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TOPICS ON KÄHLER MANIFOLDS

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1. Complex and almost complex structures

1.1. **Complex manifolds.** We begin by saying what exactly a complex manifold is. Roughly speaking it is a topological space that locally looks like some open subset in $\mathbb{C}^m$.

Definition 1. Let M be a Hausdorff topological space with a countable basis. We say that M is a **complex manifold** of dimension m iff there exists an open cover $\{U_i\}_{i\in I}$ of M and diffeomorphisms

$$\phi_i : U_i \longrightarrow V_i \subset \mathbb{C}^m,$$

where V_i is open for each i, such that

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

is a biholomorphism.

A pair of the form (U_i, ϕ_i) is called a **holomorphic coordinate neighbourhood** and a collection of such pairs is called a **holomorphic atlas** on M iff sets U_i cover M. Maps ϕ_i are called **local holomorphic coordinates** and $\phi_j \circ \phi_i^{-1}$ are referred to as **transition functions**.

It is easy to see that every complex manifold of (complex) dimension m is a smooth differential manifold of (real) dimension $2m$, since every holomorphic function is smooth.

The simplest example of a complex manifold is the space $\mathbb{C}^m$. Moreover, every open subset of a complex manifold M is itself a complex manifold, called a **submanifold** of M. If we have two complex manifolds, then their Cartesian product is also a complex manifold.

After giving those very simple examples we proceed to something less trivial following [Be].

Example 1 (Riemann sphere). Let S^2 denote a unit sphere in $\mathbb{C} \times \mathbb{R}$,

$$S^2 := \{(z, x) \in \mathbb{C} \times \mathbb{R} \mid z\bar{z} + x^2 = 1\}.$$

We define the stereographic projections as

$$\pi_N : S^2 \setminus \{(0, 1)\} \ni (z, x) \longmapsto \frac{z}{1-x} \in \mathbb{C}$$

and

$$\pi_S : S^2 \setminus \{(0,-1)\} \ni (z,x) \longmapsto \frac{\bar{z}}{1+x} \in \mathbb{C}.$$

Then we have a holomorphic atlas on S^2. Indeed, observe first that π_N^{-1} is given by

$$\pi_N^{-1} : \mathbb{C} \ni w \longmapsto \left(\frac{2w}{1+|w|^2}, \frac{-1+|w|^2}{1+|w|^2} \right) \in S^2 \setminus \{(0,1)\}.$$

Let us check, for instance, that $\left|\pi_N^{-1}(w)\right| = 1$, where $|\cdot|$ is the Euclidean norm in $\mathbb{C} \times \mathbb{R}$.

$$\left|\pi_N^{-1}(w)\right| =$$

$$= \left| \left(\frac{2w}{1+|w|^2}, \frac{-1+|w|^2}{1+|w|^2} \right) \right| = \sqrt{\left(\frac{1}{1+|w|^2} \right)^2 (4|w|^2 + (-1+|w|^2)^2)} =$$

$$= \frac{1}{1+|w|^2} \sqrt{|w|^4 + 2|w|^2 + 1} = \frac{1}{1+|w|^2} \sqrt{(1+|w|^2)^2} = 1.$$

To complete the proof we must verify that $\pi_S \circ \pi_N^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is holomorphic. We have

$$(\pi_S \circ \pi_N^{-1})(w) = \pi_S \left(\frac{2w}{1+|w|^2}, \frac{-1+|w|^2}{1+|w|^2} \right) =$$

$$= \frac{2\bar{w}}{(1+|w|^2)(1 - \frac{|w|^2-1}{1+|w|^2})} = \frac{2\bar{w}}{(1+|w|^2)(\frac{1+|w|^2-1+|w|^2}{1+|w|^2})} =$$

$$= \frac{2\bar{w}}{2|w|^2} = \frac{1}{w}.$$

We can similarly check that $\pi_N \circ \pi_S^{-1}$ is holomorphic. Thus, S^2 is a complex manifold.

Remark. If we take $\tilde{\pi}_S(z,x) = \frac{z}{1+x}$ instead of π_S we get a smooth manifold which is not holomorphic, because we have

$$(\tilde{\pi}_S \circ \pi_N^{-1})(w) = \tilde{\pi}_S \left(\frac{2w}{1+|w|^2}, \frac{-1+|w|^2}{1+|w|^2} \right) = \frac{1}{\bar{w}}$$

and this function is clearly not holomorphic.

Example 2 (**Complex projective space**)**.** Let us define an equivalence relation $\sim$ in $\mathbb{C}^{m+1} \setminus \{0\}$ by $(z_0, \ldots, z_m) \sim (w_0, \ldots, w_m)$ iff $(z_0, \ldots, z_m) = \lambda(w_0, \ldots, w_m)$, where $\lambda \in \mathbb{C} \setminus 0$. The quotient of $\mathbb{C}^{m+1} \setminus 0$ by this relation is called the **complex projective space** $\mathbb{C}P^m$. The equivalence class of a point $(z_0, \ldots, z_m)$ is denoted by $[z_0 : \ldots : z_m]$. We will show that $\mathbb{C}P^m$ is a complex manifold.

Let us define the sets U_i by

$$U_i = \{[z_0 : \ldots : z_m] \in \mathbb{C}P^m \mid z_i \neq 0\},$$

where $i = 0, \ldots, m$. Each set U_i can be seen as the family of all complex lines through 0 in $\mathbb{C}^{m+1}$, which are not contained in the hyperplane $z_i = 0$. Clearly, $\{U_i\}_{i=0}^m$ is an open cover of $\mathbb{C}P^m$.

Next, we can define $\phi_i : U_i \to \mathbb{C}^m$ by

$$\phi_i([z_0 : \ldots : z_m]) = (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i}).$$

The inverse of this map is given by

$$\phi_i^{-1}(w_1, \ldots, w_m) = [w_1 : \ldots : w_{i-1} : 1 : w_{i+1} : \ldots : w_m],$$

because of the fact that points $(z_0, \ldots, z_m)$ and $(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i})$ lie on the same complex line.

The transition maps $\phi_j \circ \phi_i^{-1}$ are holomorphic. Indeed,

$$(\phi_j \circ \phi_i^{-1})(w_1, \ldots, w_m) = \phi_j([w_1 : \ldots : w_{i-1} : 1 : w_{i+1} : \ldots : w_m]) =$$
$$(\frac{w_1}{w_j}, \ldots, \frac{w_{j-1}}{w_j}, \frac{w_{j+1}}{w_j}, \ldots, \frac{1}{w_j}, \ldots, \frac{w_m}{w_j}),$$

and the functions $\frac{w_i}{w_j}$ are holomorphic.

Example 3 (**Complex Grassmannian** $G_{r,m}$)**.** Let $G_{r,m}$ denote the set of all r-dimensional complex vector subspaces of $\mathbb{C}^m$ for $0 < r < m$. We first show that $G_{r,m}$ is a complex manifold.

Let M^* be the set of all complex linear maps $F : \mathbb{C}^r \to \mathbb{C}^m$ of rank r. A map π given by

$$\pi : M^* \longrightarrow G_{r,m}, \quad \pi(F) = [F] =: \mathrm{im} F,$$

where F denotes the complex subspace in $\mathbb{C}^m$ spanned by F (image of F), defines a canonical projection from M^* to $G_{r,m}$.

Let $B : \mathbb{C}^m \to \mathbb{C}^m$ be an isomorphism. Then the map from M^* into the space of $m \times r$ complex matrices $\mathbb{C}^{m \times r}$ given by

$$F \longmapsto \mathrm{Matrix}(BF),$$

where $\mathrm{Matrix}(BF)$ denotes the matrix of a linear map BF, is a bijection onto the open subset of $m \times r$-matrices of rank r. Injectivity is clear as we can see this map as a change of basis and the rest follows because of the equality of dimensions of M^* and $\mathbb{C}^{m \times r}$. Thus, we have a structure of a complex manifold of dimension mr on M^* and this structure does not depend on the choice of B.

