields. Other topics include uses of pseudo-Cauchy sequences, and additive polynomials in the field $\mathbb{F}_p((t))$.

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Preliminaries

We assume that the reader is familiar with the basic notions of valuation theory (ordered and valued fields; their valuations, valuation rings, groups, ideals and residue fields; real closed fields; Hensel's Lemma) and model theory. We refer for example to [2, 6, 21] for more details. We begin by setting notations and conventions used throughout these notes. Given a valued field (K, v) and $a \in K$, we denote the value of a as va and its residue class as av. We denote the valuation group of (K, v) by vK and its residue field by Kv. The valuation ring of (K, v) is denoted by $\mathcal{O}_v$ and its unique maximal ideal by $\mathfrak{M}_v$. We will often use the shorthand writing $(F/K, v)$ meaning the extension of valued fields $(K, v|_K) \subset (L, v)$. The theory of a valued field is understood to be its first order theory in the language of valued fields, with different sorts for the field, its residue field and its value group.

There are two conventions in regard to the order on the valuation group and the valuation axioms. In the first, the valuation group of a valued field is majorized by ∞, the value of 0, with the valuation $v : K \to vK \cup \{\infty\}$ satisfying the following axioms:

(A) $v(a) = \infty \iff a = 0$,

(B) $v(ab) = v(a) + v(b)$,

(C) $v(a + b) \geq \min\{v(a), v(b)\}$.

This convention is used by Krull (also Prestel, Ribenboim, Kuhlmann and others). We will be using this convention thruought these notes.

An alternative convention inverts the order on the valuation group, minorized by an element 0, the value of 0, with an appropriate set of axioms. This convention is used for example by Boer and Preiss-Crampe.

We use the standard algebraic and model theoretic notations and symbols. Below we give the definitions for the less obvious ones.

Given an ordered abelian group $(G, <)$, for $g \in G$ denote

$$|g| = \begin{cases} g & \text{if } g \geq 0, \\ -g & \text{otherwise.} \end{cases}$$

For $n \in \omega$ we write na meaning $\underbrace{a + a + \ldots + a}_{n \text{ times}}$. Given a group G, we denote by $\widetilde{G}$ its divisible hull, i.e. $\mathbb{Q} \otimes G$.

From the model theoretical symbols, if M, N, N_0 are structures, $M \prec_\exists N$ denotes that M is existentially closed in N, and $M \underset{M_0}{\hookrightarrow} N$ means that M embeds into N over M_0. If we want to explicitly name the embedding, we write $\phi : M \underset{M_0}{\hookrightarrow} N$ with ϕ being the elementary embedding.

1 Natural valuation

Definition 1.1. *Two elements $g, h \in G$ are said to be* ***archimedean equivalent*** *if and only if there is an $n \in \omega$ such that $n|a| > |b|$ and $n|b| > |a|$.*

It is easy to see that archimedean equivalence is an equivalence relation. For $g \in G$ the equivalence class of g is denoted by vg and called the **archimedean class of** g. The set of archimedean classes can be ordered as follows: for $g, h \in G$ let

$$vh < vg \iff 0 \leq g < h \text{ and } \forall n \in \omega \; ng < h.$$

Denote $vG = \{vg : g \in G, g \neq 0\}$ and $\infty = v0$.

Fact 1.2. *The map $v : G \to vG \cup \{\infty\}$ satisfies the following properties:*

(1) $vg = 0 \iff g = 0$,

(2) $v(g - h) \geq \min\{vg, vh\}$,

Remark 1.3. *(i) A map satisfying the conditions* $(1) - (2)$ *is called a* ***group valuation****. The order on the classes and the valuation axioms are consistent with the definitions for valued fields, under the Krull's convention.*

(ii) Replacing the condition (2) *with* $v(g + h) \geq \min\{vg, vh\}$ *yields a strictly weaker definition.*

(iii) It follows from the conditions that $vg = v(-g)$ *and that* $v(g - h) = \min\{vg, vh\}$ *if* $vg \neq vh$*.*

Proof of Fact 1.2. (1) is clear. For (2), without loss of generality assume $vg \leq vh$, i.e. $\exists n \in \omega\ n|g| \geq |h|$. We have

$$|g - h| \leq |g| + |h| \leq (n + 1)|g|,$$

so there is $m \in \omega$ such that $|g - h| \leq m|g|$, and so $v(g - h) \leq vg$ as needed. □

Now let $(K, <)$ be an ordered field. Take the natural valuation of its additive group and define the group operation on the set $vK = \{vg : g \neq 0\}$ by

$$vg + vh = v(gh).$$

The fact that addition is well defined and compatible with $<$ both follow from the definition of archimedean classes. We have:

Fact 1.4. *The mapping* $v : K^\times \to vK$ *is a field valuation.*

The natural valuation on K is compatible with the ordering. That is, for $a, b \in K$ we have

$$0 < a < b \implies va \geq vb.$$

Recall that a field K is **formally real** if -1 cannot be expressed as a sum of squares. By a result of Artin and Schreier, this is equivalent to saying that the field is **orderable** (i.e. admits a total order compatible with the field operations).

The following is a classic result:

Proposition 1.5. *Let* (K, v) *be a valued field. Then the following are equivalent:*

(1) v *is compatible with the field ordering.*

(2) $\mathcal{O}_v$ *is convex.*

(3) $\mathfrak{M}_v$ *is convex.*

(4) $1 + \mathfrak{M}_v$ *is convex.*

(5) The residue field Kv can be ordered if and only if Kv is formally real.

The natural valuation on K is a canonical object associated with K. We will show that it is also (in some sense) a universal object among the valuations on K that are compatible with the order.

Definition 1.6. *Let v and w be valuations on K. Then w is a* ***coarsening*** *of v if $\mathcal{O}_v \subset \mathcal{O}_w$.*

This is easily proven:

Fact 1.7. *The natural valuation v on an ordered field K is the finest valuation compatible with $<$. That is, any other valuation compatible with $<$ is a coarsening of v.*

Likewise, given a valuation v on K we can obtain a description of all orderings compatible with it. Let

$$\begin{aligned} X_K^v &= \{\text{orderings on } K \text{ compatible with } v\}, \\ X_{Kv} &= \{\text{orderings on } Kv\}. \end{aligned}$$

Proposition 1.8. *There is a bijection between X_K^v and the set $X_{Kv} \times \mathrm{Hom}(vK/2vK, \mathbb{Z}_2)$.*

Proof. Consider the group $K^\times/{K^\times}^2$. As a group with the exponent 2 it is a linear space over $\mathbb{Z}_2$. Similarly, the group $vK/2vK$ is a linear space over $\mathbb{Z}_2$. The valuation map v induces an epimorphism $r : K^\times/{K^\times}^2 \to vK/2vK$ which splits, i.e. there is a homomorphism s such that $r \circ s = \mathrm{id}$. Take a basis $B = \{\beta_i\}_i$ of $vK/2vK$ and put $b_i = s(\beta_i)$. The construction of the bijection will depend on the choice of s and B.

