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Definiowalna dynamika topologiczna i o-minimalność

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Streszczenie

W rozprawie rozważamy definiowalną dynamikę topologiczną grup definiowalnych głównie nad o-minimalnymi rozszerzeniami ciał rzeczywiście domkniętych. Dla o-minimalnego rozszerzenia ciała liczb rzeczywistych $\mathbb{R} = (\mathbb{R}, +, \cdot, <, \dots)$ oraz $R \succ \mathbb{R}$, rozważamy R-definiowalną grupę G i naturalne działanie $G(R)$ na przestrzeń $S_{G,ext}(R)$ zewnętrznych typów w G nad R.

Pierwsze dwa rozdziały poświęcone są wstępnym wynikom.

W trzecim rozdziale dowodzimy pewnych wyników które abstrahują od sytuacji o-minimalnej. Pracując nad dowolną strukturą pierwszego rzędu M nad którą wszystkie typy są definiowalne, rozważamy M-definiowalną grupę G posiadającą definiowalny rozkład $G = KH$ na podgrupy $K, H < G$ spełniające $K \cap H = \{e\}$, takie że grupa H jest definiowalnie ekstremalnie średniowalna. Pokazujemy, że pewne aspekty potoku $(G(M), S_G(M))$ są wyjaśnione w terminach indukowanego potoku $(G(M), S_K(M))$, tzn. dynamiki naturalnego działania G na podgrupę K. Konkretnie, wyniki pokazują homeomorfizm minimalnego potoku $(G(M), S_G(M))$ i minimalnego potoku $(G(M), S_K(M))$, oraz istnienie algebry obrazowej silnie generycznych podzbiorów $G(M)$ składającej się z cylindrów nad pewną algebrą obrazową silnie generycznych podzbiorów $K(M)$. Wyniki ulegają znacznemu wzmocnieniu przy założeniu, że H jest podgrupą normalną.

W czwartym rozdziale otrzymujemy bardziej konkretne wyniki dla grup definiowalnych w sytuacji o-minimalnej. Zajmujemy się głównie $\mathbb{R}$-definiowalnymi grupami spełniającymi teorio-modelowy odpowiednik rozkładu Iwasawy: rozkład zwarto-beztorsyny $G = KH$ z K zwartą, H beztorsyjną. Dla takiej grupy podajemy częściowy opis minimalnych potoków jej uniwersalnego definiowalnego potoku oraz pokazujemy, że grupa Ellisa jej uniwersalnego definiowalnego potoku jest abstrakcyjnie izomorficzna z $N_G(H) \cap K(\mathbb{R})$. To odpowiada negatywnie na wczesne pytanie Newelskiego czy (przy pewnych dobrych założeniach) grupa Ellisa jest izomorficzna z G/G^{00}. Ponownie uzyskujemy mocniejsze wyniki przy założeniu, że H jest normalną podgrupą G.

Następnie rozważamy uniwersalne nakrycia $\mathbb{R}$-definiowalnych grup, naturalnie interpretowane w pewnej dwusortowej strukturze, i uogólniamy wyniki dla minimalnych potoków i grupy Ellisa do tej szerszej sytuacji.

W końcu prezentujemy pewne wyniki dotyczące algebr obrazowych zbiorów silnie generycznych dla R-definiowalnych grup, w szczególności dla $G = SL(2, -)$.

Abstract

In the dissertation we study definable topological dynamics of groups mostly definable in o-minimal expansions of real closed fields. Given an o-minimal expansion of the field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, <, \ldots)$ and $R \succ \mathbb{R}$ we consider an R-definable group G and the natural action of $G(R)$ on the space $S_{G,ext}(R)$ of external types in G over R.

The first two chapters are devoted to preliminaries.

In the third chapter we prove some results that abstract from the predominant, o-minimal setup. Working over any first-order structure M such that all types over M are definable, we consider an M-definable group G admitting a definable decomposition $G = KH$ in terms of its subgroups K, H with $K \cap H = \{e\}$, such that the group H is definably extremely amenable. We show that certain aspects of the flow $(G(M), S_G(M))$ are explained in terms of the induced flow $(G(M), S_K(M))$, i.e. dynamics of the natural action of G on its subgroup K. Specifically, the results show a homeomorphism of a minimal subflow of $(G(M), S_G(M))$ and a minimal subflow of $(G(M), S_K(M))$, and the existence of an image algebra of strongly generic subsets of $G(M)$ that consists of cylinders over an image algebra of strongly generic subsets of $K(M)$. We also obtain much stronger results assuming that H is a normal factor.

In the fourth chapter we obtain more concrete results for groups definable in an o-minimal setting. We primarily consider $\mathbb{R}$-definable groups that admit a model-theoretic analogue of Iwasawa decomposition: the compact-torsion-free decomposition $G = KH$ with K compact, H torsion-free. For such a group, we provide a partial description of minimal subflows of its universal definable flow, and show that the Ellis group of its universal definable flow is isomorphic as an abstract group to $N_G(H) \cap K(\mathbb{R})$. This negatively answers an early question of Newelski whether (under some nice assumptions) the Ellis group is isomorphic to G/G^{00}. Again, we get stronger results when assuming that H is a normal factor.

We then proceed to consider universal covers of $\mathbb{R}$-definable groups, naturally interpreted in a certain two-sorted structure, and generalize the results on minimal subflows and the Ellis group to this more general context.

Finally, we present certain results on image algebras of strongly generic subsets for R-definable groups, in particular the case of $G = SL(2, -)$.

Contents

1 Introduction

This dissertation aims to study definable topological dynamics in the o-minimal setup. Classic topological dynamics considers the notion of a flow: an action of a group G on a topological space X by homeomorphisms. It is a well-established subject developed by Ellis, Glasner and many others. Recently, the methods of topological dynamics have been introduced to the model-theoretic context in a series of seminal papers by Newelski and further developed by Pillay and other authors. This resulted in the theory of definable topological dynamics. The fundamental notion in its framework is that of a definable flow, which is a flow in the classic sense that is definable in a certain sense. Definable topological dynamics allows to view model-theoretic objects from the viewpoint of classic topological dynamics, while on the other hand providing it with a way to use the tools from model theory.

The topological approach is not new in model theory. The original definition of the Morley rank directly referenced the Cantor-Bendixson rank. Since the advent of stability theory this approach gave way to a more combinatoric one, which considers properties such as NIP and uses methods from combinatorial set theory. Definable topological dynamics can be seen as the return to the topological approach. A canonical example of a definable flow is a definable group G acting on the space of types in G. Such actions are often studied in model theoretic investigations (for instance, describing the orbits of generic types is crucial in understanding of stable groups), so it is most natural to consider them within the framework of definable topological dynamics. Applying methods inspired by classic topological dynamics yields new results about such actions and shows connections between notions of classic dynamics and model theory, like the fact that an almost periodic type is weakly generic.

On the other hand, the recent work by Krupiński and Pillay provide new results for classic topological dynamics, showing a strong connection (epimorphisms) between objects such as the Ellis group of a flow or the generalized Bohr compactification of the underlying group G, and quotients of the model-theoretic connected components of G.

With an ample justification for the development of definable topological dynamics, it makes sense to study its behaviour in various standard model-theoretic setups. Among the staple objects considered in model theory are o-minimal structures, particularly o-minimal expansions of real closed fields. The groups definable in them carry a definable manifold structure, which in the particular case of an o-minimal expansion of the field of real numbers turns them into a (definable) real Lie group. This provides a rich environment for dynamical investigations. Moreover, the "universal" definable flow of a definable Lie group is close in character to the (classic) universal definable flow of the group. The dynamics of the latter flow have been of some interest in the past, and the results of this dissertation contain a thorough analysis of the dynamics of the former flow.

A substantial portion of this dissertation is contained in [18]. The following are the main results of the thesis:

1. Proposition 4.3.4, which describes the minimal subflows of the universal definable flows of a class of groups definable in an o-minimal expansion of the field of reals. This class properly includes all definable semisimple Lie groups. The description is complete for flows over $\mathbb{R}$ and partial for its elementary extensions.

2. Theorem 4.3.6, which shows the isomorphism between the Ellis group of the universal definable flow of a group as in Proposition 4.3.4 with a particular definable subgroup of its maximal compact component.

3. Theorem 4.4.6, which generalized the previous two results to universal covers interpretable in a certain two-sorted structure.

The dissertation is organized as follows. In Chapter 2 we outline the basics of topological dynamics, its interpretation in a definable setting, and elementary facts about o-minimal structures. In Chapter 3 we prove various results on the dynamical structure of arbitrary definable groups admitting a certain kind of decomposition. Then we apply them in Chapter 4 to groups definable in an o-minimal structure.

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2 Preliminaries

In this chapter we present the basic notions and preliminary results about the main subject of the dissertation. The reader is assumed to be familiar with basic model theoretical notions: formulas, models, definability, elementary maps, the space of types. The reference on the subject is for instance [14].

We use standard model-theoretic notations: for a definable set X, we denote by $X(A)$ the set of the realizations in A of the formula defining X. When working within a fixed, saturated model $\mathfrak{C}$, we write X for $X(\mathfrak{C})$. We write $S_X(A)$ for the space of types in X over the parameters from A.

2.1 Topological dynamics

The study of topological dynamics in the model-theoretical context was initiated by Newelski in [25] and since then it saw further development by a number of authors. There is a connection between the definable version of topological dynamics of a definable group G with the model-theoretic connected components of G. In some cases (stable, definably compact o-minimal), the Ellis group of the universal definable flow of a definable group G is shown to be isomorphic to the quotient G/G^{00}. Newelski asked whether this is always the case, at least in a sufficiently "tame" setting. In [11], the authors provided a counterexample by examining the particular case of the group $SL_2(\mathbb{R})$ defined in the structure $(\mathbb{R}, +, \cdot)$. This dissertation draws from some of the ideas from that study and provides a broader range of such counterexamples by describing the Ellis groups in a class of groups that properly includes all definable (in some o-minimal extension or the field of reals) semisimple real Lie groups.

We briefly recall some basic definitions from the classic topological dynamics. The example references here are [9, 2]. By a (point-transitive) *G-flow* we mean a left action of the group G on a compact topological space X

by homeomorphisms that contains a dense orbit. We denote such a flow by (G, X). Each $g \in G$ defines a homeomorphism $\pi_g : X \to X$, an element of the topological space X^X (with the usual product topology). We denote by $E(X)$ the closure of the set $\{\pi_g : g \in G\}$. The structure $(E(X), *)$ with $*$ denoting the operation of function composition is a semigroup and we call it the *Ellis semigroup* of the G-flow. $E(X)$ itself is also a G-flow. The semigroup operation is continuous in the first coordinate.

Important objects associated with the Ellis semigroup $E(X)$ are its minimal ideals. These ideals are sub-semigroups and they turn out to be exactly its *minimal G-subflows*: minimal nonempty closed subsets which are closed under the G-action. Every minimal ideal I is the closure of the G-orbit of p for any $p \in I$. Each $p \in E(X)$ such that $\mathrm{cl}(Gp)$ is a minimal subflow is called *almost periodic*. A point $p \in E(X)$ is called an *idempotent* if $p * p = p$.

For I a minimal subflow of $E(X)$, let $J(I)$ be the set of idempotents of I. We have that $I = \mathrm{cl}(Gu)$ for any $u \in J(I)$. Moreover,

$$I = \coprod_{u \in J(I)} uI,$$

where every $(uI, *)$ is a group (with the group identity being u), and all those groups are isomorphic to each other, even for different I. We call these groups the *ideal subgroups* and define the *Ellis group* of $E(X)$ to be their isomorphism class. We denote this canonical object by $\mathcal{H}(G, X)$.

2.2 Definable topological dynamics

Now we consider the model-theoretic setup. For the reference, see [25, 24, 26]. We work in an arbitrary first-order theory T in a language $\mathcal{L}$. Let $M \models T$ and $\mathfrak{C}$ be an $|M|^+$-saturated elementary extension of M. Consider an M-definable group G, an M-definable set X, and an M-definable, transitive (left) action $\pi : G \times X \to X$. Let E be an equivalence relation on $X(\mathfrak{C})$ which is:

1. Type-definable: $E(x, y)$ is a partial type with parameters over M.
2. Bounded: $|X(\mathfrak{C})/E| < \kappa$ for a fixed κ, independent of $\mathfrak{C}$.
3. G-invariant: for any $x, y \in X, g \in G$ we have $xEy \iff gxEgy$.

The quotient $X(\mathfrak{C})/E$ has the natural logic topology (i.e. a subset of $X(\mathfrak{C})/E$ is closed whenever its preimage in $X(\mathfrak{C})$ is a type-definable set) and the action π induces a well-defined action of $G(M)$ on $X(\mathfrak{C})/E$. This is an action by homeomorphisms which is point-transitive, as the types of elements of $X(M)$ form a dense orbit. Thus $(G(M), X(\mathfrak{C})/E)$ is a point-transitive $G(M)$-flow.