Write

$$\mathrm{Matrix}(BF) = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix},$$

where $F_0 \in \mathbb{C}^{r \times r}$ and $F_1 \in \mathbb{C}^{(m-r) \times r}$. Let U_B be the subset of $[F]$ in $G_{r,m}$ such that F_0 has rank r. We show that

$$Z_B : U_B \longrightarrow \mathbb{C}^{(m-r) \times r}, \quad Z_B([F]) = F_1 F_0^{-1},$$

is a well defined bijection.

Note first that multiplication of matrices of dimensions $(m-r)\times r$ and $r\times r$ gives a $(m-r)\times r$-matrix. Let us assume that $[F]=[G]$ for some $F,G\in M^*$. Then there exists a square r-matrix A such that $\mathrm{Matrix}(F)=\mathrm{Matrix}(GA)$. We can write

$$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} G_0A \\ G_1A \end{pmatrix}.$$

It gives

$$Z_B([F]) = F_1F_0^{-1} = G_1AA^{-1}G_0^{-1} = G_1G_0^{-1} = Z_B([G]),$$

so Z_B is well defined.

To show that Z_B is surjective let us choose a matrix $A\in\mathbb{C}^{(m-r)\times r}$ and define

$$N = \begin{pmatrix} I_r \\ A \end{pmatrix},$$

where I_r is the identity r-matrix. It is clear that $Z_B([N])=A$, where we identify the matrix N with the linear map given by N.

Assume that $Z_B([F])=A=Z_B([G])$ for some $F,G\in M^*$. As before we identify F,G with their matrices. We then have

$$F_1F_0^{-1}=A\Rightarrow F_1=AF_0,\quad G_1G_0^{-1}=A\Rightarrow G_1=AG_0.$$

From this we have the following formulas for F and G.

$$\begin{pmatrix} F_0 \\ AF_0 \end{pmatrix} = \begin{pmatrix} I_r \\ A \end{pmatrix}F_0,\quad \begin{pmatrix} G_0 \\ AG_0 \end{pmatrix} = \begin{pmatrix} I_r \\ A \end{pmatrix}G_0.$$

From this we see that $[F]$ and $[G]$ are the same subspaces in $\mathbb{C}^m$.

It remains to show that $Z_B\circ Z_C^{-1}$ is holomorphic for any isomorphisms $B,C:\mathbb{C}^m\to\mathbb{C}^m$. Take $A\in\mathbb{C}^{(m-r)\times r}$. We can assume that

$$Z_C^{-1}(A) = \left[\begin{pmatrix} I_r \\ A \end{pmatrix}\right].$$

Moreover we have

$$\begin{pmatrix} I_r \\ A \end{pmatrix} = CF$$

for some $F\in M^*$. So we have

$$F = C^{-1}\begin{pmatrix} I_r \\ A \end{pmatrix},$$

which completes the proof.

A map $f:M\to N$ between two complex manifolds is said to be **holomorphic** iff for every holomorphic coordinate neighborhood (U_i,ϕ_i) in M and (V_j,ψ_j) in N the map $\psi_j\circ f\circ\phi_i^{-1}$ is holomorphic. This, of course, is a local property.

This definition is independent of the choice of local holomorphic coordinates, since the transition functions are holomorphic.

Example 4 (**Biholomorphism between S^2 and $\mathbb{C}P^1$**)**.** We will show that S^2 and $\mathbb{C}P^1$ are biholomorphic, i.e., there exists an invertible holomorphic function $f : S^2 \to \mathbb{C}P^1$, whose inverse is also holomorphic. Let f be given by

$$f(z) = \begin{cases} [\pi_N(z,x) : 1], & (z,x) \neq (0,1), \\ [1 : \pi_S(z,x)], & (z,x) \neq (0,-1). \end{cases}$$

We use the notation introduced above.

This map is well defined. It suffices to check, that for a point $(z,x) \in S^2 \setminus \{(0,1),(0,-1)\}$ we have $[\pi_N(z,x) : 1] = [1 : \pi_S(z,x)]$. But this means that

$$[\frac{z}{1-x} : 1] = [1 : \frac{\bar{z}}{1+x}].$$

Multiplying $(\frac{z}{1-x}, 1)$ by $\frac{\bar{z}}{1+x}$ we get $(1, \frac{\bar{z}}{1+x})$, so both points are in the same equivalence class in $\mathbb{C}^2$.

It is easy to check that the map $f^{-1} : \mathbb{C}P^1 \to S^2$ defined by

$$f^{-1}([z:w]) = \left(\frac{2\frac{z}{w}}{1+|\frac{z}{w}|^2}, \frac{-1+|\frac{z}{w}|^2}{1+|\frac{z}{w}|^2} \right)$$

for $w \neq 0$ is inverse to f. Hence f is a bijection.

For a point w in $\mathbb{C}$ we have

$$(\varphi_1 \circ f \circ \pi_N^{-1})(w) = (\varphi_1 \circ f)(\pi_N^{-1}(w)) = \varphi_1([\pi_N(\pi_N^{-1}(w)) : 1]) = \varphi_1([w:1]) = w.$$

This function is clearly biholomorphic and the same holds for points in $S^2 \setminus \{(0,-1)\}$.

1.2. **Almost complex structure.** We will now pass on to introduce a complex structure of a complex manifold in terms of tangent vectors.

Denote by j_n the matrix

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix},$$

where I_n is the identity matrix of dimension n. If we view $\mathbb{C}$ as $\mathbb{R}^2$, then multiplication by the matrix j_1 of vectors in $\mathbb{R}^{2n}$ corresponds to the multiplication by the imaginary unit i in $\mathbb{C}$. Let us note, that a map $F : U \subset \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic iff $F_* j_n = j_m F_*$, where by F_* we mean a differential of the map F.

Let us now define an endomorphism J of the tangent space of a complex manifold M of dimension n. Denote by (U,ϕ) a coordinate neighbourhood, let $x \in U$ and $X \in T_x M$. We define

$$J_U(X) = (\phi)_*^{-1} \circ j_n \circ (\phi)_*(X).$$

We check, that this definition does not depend on the holomorphic coordinate neighbourhood (U,ϕ). Let (V,ψ) be another holomorphic coordinate neighbourhood of M such that $x \in V$. Denote the holomorphic transition map $\phi_V \circ \phi_U^{-1}$ by

ϕ_{VU}. We have

$$J_V(X) = (\phi_V)_*^{-1} \circ j_m \circ (\phi_V)_*(X) = (\phi_V)_*^{-1} \circ j_m \circ (\phi_{VU})_* \circ (\phi_U)_*(X) =$$
$$= (\phi_V)_*^{-1} \circ (\phi_{VU})_* \circ j_m \circ (\phi_U)_*(X) = (\phi_U)_*^{-1} \circ j_m \circ (\phi_U)_*(X) = J_U(X).$$

We conclude, that the family $\{J_U\}_{U \in \mathcal{U}}$, where $\mathcal{U}$ is an open cover of M, is a well defined $(1,1)$-tensor J on M. Moreover, this tensor satisfies $J^2 = -id_{TM}$.

We can state a general definition.

Definition 2. A $(1,1)$-tensor, J on a smooth manifold M satisfying $J^2 = -id_{TM}$ is called an **almost complex structure** on M. The pair (M, J) is called an **almost complex manifold**.

All complex manifolds thus have an almost complex structure. In this case we will call this structure a **complex structure**.

1.3. **Complexified tangent bundle.** We define the complexification $T^{\mathbb{C}}M$ of the tangent bundle to an almost complex manifold (M, J) by

$$T^{\mathbb{C}}M = TM \otimes \mathbb{C}.$$

All vectors $Z \in T^{\mathbb{C}}M$ can be written in the form $Z = X + iY$, where $X, Y \in TM$. By $\mathbb{C}$-linearity, we can extend all real operators from TM to $T^{\mathbb{C}}M$. For example, if $Z \in T^{\mathbb{C}}M$ is as before, then

$$JZ = JX + iJY.$$

Let us denote the eigenspace of J corresponding to the eigenvalue i $(-i)$ by $T^{1,0}M$ $(T^{0,1}M)$.

Lemma 1. *We have*

$$T^{1,0}M = \{X - iJX \mid X \in TM\} \quad \textit{and} \quad T^{0,1}M = \{X + iJX \mid X \in TM\}.$$

Moreover $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$.