Take an ordering $\prec$ on Kv and a character $\sigma : vK/2vK \to \mathbb{Z}_2$. We construct an ordering of K determined by $\prec$ and σ. Take any $a \in K$. For some $b_1, \dots b_n \in B$ we have

$$\begin{aligned} va + 2vK &= vb_1 + \ldots + vb_n + 2vK, \\ va &= vb_1 + \ldots + vb_n + 2vd \text{ for some } d, \\ va &= v(b_1 \ldots b_n d^2). \end{aligned}$$

So for some invertible u (with a value of 0) we have $a = b_1 \ldots b_n d^2 u$. Note that in any ordering, every square must be positive. So the sign of a depends only on the signs on b_i's and u. We let

$$\begin{aligned} u > 0 &\iff 0 \preceq uv, \\ b_i > 0 &\iff \sigma(\beta_i) = 1. \end{aligned}$$

It is left to the reader to see that the ordering is well defined and that different characters on $vK/2vK$ and orderings on Kv give different orderings on K.

To show that the construction is indeed bijective, take any ordering $<$ on K compatible with v. Let P be the positive cone of this ordering. Then $\overline{P} =$

$(P \cap \mathcal{O}_v)v$ is the positive cone of an ordering $\prec$ on Kv. Now define a character on $vK/2vK$ by

$$\sigma(\beta_i) = 1 \iff b_i > 0.$$

Then $\prec$ and σ determine $<$. □

Given (K, v) where v is natural, clearly the residue field $Kv = k$ is archimedean. This is equivalent to saying that every two nonzero elements of k have the same value. We have a classic result:

Proposition 1.9 (Hölder)**.** *An ordered field (K, v) is archimedean if and only if it is a subfield of $(\mathbb{R}, <)$. Similarly, an abelian ordered group $(G, <)$ is archimedean if and only if it is a subgroup of $(\mathbb{R}, +, <)$.*

Recall that a **real closed field** is a field K such that every polynomial of odd degree has a root in K and that for every $a \in K$, either a or $-a$ is a square. A real closed field is ordered and of characteristic 0. There are several characterizations of real closed fields. A model theoretical one is that K is a real closed field if and only if it is elementarily equivalent to $(\mathbb{R}, \cdot, +)$. The following theorem justifies the "real closed" term from the algebraic point of view:

Theorem 1.10 (Artin-Schreier)**.** *Let K be a field. The following are equivalent:*

(1) K is real closed.

(2) No proper algebraic extension of K can be ordered.

(3) $K^{acl} = K(i)$ for $i^2 = -1$.

(4) The extension K^{acl}/K is finite.

The next theorem provides a connection between the notion of real closedness and valuations on an ordered field:

Theorem 1.11 ([7, Lemmata 4, 5, Theorem 3])**.** *Let $(K, <)$ be an ordered field and v a valuation on K compatible with $<$. Then K is real closed if and only if*

(1) Kv is real closed.

(2) vK is divisible.

(3) v is henselian.

We note that if $(K, <)$ is real closed then the set of nonnegative elements is exactly the set of all squares. Thus $<$ is the unique ordering on K. In particular there is a unique ordering on $\mathbb{R}$. By Theorem 1.10, the algebraic closure $\mathbb{C}$ of $\mathbb{R}$ (the only proper algebraic extension) cannot be ordered. It is natural to ask about the orderings on nonalgebraic extensions of $\mathbb{R}$.

So consider the transcendence degree 1 extension $\mathbb{R}(X)$ and any ordering on it. We will show that such an ordering is nonarchimedean. Let

$$\begin{aligned} D &= \{r \in \mathbb{R} : r < X\}, \\ E &= \{r \in \mathbb{R} : r > X\}. \end{aligned}$$

The pair (D, E) is the **cut** of X in $\mathbb{R}$. By the cut completeness of $\mathbb{R}$, if both $D, E \neq \emptyset$ then either D has the maximum d or E has the minimum e. Then (appropriety) either $X - d$ or $e - X$ is **positive infinitesimal**, i.e. positive but smaller than any positive $r \in \mathbb{R}$.

If either D or E is empty, then either $1/x$ or $-1/x$ is positive infinitesimal. In each case, the existence of infinitesimal elements means that the order is not archimedean.

For the last thing in this section we give a way for constructing of large ordered, valued fields and ordered abelian groups. While model theoretic constructions can give arbitrarily large and saturated models of the theory of ordered valued fields, they cannot control their valuation groups or residue fields beyond their first order properties. We would like to have a better control over these canonical objects.

First we show a construction of a valued ordered field K with desired vK and Kv. Let G be an ordered abelian group and let k be a field. Take

$$K(G) := k((x^G)) = \{\sum_{g \in G} c_g x^g : \text{ the set of } g\text{'s such that } c_g \neq 0 \text{ is well-ordered}\}.$$

The set of g's in the definition is called the **support** of the corresponding sequence. The multiplication on the formal symbols x^g is defined by $x^g \cdot x^h = x^{g+h}$ and extends to $K(G)$ in a natural way. This field is equipped with the x-adic valuation. We have $vK(G) = G$ and $(K(G))v = k$. We can also define a lexicographical order on $K(G)$ as follows:

$$a = \sum c_g x^g > 0 \iff c_{va} > 0.$$

The canonical x-adic valuation on $K(G)$ is always henselian. So the construction gives the way of constructing large real closed fields, since by Theorem 1.11, if G is divisible and k is real closed, then $K(G)$ is real closed.

A related construction is that of large ordered abelian groups. Let $(I, <)$ be an ordered set. Let $\{G_i : i \in I\}$ be a family of an ordered abelian groups. We define its **Hahn product** as follows:

$$\mathbf{H}_{i \in I} G_i = \{f \in \prod_{i \in I} G_i : \text{supp}(f) \text{ is well-ordered}\}.$$

The Hahn product has a natural lexicographic ordering that makes it an ordered abelian group that contains each G_i. We note that we can assume that every G_i appearing in a Hahn product is archimedean. This is because every group G is itself a Hahn product of the group's archimedean classes: to se this, take the natural valuation $G \to \Gamma \cup \{\infty\}$ and for $\gamma \in \Gamma$ let

$$\begin{aligned} \mathcal{O}_\gamma &= \{a : va \geq \gamma\}, \\ \mathfrak{M}_\gamma &= \{a : va > \gamma\}. \end{aligned}$$

Then the quotient group $C_\gamma = \mathcal{O}_\gamma / \mathfrak{M}_\gamma$ has the induced archimedean order $<$ and we have

$$G < \mathbf{H}_{\gamma \in \Gamma} C_\gamma.$$

2 AKE Principles

Among the important model theoretical properties of theories are those concerning their model completeness, admissibility of quantifier elimination, and decidability. Their models can be furthermore classified by isomorphism, or elementary equivalence. In this section we look into the case of theory of valued fields. In this case, the two canonical objects in any model are the value group and the residue field. It is natural to ask whether the first order properties of those two objects are sufficient to describe the first order properties of the whole model. Specifically, we ask under what conditions certain statements of varying strength, collectively called the **Ax-Kochen-Ershov (AKE) Principles**, do hold. Given valued fields (K, v) and (L, v), we state:

$$vK \equiv vL \wedge Kv \equiv Lv \implies (K, v) \equiv (L, V). \qquad (AKE^{\equiv} \text{ principle})$$

Also assuming $(K, v) \subset (L, v)$:

$$vK \prec vL \wedge Kv \prec Lv \implies (K, v) \prec (L, V), \qquad (AKE^{\prec} \text{ principle})$$
$$vK \prec_{\exists} vL \wedge Kv \prec_{\exists} Lv \implies (K, v) \prec_{\exists} (L, V). \qquad (AKE^{\exists} \text{ principle})$$

In each principle, the relations between vK and vL are meant to be in the language of ordered groups and those between Kv and Lv in the language of valued fields. The $AKE^{\equiv}$ Principle has been proved for the class of henselian fields of residue characteristic zero [3, 10]. Much work has been done in showing the $AKE^{\exists}$ Principle for models of valued fields. In particular it has been shown for:

(i) models of theory of henselian valued fields with residue characteristic 0 (Ax-Kochen [3], Ershov [10]),

(ii) p-adically closed fields (Ax-Kochen [4]),

(iii) $\mathfrak{p}$-adically closed fields (Prestel-Roquette, [20]),

(iv) finitely ramified henselian fields (Ershov [12], Ziegler [23]),

(v) algebraically maximal Kaplansky fields (Ershov [11], Ziegler [23]),

(vi) tame fields (Kuhlmann [17]).