The objects (G, X, π, E) that arise in this manner will be called *definable G-flows over M*. Given two G-flows $(G, X, \pi, E), (G, X', \pi', E')$, for any type-definable (over M) function $f : X \to X'$ that induces a well-defined function $f_* : X(\mathfrak{C})/E \to X'(\mathfrak{C})/E'$ such that $f_*(gx) = gf_*(x)$ for any $g \in G, x \in X(\mathfrak{C})/E$ we call the induced function f_* a *morphism* between the flows. For a fixed G and M, the set of G-flows over M together with their morphisms forms a category.

Given G acting definably on X, there is the finest type-definable (over M), bounded and G-invariant equivalence relation E on $X(\mathfrak{C})$, namely the equality of types over M. For such E, $X(\mathfrak{C})/E$ becomes the space $S_X(M)$ of types in X over M. On the other hand, there is the largest M-definable set on which G acts transitively, i.e. G itself by left translations. The flow $(G, S_G(M))$ has the universal property: every G-flow over M is an epimorphic image of it.

In general, there is no complete description of the Ellis semigroup of this universal definable flow. However, there is a broad setup in which this semigroup is known and is in fact isomorphic (as a $G(M)$-flow) to the flow itself.

Definition 2.2.1. *An externally definable subset of M^n is a trace of some $\mathfrak{C}$-definable set, i.e. it is the set of realizations in M of some formula $\phi(x_0, \ldots, x_{n-1})$ with parameters from $\mathfrak{C}$.*

Definition 2.2.2. *Let G be an abstract group acting on a set X. A collection $\mathcal{A}$ of subsets of X is called a G-algebra if it is closed under taking intersections, unions, complements (in X) and translates by the elements of G.*

We write $S_{ext,G}(M)$ for the set of ultrafilters on the boolean $G(M)$-algebra of externally definable subsets of $G(M)$. An ultrafilter $\mathcal{U} \in S_{ext,G}(M)$ is called an *external type*. $S_{ext,G}(M)$ can be identified with the subset of finitely satisfiable (in M) types in $S_G(\mathfrak{C})$ in the following way: given an ultrafilter $\mathcal{U} \in S_{ext,G}(M)$, to each $U \in \mathcal{U}$ can be assigned a formula ϕ_U with parameters from $\mathfrak{C}$ whose trace in M is U; then the set $\{\phi_U\}$ is a partial global type which extends uniquely to a complete global type finitely satisfiable in M.

There is a canonical action of $G(M)$ on $S_{ext,G}(M)$ by translations, which is a point-transitive action by homeomorphisms in the Stone topology. So $S_{ext,G}(M)$ is a point-transitive $G(M)$-flow. One of the main results of [25] is that the Ellis semigroup $E(S_{ext,G}(M))$ of this flow is precisely $S_{ext,G}(M)$ itself, with the group operation given as follows: for $\mathcal{U}_1, \mathcal{U}_2 \in S_{ext,G}(M)$,

$$U \in \mathcal{U}_1 * \mathcal{U}_2 \Leftrightarrow \{g \in G(M) : g^{-1}U \in \mathcal{U}_2\} \in \mathcal{U}_1.$$

Remark 2.2.3. *An ultrafilter $\mathcal{U} \in S_{ext,G}(M)$ can be regarded as a $\{0,1\}$ Borel measure on $S_{ext,G}(M)$ where for an externally definable set $U \subset G(M)$ we put $\mathcal{U}(U) \iff U \in \mathcal{U}$. In this case, the $*$ operation becomes the convolution of measures.*

The flow $S_{ext,G}(M)$ is a model-theoretic analogue of the "largest" flow βG consisting of ultrafilters on all subsets of G, rather than externally definable ones.

We now point the circumstances under which $S_{ext,G}(M) = S_G(M)$ and moreover, the semigroup operation on $S_G(M)$ can be described in much simpler terms. We recall some definitions from model theory:

Definition 2.2.4. *Let $M \subset A, p \in S_G(M)$ and $p' \in S_G(A)$ extending p.*

(i) We say that p' is a coheir of p if p' is finitely satisfiable in M. That is, for any $\phi(\bar{x}, \bar{a}) \in p'$ there is $\bar{m} \in M^n$ such that $\models \phi(\bar{m}, \bar{a})$.

(ii) We say that p' is an heir of p if for any $\phi(\bar{x}, \bar{a}) \in p'$ with ϕ a formula over M there is $\bar{m} \in M^k$ such that $\phi(\bar{x}, \bar{m}) \in p$.

For the sake of brevity, we introduce the following convention:

Definition 2.2.5. *Assuming that we work over a fixed model M, we say that "$\bar{a}$ is a coheir over $\bar{b}$" if $\mathrm{tp}(\bar{a}/M\bar{b})$ is a coheir of $\mathrm{tp}(\bar{a}/M)$. Likewise for heir.*

Now we turn to one of the most important notions for our investigations:

Definition 2.2.6. *Let M be a model and $p(\bar{x}) \in S_G(M)$. We say that p is definable if for every formula without parameters $\psi(\bar{x}, \bar{y})$ there is a formula $t_\psi(\bar{y})$ with parameters from M such that $\models t_\psi(\bar{m}) \iff \psi(\bar{x}, \bar{m}) \in p(\bar{x})$ for all $\bar{m}$.*

Fact 2.2.7. *Assume that all types over M are definable. Then:*

(i) Every externally definable subset of M^n is definable. Consequently, $S_{ext,G}(M) = S_G(M)$.

(ii) Let $p \in S(M)$. Then over any $M \subset A$, p has a unique heir and a unique coheir in $S(A)$. We denote the unique heir of p over A by $p|A$. If $\bar{a}$ is a finite tuple and $p \in S(M)$, we will often write $p|\bar{a}$ meaning $p|M\bar{a}$.

Note that generally, coheirs and heirs of p over a fixed $A \supset M$ exist, but are not unique.

Thus under the assumption of definability of types, the Ellis semigroup of the universal definable G-flow $(G, S_G(M))$ is the flow itself. In fact, the category of definable G-flows over M becomes closed under taking the Ellis semigroups. The description of the semigroup operation is given by Newelski in [25]:

Fact 2.2.8. *If all types over M are definable, then the semigroup operation on $S_{ext,G}(M) = S_G(M)$ becomes*

$$p * q = \mathrm{tp}(ab/M),$$

where $a \models p, b \models q|a$ (or equivalently, $b \models q$ and a is a coheir over b realizing p).

There are particular models over which all types are definable, such as $\mathbb{R} = (\mathbb{R}, <, \cdot, +, \ldots)$, any o-minimal expansion of the field of reals. Also, the authors of [12] note a result by Shelah concerning NIP theories which allows to expand the model by additional predicates to obtain the definability of types. Consider any $M \models T$. Let $\mathcal{L}^{ext}$ be the language $\mathcal{L}$ expanded by the predicates P_X for every externally definable $P_X \subset M^n$. Let M^{ext} be the $\mathcal{L}^{ext}$-structure M with every P_X interpreted as X. Write $T^{ext} = Th(M^{ext})$. Shelah proved in [31] the following:

Proposition 2.2.9. *Assume that T has NIP and $M \models T$. Then the first order $\mathcal{L}^{ext}$-structure M^{ext} has elimination of quantifiers and all types over M^{ext} are definable.*

The setup in which all types are definable was called "tame topological dynamics" in [12].

2.3 Generic sets and their variants

Recall the definitions of some canonical objects from model theory:

Definition 2.3.1. *Let $U \subset G(M)$.*

(i) U is called (left) generic if finitely many of its (left) $G(M)$-translates cover $G(M)$.

(ii) U is said to be weakly generic if there exists a nongeneric $V \subset G(M)$ such that $U \cup V$ is generic.

(iii) A formula $\phi(x) \in \mathcal{L}(M)$ is said to be [weakly] generic if the set $\phi(M) \cap G(M)$ is [weakly] generic.

(iv) A collection p of sets (formulas) which is closed under intersections (conjunctions) is called [weakly] generic if each $U \in p$ is [weakly] generic.

Note that the property of being generic or nongeneric is preserved under elementary extensions, while that of being weakly generic is not. Generic sets can be thought of as "large" subsets of the group. A generic set is weakly generic. The last item of the definition applies in particular to types in $S_G(M)$. The work from [25] gives a connection between such types and objects from definable topological dynamics. Here we recall some of the useful results:

Proposition 2.3.2. *Let G be an M-definable group. Let* APer *be the set of almost periodic points (types) in the flow $(G(M), S_G(M))$. Let* WGen *and* Gen *be the sets of weakly generic and generic types of $S_G(M)$, respectively. Then:*

(i) $\mathrm{Gen} \subset \mathrm{APer} \subset \mathrm{WGen} = \mathrm{cl}(\mathrm{APer})$.

(ii) If $\mathrm{Gen} \neq \varnothing$, *then* $\mathrm{Gen} = \mathrm{APer} = \mathrm{WGen}$ *and* Gen *is the unique minimal subflow of $(G(M), S_G(M))$.*

(iii) Conversely, the uniqueness of a minimal flow implies $\mathrm{Gen} = \mathrm{APer} = \mathrm{WGen}$.

It is important to note that in general, the existence of generic types in $S_G(M)$ is not guaranteed. They are, however, present in several important cases such as stable groups, or definably compact o-minimal groups (more generally, fsg groups). The existence of weakly generic types is guaranteed since every almost periodic type is one. A weakly generic type does not need to be almost periodic.

It can be noted that the uniqueness of a minimal subflow possesses some purely dynamical characterizations. For a group G acting on a space X, we say that $x, y \in X$ are proximal if for some net $g_i \in G$ we have $\lim g_i \cdot x = \lim g_i \cdot y$. One can consider the proximality relation P on the product space $X \times X$. Then by [1]:

Fact 2.3.3. *The following are equivalent:*

(i) There exists a unique minimal subflow $I \subset X$.

(ii) P is closed.

(iii) P is an equivalence relation.

Thus for the universal definable flow of a definable group, any of these conditions is also equivalent to the existence of generic types.

Newelski [23] introduces another notion of generic sets:

Definition 2.3.4. *A set $U \subset G(M)$ is called strongly generic if the $G(M)$-algebra generated by it (the smallest $G(M)$-algebra containing U) consists only of generic subsets of $G(M)$ and the empty set.*

Every set that belongs to a $G(M)$-algebra generated by a strongly generic set is itself strongly generic. The empty set and $G(M)$ are both strongly generic and every $G(M)$-algebra of strongly generic sets contains them both. For a less trivial example, any union of the cosets of $H(M)$, where $H < G$ has a finite index, is strongly generic.

Externally definable strongly generic subsets of $G(M)$ turn out to be important for definable topological dynamics. Let $Def_{ext,G}(M)$ be the set of all externally definable subsets of $G(M)$, so that $S_{ext,G}(M) = S(Def_{ext,G}(M))$. This is naturally a $G(M)$-algebra. Denote by $\mathrm{End}(Def_{ext,G}(M))$ the semigroup of endomorphisms of this $G(M)$-algebra, with function composition as the operation. An external type $p \in S_{ext,G}(M)$ induces a $G(M)$-endomorphism $d_p \in \mathrm{End}(Def_{ext,G}(M))$ given by

$$d_p(U) = \{g \in G(M) : g^{-1}U \in p\}.$$

It is shown that $d_{p*q} = d_p * d_q$, so d is a semigroup homomorphism mapping $S_{ext,G}(M)$ to $\mathrm{End}(Def_{ext,G}(M))$.

Definition 2.3.5. *An image algebra is a maximal $G(M)$-algebra of externally definable strongly generic subsets of $G(M)$.*

Newelski [23] shows:

Theorem 2.3.6. *(i) Any two image algebras are isomorphic.*

(ii) Every externally definable strongly generic set belongs to some image algebra.

The following justifies the term "image algebra" [23]:

Proposition 2.3.7. *Let $p \in S_{ext,G}(M)$ be almost periodic. Then $\operatorname{im} d_p$ is an image algebra and every image algebra is of this form.*

The algebras are helpful in distinguishing the ideal subgroups of a minimal flow:

Proposition 2.3.8. *Let I be the minimal subflow of $S_{ext,G}(M)$. Then $p, q \in I$ belong to the same subgroup $u * I < I$ if and only if $\operatorname{im} d_p = \operatorname{im} d_q$.*

This establishes the link between the minimal subflows and $G(M)$-algebras of externally definable strongly generic sets:

Theorem 2.3.9. *Let I be the minimal subflow of $S_{ext,G}(M)$ and $p \in I$.*

(i) The kernel $\ker d_p$ does not depend on p and can be denoted K_I.

(ii) The image $\operatorname{im} d_p$ is isomorphic to $Def_{ext,G}(M)/K_I$ as a $G(M)$-algebra.

(iii) $S(\operatorname{im} d_p) \cong I$.

(iv) Any ultrafilter $\mathcal{U} \in S(\operatorname{im} d_p)$ extends uniquely to a type $\mathcal{U}' \in I$.

The last item in the theorem allows us to see an image algebra as a smallest $G(M)$-subalgebra of $Def_{ext,G}(M)$ from which the complete structure of a minimal subflow can be recovered.