Proof. Let us check first, that each vector $Z = X + iY$ in $T^{1,0}M$ is of the form $X - iJX$ for some $X \in TM$. Note, that

$$JZ = JX + iJY = iX - Y,$$

so $JY = X$ and $-Y = JX$. We can thus write $Z = X - iJX$. The same holds for $W \in T^{0,1}M$.

For the second part, observe, that $T^{1,0}M$ and $T^{0,1}M$ both have real dimension n and $T^{\mathbb{C}}M$ has real dimension $2n$, so it must be that $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. □

In earlier section we stated the fact that every complex manifold has almost complex structure. Now we give a partial converse of this statement, a theorem by Newlander and Nirenberg.

Theorem 2 (**Newlander-Nirenberg**). *Let (M, J) be an almost complex manifold. The almost complex structure J comes from a holomorphic structure (M is a complex manifold) iff the distribution $T^{0,1}M$ is integrable.*

By an **integrable distribution** we mean a subbundle $\mathcal{D}$ of TM such that for each $X, Y \in \Gamma(\mathcal{D})$ a Poisson bracket $[X, Y]$ is a smooth section of $\mathcal{D}$.

1.4. **Complexified exterior bundle.** Let again (M, J) be an almost complex manifold of real dimension $2n$. Consider complexifications $\Lambda^k_{\mathbb{C}} M := \Lambda^k M \otimes \mathbb{C}$ for $k \in \{1, \ldots, 2n\}$. As in the case of TM we can decompose $\Lambda^1_{\mathbb{C}} M = (T^{\mathbb{C}} M)^*$ as

$$\Lambda^{1,0} M := \{\omega \in \Lambda^1_{\mathbb{C}} M \,|\, \omega(Z) = 0 \,\forall\, Z \in T^{0,1} M\}$$

and

$$\Lambda^{0,1} M := \{\omega \in \Lambda^1_{\mathbb{C}} M \,|\, \omega(Z) = 0 \,\forall\, Z \in T^{1,0} M\}.$$

We call the sections of $\Lambda^{1,0} M$ and $\Lambda^{0,1} M$ **differential forms of type** $(1, 0)$ **and** $(0, 1)$ respectively. Corresponding spaces of differential forms are denoted by $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$.

We have the following lemma, analogous to lemma 1.

Lemma 3. *We can write $\Lambda^1_{\mathbb{C}} M = \Lambda^{1,0} M \oplus \Lambda^{0,1} M$. We have*

$$\Lambda^{1,0} M = \{\omega - i\omega \circ J \,|\, \omega \in \Lambda^1 M\}$$

and

$$\Lambda^{0,1} M = \{\omega + i\omega \circ J \,|\, \omega \in \Lambda^1 M\}$$

Proof. Let $\xi \in \Lambda^{1,0} M$, so $\xi = \omega + i\tau$ for $\omega, \tau \in \Lambda^1 M$. From the definition of $\Lambda^{1,0} M$, for all $X \in TM$ we have

$$0 = \xi(X + iJX) = (\omega + i\tau)(X + iJX) = \omega(X) - \tau(JX) + i(\omega(JX) + \tau(X)).$$

It follows, that $\tau(X) = -\omega(JX)$ and we can write $\xi = \omega + i\omega \circ J$. The same holds for forms of type $(0, 1)$.

First part of the lemma follows from the fact, that $\Lambda^1_{\mathbb{C}} M$ has real dimension equal $2n$ and both $\Lambda^{1,0} M$ and $\Lambda^{0,1} M$ have dimension equal to n. $\square$

By $\Lambda^{k,0} M$ ($\Lambda^{0,k} M$) we denote k-th exterior power of $\Lambda^{1,0} M$ ($\Lambda^{1,0} M$). Consequently, by $\Lambda^{p,q} M$ we mean the tensor product $\Lambda^{p,0} M \otimes \Lambda^{0,q} M$. As the k-th exterior power of a direct sum of two vector spaces E and F is described by

$$\Lambda^k(E \oplus F) \simeq \bigoplus_{i=1}^{k} \Lambda^k E \otimes \Lambda^{k-i} F,$$

we have, by the lemma above,

$$\Lambda^k_{\mathbb{C}} \simeq \bigoplus_{p+q=k} \Lambda^{p,q} M.$$

Sections of $\Lambda^{p,q}M$ are called forms of type (p,q). The space of all forms of type (p,q) will be denoted by $\Omega^{p,q}(M)$. One can check, that a complex-valued differential k-form ω is a section of $\Lambda^{p,q}M$ iff it vanishes on $p+1$ vectors from $T^{1,0}M$ or $q+1$ vectors from $T^{0,1}M$.

In complex geometry forms of type $(1,1)$ occure very frequently, so we will give another characterization.

Proposition 4. *Let (M,J) be a complex manifold. A 2-form ω is of type $(1,1)$ iff $\omega(X,Y)=\omega(JX,JY)$ for all $X,Y\in TM$.*

Proof. As we know, a 2-form ω is of type $(1,1)$ iff $\omega(X-iJX,Y-iJY)=0$. We can write this as

$$\omega(X,Y)-i\left(\omega(X,JY)+\omega(JX,Y)\right)-\omega(JX,JY)=0.$$

Taking just the real part we get $\omega(X,Y)=\omega(JX,JY)$.

If this equality holds it is easy to check that $\omega(X-iJX,Y-iJY)=0$. Similarly one can prove that $\omega(X+iJX,Y+iJY)=0$. $\square$

In the case when (M,J) is a complex manifold we can describe spaces $\Lambda^{p,q}M$ in terms of a local coordinate system. Let z be a local holomorphic coordinate on some $U\subset M$ and denote by $\{z_1,\dots,z_n\}$ its coordinates. We will call $\{z_1,\dots,z_n\}$ a **local holomorphic coordinate system**. As each z_j is a complex-valued function, we can write $z_j=x_j+iy_j$ for some real functions x_j,y_j. By extension by $\mathbb{C}$-linearity of the exterior differential d, we set

$$dz_j=dx_j+idy_j \quad\text{and}\quad d\bar{z}_j=dx_j-idy_j,$$

differential forms of type $(1,0)$ and $(0,1)$, respectively.

Now we can see, that $\{dz_1,\dots,dz_n\}$ is a local basis for $\Omega^{1,0}(M)$ and conjugate forms $\{d\bar{z}_1,\dots,d\bar{z}_n\}$ give a local basis for $\Omega^{0,1}(M)$. Using those facts we can write a local basis for $\Omega^{p,q}(M)$ as

$$\{dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\ldots\wedge d\bar{z}_{j_q}\},$$

for $\{i_1,\dots,i_p\}$ and $\{j_1,\dots,j_q\}$ both subsets of $\{1,\dots,n\}$.

Let now (M,J) be an almost complex manifold. We can associate to J a $(2,1)$-tensor N^J, called the **Nijenhuis tensor**, defined by

$$N^J(X,Y)=[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY],$$

where X,Y are vector fields on M and $[\cdot,\cdot]$ is a Poisson bracket of vector fields.

Proposition 5. *Let J be an almost complex structure on a real, $2m$-dimensional manifold M. The following statements are equivalent:*

(1) *J is a complex structure.*
(2) *$T^{0,1}M$ is integrable.*
(3) *$d(\Omega^{1,0}(M))\subset\Omega^{2,0}(M)\oplus\Omega^{1,1}(M)$.*
(4) *$d(\Omega^{p,q}(M))\subset\Omega^{p+1,q}(M)\oplus\Omega^{p,q+1}(M), \forall\, 0\le p,q\le m$.*

(5) $N^J = 0$.

Proof can be found in [Mo] p. 66.

We can now prove the following proposition.

Proposition 6. *Every almost complex structure on a smooth manifold of dimension* 2 *is a complex structure.*

Proof. We choose a local basis X, Y of TM such that $JX = -Y$. We have

$$N^J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] =$$
$$[X,Y] - J[Y,Y] + J[X,X] - [-Y,X] = 0.$$

□

Below we give an example of an almost complex structure on S^6 which is not a complex structure. The construction is from [Be].