Remark 2.1. *If $AKE^{\exists}$ holds for the theory of valued fields, then every model of the theory must be henselian and defectless.*

Our immediate aim is to show the $AKE^{\exists}$ Principle for a certain class of fields. First we turn our attention to a classic model theoretical embedding lemma:

Lemma 2.2. *Let T be any theory, let $A, B \models T$ with $A \subset B$, and let A^* be a $|B|^+$-saturated elementary extension of A. We have:*

(i) if $B \underset{A}{\hookrightarrow} A^*$ *, then* $A \prec_\exists B$*,*

(ii) $B \underset{A}{\hookrightarrow} A^*$ *if and only if for every finitely generated substructure* $A \subset B_0 \subset B$ *we have* $B_0 \underset{A}{\hookrightarrow} A^*$*.*

Now let (K, v) and (F, v) be valued fields, and $(K^*, v^*) \succ (K, v)$ be $|F|^+$-saturated. Note that this implies that both v^*K^* and K^*v^* are $|F|^+$-saturated. Assume that $vF \underset{vK}{\hookrightarrow} v^*K^*$ and $Fv \underset{Kv}{\hookrightarrow} K^*v^*$ (by a standard argument this implies $vK \prec_\exists vF$ and $Kv \prec_\exists Fv$). We would like to be able to lift these embeddings to the field itself, i.e. to show that $F \underset{K}{\hookrightarrow} K^*$, resulting in $(K, v) \prec_\exists (F, v)$.

We have the following useful theorem:

Theorem 2.3. *Let* (K, v) *be a valued field and let* $f(\overline{x}, \overline{y}) \in K[x_i, y_j]_{i\in I, j\in J}$*. Then* vf *is the minimum value of the* f*'s monomials.*

Definition 2.4. *The* ***rational rank*** *of an abelian group* Γ *is the maximal number of rationally independent elements in* Γ*. The rational rank of* G *is denoted* $\mathrm{rr}G$*.*

Note that $\mathrm{rr}G = \dim_{\mathbb{Q}} \mathbb{Q} \otimes G$.

Choose $\alpha_i \in vF, i \in I$ rationally independent over vK and $t_j \in Fv, j \in J$ algebraically independent over Kv. Then choose $x_i \in F$ such that $vx_i = \alpha_i, i \in I$ and $y_i \in F$ with $y_jv\ t_jj \in J$. Theorem 2.3 implies that all x_i's and y_j's are algebraically independent over K. This proves:

Proposition 2.5 (Abhyankar inequality [1])**.** *Denoting* $\rho = \rho(K, F) = \mathrm{rr}(vF/vK)$ *and* $\tau = \tau(K, F) = \mathrm{trdeg}(Fv/Kv)$*, we have* $\mathrm{trdeg}(F/K) \geq \rho + \tau$*.*

Now consider a monomial $c\overline{x}^{\overline{\nu}}\overline{y}^{\overline{\mu}}$. We have

$$v(c\overline{x}^{\overline{\nu}}\overline{y}^{\overline{\mu}}) = vc + \overline{\nu}v\overline{x} = vc + \sum_{i\in I} \nu_i vx_i.$$

From this follows that

$$vF \subset vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i.$$

Also, $Fv \subset Kv(t_j : j \in J)$. Thus if we let

$$F_0 = K(x_i, y_j : 0 \leq i \leq \rho, 1 \leq j \leq \tau).$$

then F_0 is a rational function field with value group $vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$ and residue field $Kv(y_iv : 1 \leq i \leq \tau)$. We have that vF/vF_0 is a torsion group and Fv/F_0v is an algebraic field extension.

We can now show that the embeddings $\phi : vF \underset{vK}{\hookrightarrow} v^*K^*$ and $\psi : Fv \underset{Kv}{\hookrightarrow} K^*v^*$ yield $(F_0, v) \underset{(K,V)}{\hookrightarrow} (K^*, v^*)$. For each i choose $x_i^* \in K^*$ such that $v^*(x_i^*) = \phi(vx_i)$ and for each j choose $y_j^* \in K^*$ such that $y_j^*v^* = \psi(y_jv)$. As ϕ and ψ

are embeddings, $\{x_i^*, y_j^*\}$ is a set algebraically independent over K. We define a homomorphism sending x_i to x_i^* and y_i to y_i^*, i.e.

$$f(x_i, y_i) \mapsto f(x_i^*, y_i^*).$$

Then

$$\phi(vf(x_i, y_j)) = v^* f(x_i^*, y_j^*).$$

It is easy to see that this homomorphism is an embedding.

In the special case where F is an extension without a "transcendence defect", the Abhyankar inequality becomes an equality. That is, $\mathrm{trdeg}(F/K) = \rho + \tau$ and $\{x_i : 1 \leq i \leq \rho\} \cup \{y_j : 1 \leq j \leq \tau\}$ is the transcendence basis of the extension F/K. Since F/K is a finitely generated extension, F/F_0 is a finite extension. So also both vF/vF_0, Fv/F_0v are finite extensions. Recall that

$$vF_0 = vK \oplus \bigoplus_{1 \leq i \leq \rho} \mathbb{Z} vx_i.$$

As vF is a finite extension of vF_0, we have

$$vF = \Gamma \oplus \bigoplus_{1 \leq i \leq \rho} \mathbb{Z} \beta_i$$

for $\Gamma \cong vK$ (as $vK \prec_\exists vF$). Thus we can replace the chosen x_i's in such a way that $vx_i = \beta_i$ and get $vF_0 = vF$.

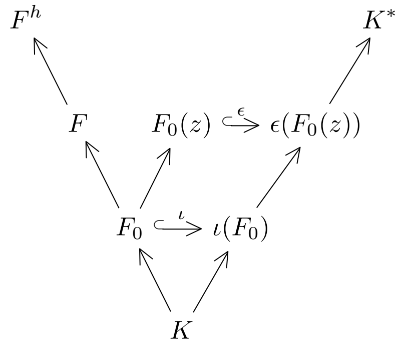
Now, still in the special case, we consider the residue extension. We have that $Kv \prec_\exists Fv \implies Fv/Kv$ is separable.

Proposition 2.6. *A finitely generated separable extension L/K is separably generated, i.e. there is a transcendence basis T such that $L/K(T)$ is a separable algebraic extension.*

By the proposition, we can choose $y_j, 1 \leq j \leq \tau$ such that $\{y_j v\}_j$ is a separating transcendence basis of Fv/Kv, meaning that Fv/F_0v is a finite separable extension. So $Fv = F_0v(\zeta)$ for some $\zeta \in F_0^{acl}$. Take $\overline{f}$ to be the minimal polynomial of ζ over F_0v and f a monic polynomial in $\mathcal{O}_{F_0}[X]$ such that $fv = \overline{f}$.