As a final note of this section, observe that it is straightforward to generalize the definitions of all variants of generic sets to the context of a group G acting on an arbitrary set X (e.g. $U \subset X$ is generic if finitely many of its G-translates cover X).

2.4 o-minimality and its variants

The main object of study in this dissertation is about topological dynamics of groups definable in an o-minimal setting. In this section we outline the definition and the basic properties of o-minimal structures, as well as the groups definable in them. For a reference, see for example [19].

Definition 2.4.1. *(i) Let $M = (M, <, \dots)$ be linearly ordered by $<$. Then M is called o-minimal if every definable subset of M is a finite union of intervals.*

(ii) A theory is called o-minimal if all of its models are o-minimal.

Note that the condition for o-minimality of a structure is equivalent to saying that every definable subset of M is definable in the language $\{<\}$ (with parameters, but without quantifiers). The property of being o-minimal is first-order, i.e. preserved by elementary equivalence. Thus the complete theory $\mathrm{Th}(M)$ of an o-minimal structure M is o-minimal.

Standard examples of o-minimal structures include:

1. Dense linear orders without endpoints.

2. Ordered divisible abelian groups (such as valuation group of a valued field).

3. Any real closed field R is o-minimal and in fact elementarily equivalent to the field $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot)$. The theory of real closed fields is denoted by RCF. The natural ordering of R is definable in the language $\{0, 1, +, \cdot\}$. This structure has quantifier elimination and every definable subset of R^n is semialgebraic.

4. $\mathbb{R}_{an} = (\mathbb{R}, 0, 1, +, \cdot, <, f)_f$, where f ranges over all analytic functions on some open neighbourhood of $[0, 1]^n$ restricted to $[0, 1]^n$. Note that this restriction is necessary: the set of zeroes of a global analytic function can be infinite and discrete, and thus cannot be definable.

5. $\mathbb{R}_{exp} = (\mathbb{R}, 0, 1, +, \cdot, <, \exp)$ with exp the exponential function. An interesting result of Miller [21] shows that exp is definable in an o-minimal expansion of $(\mathbb{R}, 0, 1, +, \cdot)$ if and only if the structure is not "polynomially bounded".

6. The combination of the previous two: $\mathbb{R}_{an,exp} = (\mathbb{R}, 0, 1, +, \cdot, f, \exp)_f$, with f as before.

As any o-minimal structure M is ordered, this order gives rise to the order topology with intervals of the form (a, b) as a base, and then the product topology on M^n. If $M = \mathbb{R}$ then this topology agrees with the usual topology on $\mathbb{R}$. Observe that the product topology is generated by the interval products $\prod (a_i, b_i)$ which are uniformly definable. From this it is easy to conclude:

Fact 2.4.2. *Let M be an o-minimal structure and $U \subset M^n$ be definable. Then the closure, interior and boundary of U are all definable.*

The last three examples from the list provide structures in which a substantial portion of analysis can be done. Indeed, given a definable function f, many of the classic analytic notions related to it are definable in this setup, such as the set of continuity points of f, the set of points at which f is differentiable, the functions $f^{(n)}$ (on an appropriate domain), or the f's limit at a given point. Consequently, many of the standard real analysis theorems are first-order statements in those structures.

The definition of o-minimal structures implies some nice properties for the functions definable in it:

Fact 2.4.3. *Let M be o-minimal and $f : M \to M$ definable. Then there are finitely many disjoint intervals $I_i, i < n$ with $\coprod I_i = M$ such that $f|I_i$ is either strictly monotone or constant for all $i < n$.*

The definition of o-minimality has consequences for definable subsets of M^n for $n > 1$ as well. We will introduce them in the later sections as needed.

An important variant of the notion of o-minimality is weak o-minimality.

Definition 2.4.4. *Let $M = (M, <, \dots)$ be linearly ordered by $<$. Then M is called weakly o-minimal if every definable subset of M is a finite union of convex sets.*

Observe that a convex subset of M is a trace of an interval with endpoints from a sufficiently saturated extension M. On the other hand, a trace in M of an interval in any extension of M is a convex set. As a consequence, the condition for weak o-minimality of M is equivalent to saying that every definable subset of M is externally definable in the language $\{<\}$, similarly to the case of o-minimality.

The importance of weak o-minimality comes from the fact that every o-minimal theory is NIP, and thus its models are the subject of Proposition 2.2.9.

Corollary 2.4.5. *For an o-minimal $\mathcal{L}$-structure M, denote by M^{ext} the structure M in the language $\mathcal{L}$ expanded by the predicates for every externally definable subset of M. Then M^{ext} is a weakly o-minimal structure, has elimination of quantifiers and all types over M^{ext} are definable.*

We close the preliminaries on o-minimality with the description of types over a fixed model. Let M be an o-minimal structure and $M^* \succ M$ be sufficiently saturated. Recall the definitions of definable and algebraic closures over M, denoted $\mathrm{dcl}_M(-)$ and $\mathrm{acl}_M(-)$ respectively. Observe that since M^* is ordered, for any finite M-definable subset $\{a_i\}_{i<n} \subset M^*$, each of $\{a_i\}$ is also M-definable and hence $\mathrm{dcl}_M(\bar{a}) = \mathrm{acl}_M(\bar{a})$ for any $\bar{a}$.

Recall that for an ordered set X, a *Dedekind cut* of X is a pair (C, D) such that $C \cap D = \varnothing, C \cup D = X$ and $C < D$. If $X \subset M^*$, then a Dedekind cut (C, D) of X with $C, D \neq \varnothing$ determines a partial type over X consisting of all formulas of the form "$c \leqslant x$" and "$x \leqslant d$" with $c \in C, d \in D$.

Fact 2.4.6. *Let $\bar{b} \in (M^*)^n$. Then every partial type determined by cut in $\mathrm{dcl}_M(\bar{b})$ extends uniquely to a type in $S_1(M\bar{b})$ and every type in $S_1(M\bar{b})$ can be obtained this way.*

We say that M is *Dedekind complete* if for any Dedekind cut (C, D) of M with $C, D \neq \varnothing$, C has the least upper bound. Assume that M is Dedekind complete and let $a \in M^*$ be such that $\mathrm{tp}(a/M)$ is the unique extension of the type determined by a cut (C, D). We define the standard part (over M) of a to be the least upper bound of C provided that $C, D \neq \varnothing$. We denote this standard part by $\mathrm{st}_M(a)$. For a tuple $(a_0, \dots, a_n)$ we let $\mathrm{st}_M(a_0, \dots, a_n) = (\mathrm{st}_M(a_0), \dots, \mathrm{st}_M(a_n))$.

If $M = \mathbb{R}$ extends the group $(\mathbb{R}, +)$, we say that $a \in M^*$ is *infinitesimal* if $\mathrm{st}_{\mathbb{R}}(a) = 0$. We say that a is *infinitesimally close* to $b \in M^*$ if $\mathrm{st}_{\mathbb{R}}(a - b) = 0$.

3 Dynamics and subgroup decomposition

In the dissertation we aim to describe topological dynamics of groups definable in o-minimal structures. In particular, we will work with their decompositions in terms of their certain subgroups and use the dynamical properties of these subgroups. In this chapter we prove abstract versions of the results we will need. For the remainder of the chapter, fix an M with all types over M definable. Consider an M-definable group G with some subgroups K and H such that $G = KH$. We say that $G = KH$ is a *definable decomposition* (over M) if K, H are M-definable and $K \cap H = \{e\}$.

Consider a definable decomposition $G = KH$. Then for every $g \in G$ there are unique $k, k' \in K$, $h, h' \in H$ such that $g = hk = k'h'$. Also, g is interdefinable with both (h, k) and (k', h'). For a fixed $h \in H$ consider the mapping $\phi_h : K \to K$ given by $\phi_h(k) = k' \iff hk = k'h'$ for $k' \in K, h' \in H$. Since $K \cap H$ is trivial, K is a section of G/H and can be identified with this quotient as a set. Under this identification, $\phi_h(k) = (hk)/H$. By calculation, one verifies that the assignment $h \mapsto \phi_h$ gives a group action of H on K. As all types over M are definable, this action can be extended to a semigroup action of $S_H(M)$ on $S_K(M)$: for a $p \in S_H(M)$ we let $\phi_p(q) = \mathrm{tp}(\phi_h(k)/M)$ for $h \models p, k \models q|h$. This action in turn restricts to an action of $H(M)$ on $S_K(M)$. The $H(M)$-action on $S_K(M)$ combined with the usual $K(M)$-action on $S_K(M)$ yields a $G(M)$-action on $S_K(M)$. In each case, we will be calling such an action a *natural action*.

Given that (as a set) $G = K \times H$, there are natural projection functions $\pi_H : G \to H$ and $\pi_K : G \to K$. Each of those functions induces a continuous projection between the respective type spaces, e.g. $\pi_H : S_G(M) \to S_H(M)$.

3.1 Minimal flows

Given an M-definable group G acting definably on an M-definable set X, we say that $p \in S_X(M)$ is *G-invariant* (or simply *invariant*) if it is a fixed point of the $G(M)$-action on $S_X(M)$. This should not be confused with other model-theoretic notions of invariance.

We begin with the following result:

Lemma 3.1.1. *Let G be an M-definable group acting definably on an M-definable set X, $p \in S_X(M)$ have a finite $G(M)$-orbit, $M \prec N$ and $p' = p|N$ be the heir extension. Then the $G(N)$-orbit of p' consists precisely of heir extensions of the $G(M)$-translates of p. In particular, the heir of an invariant type is again invariant.*

Dowód. First we show that for $g \in G(M)$ the type $g \cdot p'$ is the heir of $g \cdot p$. So take a formula $\phi(x, \bar{n}, \bar{m}) \in g \cdot p$ with $\bar{n} \in N, \bar{m} \in M$. Then $\phi(g^{-1}x, \bar{n}, \bar{m}) \in p'$ and so for some $\bar{m}' \in M$ we have $\phi(g^{-1}x, \bar{m}', \bar{m}) \in p$. Then $\phi(x, \bar{m}', \bar{m}) \in g \cdot p$.

It remains to be shown that the lengths of the $G(N)$-orbit of p' and the $G(M)$-orbit of p are the same. Assume that $|G(N) \cdot p'| \geqslant k$. Then for $i = 1, \ldots, k$ there are pairwise inconsistent formulas $\phi_i(x, \bar{n}_i)$ with $\bar{n}_i \in N$ each satisfied by some translate of a realization of p'. So the formula

$$\chi(x, \bar{n}_1, \ldots, \bar{n}_k) = \bigwedge_{i \neq j} \neg \exists y (\psi_i(y, \bar{n}_i) \wedge \psi_j(y, \bar{n}_j)) \wedge \bigwedge_{i} \exists y \psi_i(yx, \bar{n}_i)$$

belongs to p'. Then for some $\bar{m}_1, \ldots, \bar{m}_k \in M$ we have $\chi(x, \bar{m}_1, \ldots, \bar{m}_k) \in p$ witnessing that $|G(M) \cdot p| \geqslant k$. □

The assumption of finite orbit cannot be weakened, even to bounded orbit.

The following is a classic notion in topological dynamics:

Definition 3.1.2. *We say that a group G is extremely amenable if for every flow (G, X) (here X is not necessarily compact) there is a G-invariant $\{0, 1\}$ Borel measure on X.*

There is a model-theoretic variant of this notion [28]:

Definition 3.1.3. *We say that a definable group G is definably extremely amenable if there is a G-invariant $p \in S_G(M)$.*

To compare this definition with the usual definition of extreme amenability, note that a type can be considered as a measure as in Remark 2.2.3. By the universal property of $S_G(M)$, it is easy to see that if an extremely definably amenable group acts definably on a definable set X, then there is a G-invariant type in $S_X(M)$ (e.g. the image of any G-invariant type in $S_G(M)$).

We now turn to consider the dynamics of groups that can be decomposed in terms of its two subgroups, such that one of the terms is definably extremely amenable.

Proposition 3.1.4. *Let G be an M-definable group with a definable decomposition $G = KH$ such that H is definably extremely amenable, and let $p \in S_H(M)$ be $H(M)$-invariant. Then for every minimal subflow I of $(G(M), S_K(M))$, the set $I * p$ is a minimal subflow of $(G(M), S_G(M))$. Moreover, the projection $\pi_K : S_G(M) \to S_K(M)$ restricted to $I * p$ maps it homeomorphically to I.*

Dowód. We show that for any $q * p \in I * p$, the ideal $S_G(M) * q * p$ is $I * p$.