Example 5 (**Almost complex structure on** S^6)**.** Let $\mathbb{O}$ denote the normed algebra of Cayley numbers. Each element of $p \in \mathbb{O}$ can be written as 8-tuple $(p_0, \dots, p_7)$ of real numbers p_i, $i = 0, \dots, 7$. If $p_0 = 0$, we refer to p as a purely imaginary Cayley number. The norm in $\mathbb{O}$ is just the standard Euclidean norm in $\mathbb{R}^8$.

Consider the sphere S^6 of purely imaginary Cayley numbers of unit length. It can be proved that S^6 is a smooth manifold of dimension 6. For a point $p \in S^6$ and a tangent vector $X \in T_pS^6$, define

$$J_p(X) = p \cdot X,$$

where the dot is the multiplication in $\mathbb{O}$. Because p is purely imaginary and has norm one, we have $p \cdot (p \cdot x) = -x$ for all $x \in \mathbb{O}$. Then $J_p(X) \in T_pS^6$ and

$$J_p^2(X) = p \cdot (p \cdot X) = -X,$$

so $J = \{J_p\}_{p \in S^6}$ is an almost complex structure on S^6. One can show ([Be] p.18) that

$$N^J(X,Y) = 4[(p \cdot X) \cdot Y - p \cdot (X \cdot Y)].$$

Since $\mathbb{O}$ is not associative, $N^J \neq 0$ and the almost complex structure J is not integrable.

A simple computation, using Proposition 5, gives a decomposition of the exterior differential d on a complex manifold M. Let $\phi \in \Omega^{p,q}(M)$ and K, L denote subsets of $\{1,\dots,n\}$ of cardinality p and q respectively. We have

$$\begin{aligned} d\phi &= d\sum_{K,L}\phi_{K,L}dz_K\wedge d\bar{z}_L \\ &= \sum_{K,L}\sum_{k=1}^{n}\left(\frac{\partial\phi_{K,L}}{\partial x_k}dx_k+\frac{\partial\phi_{K,L}}{\partial y_k}dy_k\right)dz_K\wedge d\bar{z}_L \\ &= \sum_{K,L}\sum_{k=1}^{n}\left(\frac{\partial\phi_{K,L}}{\partial z_k}dz_k+\frac{\partial\phi_{K,L}}{\partial \bar{z}_k}d\bar{z}_k\right)dz_K\wedge d\bar{z}_L \\ &=: \partial\phi+\bar{\partial}\phi. \end{aligned}$$

Operator ∂ rises type of a form by one in p and $\bar{\partial}$ in q so, for example $\partial\phi$ is of type $(p+1,q)$. Since $d=\partial+\bar{\partial}$ and $d^2=0$, we have

$$\partial^2=\bar{\partial}^2=0 \quad\text{and}\quad \partial\bar{\partial}+\bar{\partial}\partial=0.$$

1.5. **Holomorphic objects.** Through all this subsection by (M,J) we mean a complex manifold of dimension m.

Lemma 7. *Let $f : M\to\mathbb{C}$ be a smooth complex-valued function. The following statements are equivalent:*

(1) *f is holomorphic.*
(2) *$Zf=0$ for all vector fields Z of type $(0,1)$.*
(3) *df is a 1-form of type $(1,0)$.*

Proof. (2) $\Leftrightarrow$ (3). Let us assume, that $df\in\Omega^{1,0}(M)$. This is equivalent with $df(Z)=0$ for all vector fields of type $(0,1)$ which, in turn, is equivalent with $Zf=0$ for all vector fields as before.

(1) $\Leftrightarrow$ (3). By the definition, the function f is holomorphic iff $f\circ\phi_U^{-1}$ is holomorphic for every chart (U,ϕ_U) of M. This is equivalent to $f_*\circ J=if_*$. But this last equation means, that for every $X\in TM$ we have $df(JX)=idf(X)$. We get $idf(X+iJX)=0$ for all $X\in TM$ which is equivalent to $df\in\Omega^{1,0}(M)$. □

Besides holomorphic functions the most basic objects on a complex manifold are holomorphic vector fields and holomorphic forms. We say that a vector field Z of type $(1,0)$ is a **holomorphic vector field** iff for every locally defined holomorphic function f its derivative in Z-direction, Zf, is holomorphic. We call a form ω of type $(p,0)$ a **holomorphic form** iff $\bar{\partial}\omega=0$.

We can also define a **real holomorphic vector field** as a vector field $X\in TM$ such that $X-iJX$ is a holomorphic vector field. We can now state the following lemma, which characterizes such vector fields. Let L_X denote the Lie derivative along vector field X.

Lemma 8. *Let X be a real vector field on a complex manifold (M,J). The following statements are equivalent:*

(1) *X is real holomorphic.*
(2) *$L_X J = 0$, i.e. X is an **automorphic vector field**.*
(3) *The flow of X consists of holomorphic transformations of M.*

Proof. (2) $\Leftrightarrow$ (3). For all $X, Y \in TM$ we have $(L_X J)Y = L_X(JY) - JL_XY$ so (2) is equivalent to $L_X(JY) = JL_XY$. From the definition of the Lie derivative we then get

$$-\frac{d}{dt}|_{t=0}(\varphi_t)_*(JY) = J(-\frac{d}{dt}|_{t=0}(\varphi_t)_*(Y)).$$

As J does not depend on t we have, after integration,

$$(\varphi_t)_*(JY) = J((\varphi_t)_*(Y)),$$

which is precisely the condition for φ_t being holomorphic.

We now show (1) $\Leftrightarrow$ (2). Let X be an automorphic vector field, i.e. $L_XJ = 0$. For an arbitrary real vector field Y it is equivalent to $[JY, X] = J[Y, X]$, as $(L_XJ)Y = L_X(JY) - J(L_XY)$. We get that $[Y + iJY, X]$ is of type $(0, 1)$, because we can write it as $[Y, X] + i[JY, X] = [Y, X] + iJ[Y, X]$. From Lemma 7 we get that for each locally defined holomorphic function f the function $[Y + iJY, X](f)$ is 0 everywhere. We write the Poisson bracket

$$0 = [Y + iJY, X](f) = (Y + iJY)(X(f)) - X((Y + iJY)(f)).$$

The second part on the right vanishes, because $Y + iJY$ is of type $(0, 1)$. We have then, that $(Y + iJY)(X(f)) = 0$ which is equivalent to saying that $X(f)$ is holomorphic. We can write

$$2X(f) = (X - iJX)(f) + (X + iJX)(f).$$

The second summand is zero on each locally defined holomorphic function, so $(X - iJX)(f)$ is holomorphic for each locally defined holomorphic function f. From the definition, we get that the field $X - iJX$ is holomorphic, so the field X is real holomorphic.

The converse is similar. □

Remark. The space of automorphic vector fields $\mathfrak{a}(M)$ on a complex manifold is a Lie algebra with respect to the Poisson bracket of vector fields. The complex structure J of M turns $\mathfrak{a}(M)$ into a complex Lie algebra. If M is compact, then its automorphism group (consisting of holomorphic diffeomorphisms) is a Lie group, whose Lie algebra $\mathfrak{a}(M)$. Moreover, if M is compact, then $\mathfrak{a}(M)$ is finite-dimensional.

We can now prove the following, very important proposition. Let M be again a complex manifold of complex dimension m.