Lemma 2.7 (Weak Hensel's Lemma)**.** *If $f \in \mathcal{O}_{F_0}[X]$ and $\zeta \in Fv$ are such that ζ is a simple root of f, then in any henselization F^h of F there is a $z \in F^h$ such that $zv = \zeta$ and $f(z) = 0$.*

Now assume that (K, v) is henselian. Thus also (K^*, v^*) is henselian and if we can embed a henselization F^h of F, we can embed F as well. Let ι and ϵ be the respective embeddings of F_0 and $F_0(z)$ into K^* over K. So we have the following diagram:



The polynomial ιf is an element of $\mathcal{O}_{\iota F_0}[X]$. The residue polynomial $(\iota f)v^* = \psi(fv)$ has a simple zero $\psi(\zeta)$. By Weak Hensel's Lemma we can choose $z^* \in K^*$ such that $z^* v^* = \psi(\zeta)$.

$F_0(z)v \ni \zeta$, so

$$\begin{aligned}[F_0(z)v : F_0v] \geq [F_0v(\zeta) : F_0v] = \deg \overline{f} = \deg f \geq \\ \geq [F_0(z) : F_0] \geq (vF_0(z) : vF_0)[F_0(z)v : F_0v] \geq [F_0(z)v : F_0v].\end{aligned}$$

The first and the last terms are the same, so the equality holds everywhere and $[F_0(z) : F_0] = \deg f$. This means that f is the minimal polynomial of z, so it is irreducible. It follows that $F^h/F_0(z)$ is an immediate extension. Necessarily then, the extension $F^h/F_0(z)^h$ is also immediate. As an embedding can be extended to henselization, we have $F_0(z)^h \underset{K}{\hookrightarrow} K^*$. We are done if we can show that $F^h = F_0(z)^h$. Let $d := [F^h : F_0(z)^h]$ (the **defect** of an immediate extension). If $F^h \neq F_0(z)$ then there is a nontrivial finite immediate extension $L/F_0(z)$ such that $[L/F_0(z)^h] = d$. If $d > 1$, this is only possible if $\mathrm{char} Kv > 0$. So if $\mathrm{char} Kv = 0$ we have that $F^h = F_0(z)^h$ and we are done in our special case.

3 The defect and immediate extensions

Let (K, v) be a valued field. Let $(L/K, v)$ be a finite extension of valued fields. Then

$$[L : K] \geq (vL : vK)[Lv : Kv].$$

The following makes this relation more clear:

Definition 3.1. *Let K be a field. The **characteristic exponent** of K is defined as*

$$\mathrm{charexp} K = \begin{cases} 1 & \text{if } \mathrm{char} K > 0, \\ \mathrm{char} Kv & \text{otherwise.} \end{cases}$$

Lemma 3.2 (Lemma of Ostrowski)**.** *Let K, L be valued fields, v a valuation on K and $\{v_0, \ldots, v_g\}$ valuations on L extending v. Then*

$$[L : K] = d \cdot \sum_i e_i \cdot f_i,$$

where $e_i = (v_iL : vK), f_i = [Lv_i : Kv]$ and $d = p^n$ where $p = \text{charexp}Kv$ $n \in \omega$.

This motivates the definition of an important property of a valued field extension:

Definition 3.3. *The value of d in the Lemma of Ostrowski is called the* ***defect*** *of the extension L/K.*

Note that if L/K is a normal algebraic extension of a valued field (K, v) then all valuations $\{v_1, \ldots, v_g\}$ on L extending v are conjugates. So all the e_i's and f_i's are all equal and (denoting them as e, f respectively) $[L : K] = dgef$. If K is henselian, the extending valuation is unique and $[L : K] = def$.

Definition 3.4. *A valued field (K, v) is called* ***defectless*** *if for every finite immediate extension L/K, the defect of the extension is 1.*

Remark 3.5. *The property of being defectless is elementary.*

We have a number of results that allows to transfer the property of being defectless:

Proposition 3.6. *(K, v) is defectless if and only if (K^h, v) is defectless.*

Theorem 3.7. *Let $(F/K, v)$ be an extension without transcendence defect (i.e. the Abhyankar inequality for F/K becomes equality). If (K, v) is defectless, (F, v) is also defectless.*

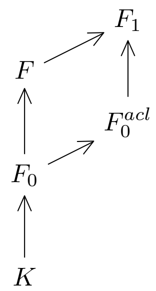
Now recall the case of transcendence defectless value field extensions from the previous section, along with all the notation used. So we work with an extension $(F/K, v)$ with trdeg$F/K = \rho + \tau$ and assume that vF and Fv are existentially closed in vK^* and K^*v respectively. We have that $vF = vK \oplus \bigoplus_i \alpha_i$ and that Fv/Kv is a separably generated function field. Note that in general, the value group extension vF/vK does not have to be finitely generated:

Proposition 3.8 (MacLane, Schilling [19])**.** *Given any subgroup $\Gamma < \mathbb{Q}$ there is a valuation on $\mathbb{Q}(X, Y)$ which is trivial on $\mathbb{Q}$ such that $v\mathbb{Q}(X, Y) = \Gamma$.*

A similar result can be found for residue field extension.

We additionally recall the results about F_0: vF/vF_0 is a torsion group and Fv/F_0v is an algebraic extension. Denote by F_1 the field generated by F and

F_0^{acl}:



Recall that for any valued field E, $vE^{acl} = \widetilde{vE}$, the divisible hull of vE. So we have:

$$\begin{aligned} vF_0^{acl} &= \widetilde{vF_0}, \\ F_0^{acl}v &= (F_0 v)^{acl}. \end{aligned}$$

Since F_1/F is an algebraic extension,

$$vF_1 \subset \widetilde{vF} = \widetilde{vF_0} = vF_0^{acl} \subset vF_1,$$
$$F_1 v \subset (Fv)^{acl} \subset (F_0 v)^{acl} = F_0^{acl} v = F_1 v,$$

and so we have equalities everywhere. Thus

$$\begin{aligned} vF_1 &= vF_0^{acl}, \\ F1_v &= F_0^{acl} v. \end{aligned}$$

This means that $(F_1/F_0^{acl}, v)$ is an immediate extension.

Now consider a maximal immediate extension (M, v) of (K, v). One can show that

$$(K, v) \prec_\exists (M, v) \iff (K, v) \text{ is henselian and defectless.}$$

Therefore if $AKE^\exists$ is true, then (K, v) is henselian and defectless. So it makes sense to ask about the existence (and uniqueness) of such a maximal extension.

Definition 3.9. *A valued field (K, v) is **maximal** if it has no proper immediate extensions.*

Lemma 3.10 (Krull)**.** *If $(L/K, v)$ is an immediate extension, then*

$$|L| \leq |Kv|^{|vK|}.$$

The lemma implies that all immediate extensions form a set, so a maximal one can be found by Zorn's Lemma.

The uniqueness of a maximal extension can be proved in the case of $\mathrm{char} Kv = 0$. In general, we need some additional assumptions.

Definition 3.11 (Kaplansky Hypothesis A [14]). *Let (K, v) be a valued field and $p = \text{charexp} Kv$. The field (K, v) is called **Kaplansky field** if and only if*

(1) vK is p-divisible,

(2) for every finite extension F/Kv, p does not divide $[F : Kv]$.