Let $r \in S_G(M)$. The type $r*q*p$ is realized by some $k'h'kh$ for $k', k \in K, h', h \in H$ with

$$k'h' \models r, k \models q|(k', h') \text{ and } h \models p|(k', h', k).$$

Writing $h'k = \phi_{h'}(k)h''$ for some $h'' \in H$, we have $k'h'kh = k'\phi_{h'}(k)h''h$. As the pair (h', k) is interdefinable with $(\phi_{h'}(k), h'')$, we have

$$h \models p|(k', \phi_{h'}(k), h'')$$

and so by Lemma 3.1.1 also

$$h''h \models p|(k', \phi_{h'}(k), h'').$$

Thus

$$k'\phi_{h'}(k)h''h \models \mathrm{tp}(k'\phi_{h'}(k)/M) * p.$$

We show that $\mathrm{tp}(k'\phi_{h'}(k)/M) \in I$. If not, by closedness of I we can find a formula $\chi(x)$ over M with $\models \chi(k'\phi_{h'}(k))$ and $[\chi] \cap I = \varnothing$. Then the formula (in variables x_k, y_h) $\chi(x_k\phi_{y_h}(k))$ is in $\mathrm{tp}(k', h'/Mk)$ which is finitely satisfiable in M. So there are $k_m \in K(M), h_m \in H(M)$ such that $\mathrm{tp}(k_m\phi_{h_m}(k)/M) \notin I$. This contradicts the assumption that I is closed under both $K(M)$ and $H(M)$ actions on $S_K(M)$, hence $S_G(M) * q * p \subset I * p$.

Now we show that the closure of the orbit $G(M)(q * p)$ is $I * p$. Let $k_0 \models q, h_0 \models p|k_0$ so that $k_0h_0 \models q * p$. For any $k_m \in K(M), h_m \in H(M)$ we have $k_mh_m(q * p) = \mathrm{tp}(k_mh_mk_0h_0/M) = \mathrm{tp}(k_m\phi_{h_m}(k_0)h_1h_0/M)$ for some $h_1 \in H(M)$ with $h_0 \models p|k_0h_1$. Hence $k_mh_m(q*p) = \mathrm{tp}(k_m\phi_{h_m}(k_0)/M)*p$ and consequently

$$G(M)(q * p) = \{\mathrm{tp}(k_m\phi_{h_m}(k_0)/M) : k_m \in K(M), h_m \in H(M)\} * p.$$

By minimality of I (under the G-action) the closure of the set

$$\{\mathrm{tp}(k_m\phi_{h_m}(k_0)/M) : k_m \in K(M), h_m \in H(M)\}$$

is I. Then by the fact that the semigroup operation is continuous in the first coordinate, $\mathrm{cl}(G(M)(q * p)) = I * p$.

The "moreover" part is clear as $S_K(M) * p$ is homeomorphic to $S_K(M)$ by the projection map. □

We can get more if we assume that the definably extremely amenable factor is a normal subgroup.

Fact 3.1.5. *Let G be any group with a decomposition $G = KH$. The following are equivalent:*

(i) $G = K \ltimes H$, *i.e.* $H \lhd G$.

(ii) $N_G(H) \cap K = K$.

(iii) The natural H-action on K is trivial.

Dowód. Simply note that $N_G(H) \cap K$ is the set of fixed points of the H-action. □

If the natural action of H on K is trivial, then a subset $I \subset S_K(M)$ is G-invariant if and only if it is K-invariant, i.e. if it is a minimal subflow of the universal definable flow of K.

Corollary 3.1.6. *Let G be an M-definable group with a definable decomposition $G = K \ltimes H$ such that H is definably extremely amenable, and let $p \in S_H(M)$ be $H(M)$-invariant. Then for every minimal subflow I of $(K(M), S_K(M))$, the set $I * p$ is a minimal subflow of $(G(M), S_G(M))$. Particularly, if K is also definably extremely amenable, then so is G.*

The previous results allow us to describe the minimal subflows of the universal definable flow of $G = KH$ in terms of the G action on the group K. In a sense, the definably extremely amenable subgroup "factors out".

3.2 Strongly generic subsets and idempotents

Again we work with a definable group decomposition $G = KH$ with H definably extremely amenable. In this section, we aim to show that an image algebra of definable strongly generic subsets of $G(M)$ can be described in terms of strongly generic subsets for the $G(M)$-action on $K(M)$.

By Proposition 3.1.4 we fix a minimal subflow I of $(G(M), S_K(M))$ and $p \in S_H(M)$ an H-invariant type, so that $I * p$ is a minimal subflow of $(G(M), S_G(M))$.

Proposition 3.2.1. *There is $q * p \in I * p$ such that $\operatorname{im} d_{q*p} = \operatorname{im} d_q \cdot H(M)$, where $d_q \in \operatorname{End}(Def_K(M))$. In particular, every $U \in \operatorname{im} d_{q*p}$ is a cylinder over a definable subset of $K(M)$ with regards to the projection map $\pi_K : G(M) \to K(M)$.*

Dowód. As H is definably extremely amenable, there is an $H(M)$-invariant type $q * p \in I * p$. Let $U \subset G(M)$ be M-definable. Then for any $g \in G(M)$, $h \in H(M)$, we have $g \in d_{q*p}(U) \iff gh \in d_{h^{-1}(q*p)}(U) \iff gh \in d_{q*p}(U)$ and so $d_{q*p}(U)$ is a union of cosets of $H(M)$. Now by [23], d_{q*p} restricted to its own image is a permutation of that image. Thus $d_{q*p}(U) = d_{q*p}(V \cdot H(M))$ for some M-definable $V \subset K(M)$. Now we see that

$$k \in d_{q*p}(U) \iff k^{-1}V \cdot H(M) \in q * p \iff k^{-1}V \in q \iff k \in d_q(V),$$

and so $U = d_q(V) \cdot H(M)$ as wanted. □

An $H(M)$-invariant type of the form $q * p$ can be selected in a particular way. We begin with a technical claim:

Lemma 3.2.2. *Let G be an M-definable group acting definably on an M-definable set X, inducing an action of $G(M)$ on $S_X(M)$. Then for any $r \in S_X(M)$, the stabilizer $\operatorname{Stab}_{G(M)}(r)$ is type-definable over M.*

Dowód. This follows from definability of types over M. It is sufficient to show that for any $\phi(x, \bar{y}) \in \mathcal{L}$, the set $U \subset G(M)$ of all $g \in G(M)$ such that for all $\bar{b} \subset M$

$$\phi(x, \bar{b}) \in r \iff \phi(g^{-1}x, \bar{b}) \in r$$

is definable over M. So we take a formula $\psi(x, \bar{y}) \in \mathcal{L}(M)$ such that

$$M \models \psi(g, \bar{b}) \iff \phi(g^{-1}x, \bar{b}) \in r$$

and it is easy to see that U is the set of realizations of the formula

$$(\forall \bar{y})(\psi(x, \bar{y}) \leftrightarrow \psi(1, \bar{y})).$$

□

For the remainder of the section, denote by $\cdot_1$ the natural action of H on K and by $*_1$ the induced semigroup action of $S_H(M)$ on $S_K(M)$.

Lemma 3.2.3. *Let $r \in I$ be such that $p *_1 r = r$. Then both r and $r * p \in I * p$ are $H(M)$-invariant.*

Dowód. By Lemma 3.2.2, $\mathrm{Stab}_{H(M)}(r)$ is a partial type in H over M. By assumption, p extends this partial type. Assume $\mathrm{Stab}_{H(M)}(r) \lneq H(M)$. Then by compactness there is a formula $\phi(x) \in \mathrm{Stab}_{H(M)}(r)$ such that for some $h \in H(M)$ from a non-trivial coset of $\mathrm{Stab}_{H(M)}(r)$ we have $h \cdot \phi(x) \notin \mathrm{Stab}_{H(M)}(r)$. This contradicts the invariance of p. Thus r is $H(M)$-invariant.

Now if $k \models r$, $h \models p|k$ (i.e. $kh \models r * p$) and $h' \in H(M)$, we have

$$h'kh = \phi_{h'}(k) \cdot h_0 h$$

for some h_0 definable over $h'k$, so that $h_0 h \models p|\phi_{h'}(k)h_0$. Then by $H(M)$-invariance of r we have $\phi_{h'}(k) \models r$ and so $h' \cdot r * p = r * p$ as required. □

We conclude with a proposition that allows us to find fixed points of $a \mapsto p *_1 a$ with additional properties.

Proposition 3.2.4. *There is $u \in I$ such that:*

(i) u is an idempotent in $S_K(M)$.

(ii) $p *_1 u = u$.

(iii) $u * p$ is an idempotent in $S_G(M)$.

Dowód. Since $I * p$ is a minimal subflow of $(G(M), S_G(M))$, it contains an idempotent of the form $r * p$ with $r \in S_K(M)$. We claim that $u = p *_1 r$ is as desired. First we show that for any $a \in S_K(M)$ we have

$$(p *_1 a) * p = p * a * p.$$

So take any $h' \models p, k \models a|h', h \models p|h'k$ so that $(h' \cdot_1 k)h \models (p *_1 a) * p$. We have $h'k = \phi_{h'}(k)h_0$ for some $h_0 \in \operatorname{acl}(h', k)$, so also $\phi_{h'}(k) = h'kh_0^{-1}$ and

$$(h' \cdot_1 k)h = h'kh_0^{-1}h.$$

Then $h \models p|h'kh_0^{-1}$ and so also $h_0^{-1}h \models p|h'kh_0^{-1}$, hence

$$h'kh_0^{-1}h \models p * a * p.$$

From this we then get:

(i) It is sufficient to show that $u * u * p = u * p$. But

$$\begin{aligned} u * u * p &= (p *_1 r) * (p *_1 r) * p \\ &= p * r * p * r * p \\ &= p * r * p \\ &= (p *_1 r) * p \\ &= u * p \end{aligned}$$

as needed.

(ii) $p *_1 u = p *_1 (p *_1 r) = (p * p) *_1 r = p *_1 r = u$, since p is invariant and thus necessarily an idempotent in $S_H(M)$.

(iii) Similarly, $(u * p) * (u * p) = u * (p *_1 u) * p$, and now by (i) and (ii) we have $u * (p *_1 u) * p = u * u * p = u * p$. □

4 Topological dynamics and o-minimality

In this chapter we will discuss topological dynamics of groups definable in an o-minimal expansion of the field of reals, interpreted in its elementary extensions. We will also deal with their universal covers interpreted in a certain two-sorted structure.

We begin with recalling some of the basic results on groups definable in o-minimal structures. Our starting point is a fundamental result by Pillay:

Proposition 4.0.5 ([27, Proposition 2.5])**.** *Let M be o-minimal and let G be M-definable. The group G can be equipped with a definable manifold structure making G a topological group. That is, there is a finite collection of definable partial functions $f_i : M^n \to U_i$ whose domains are open subsets of M^n, with $U_i \subset G$ and $\bigcup U_i = G$ such that the group operations $\cdot$ and $^{-1}$ are continuous on $G(M)$, considered with the topology induced by the maps f_i.*

The reader is referred to [27] for a more detailed explanation of definable manifolds.

The order-induced topology on M^n gives rise to the notion of topological dimension of a set. For the remainder of this section, assume that M is a weakly o-minimal structure.

Definition 4.0.6 ([20, Section 4])**.** *Let $M \prec M^*$. An n-block over M is a product of n open intervals with endpoints in M. We define $\dim_M(-)$, the dimension over M, as follows:*

(i) *For a subset $U \subset (M^*)^{<\omega}$, $\dim_M(U)$ is the maximum n such that for some coordinates $\{i_0, \ldots, i_{n-1}\} \in \omega$, the projection $\pi_{\{i_0,\ldots,i_{n-1}\}}(U)$ contains an n-block over M.*

(ii) *For a formula $\phi(\bar{x})$ over M, $\dim_M(\phi) = \dim_M(\phi(M^n))$.*

(iii) *For a partial type p over M, $\dim_M(p) = \min\{\dim_M(\phi) : \phi \in p\}$.*

(iv) For a tuple $\bar{m} \in M^*$, $\dim_M(\bar{m}) = \dim_M(\mathrm{tp}(\bar{m}/M))$.

An important fact is that in (weakly) o-minimal structures, the dimension of a tuple (set, formula, ...) is invariant under taking images by nondegenerate, definable functions:

Fact 4.0.7 ([32, Theorem 2.13])**.** *Let f be an M-definable function with $\bar{m}$ in its domain. Then $\dim_M(f(\bar{m})) \leqslant \dim_M(\bar{m})$. If f is a bijection in some M-definable neighbourhood of $\bar{m}$, then $\dim_M(f(\bar{m})) = \dim_M(\bar{m})$.*

Corollary 4.0.8. *Let $X, Y \subset (M^*)^n$ be M-definable. Then $\dim_M(X \cup Y) = \max\{\dim_M(X), \dim_M(Y)\}$.*

For an o-minimal structure $\mathbb{R} = (\mathbb{R}, <, \ldots)$, the topological dimension over $\mathbb{R}$ is the usual notion of the dimension in an Euclidean space. The preservation of dimension prohibits the existence of objects such as definable space-filling curves. Recalling the structures $\mathbb{R}_{an}, \mathbb{R}_{exp}, \mathbb{R}_{an,exp}$, the o-minimal setup can be seen as "analysis minus Peano curve".

Observe that a subset $U \subset G(M)$ is open if and only if $\dim_M(U) = \dim_M(G)$.