Proposition 9 (The local $i\partial\bar{\partial}$-lemma). *Let $\omega \in \Lambda^{1,1}M \cap \Lambda^2 M$ be a real 2-form of type $(1, 1)$ on a complex manifold M. Then ω is closed iff every point x in M has a neighbourhood U such that the restriction of ω to U equals $i\partial\bar{\partial}u$ for some real function u on U.*

Proof. One implication follows from the fact, that $\partial^2 = \bar{\partial}^2 = 0$. Namely, we have

$$d(i\partial\bar{\partial}) = i(\partial + \bar{\partial})\partial\bar{\partial} = i(\partial^2\bar{\partial} - \partial\bar{\partial}^2) = 0.$$

Let ω be a closed real form of type $(1,1)$. From the Poincaré Lemma, there locally exists a real 1-form τ such that $d\tau = \omega$. Let $\tau^{1,0}$ and $\tau^{0,1}$ denote the decomposition of τ in $(1,0)$- and $(0,1)$-part respectively. Clearly we have $\tau^{1,0} = \overline{\tau^{0,1}}$. By comparison of types in

$$\omega = d\tau = \bar{\partial}\tau^{0,1} + (\partial\tau^{0,1} + \bar{\partial}\tau^{1,0}) + \partial\tau^{1,0}$$

we get $\bar{\partial}\tau^{0,1} = 0$ and $\omega = \partial\tau^{0,1} + \bar{\partial}\tau^{1,0}$. By the Dolbeault Lemma below, there exists a locally defined function f such that $\tau^{0,1} = \bar{\partial}f$. By complex conjugation we get $\tau^{1,0} = \partial\bar{f}$. So we have

$$\omega = \partial\tau^{0,1} + \bar{\partial}\tau^{1,0} = \partial\bar{\partial}f + \bar{\partial}\partial\bar{f} = i\partial\bar{\partial}(2\mathrm{Im}(f)),$$

and the thesis follows with $u := 2\mathrm{Im}(f)$. $\square$

Proposition 10 (**Dolbeault Lemma**). *A $\bar{\partial}$-closed, $\mathcal{C}^1$ form ω of type (p,q), $q > 0$, is locally $\bar{\partial}$-exact that is, there locally exists a form τ of type $(p, q-1)$ such that $\bar{\partial}\tau = \omega$.*

Proof. In the first step we reduce this theorem to the case where $p = 0$. Let I, J be subsets of $\{1, \ldots, m\}$ of cardinality p and q respectively. Let dz_I denote the form $dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, where $i_1, \ldots i_p$ are natural numbers from 1 to m.In local holomorphic coordinates $z_1, \ldots, z_m$ we write the form α as

$$\omega = \sum_{I,J} \omega_{I,J} dz_I \wedge d\bar{z}_J.$$

In this setting we have

$$\bar{\partial}\omega = \sum_{I,J} \bar{\partial}\omega_{I,J} dz_I \wedge d\bar{z}_J$$

and if $\bar{\partial}\alpha = 0$, then for every set I of cardinality p the form

$$\alpha_I = \sum_J \alpha_{I,J} d\bar{z}_J$$

is of type $(0,q)$ and $\bar{\partial}$-closed. If the proposition is proved for the forms of type $(0,q)$ then, locally, we have $\alpha_I = \bar{\partial}\beta_I$, where β_I is defined by

$$\alpha = (-1)^p \bar{\partial}\left(\sum_I dz_I \wedge \beta_I\right).$$

It remains to prove the proposition for forms of type $(0,q)$. We can write such a form in local holomorphic coordinates as

$$\alpha = \sum_J \alpha_J d\bar{z}_J.$$

The proof will follow by induction on the largest integer k such that there exists J with $k \in J$ and $\alpha_J \neq 0$. Thus we have $k \geq q$. When $k = q$, we have $\alpha = f d\bar{z}_J$. The condition that α is $\bar{\partial}$-closed is now equivalent to the function f being holomorphic in variables z_l for $l > q$. We can now apply the following result.

Proposition 11. *Let f be a differentiable (in real sense) function of n complex variables $z_1, \dots, z_n$. Let us further assume, that f is holomorphic in variables z_l for $l > q$. Then there locally exists a differentiable function g, holomorphic in variables z_l, $l > q$, such that $\frac{\partial g}{\partial \bar{z}_q} = f$.*

Proof can be found in [Vo] p. 35.

Suppose that g is as in the proposition. Since $\frac{\partial g}{\partial \bar{z}_l} = 0$ for $l > q$, we have

$$\bar{\partial}\left(g d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_{q-1}\right) = (-1)^{q-1} f d\bar{z}_J.$$

Thus we proved the case $k = q$.

Now, suppose that the proposition is proved for $k - 1 \geq q$. Let us write α as

$$\alpha = \alpha_1 + \alpha_2 \wedge d\bar{z}_k,$$

where α_1 and α_2 have only coordinates strictly less than k. Write $\alpha_2 = \sum_J \alpha_{2,J} d\bar{z}_J$ where the set J is a subset of $\{1, \dots, k-1\}$ of cardinality $q - 1$.

If α is $\bar{\partial}$-closed, then the functions $\alpha_{2,J}$ are holomorphic in variables z_l for $l > k$. Using Proposition 11 we can write

$$\alpha_{2,J} = \frac{\partial \beta_{2,J}}{\partial \bar{z}_k},$$

where $\beta_{2,J}$ is holomorphic in variables z_l, $l > k$. We have

$$\bar{\partial}\left(\sum_J \beta_{2,J} d\bar{z}_J\right) = (-1)^{q-1} \alpha_2 \wedge d\bar{z}_k + \alpha_1'$$

with the form α_1' involving only coordinates with index strictly less than k. So we have written

$$\alpha = \alpha_1'' + \bar{\partial}\beta$$

where in α_1'' we have, as before, only the coordinates z_l, $l < k$. So $\bar{\partial}\alpha_1'' = 0$, since α and $\bar{\partial}\beta$ are $\bar{\partial}$-closed. Applying the induction hypothesis to α_1'' we get local exactness of α, since α_1'' is locally exact. □

1.6. **Holomorphic bundles.** Let M be a smooth manifold. A vector bundle $\pi : E \to M$ is called a **complex vector bundle of rank** k iff for each $x \in M$ the fiber $\pi^{-1}(x)$ is a k-dimensional complex vector space.

Definition 3. Let $\pi : E \to M$ be a smooth complex vector bundle over a complex manifold M and let $\mathcal{U}$ be an open atlas on M. We call $\pi : E \to M$ a **holomorphic vector bundle** iff for any $U_i \in \mathcal{U}$ the trivializations

$$\psi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^k$$

are such that the **transition functions** $g_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{C}) \subset \mathbb{C}^{k^2}$, defined by $\psi_i \circ \psi_j^{-1}(x, z) = (x, g_{ij}(x)z)$, are holomorphic.

Remark. We call $\psi_i \circ \psi_j^{-1}$ the **gluing functions** of the vector bundle.

Example 6 (**Holomorphic tangent bundle**)**.** Let M be a complex manifold and let $\{(U_i, \phi_i)\}_{i\in I}$ be a holomorphic atlas on M. Because $\phi_i \circ \phi_j^{-1}$ is holomorphic the same holds for $(\phi_i \circ \phi_j^{-1})_*$. We define the transition functions g_{ij} by assigning to each point $x \in M$ the Jacobi matrix of $\phi_i \circ \phi_j^{-1}$ at this point.

We will now prove a proposition describing holomorphic vector bundles.

Proposition 12. *If a complex vector bundle $\pi : E \to M$ over a complex manifold M is holomorphic, then there exists a complex structure on E such that π is a holomorphic map with respect to this structures.*

Proof. Let $\{(U_i, \phi_i)\}_{i\in I}$ be a holomorphic atlas on M. As $\pi : E \to M$ is a holomorphic bundle, the gluing functions $\psi_i \circ \psi_j^{-1}$ of E are holomorphic. This is equivalent to

$$(\phi_i, \mathrm{id}_{\mathbb{C}^k}) \circ \psi_i \circ \psi_j^{-1} \circ (\phi_i^{-1}, \mathrm{id}_{\mathbb{C}^k})$$

being holomorphic. We can define a holomorphic atlas on E by

$$\{(\pi^{-1}(U_i), (\phi_i, \mathrm{id}_{\mathbb{C}^k}) \circ \psi_i)\}_{i\in I}.$$

It is now easy to see that π is holomorphic. By definition π is holomorphic iff it is holomorphic in local holomorphic coordinates, i.e. the map

$$\phi_i \circ \pi \circ \psi_i^{-1} \circ (\phi_i^{-1}, \mathrm{id}_{\mathbb{C}^k})$$

is holomorphic for each $i \in I$. As $\pi = \mathrm{proj}_i \circ \psi_i$, where $\mathrm{proj}_i : U_i \times \mathbb{C}^k \to U_i$ is the projection on U_i, we have

$$\phi_i \circ \mathrm{proj}_i \circ (\phi_i^{-1}, \mathrm{id}_{\mathbb{C}^k}) = \phi_i \circ \phi_i^{-1} = \mathrm{id}_{U_i}$$

which is clearly holomorphic. □

This proposition suggests the way to define a general holomorphic fibre bundle. We say that a locally trivial fibre bundle $\pi : E \to M$ is **holomorphic** iff there exists a complex structure on the total space E such that the local trivializations are holomorphic.