Originally, this condition asserted that Kv is p-closed: every non-constant p-polynomial has a root in Kv. It was proved that this condition implies (2) (Whaples [22], Delon [8]). The condition also implies that Kv is perfect. We can now state:

Theorem 3.12 (Kaplansky [14]). *Let (K, v) be a Kaplansky field. Then all maximal immediate extensions of K are isomorphic over K as valued fields.*

The converse of this statement is an open problem. In this regard, we have

Proposition 3.13 (Kuhlmann). *If (K, v) is not a Kaplansky field, then there is a finite extension F/K such that F does not have a unique (up to isomorphism over F) maximal immediate extension.*

The following partial characterizations have been established:

$$\begin{aligned} \text{algebraically maximal Kaplansky fields} &= \text{henselian defectless Kaplansky fields,} \\ \text{algebraically maximal perfect fields} &= \text{henselian defectless perfect fields,} \end{aligned}$$

the second equality only for fields of positive characteristic.

Kaplansky fields are known to have "well-behaved" model theory. It is suspected that the uniqueness of the maximal immediate extension is an important factor in that. The uniqueness was proved in a number of other cases:

(i) p-adic fields,

(ii) $\mathfrak{p}$-adic fields,

(iii) finitely ramified fields (i.e. field with the valuation group having the smallest element).

4 Pseudo-Cauchy sequences

In this section we describe a notion of pseudo-Cauchy sequences and their pseudolimits. As will be seen later, it is a useful notion that allows a characterization of maximal valued fields. Consider an immediate extension $(L/K, v)$ and $a \in L$. Since $vK = vL$, there is some $b \in K$ such that $va = vb$, so $v(a/b) = 0$. Also $(a/b)v \in Lv = Kv$ so there is some $d \in K$ such that $(a/b)v = dv$. We have $(a/b - d)v = 0$, hence $v(a/b - d) > 0$, so also $v(a - bd) > vb = va$. Overall, this shows that for a given $a \in L$ there is a $c \in K$ such that $v(a - c) > va$.

Lemma 4.1. *$(L/K, v)$ is immediate if and only if $\forall a \in L\ \exists c \in K\ v(a-c) > va$.*

Proof. The left to right direction has already been shown. For the converse, note that if $a \in L, c \in K$ with $v(a-c) > va$, then $va = vc$ so $va \in vK$. Similarly we can show $Kv = Lv$. □

The element $c \in K$ satisfying $v(a-c) > va$ can be seen as an approximation of a inside the base field K. Let $a_0 = c$. The element $a - a_0$ can be further approximated by another $c' \in K$, i.e. $v(a - a_0 - c') > v(a - a_0)$. Define $a_1 = a_0 + c'$. Continuing this, we get a sequence $(a_\gamma)_{\gamma<\lambda}$, with λ a limit ordinal, such that

$$v(a - a_0) < v(a - a_1) < \ldots < v(a - a_i) < \ldots, i \in \lambda.$$

The terms of such approximating sequences satisfy a certain weak convergence property.

Definition 4.2. *Let (K, v) be a valued field.*

*(i) A sequence $(a_\gamma)_{\gamma<\lambda} \in K$, $\lambda \in \mathrm{Lim}$ is called a **pseudo-Cauchy sequence** if $\forall \rho < \sigma < \tau\ v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$.*

*(ii) Let $(L/K, v)$ be an extension and $(a_\gamma)_{\gamma<\lambda} \in K$ be a pseudo-Cauchy sequence. The **pseudolimit** of $(a_\gamma)_{\gamma<\lambda}$ is an $a \in L$ such that $v(a - a_\nu) = v(a_\nu - a_{\nu+1})$ for each $\nu < \lambda$.*

*(iii) (K, v) is **pseudo-Cauchy complete** if every pseudo-Cauchy sequence in K has a pseudolimit in K.*

Remark 4.3. *By the construction preceding Lemma 4.1, one can see that the right side condition can be replaced with "for each $a \in L$ there is a pseudo-Cauchy sequence in K having a pseudolimit a."*

The "pseudo" in the name comes from the fact that a pseudo-Cauchy sequence does not need to have a unique pseudolimit (such a sequence would just be a regular **Cauchy sequence**). The following can be easily shown:

Lemma 4.4. *Let $(a_\gamma)_{\gamma<\lambda} \in K$ be a pseudo-Cauchy sequence. Then $\forall \rho < \sigma < \lambda\ v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_{\rho+1})$.*

We give examples of some pseudo-Cauchy sequences in different contexts:

Example 4.5. *Let k be a field. Consider an extension $k(t) \subset k((t))$ (fields with the t-adic valuation) and $N \in \omega$. Then $(\sum_{i=N}^{m} at^i)_{m\in\omega} \in k(t)$ is a pseudo-Cauchy sequence (in fact a Cauchy sequence) with pseudolimit $\sum_{i=N}^{\infty} at^i \in k((t))$.*

Example 4.6. *Let k be a field and $K = k(t^{\mathbb{Z}\times\mathbb{Z}})$. Then $(\sum_{i=0}^{m} t^{(0,i)} + t^{(1,0)})_{m\in\omega}$ is a pseudo-Cauchy sequence without a pseudolimit.*

Example 4.7. *Let k be a field and $K = k(t^{\mathbb{Q}})$.*

(i) *Consider the sequence* $a_n = \sum_{i=1}^{N} t^{1-1/i}, n \in \omega$. *This is a pseudo-Cauchy sequence with a pseudolimit* $x = \sum_{i=1}^{\infty} t^{1-1/i}$. *However, the sequence*

$$v(x - a_0), v(x - a_1), \ldots$$

is not cofinal in vK. *For this reason, the sequence has more pseudolimits in* K*: for example, every* $x + t^q$ *with* $q \geq 1$ *is a pseudolimit.*

(ii) *On the other hand, consider the sequence* $\sum_{q \in \mathbb{Q}} c_q t^q$ *where*

$$c_q = \begin{cases} 1 & \text{if } q = n - 1/l \text{ for some } n, l \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

This is an element of K *since its support is well ordered (of order type* $\omega \times \omega$*). This sequence is Cauchy.*

Take a pseudo-Cauchy sequence $(a_\nu)_{\nu \in \lambda}$ in K. The collection of formulas $\{v(x - a_\nu) = v(a_{\nu+1} - a_\nu) : \nu \in \lambda\}$ is a partial type $p(x)$ over K. Given an extension $(L/K, v)$, the set $p(L)$ consists of all pseudolimits of the sequence in L. Note that if L is $|\lambda|^+$-saturated, this set is nonempty.

Theorem 4.8 (Kaplansky; cf. [13, Lemma 2.14])**.** *Let* (K, v) *be a valued field and* $a \notin K$ *such that* $(K(a)/K, v)$ *is immediate. Then* a *is a pseudolimit of a pseudo-Cauchy sequence* $(a_i)_i$ *that does not have a pseudolimit in* K.

Proof. By Remark 4.3, there is a pseudo-Cauchy sequence in K approximating a. It is easy to verify that the class of pseudo-Cauchy sequences in K approximating a (ordered by "being the initial segment") is a set and the union of a chain of pseudo-Cauchy sequences is again a pseudo-Cauchy sequence. So by Zorn's Lemma there is a maximal pseudo-Cauchy sequence $(a_\nu)_{\nu < \lambda}$ approximating a in K. This sequence cannot have a limit in K: otherwise it can be extended by finding $a_\lambda, a_{\lambda+1}, \ldots$ such that for all $\rho < \lambda$, $v(a_\rho - a) < v(a_\lambda - a) < v(a_{\lambda+1} - a), \ldots$, contradicting the maximality. □

We can distinguish two kinds of pseudo-Cauchy sequences:

Definition 4.9. *Let* (K, v) *be a valued field and* $(a_\nu)_{\nu < \lambda}$ *a pseudo-Cauchy sequence in* K.