The notion of dimension will be useful for characterization of dynamical objects in definable flows of groups definable over M. Assume that G is an M-definable group:

Proposition 4.0.9 ([27, Lemma 2.4])**.** *Let $X \subset G$ be M-definable such that $\dim_M(G \setminus X) < \dim_M(G)$. Then finitely many translates of X cover G.*

Corollary 4.0.10. *Let $p \in S_G(M)$ be weakly generic. Then $\dim_M(p) = \dim_M(G)$.*

Dowód. Take $\phi(\bar{x}) \in p$ and let $U = \phi(M^n) \cap G$. Then there exists a nongeneric $V \subset G$ and $g_0, \ldots, g_n \in G(M)$ such that $\bigcup_{i=0}^n g_i(U \cup V) = G$. We must have $\dim_M(G \setminus \bigcup_{i=0}^n g_i V) = \dim_M(G)$ and so $\dim_M(\bigcup_{i=0}^n g_i U) = \dim_M(G)$. Then $\dim_M(U) = \dim_M(G)$. □

The converse to the above corollary is usually not true: consider the group $(\mathbb{R}, +)$ and the type of a positive element infinitesimally close to 0: it is generated by bounded intervals, which are not weakly generic but of dimension 1.

In the dissertation, we will mostly deal with $\mathbb{R} = (\mathbb{R}, <, +, \cdot, \ldots)$, an o-minimal expansion of the field of reals. In this case, an $\mathbb{R}$-definable group G

is a real Lie group. Not all real Lie groups are definable in some o-minimal structure: for example, any group with infinite, discrete center is not definable. However, important classes of real Lie groups, such as linear, simple or compact are known to be definable this way.

From now on, we fix $\mathbb{R} = (\mathbb{R}, <, +, \cdot, \ldots)$, an o-minimal expansion of the ordered field of reals. We let R to be an $\aleph_1$-saturated extension of $\mathbb{R}$. We consider the structure R^{ext} as in Corollary 2.4.5. Since all types over R^{ext} are definable, for every R-definable group G we have $S_{ext,G}(R) = S_G(R^{ext})$.

Every external type can be restricted to its fragment consising of definable sets. After composing this restriction with the natural parameter restriction map $r : S_G(R) \to S_G(\mathbb{R})$, we obtain a well-defined restriction map $r' : S_G(R^{ext}) \to S_G(\mathbb{R})$. Thus it makes sense to consider an extension of a type $p \in S_G(\mathbb{R})$ to $p' \in S_G(R^{ext})$.

The remainder of the chapter is organized as follows. In sections 4.1 and 4.2 we discuss the dynamics (minimal flows, almost periodic types, Ellis groups) of torsion-free and compact groups respectively. In 4.3, we present results on dynamics of groups admitting a definable compact-torsion-free decomposition, the definable analogue of Iwasawa decomposition. In 4.4 we extend these results to the universal covers of groups from 4.3. Section 4.5 is devoted to results on strongly generic subsets of previously considered groups.

4.1 Torsion-free groups

Torsion-free groups definable in an o-minimal expansion of a real closed field have been extensively studied by many authors. An example of study of the subject is [29]. These groups can be generally complicated and have not been classified, even in low dimensions. Peterzil shows:

Proposition 4.1.1 ([22])**.** *Assume that R is polynomially bounded. Then every definable 1-dimensional group is definably isomorphic to either $(R, +)$ or $(R_{>0}, \cdot)$.*

It is not known whether the assumption on polynomial boundedness is necessary.

However, torsion-free groups are not very complex from the dynamical point of view. For the remainder of this section, assume that G is an R-definable torsion-free group.

Proposition 4.1.2. *The flow* $(G(R), S_G(R^{ext}))$ *has a one-point minimal subflow.*

In other words, the group $G(R^{ext})$ is definably extremely amenable. The proposition is essentially the same as [7, Proposition 4.7], which asserts the result for the flow $(G(R), S_G(R))$. The proof for Proposition 4.1.2 requires only minor changes.

Remark 4.1.3. *The proof of [7, Proposition 4.7] gives an explicit construction for* $2^{\dim_R(G)}$ *distinct* $G(R)$*-invariant types. In general, there are unboundedly many such types unless* $\dim_R(G) \leqslant 1$.

Under additional assumptions, an invariant type $p \in S_G(R^{ext})$ can be taken as a product of one-dimensional types.

Proposition 4.1.4. *Assume that* G *with* $\dim_R(G) = n$ *is a union of definable,* 1*-dimensional subgroups. Then there are* 1*-dimensional subgroups* $H_0, H_1, \ldots, H_{n-1} < G$ *with* $G = H_0 H_1 \ldots H_{n-1}$ *and types* $p_i \in S_{H_i}(R^{ext})$ *for* $i < n$ *such that the type* $p_0 * p_1 * \ldots * p_{n-1} \in S_G(R^{ext})$ *is* $G(R)$*-invariant.*

Dowód. We proceed by induction on n. By Corollary 2.12 of [29], G has a normal, definable subgroup H of codimension 1. By assumption, H has an $H(R)$-invariant type p that is a product of 1-dimensional types. Since G is a union of definable, 1-dimensional subgroups, there is such a subgroup K with $K \subsetneq H$. Then we must have $K \cap H = \{e\}, G = KH$ and so by Corollary 3.1.6 taking any $K(R)$-invariant $r \in S_K(R^{ext})$ we get a $G(R)$-invariant type $r * p \in S_G(R^{ext})$. □

The assumption of Proposition 4.1.4 is satisfied in a number of cases.

1. In [29], the authors show that under certain assumptions on R, every definable torsion-free abelian group is a direct product of 1-dimensional subgroups.

2. By for example [6], a definably simple definable torsion-free group G is definably isomorphic to a linear group $G' < GL_n(R)$ with $G' = AN$, where A is a subgroup of $D_n(R)$, the group of diagonal matrices, and N is a subgroup of $UT_n(R)$, the group of upper-triangular matrices with 1's on the diagonal. The group A is abelian and a union of 1-dimensional subgroups, while N is nilpotent.

3. The group $UT_n(R)$ from the previous item, as well as its definable subgroups, is also a union of 1-dimensional subgroups. The Lie algebra $\mathfrak{l}$ of $UT_n(R)$ consists of all upper-triangular matrices with 0's on the diagonal. Every such a matrix is nilpotent with degree $\leqslant n$, and so the matrix exponential function restricted to $\mathfrak{l}$ is given by a polynomial, hence it is definable in the language of fields. Every linear subspace in $\mathfrak{l}$ is definable and its image by the matrix exponential function is a 1-dimensional subgroup of $UT_n(R)$. These subgroups cover $UT_n(R)$.

We close the section with an example.

Example 4.1.5. 1. Let $G = (R, +)$. Then the type "at infinity" of an element $c > R$ is $G(R)$-invariant. Denote this type by p_∞.

2. Let $G = (R, +) \ltimes (R, +)$. Then by Proposition 4.1.4 the type $p_\infty * p_\infty$ is $G(R)$-invariant. A pair (a, b) realizing $p_\infty * p_\infty$ satisfies $a > R$, $b > \mathrm{dcl}_R(a)$.

4.2 Compact groups

In this section we turn our attention to R-definable groups that satisfy the definable analogue of compactness [30]:

Definition 4.2.1. *We say that an R-definable group is definably compact if every definable curve $f : R \to G$ can be completed, i.e. there is some $g \in G(R)$ such that* $\lim_{r\to\infty} f(r) = g$.

For the remainder of the section, assume that G is a compact $\mathbb{R}$-definable group. By the classification of compact real Lie groups, without loss of generality we can assume that $G(\mathbb{R}) \subset \mathbb{R}^n$ for some n, and the group action is continuous with respect to the order topology inherited from $\mathbb{R}^n$. As $G(\mathbb{R})$ is bounded, for every $a \in G(R)$, a has its standard part $\mathrm{st}_\mathbb{R}(a) \in G(\mathbb{R})$. It follows from the continuity of group operation that the function $\mathrm{st}_\mathbb{R} : G(R) \to G(\mathbb{R})$ is a group homomorphism. Its kernel consists of all elements of $G(R)$ infinitesimally close to the group's identity 1_G; by [15] for example, this is $G^{00}(R)$, the smallest type-definable subgroup of $G(R)$ of bounded index. We observe that the standard part map naturally induces the semigroup epimorphism between $S_G(\mathbb{R})$ and $G(\mathbb{R})$ by letting $p \mapsto \mathrm{st}_\mathbb{R}(a)$ where a is any realization of p. This epimorphism will also be denoted $\mathrm{st}_\mathbb{R}$.

An important result on definably compact groups is the existence of generic types, as well as external generic types.

Lemma 4.2.2. *Let G be an $\mathbb{R}$-definable compact group. Then the following are equivalent for the subset $U \subset G(R)$:*

(i) U contains an open $\mathbb{R}$-definable neighbourhood.

(ii) U is generic in $G(R)$.

(iii) $\dim_{\mathbb{R}}(U) = \dim_{\mathbb{R}}(G)$.

Dowód. (i) implies (ii) by compactness of $G(\mathbb{R})$, (ii) implies (iii) by preservation of dimension and (iii) implies (i) by definition. □

Observe that by Corollary 4.0.8, the collection of formulas

$$\{U^c \subset G(\mathbb{R}) : U \text{ is definable and } \dim_{\mathbb{R}}(U) < \dim_{\mathbb{R}}(G)\}$$

is consistent. Clearly, every extension of this collection is a generic type in $S_G(\mathbb{R})$. The collection remains consistent if we replace $G(\mathbb{R})$ with $G(R)$ and "definable" with "externally definable". This shows that every generic type in $S_G(\mathbb{R})$ extends uniquely to a generic type in $S_G(R)$ and then a generic type in $S_{ext,G}(R) = S_G(R^{ext})$. These extensions are finitely satisfiable in $\mathbb{R}$. This establishes bijections between generic types in each type space.

Let S be either of the flows $(G(\mathbb{R}), S_G(\mathbb{R}))$ or $(G(R^{ext}), S_G(R^{ext}))$. Since there is a generic type in S, by 2.3.2(ii), the unique minimal subflow of S is the set Gen of all generics. In fact, this minimal subflow is completely described by the results from [24, Section 4]. Here we summarize the results:

Proposition 4.2.3. *Let* Gen *be the unique minimal subflow of S. Then*

(i) the idempotents $J(\mathrm{Gen})$ are precisely the generic types with realizations from G^{00},

(ii) for every $u \in J(\mathrm{Gen})$ the ideal subgroup $u * \mathrm{Gen}$ is a section of Gen/G^{00} and is isomorphic to $G(\mathbb{R})$.

Note that the Ellis group does not change when considering the flow over R instead of the flow over $\mathbb{R}$.

The isomorphism mentioned in item (ii) is the map $\mathrm{st}_{\mathbb{R}}$ restricted to $u * \mathrm{Gen}$. Now fix any $u \in J(\mathrm{Gen})$. For $g \in G(\mathbb{R})$, let $u(g) \in u * \mathrm{Gen}$ be such that $\mathrm{st}_{\mathbb{R}}(u(g)) = g$. That is, the mapping $G(\mathbb{R}) \ni g \mapsto u(g) \in u * \mathrm{Gen}$ is the inverse of $\mathrm{st}_{\mathbb{R}}|u * \mathrm{Gen}$ and an isomorphism. Since Gen is the disjoint union of its ideal subgroups, every $r \in \mathrm{Gen}$ is of the form $u(k)$ for some $u \in J(\mathrm{Gen}), g \in G(\mathbb{R})$. The semigroup operation on Gen can be described as follows:

Lemma 4.2.4. *Let $r, r' \in \mathrm{Gen}$ such that $r = u(g), r' = u'(g')$. Then $r * r' = u(gg')$.*

Dowód. We have $r * r' = u(g) * u'(g') \in u * \mathrm{Gen}$, so $r * r' = u(g'')$ for some $g'' \in G(\mathbb{R})$. But $g'' = \mathrm{st}_{\mathbb{R}}(r * r') = gg'$. □

In particular, $r * r'$ depends only on r and the standard part of r'.

4.3 Definable compact-torsion-free decomposition

In this section, we broaden our investigations to a large class of groups that can be decomposed into subgroups of the types considered in the previous sections, and describe their dynamics in terms of this decomposition. We will work with groups definable over $\mathbb{R}$, but consider their universal definable flows both over $\mathbb{R}$ and R^{ext}.