Example 7 (**Holomorphic principal bundle over $G_{r,m}$**)**.** Let $\pi : M^* \to G_{r,m}$ be as in the Example 3. We have already shown that M^* and $G_{r,m}$ are complex manifolds. Moreover, the group $GL(r, \mathbb{C})$ is a complex Lie group, so each fibre, which is isomorphic to $GL(r, \mathbb{C})$, is a complex submanifold of M^* identified with the set of all complex matrices of rank r.

It is easy to see that π is holomorphic from

$$Z_B(\pi(A)) = Z_B([A]) = A_1 A_0^{-1},$$

for some $A \in M^*$.

The group $GL(r, \mathbb{C})$ acts on M^* on the right

$$M^* \times GL(r, \mathbb{C}) \to M^* \quad \text{by} \quad (F, A) \longmapsto FA.$$

This is a holomorphic map and it turns $\pi : M^* \to G_{r,m}$ into a holomorphic principal bundle.

We will now state a theorem characterizing holomorphic vector bundles. For the proof we send the reader to [Mo] pp. 72-74. We will need the following definition.

Definition 4. A **holomorphic structure** on a complex vector bundle $\pi : E \to M$ is an operator $\bar{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ which satisfies the condition $\bar{\partial}_E^2 = 0$ and the Leibniz rule

$$\bar{\partial}_E(\omega \wedge \sigma) = (\bar{\partial}\omega) \wedge \sigma + (-1)^{p+q}\omega \wedge (\bar{\partial}_E\sigma),$$

where $\omega \in \Omega^{p,q}(M)$ and $\sigma \in \Omega^{p,q}(E)$.

Remark. We say that a section σ of a holomorphic vector bundle is **holomorphic** iff $\bar{\partial}\sigma = 0$.

Example 8 (**Holomorphic structure on** TM**).** The holomorphic structure of the holomorphic bundle TM of a Hermitian manifold (see Definition 6 below) is given by

$$2\bar{\partial}_{TM}Y(X) := \nabla_X Y + J\nabla_{JX}Y - J(\nabla_Y J)X,$$

in terms of the Levi-Civita connection on M. The proof can be found in [Mo] p. 85.

Theorem 13. *A complex vector bundle $\pi : E \to M$ is holomorphic iff it has a holomorphic structure $\bar{\partial}$.*

2. Hermitian and Kähler geometry

2.1. **Hermitian structures.** Let $\pi : E \to M$ be a complex vector bundle over some smooth manifold M.

Definition 5. A **Hermitian structure** H on $\pi : E \to M$ is a smooth assignment of a Hermitian product to each fibre E_x, $x \in M$. More precisely, $H : E_x \times E_x \to \mathbb{C}$ satisfies

- $H(\cdot, Y)$ is $\mathbb{C}$-linear for a fixed $Y \in E_x$,
- $H(X, Y) = \overline{H(Y, X)}$, for all $X, Y \in E_x$,
- $H(X, X) > 0$, for $X \neq 0$,
- $H(\sigma, \omega)$ is a smooth function on M for every smooth sections σ and ω of E.

If we have a Hermitian structure on a complex vector bundle, we call this bundle a **Hermitian vector bundle**.

Every complex vector bundle $\pi : E \to M$ of rank k is Hermitian. To see this, take trivializations ψ_i of E over some open cover $\{U_i\}_{i\in I}$ of M and a partition of unity $\{f_i\}_{i\in I}$ subordinate to this open cover. For every $x \in U_i$, let us define H_i^x as a pull-back of the standard Hermitian metric on $\mathbb{C}^k$ by the $\mathbb{C}$-linear map $\psi_i|_{E_x}$. The Hermitian structure H on $\pi : E \to M$ is defined by $H = \sum f_i H_i^x$.

Recall that a connection ∇ on a real/complex smooth vector bundle is an $\mathbb{R}/\mathbb{C}$-linear map

$$\nabla : C^\infty(E) \longrightarrow \Omega^1(E),$$

satisfying Leibniz' rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

For a Riemannian manifold (M, g) we have a unique connection ∇ on the tangent bundle TM, called the Levi-Civita connection, which is metric and torsion-free. By metric we mean $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ and by torsion-free $\nabla_X Y - \nabla_Y X = [X, Y]$ for any tangent vector fields X, Y, Z on M.

Let $\pi : E \to M$ be a holomorphic vector bundle over a complex manifold M. Suppose that $\pi : E \to M$ is equipped with a Hermitian structure H. From Theorem 13 we know that there is an operator $\bar{\partial}_E$ on E satisfying Leibniz' rule with respect to the operator $\bar{\partial}$ on M.

Let ∇ be a $\mathbb{C}$-linear connection on E. The operator

$$\nabla^{0,1} : C^\infty(E) \longrightarrow \Omega^{0,1}(E)$$

obtained by composing ∇ with the projection $\pi^{0,1} : \Omega^1(E) \to \Omega^{0,1}(E)$ also satisfies Leibniz' rule with respect to the operator $\bar{\partial}$ on functions.

Theorem 14. *For each Hermitian structure H on a holomorphic vector bundle $\pi : E \to M$ over a complex manifold there exists a unique $\mathbb{C}$-linear connection ∇ which satisfies*

- *H viewed as a field of $\mathbb{C}$-valued, bilinear forms on E is parallel with respect to ∇ (∇ is a* ***Hermitian connection****),*
- *$\nabla^{0,1} = \bar{\partial}_E$.*

We call such a connection the ***Chern connection*** *of the bundle $\pi : E \to M$ with respect to H.*

Proof. Note that the dual vector bundle $\pi^* : E^* \to M$ is also holomorphic with holomorphic structure $\bar{\partial}_{E^*}$. Note also that any connection ∇ on $\pi : E \to M$ induces a connection, denoted also by ∇ on the dual vector bundle by

$$(\nabla_X \sigma^*)(\sigma) := X(\sigma^*(\sigma)) - \sigma^*(\nabla_X \sigma),$$

where $X \in TM$, σ and σ^* are sections of $\pi : E \to M$ and $\pi^* : E^* \to M$ respectively.

Let us remark that the condition $\nabla^{0,1} = \bar{\partial}_E$ means that $\nabla\sigma \in \Omega^{1,0}(E)$ for each holomorphic $\sigma \in C^\infty(E)$. From the definition of the connection on the dual bundle we see that if this equality holds on $\pi : E \to M$ then it holds on $\pi^* : E^* \to M$.

Suppose now that ∇ is an Hermitian connection with $\nabla^{0,1} = \bar{\partial}_E$. The $\mathbb{C}$-anti-linear isomorphism $H : E \to E^*$ is then ∇-parallel, so for every $\sigma \in C^\infty(E)$ and every real vector field X on M we get

$$\nabla_X(H(\sigma)) = (\nabla_X H)(\sigma) + H(\nabla_X \sigma) = H(\nabla_X \sigma).$$

By the $\mathbb{C}$-anti-linearity of H we have $\nabla_Z(H(\sigma)) = H(\nabla_{\bar{Z}}\sigma)$ for every $Z \in T^{\mathbb{C}}M$. For $Z \in T^{1,0}M$, this shows that

$$\nabla^{1,0}\sigma = H^{-1} \circ \nabla^{0,1}(H(\sigma)) = H^{-1}(\bar{\partial}_E(H(\sigma))).$$

Hence, we get $\nabla = \bar{\partial}_E + H^{-1} \circ \bar{\partial}_E \circ H$, which proves the claim. □

2.2. **Hermitian metrics.**

Definition 6. We call a Riemannian metric h on an almost complex manifold (M, J) a **Hermitian metric** iff $h(X, Y) = h(JX, JY)$ for any $X, Y \in TM$. The 2-form ω defined by $\omega(X, Y) = h(JX, Y)$ is called the **fundamental form** of the metric h.