(i) *If* f *is a polynomial over* K*, we say that* $(a_\nu)_{\nu < \lambda}$ ***fixes*** f *if there is* $\nu_0 < \lambda$ *such that* $vf(a_\nu)$ *is fixed for* $\nu_0 \geq \nu < \lambda$.

(ii) *The sequence* $(a_\nu)_{\nu < \lambda}$ *is called* ***transcendental*** *if it fixes all* $f \in K[X]$. *Otherwise it is called* ***algebraic***.

Fact 4.10. *If* $(a_\nu)_{\nu < \lambda}$ *does not fix the value of some* $f \in K[X]$ *then for sufficiently large* ν*'s,* $vf(a_\nu)$ *is strictly increasing.*

Take a pseudo-Cauchy sequence $(a_\nu)_{\nu < \lambda}$ in K of algebraic type. Consider a polynomial of the smallest degree whose value is not fixed.

Fact 4.11. *Such a polynomial is irreducible over K.*

Pseudolimits of transcendental pseudo-Cauchy sequences can be easily realized in transcendental extensions:

Theorem 4.12 (Kaplansky [14, Theorem 2])**.** *Let $(a_\nu)_{\nu<\lambda}$ be a transcendental pseudo-Cauchy sequence in (K, v). Then there is a simple transcendental extension $K(X)/K$ and an extension of v to $K(X)$ such that X becomes a pseudolimit of the sequence in $(K(X), v)$.*

Proof. For $f(x) \in K[X]$, we set $vf(x)$ to be the fixed (for sufficiently large i's) value of $vf(a_\nu)$. One can check that this indeed yields a valuation on $K(X)$. We also have that for all sufficiently large $\nu_0 > \nu$,

$$v(x - a_\nu) = v(a_{\nu_0} - a_\nu) = v(a_{\nu+1} - a_\nu).$$

Hence x is a pseudolimit of the sequence. □

Now we consider a pseudo-Cauchy sequence in K of algebraic type. There is a polynomial $f \in K[X]$ not fixed by the sequence having the smallest degree. By Fact 4.11, f is irreducible. Without loss of generality, f can be assumed to be monic.

Theorem 4.13 (Kaplansky [14, Theorem 3])**.** *Let $(a_\nu)_{\nu<\lambda}$ be an algebraic pseudo-Cauchy sequence in (K, v) and $f \in K[X]$ irreducible and not fixed by $(a_\nu)_{\nu<\lambda}$. Then there is an immediate extension of valued field $(K, v) \subset (K(z), v)$ such that z becomes a pseudolimit of $(a_\nu)_{\nu<\lambda}$ in $K(z)$. Moreover, if z' is a root of f and there is an extension of v to a valuation on $K(z')$ such that z' is a pseudolimit of $(a_\nu)_{\nu<\lambda}$ in $K(z')$, then the assignment $z \mapsto z'$ induces a valuation preserving isomorphism between $K(z)$ and $K(z')$.*

With regard to pseudo-Cauchy completeness, we can establish the following:

Proposition 4.14 (Kaplansky [14])**.** *Let (K, v) be a valued field. Then there is an immediate extension $(L/K, v)$ such that L is pseudo-Cauchy complete.*

Proof. Take $\lambda \in \mathrm{Ord}$ with $|vK| < \mathrm{cf}\lambda$. By Theorems 4.12 and 4.13 it can be seen that for any valued field (E, v), there is an immediate extension $(F/E, v)$ such that every pseudo-Cauchy sequence in E has a limit in F. Since a union of tower of immediate extensions is again an immediate extension, we can indictively build a chain of valued fields $(K_\nu, v), \nu \leq \lambda$ such that

(i) $K_0 = K$,

(ii) for all $\rho < \tau \leq \lambda$, $(K_\tau/K_\rho, v)$ is an immediate extension,

(iii) for all $\tau < \lambda$, all pseudo-Cauchy sequences in K_τ have pseudolimits in $K_{\tau+1}$.

Then K_λ is pseudo-Cauchy complete and $(K_\lambda/K, v)$ is an immediate extension. □

Finally, we have:

Theorem 4.15 (Kaplansky [14, Theorem 4])**.** *A valued field (K, v) is maximal if and only if K is pseudo-Cauchy complete.*

5 Algebraically maximal and tame fields

A concept related to pseudo-Cauchy sequences is that of ultrametric balls. Recall that a given valuation $v : K \to (vK, <)$ can be associated with the ultrametric $u : K \times K \to (vK, <^*)$, where $<^*$ is the inverted ordering $<$, by placing $u(x, y) = v(x - y)$. So we have $u(x, y) <^* u(x, z) \iff v(x - y) > v(x - z)$. The natural definition of closed balls for that ultrametric can be given in the language of valued fields. Given a value field (K, v), an **ultrametric ball** with a centre $a \in K$ and a radius $\alpha \in vK$ is defined as

$$B_\alpha(a) = \{b \in K : v(a - b) \geq \alpha\}.$$

Definition 5.1. *A descending chain of (ultrametric) balls is called a* ***nest*** *of (ultrametric) balls. The field (K, v) is* ***spherically complete*** *if every nonempty nest of balls in K has a nonempty intersection.*

All ultrametric balls satisfy an equivalent of the "triangle law" for ultrametric spaces:

$$b \in B_\alpha(a) \implies B_\alpha(a) = B_\alpha(b),$$

that is, every point of a ball is its centre. It follows that if B, B' are two ultrametric balls, then

$$B \cap B' \neq \emptyset \implies B \subset B' \vee B' \subset B.$$

There is a correspondence between nests of balls and pseudo-Cauchy sequences. Given a pseudo-Cauchy sequence $(a_\nu)_{\nu \in \lambda}$, an element a_ν can be identified with the ball $B_{\gamma_\nu}(a_\nu)$ with $\gamma_\nu = v(a_{\nu+1} - a_\nu)$. Thus the entire sequence can be assigned a nest of balls. The intersection of the nest is the set of pseudolimits of $(a_\nu)_{\nu \in \lambda}$. Conversely, a nest of balls $\{B_{\gamma_\nu}(a_\nu) : \nu \in \lambda\}$ can be assigned a sequence $(a_\nu)_{\nu \in \lambda}$. Since the intersection of a nest is the set of pseudolimits of the corresponding pseudo-Cauchy sequence, this establishes (along with Theorem 4.15):

$$\begin{aligned} (K, v) \text{ is spherically complete} \quad &\iff \quad (K, v) \text{ is pseudo-Cauchy complete} \\ &\iff \quad (K, v) \text{ is maximal.} \end{aligned}$$

We return now to the issue of $AKE^\exists$ Principle. We consider an immediate extension $(F/K, v)$ and an extension $(K, v) \subset (K^*, v^*)$ for an $|F|^+$-saturated K^*. Our goal is to embed F into K^* over K. If $x \in F \setminus K$ then x is a pseudolimit of a pseudo-Cauchy sequence in K without a pseudolimit in K. We ask whether we can embed $K(x)$ into K^* over K. One obstacle is that pseudo-Cauchy sequences converging to x can be of algebraic type. We need to extend our assumptions to ensure it does not happen.