The following results are due to Conversano [5, 6] and hold for any model of an o-minimal expansion of RCF, the theory of real closed ordered fields:

Definition 4.3.1. *Let G be a group definable over a model of an o-minimal expansion of RCF. We say that G has a definable compact-torsion-free decomposition if for some definable, definably compact $K < G$ and a definable torsion-free $H < G$ we have $G = K \cdot H$ and $K \cap H = \{e\}$.*

Definable compact-torsion-free decomposition is a model theoretic analogue of Iwasawa decomposition for real Lie groups. If $G = KH$ is such a decomposition, then K and H are the maximal definable subgroups which are respectively compact and torsion-free. The work done in [5] shows that every definable group G contains a maximal definable torsion-free subgroup, however the existence of a maximal definable, definably compact one is not guaranteed: indeed it is equivalent to G having a definable compact-torsion-free decomposition (see [5, Proposition 5.5]). The papers [5, 6] show that this condition holds for some large classes of definable groups, such as semisimple or linear. Moreover,

Proposition 4.3.2 ([5, Corollary 1.6])**.** *Let G be a definable definably connected group. Then there exists a definable $\mathcal{A}(G) < G$ which is central and invariant under definable automorphisms of G such that $G/\mathcal{A}(G)$ has a definable compact-torsion-free decomposition, and it is the maximal definable quotient of G with a definable compact-torsion-free decomposition.*

For the remainder of this section, let G be an $\mathbb{R}$-definable group with a compact-torsion-free decomposition $G = KH$. We consider the universal definable flow $(G(R^{ext}), S_G(R^{ext}))$. We write $\mathrm{Gen}_K(R^{ext})$ for the set of generic types in $S_K(R^{ext})$ with regards to the action of $K(R)$, i.e. the types of maximum dimension over $\mathbb{R}$. We also write $\mathrm{MaxDim}_K(R^{ext})$ for the set of types in $S_K(R^{ext})$ with maximum dimension over R^{ext} (equivalently, over R). Clearly, $\mathrm{Gen}_K(R^{ext}) \subset \mathrm{MaxDim}_K(R^{ext})$.

Recall the notion of the natural action $\cdot_1$ of $G(R^{ext})$ on $K(R^{ext})$ and the induced action $*_1$ on the space $S_K(R^{ext})$.

Proposition 4.3.3. *(i) The set* $\mathrm{MaxDim}_K(R^{ext})$ *is* $G(R^{ext})$*-invariant and closed.*

(ii) $G(R^{ext}) \cdot_1 \mathrm{Gen}_K(R^{ext}) \subset \mathrm{MaxDim}_K(R^{ext})$.

(iii) $\mathrm{Gen}_K(R^{ext}) = \mathrm{MaxDim}_K(R^{ext})$ *if and only if* $R = \mathbb{R}$.

Dowód. (i) Invariance: since $G(R^{ext})$ acts on $K(R^{ext})$ by bijections, it preserves the dimensions over R^{ext}. Closedness: for any $p \notin \mathrm{MaxDim}_K(R^{ext}), p \in [\phi]$ for any ϕ witnessing $\dim_R(p) < \dim_R(G)$. (i) immediately implies (ii). For (iii), if $R = \mathbb{R}$ then the other equality is clear. If $R \neq \mathbb{R}$, then it is easy to construct a type of maximum dimension over R that contains an n-block of an infinitesimal size. □

This allows us to give the following description of a collection of minimal flows of the group G.

Proposition 4.3.4. *There is a unique set* $I \subset S_K(R^{ext})$ *such that for any* $H(R^{ext})$*-invariant type* $p \in S_H(R^{ext})$*, the set* $I * p$ *is a minimal subflow of the flow* $(G(R^{ext}), S_G(R^{ext}))$*. The set* I *satisfies* $\mathrm{Gen}_K(R^{ext}) \subset I \subset \mathrm{MaxDim}_K(R^{ext})$*. The projection map* $\pi_K : S_G(R^{ext}) \to S_K(R^{ext})$ *restricted to* $I * p$ *maps it homeomorphically to* I*. If* $R = \mathbb{R}$*, then* $I = \mathrm{Gen}_K(R^{ext}) = \mathrm{MaxDim}_K(R^{ext})$*.*

Dowód. Since $\mathrm{Gen}_K(R^{ext})$ is a minimal subflow of $S_K(R^{ext})$ with respect to $K(R^{ext})$-action, any minimal subflow $S_K(R^{ext})$ with respect to the natural $G(R^{ext})$-action must contain an arbitrary generic $q \in \mathrm{Gen}_K(R^{ext})$. Therefore the set, $I = \mathrm{cl}(G(R^{ext}) \cdot_1 q)$ is a minimal subflow of $(G(R^{ext}), S_K(R^{ext}))$. By Proposition 4.3.3(i), $I \subset \mathrm{MaxDim}_K(R^{ext})$. Use Proposition 3.1.4 to conclude. □

Corollary 4.3.5. *If $G = K \ltimes H$, then the set I from Proposition 4.3.4 equals $\mathrm{Gen}_K(R^{ext})$.*

Dowód. Apply Corollary 3.1.6. □

By Proposition 4.3.3(iii), if $R \neq \mathbb{R}$, $\mathrm{Gen}_K(R^{ext}) \neq \mathrm{MaxDim}_K(R^{ext})$. Generally, the set I from Proposition 4.3.4 can lie between $\mathrm{Gen}_K(R^{ext})$ and $\mathrm{MaxDim}_K(R^{ext})$ in an arbitrary way. We give examples to illustrate various cases.

1. As indicated in Corollary 4.3.5 (e.g. when $H = \{e\}$), $I = \mathrm{Gen}_K(R^{ext})$ is a possibility.
2. The example of $SL(2, \mathbb{R}) = KH$ from [11] is easily generalized to $SL(2, R)$. In this case, $I = \mathrm{MaxDim}_K(R^{ext})$.
3. Taking G to be the product of a definably compact $\mathbb{R}$-definable group K' with $SL(2, \mathbb{R}) = KH$ yields a natural compact-torsion-free decomposition $G = K'K \cdot H$. Computing the minimal subflow of the flow $(G(R^{ext}), S_G(R^{ext}))$ gives $I = \mathrm{Gen}_{K'}(R^{ext}) * \mathrm{MaxDim}_{K'}(R^{ext})$. This set lies strictly between $\mathrm{Gen}_{K'K}(R^{ext})$ and $\mathrm{MaxDim}_{K'K}(R^{ext})$ whenever $R \neq \mathbb{R}$.

In the remainder of this section, we calculate the Ellis group of the universal definable flow of G over $\mathbb{R}$. Our goal is to prove:

Theorem 4.3.6. *Let G be an $\mathbb{R}$-definable group with a definable compact-torsion-free decomposition $G = KH$. Then the Ellis group of the universal definable flow of G is abstractly isomorphic to $N_G(H) \cap K(\mathbb{R})$.*

Yao [33] used our method of calculating the Ellis group in the proof of Theorem 4.3.6 in a more general setting of the universal flow of G over R, generalizing Theorem 4.3.6 to this broader context.

Having fixed a group $G = KH$, we choose an $H(\mathbb{R})$-invariant type $p \in S_H(\mathbb{R})$ and write I for $\mathrm{Gen}_K(\mathbb{R}) * p$, the associated minimal subflow of $(G(\mathbb{R}), S_G(\mathbb{R}))$. We will also write st instead of $\mathrm{st}_{\mathbb{R}}$ and dim instead of $\dim_{\mathbb{R}}$.

We will describe the semigroup operation on I. For this, we need the following:

Lemma 4.3.7. *Let $\bar{a}$ be a coheir over $\bar{b}$. Then for any $\bar{a}$-definable continuous $f : X \to Y$ where X, Y are compact, $\mathbb{R}$-definable and with $\bar{b} \in X$, we have $\mathrm{st}(f(\bar{b})) = \mathrm{st}(f(\mathrm{st}(\bar{b})))$.*

Dowód. Assume for a contradition $\mathrm{st}(f(\bar{b})) \neq \mathrm{st}(f(\mathrm{st}(\bar{b})))$. Then we can find an $\mathbb{R}$-definable, open V containing $f(\bar{b})$ but not $f(\mathrm{st}(\bar{b}))$. The set $U = f^{-1}[V]$ is open $\bar{a}$-definable with $\bar{b} \in U, \mathrm{st}(\bar{b}) \notin U$. Since the property of being open is $\varnothing$-definable and $\bar{b}$ is an heir over $\bar{a}$, there is an open, $\mathbb{R}$-definable V' containing $\bar{b}$ but not its standard part, a contradiction. □

Let $\psi : K(\mathbb{R}) \to K(\mathbb{R})$ be given by $\psi = \mathrm{st} \circ \phi_p | K(\mathbb{R})$ with ϕ_p given by the natural action of $S_H(\mathbb{R})$ on $S_K(\mathbb{R})$ and $K(\mathbb{R})$ naturally identified with the set of algebraic types in $S_K(\mathbb{R})$. This function and its image will be crucial for the understanding of the semigroup operation on I. Note that $\psi = \mathrm{st} \circ \phi_h | K(\mathbb{R})$ for any $h \models p$.

Lemma 4.3.8. *The map ψ is $\mathbb{R}$-definable.*

Dowód. Given $k \in K(\mathbb{R})$, $\psi(k) = \mathrm{st}(\phi_h(k))$ for any $h \models p$. So fix such an h and let $\mathbb{R} \prec \mathbb{R}^* \prec \mathfrak{C}$ be elementary extensions such that $h \in \mathbb{R}^*$ and that there is a $\mathfrak{C}$-definable X containing $K^{00}(\mathbb{R}^*)$ but disjoint from $rK^{00}(\mathbb{R}^*)$ for every $r \in K(\mathbb{R})$ (e.g. X is a ball with infinitesimal radius centered at the identity). Then the set

$$\{(x, y) \in K \times K : \phi_h(x) \in yX\} \cap (K(\mathbb{R}) \times K(\mathbb{R}))$$

coincides with the graph of ψ and is externally definable, hence definable. □

Now write $F = N_G(H) \cap K$, an $\mathbb{R}$-definable subgroup of K. Under the identification $K = G/H$ one can view F as the quotient $N_G(H)/H$. We will eventually show that the ideal subgroups of I are all algebraically isomorphic to $F(\mathbb{R})$.

Proposition 4.3.9. *The function ψ is an idempotent and* $\mathrm{im}\,\psi = F(\mathbb{R})$.

Dowód. First we show that if $k \in K(\mathbb{R})$, then $\psi(k) \in F$. Let $h \models p$ so that $\psi = \mathrm{st} \circ \phi_h$. For any $h' \in H$, the value of $\mathrm{st}(\phi_{h'}(k))$ is determined by $\mathrm{tp}(h'/\mathbb{R})$. Since p is $H(\mathbb{R})$-invariant, for any $h_r \in H(\mathbb{R})$ we have $\mathrm{st}(\phi_h(k)) = \mathrm{st}(\phi_{h_r h}(k))$. Now $\mathrm{st}(\phi_{h_r h}(k)) = \mathrm{st}(\phi_{h_r}(\phi_h(k)))$ and by 4.3.7 the standard part of ϕ_{h_r} depends only on the standard part of the argument. So $\mathrm{st}(\phi_{h_r}(\phi_h(k))) = \mathrm{st}(\phi_{h_r}(\mathrm{st}(\phi_h(k)))) = \mathrm{st}(\phi_{h_r}(\psi(k))) = \phi_{h_r}(\psi(k))$. So ultimately, $\psi(k) = \phi_{h_r}(\psi(k))$ for every $h_r \in H(\mathbb{R})$. By the definition of ϕ_{h_r} this implies $\psi(k) \in F(\mathbb{R})$ as required. Also, if $k \in F(\mathbb{R})$, then $\psi(k) = k$. □

An alternative way to see that ψ is an idempotent is by noting that p is an idempotent. Then $\phi_p \circ \phi_p = \phi_{p*p} = \phi_p$ and the same result for ψ can be obtained by an application of Lemma 4.3.7.

Recall that an element of $\mathrm{Gen}_K(\mathbb{R})$ is uniquely represented as $u(k)$ for some $u \in \mathrm{Gen}_K(\mathbb{R}) \cap K^{00}$ and $k \in K(\mathbb{R})$.

Proposition 4.3.10. *Let $r * p, r' * p \in I$ where $r = u(k)$ and $r' = u'(k')$. Then $r * p * r' * p = u(k\psi(k')) * p$.*

Dowód. Let $u(k) * p * u'(k') * p$ be realized by $fhf'h'$ where $f \models u(k)$, $h \models p|f$, $f' \models u'(k')|(f, h)$ and $h' \models p|(f, h, f')$. As in the proof of Proposition 3.1.4 we write $hf' = \phi_h(f')h''$ for some $h'' \in H$ and note that $h''h'$ realizes $p|(f, \phi_h(f'), h'')$. Now f is still a coheir over $(\phi_h(f'), h'', h)$ so $fhf'h' = f\phi_h(f')h''h'$ realizes $\mathrm{tp}(f/\mathbb{R}) * \mathrm{tp}(\phi_h(f')/\mathbb{R}) * p = u(k) * \mathrm{tp}(\phi_h(f')/\mathbb{R}) * p$.

We consider $\mathrm{tp}(\phi_h(f')/\mathbb{R})$. Note that h realizes a weakly generic type p while f' realizes a generic, so their acl-dimensions are maximal. h is a coheir over f and since coheirs are nonforking extensions, $\dim(h, f) = \dim K + \dim H$. Then also $\phi_h(f')$ and h'' must have the maximal acl-dimensions, so $\phi_h(f')$ realizes a generic type. Thus $\mathrm{tp}(\phi_h(f')/\mathbb{R})$ is of the form $u''(k'')$ for some $u'' \in J(\mathrm{Gen}_K(\mathbb{R}))$ and $k'' = \mathrm{st}(\phi_h(f'))$. Since h is a coheir over f', by Lemma 4.3.7 we have $\mathrm{st}(\phi_h(f')) = \mathrm{st}(\phi_h(\mathrm{st}(f'))) = \mathrm{st}(\phi_h(k')) = \psi(k')$. So $u(k) * p * u'(k') * p = u(k) * u''(\psi(k')) * p$ and we obtain the result by Lemma 4.2.4 □

Corollary 4.3.11. *The set of idempotents $J(I)$ is equal to*

$$\{g * p \in I : \psi(\mathrm{st}(g)) = 1_K\}.$$

It is a closed subsemigroup of I.