Let us first remark that if a Riemannian manifold (M, g) has an almost complex structure J, then one can define a Hermitian metric on M by $h(X, Y) = g(X, Y) + g(JX, JY)$.

If h is a Hermitian metric on an almost complex manifold (M, J) we can define a Hermitian structure H on TM thanks to the fact that TM is a complex vector bundle. It is straightforward to check that $H(X, Y) = (h - i\omega)(X, Y)$ satisfies the conditions from the definition of a Hermitian structure.

Conversely, if H is a Hermitian structure on the tangent bundle TM to (M, J), then $h = \mathrm{Re}(H)$ defines a Hermitian metric on M.

Let (M, J) be a complex manifold of complex dimension m with Hermitian metric h. Denote by z_k the local holomorphic coordinates of M. We denote by $h_{k\bar{l}}$ the coefficients $h(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l})$ of the metric tensor h. In this setting we have the following lemma.

Lemma 15. *The fundamental form ω of h is of type $(1, 1)$ and is given by*

$$\omega = i \sum_{k,l=1}^{m} h_{k\bar{l}} dz_k \wedge d\bar{z}_l.$$

Proof. By Proposition 4 we just have to check that $\omega(X, Y) = \omega(JX, JY)$ for any $X, Y \in TM$. We compute

$$\omega(X, Y) = h(JX, Y) = h(JJX, JY) = \omega(JX, JY).$$

That we can write ω as above follows from the fact that $J\frac{\partial}{\partial z_k} = i\frac{\partial}{\partial z_k}$. So we have

$$h(J\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}) = ih(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}).$$

□

2.3. **Kähler metrics.** If the fundamental form ω of a Hermitian metric h on a complex manifold is closed, then by Local $i\partial\bar{\partial}$-Lemma there locally exists a real function u such that $\omega = i\partial\bar{\partial}u$. In local holomorphic coordinates this gives a very simple expression for the metric tensor h

$$h_{k\bar{l}} = \frac{\partial^2 u}{\partial z_k \partial \bar{z}_l}.$$

Definition 7. A Hermitian metric h on an almost complex manifold (M, J) is called a **Kähler metric** iff J is a complex structure and the fundamental form ω is closed. The local function u satisfying $\omega = i\partial\bar{\partial}u$ is called the **Kähler potential** of h.

We now give a characterization of Kähler metrics following [Be]. We start with the following proposition.

Proposition 16. *Let (M, J) be a complex manifold with a Hermitian metric h and associated Levi-Civita connetcion ∇. For a fundamental form ω we have*

$$d\omega(X, Y, Z) = h((\nabla_X J)Y, Z) + h((\nabla_Y J)Z, X) + h((\nabla_Z J)X, Y),$$
$$2h((\nabla_X J)Y, Z) = d\omega(X, Y, Z) - d\omega(X, JY, JZ).$$

Proof. Because M is a complex manifold we can assume, with no loss of generality, that X, Y, Z, JY and JZ commutes. We then have

$$\begin{aligned} d\omega(X, Y, Z) &= X\omega(Y, Z) + Y\omega(Z, X) + Z\omega(X, Y) \\ &= Xh(JY, Z) + Yh(JZ, X) + Zh(JX, Y), \\ d\omega(X, JY, JZ) &= X\omega(JY, JZ) + JY\omega(JZ, X) + JZ\omega(X, JY) \\ &= -Xh(Y, JZ) - JYh(Z, X) + JZh(JX, JY). \end{aligned}$$

Because ∇ is metric, we get

$$\begin{aligned} d\omega(X, Y, Z) = {} & h(\nabla_X(JY), Z) + h(JY, \nabla_X Z) + h(\nabla_Y(JZ), X) + h(JZ, \nabla_Y X) \\ & + h(\nabla_Z(JX), Y) + h(JX, \nabla_Z Y). \end{aligned}$$

By $h(JX, Y) = -h(X, JY)$ and $(\nabla_X J)Y = \nabla_X(JY) - J\nabla_X Y$ this gives

$$\begin{aligned} d\omega(X, Y, Z) = {} & h((\nabla_X J)Y), Z) + h(\nabla_Y(JZ), X) + h(\nabla_Z(JX), Y) \\ & + h([Z, Y], JX) + h([X, Z], JY) + h([Y, X], JZ). \end{aligned}$$

Again from the fact that X, Y and Z commutes we get the first equality.

To prove the second one note first, that

$$h((\nabla_X J)Y, Z) = h(\nabla_X(JY), Z) - h(J\nabla_X Y, Z) = h(\nabla_X(JY), Z) + h(\nabla_X Y, JZ).$$

Using Koszul formula and the definition of ω we see that

$$\begin{aligned} 2h(\nabla_X(JY), Z) &= Xh(JY, Z) + JYh(X, JZ) - Zh(X, JY) \\ &= X\omega(Y, Z) - JY\omega(JZ, X) + Z\omega(X, Y) \end{aligned}$$

and

$$\begin{aligned} 2h(\nabla_X Y, JZ) &= Xh(Y, JZ) + Yh(X, JZ) - JZh(X, Y) \\ &= -X\omega(JY, JZ) + Y\omega(Z, X) - JZ\omega(X, JY), \end{aligned}$$

where we use the fact that X, Y, Z, JY, JZ commutes. Adding those two equalities we get the thesis (see the formula for $d\omega(X, Y, Z)$ and $d\omega(X, JY, JZ)$. □

Theorem 17. *Let (M, J) be a complex manifold of dimension m with a Hermitian metric h and the Levi-Civita connection ∇ of h. The following statements are equivalent.*

(1) *h is a Kähler metric.*
(2) *$d\omega = 0$.*
(3) *$\nabla J = 0$.*
(4) *In local holomorphic coordinates we have*

$$\frac{\partial h_{j\bar{k}}}{\partial z_l} = \frac{\partial h_{l\bar{k}}}{\partial z_j} \quad \text{or equivalently} \quad \frac{\partial h_{j\bar{k}}}{\partial \bar{z}_l} = \frac{\partial h_{j\bar{l}}}{\partial \bar{z}_k}.$$

(5) *The Chern connection D of the Hermitian structure H associated to h is equal to the Levi-Civita connection ∇.*
(6) *For any point $p \in M$ there exists a local real function u such that $\omega = i\partial\bar{\partial}u$ in this neighbourhood.*
(7) *For any point $p \in M$ there exists local holomorphic coordinates z such that $h(z) = 1 + O(|z|^2)$.*

Proof. The equivalence of the first two statements follows from the definition of a Kähler metric. By Proposition 16 (2) and (3) are equivalent.

The two statements from (4) are equivalent because we can get each equation by barring the other. As for (2) $\Leftrightarrow$ (4) by Lemma 15 and $d\omega = 0$ we have

$$\sum_{k,l=1}^{m} dh_{k\bar{l}} \wedge dz_k \wedge d\bar{z}_l = 0.$$

The rest follows from the definition of ∂ and $\bar{\partial}$ and the equality $d = \partial + \bar{\partial}$.

To prove (3) $\Leftrightarrow$ (5) take local vector fields X, Y and assume that X is holomorphic. Then

$$\nabla_{JY} X = \nabla_X(JY) + [JY, X] = \nabla_X(JY) + J[Y, X],$$

where we use the fact that $[JY, X] = J[Y, X]$ for vector fields as above. This fact was proved along the proof of Lemma 8. We also have

$$J\nabla_Y X = J(\nabla_X Y + [Y, X]) = J\nabla_X Y + J[Y, X].$$

Now subtract the second formula from the first

$$\nabla_X(JY) - J\nabla_X Y = \nabla_{JY} X - J\nabla_Y X.$$

As $(\nabla_X J)Y = \nabla_X(JY) - J\nabla_X Y$ we get that J is ∇-parallel iff ∇X is a form of type $(1, 0)$ for all holomorphic vector fields X. If the Levi-Civita connection ∇ is equal to the Chern connection, then the $(0, 1)$ part of ∇X vanish, as X is holomorphic. So we get (5) $\Rightarrow$ (3).