Definition 5.2. *A valued field (K, v) is* ***algebraically maximal*** *if it has no proper immediate algebraic extensions.*

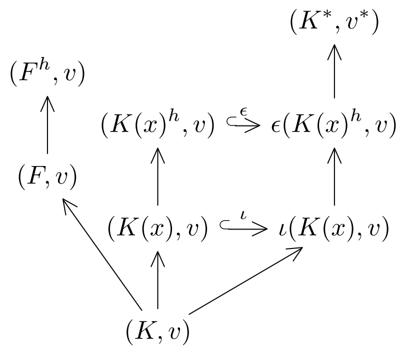
Theorem 5.3. *Let (K, v) be a valued field. Then (K, v) is algebraically maximal if and only if every pseudo-Cauchy sequence in K without a pseudolimit in K is of transcendental type.*

So our assumption is that K is algebraically maximal. Note that we have:

Proposition 5.4. *Let (K, v) be a field. If K is henselian and defectless, then K is algebraically maximal.*

Conversely, if K is algebraically maximal, then K is henselian, since henselization is an immediate algebraic extension.

Now, as x is a limit of a pseudo-Cauchy sequence in K without a limit in K, by saturation this sequence has a pseudolimit $y \in K^*$. By Theorem 4.12 there is an isomorphism $(K(x), v) \to (K(y), v)$ over K. So $K(x) \underset{K}{\hookrightarrow} K^*$ and the same is true for $K(x)^h$.



In the residue characteristic zero, we can relax the assumption on K:

Fact 5.5. *If (K, v) is a valued field with* $\mathrm{char} Kv = 0$*, then*

$$\begin{aligned} (K, v) \text{ is henselian and defectless} &\iff (K, v) \text{ is algebraically maximal} \\ &\iff (K, v) \text{ is henselian.} \end{aligned}$$

In this case, $K(x)^h$ is algebraically maximal and we can continue the process replacing by K with $K(x)$. By an inductive argument, we get $(F, v) \underset{K}{\hookrightarrow} (K^*, v^*)$.

We have considered the question of embedding $(F, v) \underset{K}{\hookrightarrow} (K^*, v^*)$ in the case of residue characteristic zero. In the nonzero case, we look for possible reductions of the problem. Consider an elementary class $\mathcal{C}$ of valued fields.

Question 5.6. *Let $(K, v) \in \mathcal{C}$ and (F, v) be an immediate function field over K. Is there an algebraic extension (L, v) of (F, v) in $\mathcal{C}$ and a chain of extensions $(L_i, v) \in \mathcal{C}$ such that*

$$(K, v) = (L_0, v) \subset (L_1, v) \subset \ldots \subset (L_n, v) = (L, v),$$

with $\mathrm{trdeg} L_{i+1}/L_i = 1$ *for all i?*

It turns out that this is not the case: a counterexample is the class of models elementarily equivalent to $\mathbb{F}_p(t)$. However, it does hold for a class of "tame fields."

Definition 5.7. *Let (K, v) be a henselian field.*

(i) An algebraic extension $(L/K, v)$ is a ***tame extension*** *if if for every finite subextension $K \subset F \subset L$ we have, F/K is defectless and tamely ramified (i.e. the ramification index $(vF : vK)$ is prime to $p = \mathrm{charexp} Kv$ and Ev/Kv is a separable extension).*

(ii) (K, v) is ***tame*** *if all of its algebraic extensions are tame.*

We have the following results for tame fields:

Lemma 5.8. *Let (L, v) be a tame field and K relatively algebraically closed in L with Lv/Kv algebraic. Then (K, v) is a tame field, Kv is relatively separably algebraically closed in Lv and the order of every torsion element in vL/vK is a power of $p = \mathrm{charexp} Kv$.*

Theorem 5.9. *Let (K, v) be a tame field and $(F/K, v)$ an immediate function field extension with $\mathrm{trdeg} F/K = 1$. Then F is henselian rational, i.e. there is $y \in F$ such that $F^h = K(y)^h$.*

Investigating $AKE^{\exists}$ Principle in the previous sections, we have considered extensions F/K that are either Abhyankar (i.e. $\mathrm{trdeg} F/K = \mathrm{rr} vF/vK + \mathrm{trdeg} vF/vK$) or immediate. More generally, we considered a subfield $K \subset F_0 \subset F$ generated by the preimages of elements in vF, Fv witnessing $\mathrm{rr} vF/vK$ and $\mathrm{trdeg} vF/vK$. In this case, we have that vF/vF_0 is a torsion group and Fv/F_0v is an algebraic extension.

So take any extension $(F/K, v)$. Assume that $\mathrm{char} K = p > 0$. Define F_0 as before and consider an extension L/F such that L is a tame field and $vK \prec_{\exists} vF \implies vK \prec_{\exists} vL$. Let L_0 be the relative algebraic closure of F_0 in L. By Lemma 5.8, L_0 is a tame field.

Fact 5.10. *Let (K, v) be a tame field and $\mathrm{char} K = p > 0$. Then K is perfect.*

So we have that L is a perfect field. Thus L_0 is also perfect and by Lemma 5.8, both vL/vL_0 and Lv/L_0v are trivial. This means that $(L/L_0, v)$ is an immediate extension. Proceeding like in the case of Abhyankar extensions in Section 2 with L_0 in place of F_0, we get that $K \prec_{\exists} L_0$.

6 $\mathbb{F}_p((t))$ and additive polynomials

In this section we describe some results on the field $\mathbb{F}_p((t))$ with t-adic valuation. It exhibits a number of superficial similarities to the well known p-adic field $\mathbb{Q}_p$: they are both complete, discrete, henselian and defectless valued fields with the same residue field $\mathbb{F}_p$. There are many results concerning the theory of $(\mathbb{Q}_p, v)$:

it is model complete and decidable (Ax-Kochen [4], Ershov [10]); it admits quantifier elimination if the language is extended by the so-called Macintyre predicates (Macintyre [18]); Skolem functions are definable in this extended language (van den Dries [9]). In contrast, the theory of $(\mathbb{F}_p((t)), v)$ is not as well developed. Numerous authors contributed to improve the understanding of it.

The theory of $(\mathbb{F}_p((t)), v)$ has the following axiomatization:

(T1) K is a henselian, defectless valued field.

(T2) vK is a $\mathbb{Z}$-group, $Kv \cong \mathbb{F}_p$.

(T3) char$K = 0$.

The theory of $(\mathbb{Q}_p, v)$ is axiomatized by the axioms (T1), (T2) and

(T3') char$K = p$.

Both fields are complete and discrete (as valued fields). However, these properties are not first order.

We now take a look on the field $K = \mathbb{F}_p((t))$. Its p-degree is 1, so we can write it as a direct sum

$$K = K^p \oplus tK^p \oplus t^2K^p \oplus \ldots \oplus t^{p-1}K^p. \qquad (\dagger)$$

Every summand is the image of a polynomial of the form $t^iX^p \in K[X]$. All such polynomials are **additive**, i.e. satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in K^{acl}$. Note that being a direct sum by images of additive polynomial is a first order property.