Since ψ is the identity function on F, in particular any idempotent $u = u(1) \in J(\mathrm{Gen}_K(\mathbb{R}))$ gives the idempotent $u * p$.

We are now ready to prove the main result of the section.

Proof of Theorem 4.3.6. Take any $u \in J(\mathrm{Gen}_K(\mathbb{R}))$. Then $u(1) * p$ is an idempotent in I. Let $E = u(1) * p * I$ be the ideal subgroup generated by it. We will show that E is isomorphic to $F(\mathbb{R})$. We have

$$\begin{aligned} E &= \{u(1) * p * u'(k) * p : u' \in J(\mathrm{Gen}_K(\mathbb{R})), k \in K(\mathbb{R})\} \\ &= \{u(\psi(k)) * p : k \in K(\mathbb{R})\}. \end{aligned}$$

So we have a bijection $i_u : F(\mathbb{R}) \to E$, $i_u(k) = u(k) * p$. But for any $a, b \in F(\mathbb{R})$ we have $i_u(a) * i_u(b) = i_u(a\psi(b)) = i_u(ab)$ since $\psi|F(\mathbb{R})$ is the identity. □

Note that this result is similar to one obtained in [4], where the authors consider the much smaller flow $(G(\mathbb{R}), K(\mathbb{R}))$ rather than $(G(\mathbb{R}), S_G(\mathbb{R}))$.

Corollary 4.3.12. *If $Z(G)$ is finite, then it embeds abstractly in $\mathcal{H}(G(\mathbb{R}), S_G(\mathbb{R}))$.*

By Lemma 3.1.5, we also establish:

Corollary 4.3.13. *The following are equivalent:*

(i) $G = K \ltimes H$.

(ii) $F(\mathbb{R}) = K(\mathbb{R})$.

We now give some examples of linear Lie groups, where the Ellis group can be explicitly calculated. In the second example, we compare the results with G/G^{00}:

Example 4.3.14. Let $G(\mathbb{R}) = GL_n(\mathbb{R})$. Its definable compact-torsion-free decomposition is $GL_n = O_n T_n^+$ where O_n is the subgroup of all orthogonal matrices, and T_n^+ the subgroup of all upper-triangular matrices with positive elements on the diagonal. One checks that $F(\mathbb{R})$ is precisely $O_n(\mathbb{R}) \cap T_n(\mathbb{R}) = O_n(\mathbb{R}) \cap D_n(\mathbb{R})$, the finite subgroup of matrices with ± 1 on the diagonal and 0 elsewhere. This subgroup is isomorphic to $\mathbb{Z}_2^n$.

Example 4.3.15. Similarly for $G(\mathbb{R}) = SL_n(\mathbb{R}) = SO_n(\mathbb{R})T_n^+(\mathbb{R})$ where $SO_n(\mathbb{R})$ are the matrices from $O_n(\mathbb{R})$ with determinant 1. Here we obtain $F(\mathbb{R}) = SO_n(\mathbb{R}) \cap D_n(\mathbb{R})$. This is isomorphic to $\mathbb{Z}_2^{n-1}$. But by Lemma 2.5 of [8], $SL_3^{00} = SL_3$ (for instance), so the quotient G/G^{00} is trivial in this case. This shows that the Ellis group may not be related to the center of the group.

4.4 Universal covers

In this section, we use some of the results obtained in [16] concerning universal covers of groups definable over o-minimal expansions of the field of reals. This allows us to extend the results regarding Ellis groups beyond the strictly o-minimal setting. As previously we will work over the structure $\mathbb{R}$. Let G be a definably connected $\mathbb{R}$-definable group and let $\widetilde{G}$ denote its universal cover as a topological group. Such a group might not be $\mathbb{R}$-definable even as an abstract group. Let Γ denote the kernel of the projection map. It is the fundamental group $\pi_1(G(\mathbb{R}))$ of $G(\mathbb{R})$. This is a central, closed and discrete subgroup of $\widetilde{G}$, giving the central extension $1 \to \Gamma \to \widetilde{G} \to G(\mathbb{R}) \to 1$.

Proposition 4.4.1 ([16, proof of Theorem 8.5])**.** *The group $\widetilde{G}$ is definable (as an algebraic group) in the two-sorted structure $\mathcal{M} = ((\mathbb{R}, <, +, \cdot, \ldots), (\Gamma, +))$ as a definable subset $G(\mathbb{R}) \times \Gamma$ with the group operation $(x, y) \cdot (x', y') = (xx', y + y' + h(x, x'))$ for some $\mathcal{M}$-definable function $h : G(\mathbb{R}) \times G(\mathbb{R}) \to \Gamma$ with a finite image.*

We will write $G(\mathbb{R}) \times_h \Gamma$ to denote the group structure described in Proposition 4.4.1. The function h that gives rise to this group operation is called a 2-cocycle. It is given by an $\mathbb{R}$-definable function $f : G(\mathbb{R}) \times G(\mathbb{R}) \to X$ with a finite image X and an identification of X with a subset of Γ. This way, h can be naturally extended to $h' : S_G(\mathbb{R}) \times S_G(\mathbb{R}) \to \Gamma$ by $h'(p, q) = h(a, b)$ where $a \models p, b \models q|a$. Its image is exactly the finite image of h. We will still use h to denote h'. This induced cocycle also induces a semigroup operation on the product $S_G(\mathbb{R}) \times S(\Gamma)$ in a natural way. We will similarly denote this semigroup structure as $S_G(\mathbb{R}) \times_h S(\Gamma)$.

With tools to deal with the group $G(\mathbb{R})$ in the o-minimal sort, we need to know more about the group Γ in order to work out the topological dynamics of the covering group $\widetilde{G}$.

Lemma 4.4.2. *Let G be a real connected Lie group. Then its fundamental group $\pi_1(G)$ is abelian and therefore stable as a pure group. Moreover, it is interpretable in the structure $(\mathbb{Z}, +)$.*

Dowód. It is folklore that $\pi_1(G)$ is abelian (it follows from the fact that G is a group object in the category of topological spaces). Commutativity of $\pi_1(G)$ is enough for its stability. For the "moreover" part: by [17], G retracts to its maximal compact subgroup K and thus $\pi_1(G) = \pi_1(K)$. Now, K is a compact manifold and by [13] its fundamental group is finitely generated and finitely presented. Thus $\pi_1(G)$ is a quotient of the finitely generated free abelian group $(\mathbb{Z}^n, +)$ by a finitely generated subgroup $N = \langle r_1, \ldots, r_n \rangle$. From commutativity, $N = \langle r_1 \rangle \ldots \langle r_n \rangle$ and each of the cyclic factors is definable in $(\mathbb{Z}, +)$. □

The following is a consequence of the stability of Γ:

Fact 4.4.3. *Every externally definable subset in $\mathcal{M}$ is definable and since there are no relations between the sorts in the language,*

$$S_{\widetilde{G}}(\mathcal{M}) \cong S_G(\mathbb{R}) \times S(\Gamma)$$

as a topological space.

Topological dynamics of stable groups have been described in detail in [26]. Here we recall some of the results that we will use:

Proposition 4.4.4. *Let H be a stable group (not necessarily abelian). Then the universal definable flow $(H(M), S_H(M))$ has a unique minimal subflow* $\mathrm{Gen}(H(M), S_H(M))$ *consisting of generic types of $S_H(M)$. This subflow contains a unique idempotent u_h, the generic type of H^{00}, and so this subflow is also the unique ideal subgroup.*

Because in a stable theory, heirs and coheirs coincide, we get the following:

Fact 4.4.5. *If an M-definable group H is stable and abelian, then the semigroup $S_H(M)$ is also abelian.*

Note that in the stable case, since the ideal subgroup coincides with the minimal subflow, it is invariant under the group translations. The latter property is also true for any ideal subgroup when the group is abelian. In particular, such a subgroup is invariant under translations by the image of h. By Fact 4.4.5, we can write the semigroup operation on $S(\Gamma)$ additively. Thus the semigroup operation on $S_G(\mathbb{R}) \times S(\Gamma)$ induced by the isomorphism $S_{\widetilde{G}}(\mathcal{M}) \cong S_G(\mathbb{R}) \times S(\Gamma)$ becomes

$$(p, q) * (p', q') = (p * p', q + q' + h(p, q)).$$

This is exactly $S_G(\mathbb{R}) \times_h S(\Gamma)$. We can now proceed to transfer the results on minimal subflows and Ellis groups of the universal flows for $G(\mathbb{R})$ and Γ to the universal flow $(\widetilde{G}(\mathcal{M}), S_{\widetilde{G}}(\mathcal{M}))$.

Theorem 4.4.6. *Let I_G be a minimal subflow of $(G(\mathbb{R}), S_G(\mathbb{R}))$, $u_G \in J(I)$, I_Γ the unique minimal subflow of $(\Gamma, S(\Gamma))$ and u_Γ its unique idempotent. Then*

(i) $I = I_G \times I_\Gamma$ *is a minimal subflow of* $(\widetilde{G}(\mathcal{M}), S_{\widetilde{G}}(\mathcal{M})) = (G(\mathbb{R}) \times_h \Gamma, S_G(\mathbb{R}) \times S(\Gamma))$.

(ii) $u_{\widetilde{G}} = (u_G, u_\Gamma - h(u_G, u_G))$ *is an idempotent in I.*

(iii) The ideal subgroup $u_{\widetilde{G}} * I$ is isomorphic to $(u_G * I_G) \times_h I_\Gamma$ (seen as a subsemigroup of $S_G(\mathbb{R}) \times_h S(\Gamma)$).

Dowód. (i) Take any $(p, q) \in I_G \times I_\Gamma = I$. Our goal is to show

$$(S_G(\mathbb{R}) \times_h S(\Gamma)) * (p, q) = I.$$

So take any $(p', q') \in S_G(\mathbb{R}) \times_h S(\Gamma)$. Then

$$(p', q') * (p, q) = (p' * p, q' + q + h(p', q'))$$

is in $I_G \times I_\Gamma$ because $I_\Gamma + h(p', q') = I_\Gamma$. So $(S_G(\mathbb{R}) \times_h S(\Gamma)) * (p, q) \subset I$. On the other hand, given any $p' \in I_G$ and $q' \in I_\Gamma$ there are $p'' \in I_G$ and $q'' \in I_\Gamma$ such that $p'' * p = p', q'' + q = q'$. Then

$$(p'', q'' - h(p'', p)) * (p, q) = (p', q').$$

Item (ii) is trivial. For (iii), simply note that $u_{\widetilde{G}} * I$ and $(u_G * I_G) \times I_\Gamma$ coincide as sets. □

Part (iii) of the proposition allows us to recover the Ellis group of $\widetilde{G}$ by examining the values of h on the elements of a fixed ideal subgroup of $(G(\mathbb{R}), S_G(\mathbb{R}))$. We conclude this section by providing examples.

Example 4.4.7. Let $G(\mathbb{R}) = SL_2(\mathbb{R}) = SO_2(\mathbb{R})T_2^+(\mathbb{R})$. Its universal cover $\widetilde{G}$ constructed by means of a cocycle as in Proposition 4.4.1 has been studied for example in [3] and also model-theoretically in [10]. The sort $(\Gamma, +)$ here is just $(\mathbb{Z}, +)$ and its Ellis group is the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$, the inverse limit $\varprojlim_n \mathbb{Z}/n\mathbb{Z}$. Both papers provide an explicit form of the cocycle h. This is taken from [10]: for $c, d \in \mathbb{R}$ write $c(d) = \begin{cases} c & \text{, if } c \neq 0 \\ d & \text{, if } d = 0 \end{cases}$. Then for $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in SL_2(\mathbb{R})$ write $\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then

$$h\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = \begin{cases} 1 & \text{, if } c_1(d_1) > 0, c_2(d_2) > 0, c_3(d_3) < 0, \\ -1 & \text{, if } c_1(d_1) < 0, c_2(d_2) < 0, c_3(d_3) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Either by the results from Section 4.3 of this paper (particularly Example 4.3.15) or the study from [11], we find an ideal subgroup of $(G(\mathbb{R}), S_G(\mathbb{R}))$ consisting of the types $q_0 = q * p$ and $q_1 = q' * p$, where $p = \text{tp}\left(\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix} / \mathbb{R}\right)$

for b infinite and $c > \mathrm{dcl}(\mathbb{R}, b)$, $q = \mathrm{tp}\left(\begin{pmatrix} 1-x & -y \\ y & 1-x \end{pmatrix}/\mathbb{R}\right)$ for x positive infinitesimal, y positive and $(x-1)^2+y^2 = 1$; and $q' = \mathrm{tp}\left(\begin{pmatrix} x-1 & -y \\ y & x-1 \end{pmatrix}/\mathbb{R}\right)$ for x positive infinitesimal, y negative and $(x-1)^2 + y^2 = 1$. q_0 is the identity of the group $\{q_0, q_1\}$. Direct calculation shows $h(q_i, q_j) = -i \cdot j$ for $i, j \in \{0, 1\}$. Thus we recover the Ellis group $\mathcal{H}(\widetilde{G}(\mathcal{M}), S_{\widetilde{G}}(\mathcal{M}))$ as the group whose underlying set is $\mathbb{Z}_2 \times \widehat{\mathbb{Z}}$ with the group operation given by $(x, n)(x', n') = (x +_2 x', n + n' - xx')$. This is isomorphic to $\widehat{\mathbb{Z}}$ by sending $(x, n) \mapsto x - 2n$.