Remark 6.1. *If $f \in K[X_1, \ldots, X_n]$ is additive, then $f(x_1, \ldots, x_n) = f(x_1, 0, \ldots, 0) + f(0, x_2, 0, \ldots, 0) + \ldots + f(0, \ldots, 0, x_n)$.*

Now consider an Artin-Schreier polynomial $\mathfrak{p}(x) = x^2 - x$. It is also an additive polynomial (as a combination of the Frobenius endomorphism and the identity). By replacing the first summand in (†) with the image of $\mathfrak{p}$, and adding a corrective summand D equal to either $\mathbb{F}_p$ or $\mathcal{O}_v$, we obtain

$$K = \mathfrak{p}(K) \oplus tK^p \oplus t^2K^p \oplus \ldots \oplus t^{p-1}K^p \oplus D.$$

So we have that the following sentence over t holds in $\mathbb{F}_p((t))$:

$$\forall x \exists x_0, \ldots, x_{p-1} \exists y \ (x = x_0^p - x_0 + tx_1^p + \ldots + t^{p-1}x_{p-1}^p + y \wedge vy \geq 0). \qquad (*)$$

A question arises: given any additive polynomials $f_1, \ldots f_n$, when does $K = f_1(K) \oplus \ldots \oplus f_n(K)$ hold? And when it does not (i.e. when the right side is a proper definable subgroup of K^+), is there an algorithm that determines the corrective summand D such that $K = f_1(K) \oplus \ldots \oplus f_n(K) \oplus D$?

In an effort to answer this question, the following notion has been developed (Kuhlmann, van den Dries). Given a definable subgroup $S < K^+$, we say that S has the **Optimal Approximation Property** if

$$\forall x \exists y \in S \ \forall z \in S \ v(x-y) \geq v(x-z). \qquad (OAP)$$

The property means that the point y is the closest to x among all the points in S in the sense of ultrametric associated with v. Note that by the properties of ultrametric, this point is usually not unique. We ask for conditions on the polynomials f_i such that $f_1(K) \oplus \ldots \oplus f_n(K)$ satisfies OAP.

Definition 6.2. *Let $G_1, \ldots G_n$ be subgroups of K^+. Then the sum $G = G_1 \oplus \ldots \oplus G_n$ is **pseudo-direct** if for all $a \in G$ there are some $a_i \in G_i$ such that $a = a_1 + \ldots + a_n$ with $va_i \geq va$.*

Remark 6.3. *(1) Being a pseudo-direct sum is an elementary condition. In every maximal field of characteristic $p > 0$, if $f_1(K) \oplus \ldots \oplus f_n(K)$ is pseudo-direct, then $f_1(K) \oplus \ldots \oplus f_n(K)$ has OAP. This leads to an axiom scheme "sum is pseudo-direct $\implies$ sum has OAP" for every sum of this form.*

(2) There is an extension $(L/\mathbb{F}_p((t)), v)$ of transcendence degree 1 that satisfies the axioms (T1)-(T3) but does not satisfy $(*)$. So the sentence is independent from the axioms.

Recall the definition of spherical completeness (Definition 5.1). We note the following result:

Lemma 6.4 (Kuhlmann [16]). *Let $S \subset K$ be spherically complete. Then S satisfies OAP.*

So now we will look for the conditions for the polynomials $f_1, \ldots f_n$ such the direct sum of their images is spherically complete. We have:

Theorem 6.5 (Kuhlmann, [15]). *Let (K, v) be a valued field and $G_1, \ldots, G_n < K^+$ be spherically complete. If $G_1 \oplus \ldots \oplus G_n$ is pseudo-direct, then it is spherically complete.*

Sketch of proof. Each G_i is a spherically complete ultrametric space. So the product $\prod_i G_i$ is also a spherically complete ultrametric space. We consider the mapping

$$\prod_i G_i \ni (g_1, \ldots, g_n) \mapsto \sum_i g_i \in G_1 \oplus \ldots \oplus G_n.$$

Then the condition that $G_1 \oplus \ldots \oplus G_n$ is pseudo-direct is equivalent to the fact that this mapping is "immediate", i.e. satisfies conditions for a version of attractor theorem for ultrametric spaces. From this theorem it follows that $G_1 \oplus \ldots \oplus G_n$ is spherically complete. □

Since $\mathbb{F}_p((t))$ is spherically complete, for every polynomial $g \in \mathbb{F}_p((t))[X]$ we have that $g(\mathbb{F}_p((t)))$ is spherically complete ultrametric space. So we get that in $\mathbb{F}_p((t))$, images of additive polynomials have OAP.

We can give a description of additive polynomials in $K = \mathbb{F}_p((t))$. Consider the ring $\mathrm{End}(K)$ and the its subring $K[\phi]$, where ϕ denotes the Frobenius endomorphism $x \mapsto x^p$. As ϕ is a polynomial endomorphism, $K[\phi]$ can be seen as a "skewed polynomial ring":

$$K[\phi] = \{\sum_{i=0}^{n} c_i X^{p^i} : c_i \in K, n \in \omega\},$$

with the operations of addition and function composition. We have the identification $\sum_{i=0}^{n} c_i X^{p^i} \leftrightarrow \sum_{i=0}^{n} c_i \phi^i$. This skewed polynomial ring turns out to be exactly the ring of additive polynomials.

The last thing we will discuss in this section is the notion of extremal fields. Note that we have:

Theorem 6.6 (Bishnoi-Khanduja [5]). *Let (K, v) be a valued field. Then (K, v) is algebraically maximal if and only if for every $f \in K[X]$ the set $\{fv(x) : x \in K\}$ has a maximum in $vK \cup \{\infty\}$.*

We can consider variants of the right hand condition in the theorem above.

Definition 6.7. *Let (K, v) be a valued field.*

*(i) (K, v) is called K**-extremal** if $\forall f \in K[X_1, \ldots, X_n]$ $vf(K^n)$ has a maximum in $vK \cup \{\infty\}$.*

*(ii) (K, v) is called $\mathcal{O}_v$**-extremal** if $\forall f \in K[X_1, \ldots, X_n]$ $vf(\mathcal{O}_v^n)$ has a maximum in $vK \cup \{\infty\}$.*

This basic fact gives a reason for considering K-extremality in several variables along with its weakening to $\mathcal{O}_v$-extremality:

Fact 6.8. *A valued field (K, v) is K-extremal if and only if K is algebraically closed.*

The next two theorems show the use of $\mathcal{O}_v$-extremality for characterization of fields with more canonical poperties:

Theorem 6.9. *Let (K, v) be a valued field with* $\mathrm{char}Kv = 0$. *Then (K, v) is $\mathcal{O}_v$-extremal if and only if either vK is a $\mathbb{Z}$-group or vK is divisible and Kv is "large".*

Theorem 6.10. *et (K, v) be a valued field with* $\mathrm{char}Kv > 0$. *If (K, v) is $\mathcal{O}_v$-extremal, then (K, v) is henselian, defectless and either vK is a $\mathbb{Z}$-group, or vK is divisible and Kv is "large".*

The converse to the above theorem holds if $\mathrm{char}K = \mathrm{char}Kv = p$ and $K = K^p$. We also have

Theorem 6.11. *If (K, v) is a henselian, defectless valued field with $vK \cong \mathbb{Z}$, then (K, v) is $\mathcal{O}_v$-extremal.*

It can be shown that $(\mathbb{F}_p((t)), v)$ is $\mathcal{O}_v$-extremal, but it is not K-extremal since it is not algebraically closed. However:

Theorem 6.12. *$(\mathbb{F}_p((t)), v)$ is K-extremal with respect to the set of additive polynomials in several variables.*

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