Example 4.4.8. Let $G = [0, 1)$ with addition modulo 1. This is a compact group and its universal cover is a line. When realized in the two-sorted structure, the sort $(\Gamma, +)$ is again $(\mathbb{Z}, +)$ and the cocycle is given by $h(a, b) = \lfloor a + b \rfloor = \begin{cases} 1 & , \text{if } a + b \geqslant 1 \\ 0 & , \text{if } a + b < 1 \end{cases}$. The set $G(\mathbb{R}) \times \{n\}$ can be seen as the interval $[n, n + 1)$. The $\mathcal{M}$-structure on $G(\mathbb{R}) \times \mathbb{Z}$ is neither more or less rich than the $\mathbb{R}$-structure of the line seen in the o-minimal sort.

Write 0^+ for the type over $\mathbb{R}$ of a positive infinitesimal, then for $a \in G(\mathbb{R})$ let $a^+ = a + 0^+$. The set $\{a^+ : a \in G(\mathbb{R})\}$ is an ideal subgroup of $(G(\mathbb{R}), S_G(\mathbb{R}))$ isomorphic to $G(\mathbb{R})$ and it is easy to see that $h(a^+, b^+) = h(a, b)$. Thus the Ellis group $\mathcal{H}(\widetilde{G}(\mathcal{M}), S_{\widetilde{G}}(\mathcal{M}))$ is $G(\mathbb{R}) \times \widehat{\mathbb{Z}}$ with the group operation $(x, n) * (x', n') = (x +_1 x', n + n' + \lfloor x + x' \rfloor)$.

4.5 Strongly generic subsets

In this section, we discuss the notion of externally definable strongly generic subsets of an R-definable group. We will give some preliminary results on the structure of these sets and devote a subsection for an example of a direct computation of an image algebra in the group $SL(2, R)$. This allows for the comparison of image algebras of $SL(2, -)$ interpreted over elementary extensions. Such comparison is one of the interesting problems related to strongly generic sets and have a connection with the problem of extension of weakly generic types.

We begin with a general discussion on definable subsets of an o-minimal structure M. The following material is classic, see for instance [19].

The definition of o-minimality concerns only definable subsets of M, it nevertheless implies good structural results for definable subsets of M^n as well.

Definition 4.5.1. *Let M be o-minimal. We inductively define the set of cells, that are certain definable subsets of $M^{<\omega}$:*

(1) A point is a cell.

(2) If $X \subset M^n$ is a cell and $f : X \to M$ is a definable function, then its graph $\Gamma(f) \subset M^n \times M$ is a cell.

(3) If $X \subset M^n$ is a cell and $f, g : X \to M$ are definable functions, then the set $\{(\bar{m}, y) \in M^{n+1} : f(\bar{m}) < y < g(\bar{m})\}$ is a cell.

(4) A set obtained by permuting the coordinates of a cell is also a cell.

Remark 4.5.2. *If $U \subset M^{<\omega}$ is (up to a permutation of coordinates) a cell constructed using steps $(1)-(3)$ in the above definition, then $\dim_M(U)$ is the number of times step (3) was used in the construction.*

Intuitively, a cell is a homeomorphic image of some M^n by a definable function. The notion of a cell allows us to state:

Theorem 4.5.3 (Cell Decomposition Theorem)**.** *Let M be o-minimal. Then any definable $U \subset M^n$ is a finite union of disjoint cells.*

As a corollary, we get that given an externally definable subset $U \subset M^n$, U is a finite union of traces of cells definable in some $M^* \succ M$.

Fact 4.5.4. *Let $M \prec M^*$ and $U \subset (M^*)^n$ a cell. Then*

$$\dim_M(U \cap M^n) \leqslant \dim_M(U).$$

This inequality can be strict. In fact, the trace in M of a non-empty, M^*-definable $U \subset (M^*)^n$ can be empty.

This concludes the exposition. Now we return to considering the structures $R \succ \mathbb{R}$ as discussed at the beginning of the chapter. Let G be an R-definable group.

Proposition 4.5.5. *Let $p \in \mathrm{MaxDim}_G(R^{ext})$. Then the image algebra* $\operatorname{im} d_p$ *consists of all finite unions of the sets of the form $d_p(U)$ with $U \subset G(R)$ a trace in R of a cell definable in some $R^* > R$ such that* $\dim_R(U) = \dim_R(G)$. *In particular, if $R = \mathbb{R}$, the sets U can be taken to be open $\mathbb{R}$-definable subsets of $G(\mathbb{R})$.*

Dowód. Take any $X \subset R^n$, a trace in R of some R^*-definable X'. Then X' decomposes into a finite union $X' = \bigcup_i (X'_i)$, with each X'_i an R^*-definable cell. As d_p is an endomorphism of $G(R)$-algebra $Def_{ext,G}(R)$, $d_p(X) = \bigcup_i d_p(X'_i \cap G(R))$. So we can assume that X' is a cell. Since $\dim_M(p) = \dim_M(G)$, from the definition of d_p it follows that $d_p(X' \cap G(R)) = \varnothing$ whenever $\dim(X' \cap G(R)) \neq \dim_M(G)$. In particular, if $R = \mathbb{R}$ then $X' \cap G(\mathbb{R})$ is $\mathbb{R}$-definable and by maximality of the dimension it is open. □

In the following subsection we will give an example of the direct computation of an image algebra in the group $SL_2(-)$ interpreted both over the standard model $\mathbb{R}$ and its elementary extension R. We close this introduction to give an example that we will later contrast with the results for $SL_2(-)$.

Example 4.5.6 ([23])**.** Let $G = SO(2, -)$ be the circle group, definable over $\mathbb{R}$ and denote its identity by 0. This is a definably compact group. Working in either $G(\mathbb{R})$ or $G(R)$, we define an arc to be a definably connected subset of $G(-)$. By fixing an orientation on G, we can also consider intervals in G of the form $[a, b], [a, b)$ etc. An interval is an arc, but inside $G(R)$ there are arcs without endpoints.

(i) When working over $\mathbb{R}$, every arc is an interval of some form. The collection of intervals of the form $(0, a), a \in G(\mathbb{R}), a \neq 0$ implies a complete, generic type $q_+ \in S_G(\mathbb{R})$ of an element infinitesimally close to 0 on the "positive" side. One checks that the $G(\mathbb{R})$-algebra $\mathcal{A}_+$ consisting of all finite unions of the intervals of the form $[a, b)$ with $a, b \in G(\mathbb{R})$ is precisely the image algebra $\operatorname{im} d_{q_+}$. Similarly, the $G(\mathbb{R})$-algebra $\mathcal{A}_-$ consisting of all finite unions of the intervals of the form $(a, b]$ with $a, b \in G(\mathbb{R})$ is also an image algebra, and the set of all (externally) definable strongly generic subsets is the union $\mathcal{A}_+ \cup \mathcal{A}_+$.

(ii) Working over the extension R, it is no longer true that an interval of the form $[a, b), a, b \in G(R)$ is strongly generic. In fact, it is not even generic unless it contains a subinterval with endpoints from $G(\mathbb{R})$. The generic type $q_+ \in S_G(\mathbb{R})$ from the previous item extends uniquely to a generic type $q'_+ \in S_G(R^{ext})$. Then the image algebra $\mathcal{B}_+ = \operatorname{im} d_{q'_+}$ consists of all finite unions of the sets of the form $[a,b) \cdot G^{00}(R)$ with $a, b \in G(\mathbb{R})$, where $G^{00}(R)$ is the set of elements of $G(R)$ infinitesimally close to 0, a normal subgroup of $G(R)$. Thus a set of the form $[a,b) \cdot G^{00}(R)$ consists of all points infinitesimally close to the real interval $[a,b)$. Such an arc is not an interval. With an algebra $\mathcal{B}_-$ defined in an analogous way, the union $\mathcal{B}_+ \cup \mathcal{B}_+$ is the set of all externally definable strongly generic subsets of $G(R)$.

4.5.1 $SL(2,-)$

In this subsection we explicitly calculate a maximal image algebra of externally definable subsets of $G = SL(2,-)$ over $\mathbb{R}$ and over R. G is an $\mathbb{R}$-definable group and its compact-torsion-free decomposition is $SL(2,-) = SO(2,-)T^+(2,-)$ where $SO(2,-)$ is the circle group as in Example 4.5.6, and $T^+(2,-)$ is the group of triangular matrices with positive elements on the diagonal with determinant 1. The dynamics of the group G has been thoroughly studied in [11]. The results concerned the dynamics over $\mathbb{R}$, but the authors note they generalize to dynamics over R.

Denote $K = SO(2,-)$ and $H = T^+(2,-)$ so that $G = KH$. Denote by 0_K the identity of $SO(2,-)$ (i.e. the identity matrix). As in Example 4.5.6, we fix an orientation on K and consider arcs and intervals in K.

The collection of arcs of the form $(0_K, b)$ with $b \in K(\mathbb{R}), b \neq 0_K$ implies a complete type $q_0 \in S_K(\mathbb{R})$. Similarly, the collection of arcs of the form $(0_K, b)$ with $b \in K(R), b \neq 0_K$ implies a complete type $q \in S_K(R^{ext})$.

The type q_0 is the generic type q_+ discussed in Example 4.5.6(i). It extends to the type q. However, q is not the type q'_+ from Example 4.5.6(ii). This distinction can be made more precise: the extension $q_+ \subset {q'}_+$ from Example 4.5.6 is the (external) coheir extension, while the extension $q_0 \subset q$ is the (external) heir extension. We still have that $\dim_R(q) = \dim_R(G) = 1$, i.e. $q \in \operatorname{MaxDim}_K(R^{ext})$.

With the notation as above, we will now state the results from [11] that we need.

Lemma 4.5.7. *There is an H-invariant type $p \in S_H(R^{ext})$ such that:*

(i) $\operatorname{MaxDim}_K(R^{ext}) * p$ *is a minimal subflow of $S_G(R^{ext})$. In particular, $q * p$ is almost periodic.*

(ii) $p *_1 q = q$.

We will now calculate the image algebra $\operatorname{im} d_{q*p}$. By Lemma 4.5.7(ii) and Lemma 3.2.3, the type $q * p \in S_G(R^{ext})$ is $H(R^{ext})$-invariant. By the proof of Proposition 3.2.1, the image algebra $\operatorname{im} d_{q*p}$ is fully determined by the set $\operatorname{im} d_q$. Specifically, every $U \in \operatorname{im} d_{q*p}$ is of the form $d_q(V) \cdot H(\mathbb{R}^{ext})$ for some definable $V \subset R^{ext}$. We now calculate the set $\operatorname{im} d_q$.

Proposition 4.5.8. *The set* $\operatorname{im} d_q$ *consists of all finite unions of the arcs* $I \subset K(R^{ext})$ *such that:*

(a) If I *has a left endpoint* $e_l \in K(R)$*, then* $e_l \in I$*.*

(b) If I *has a right endpoint* $e_r \in K(R)$*, then* $e_r \notin I$*.*

Dowód. Observe that any arc is a definable subset of $K(R^{ext})$. By Proposition 4.5.5, it is sufficient to show that if I' is an interval with endpoints in some $R^* \succ R$ such that $I_0 = I' \cap K(R)$ has dimension 1, then $d_q(I_0)$ is an arc with the properties (a)-(b). Clearly I_0 itself is an arc.

Bearing in mind that q is the type of an element "infinitesimally close" to 0_K from the right side, one checks for a $k \in K(R^{ext})$:

(i) If $k \in I_0$ is not an endpoint of I_0, then $k \in d_q(I_0)$.

(ii) If k is a left endpoint of I_0, then $k \in d_q(I_0)$.

(iii) If k is a right endpoint of I_0, then $k \notin d_q(I_0)$.

(iv) Otherwise, $k \notin d_q(I_0)$.

The result follows. □

Observe that, in contrast with Example 4.5.6, the sets of the form

$$X = I \cdot H(R^{ext})$$

with I an infinitesimal arc are still generic (direct calculations shows that this is witnessed by at most three translates of X) and even strongly generic if I satisfies the conditions (a)-(b) from Proposition 4.5.8.

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