To prove the converse assume that J is ∇-parallel. Then ∇ is $\mathbb{C}$-linear. We check that ∇ is Hermitian.

$$\nabla H = \nabla(h - i\omega) = \nabla h - i\nabla\omega = 0,$$

since ∇ is metric with respect to h and $\nabla J = 0$.

It remains to show that $\nabla^{0,1} = \bar{\partial}_{TM}$. Observe, that $\nabla^{0,1}_X = \frac{1}{2}(\nabla_X + J\nabla_{JX})$. The equality of the holomorphic structure of TM and the $(0, 1)$-part of the connection follows from

$$2\bar{\partial}_{TM} Y(X) := \nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X,$$

where Y is a section of TM and X a vector in TM. The proof of the above formula can be found in [Mo] on p. 85.

We now show that (2) $\Rightarrow$ (6). Since ω satisfies assumptions of Lemma 9, we locally have $\omega = d\alpha$ for some real 1-form α. We can write $\alpha = \beta + \bar{\beta}$, where β is a 1-form of type $(1, 0)$. We have

$$\partial\beta = 0, \quad \bar{\partial}\bar{\beta} = 0, \quad \omega = \bar{\partial}\beta + \partial\bar{\beta},$$

since ω is of type $(1, 1)$. So, by the Dolbeault Lemma, $\beta = \partial\phi$ for some local, smooth, complex function ϕ. Then $\bar{\beta} = \bar{\partial}\bar{\phi}$ and

$$\omega = \bar{\partial}\partial\phi + \partial\bar{\partial}\bar{\phi} = \partial\bar{\partial}(\bar{\phi} - \phi) = i\partial\bar{\partial} f$$

where $f = i(\phi - \bar{\phi})$.

It is easy to see that (6) $\Rightarrow$ (2). As $d = \partial + \bar{\partial}$ and $\partial^2 = \bar{\partial}^2 = 0$ we must have $d\omega = 0$ if $\omega = i\partial\bar{\partial} f$ for some local function f.

Denote by δ_{ij} a standard Hermitian metric. To see that (7) $\Rightarrow$ (2) let us assume, that $z_1, \ldots, z_m$, $z_j = x_j + iy_j$, are local holomorphic coordinates at p such that $h_{i\bar{j}} = \frac{1}{2}\delta_{ij} + r_{ij}$ and $r_{ij}(p) = \frac{\partial r_{ij}}{\partial x_k}(p) = \frac{\partial r_{ij}}{\partial y_k}(p)$ for every $i, j, k = 1, \ldots, m$. We have

$$d\omega = i \sum_{j,k,l=1}^{m} \left(\frac{\partial h_{j\bar{k}}}{\partial x_l} dx_l + \frac{\partial h_{j\bar{k}}}{\partial y_l} dy_l \right) dz_j \wedge d\bar{z}_k.$$

By our assumptions and fact that p was arbitrary, we have $d\omega = 0$.

For the proof of how (7) is implied by the fact the the metric h is Kähler we refer the interested reader to [Mo] pp. 84-85. □

If (M, g, J) is a Kähler manifold of complex dimension m with the Levi-Civita conection ∇ we have some additional symmetries in the Riemannian curvature tensor R. Since J is ∇-parallel the $(3,1)$-curvature tensor satisfies

$$R(X,Y)JZ = JR(X,Y)Z$$

for all vector fields X, Y, Z on M. This implies that

$$R(X,Y,JZ,JT) = R(X,Y,Z,T) = R(JX,JY,Z,T).$$

If $\{e_1, \ldots, e_{2m}\}$ is a local orthonormal basis of TM, we have

$$\mathrm{Ric}(JX,JY) = \sum_{i=1}^{2m} R(e_i, JX, JY, e_i) = \sum_{i=1}^{2m} R(Je_i, X, Y, Je_i) = \mathrm{Ric}(X,Y),$$

since $\{Je_i\}_{i=1}^{2m}$ is also an orthonormal basis. This equation shows that $\mathrm{Ric}(JX, Y)$ is skew-symmetric in X and Y, hence we can state the following definition.

Definition 8. The **Ricci form** ρ of a Kähler manifold is defined by

$$\rho(X,Y) = \mathrm{Ric}(JX,Y), \quad \forall X, Y \in TM.$$

We now prove the following proposition.

Proposition 18. *The Ricci tensor of a Kähler manifold satisfies*

$$\mathrm{Ric}(X,Y) = \frac{1}{2}\mathrm{Tr}(R(X,JY) \circ J).$$

Moreover, the Ricci form ρ is closed.

Proof. Let $\{e_i\}_{i=1}^2 m$ be a local orthonormal basis of TM. Using the first Bianchi identity we have

$$\begin{aligned}
\mathrm{Ric}(X,Y) = \sum_{i=1}^{2m} R(e_i, X, Y, e_i) &= \sum_{i=1}^{2m} R(e_i, X, JY, Je_i) = \\
= \sum_{i=1}^{2m} (-R(X, JY, e_, Je_i) &- R(JY, e_i, X, Je_i)) = \\
= \sum_{i=1}^{2m} (R(X, JY, Je_i, e_i) &+ R(Y, Je_i, X, Je_i)) = \\
= \mathrm{Tr}(R(X,JY) &\circ J) - \mathrm{Ric}(X,Y).
\end{aligned}$$

For the second part of the proposition note, that because of the first equality we can write $2\rho(X,Y) = \mathrm{Tr}(R(X,Y) \circ J)$. We then have

$$\begin{aligned}
2d\rho(X,Y,Z) &= 2((\nabla_X \rho)(Y,Z) + (\nabla_Y \rho)(Z,X) + (\nabla_Z \rho)(X,Y)) = \\
&= \mathrm{Tr}(((\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y)) \circ J) = 0,
\end{aligned}$$

where in the last equality we used the second Bianchi identity. □

2.4. **Examples of Kähler manifolds.**

Example 9 ($\mathbb{C}^m$)**.** The first and simplest example of a Kähler manifold is the space $\mathbb{C}^m$, $m \geq 0$ with the standard Euclidean metric. Namely, for $1 \leq j, k \leq m$,

$$h_{j\bar{k}} = h(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}) = \frac{1}{4}h(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}) = \frac{1}{2}\delta_{ij},$$

where $z_j = x_j + iy_j$ are standard holomorphic coordinates on $\mathbb{C}^m$. By Lemma 15 we see that the Kähler form ω is given by

$$\omega = i\frac{1}{2}\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j = i\frac{1}{2}\partial\bar{\partial}|z|^2,$$

where $z = (z_1, \ldots, z_m)$ is a local holomorphic coordinate system as above. We see that $\frac{1}{2}|z|^2$ is a Kähler potential for the canonical Hermitian metric on $\mathbb{C}^m$.

As we have our first explicit Kähler manifold we will now say how we can get some more examples from a given one.

- Products of Kähler manifolds with the product complex structure and product metric are Kähler.
- A complex submanifold of a Kähler manifold is Kähler with the induced metric. More generally, all holomorphic immersions of a complex manifold N in Kähler manifold M give a Kähler structure on N with induced metric.
- All covering spaces of Kähler manifolds are Kähler with the induced metric.
- All quotients of a Kähler manifold by a properly discontinuous and free group actions by holomorphic and isometric transformations are Kähler manifolds.

Remark. Let G be a group. The group G **acts freely** on a set X iff for all $x \in X$ and $g \in G$ the equality $gx = x$ implies that g is the identity in G.

Now let G be a topological group acting on a topological space X. We say that G acts **properly discontinuously** on X iff for any $x \in X$ there is a neighbourhood $U \subset X$ such that the set of all $g \in G$ with the property that $gU \cap U \neq \emptyset$ is only the identity.

We will now give the last example of a Kähler manifold – the complex Grassmannian – following [Be].

Example 10. Here we gather the results from examples 3 and 7. Let Z be a holomorphic section of $\pi : M^* \to G_{r,m}$ over some open set $U \subset G_{r,m}$. Define a form ω by

$$\omega = i\partial\bar{\partial}\ln\det(\bar{Z}^T Z),$$

where Z^T means that we transpose the values of Z. It is clear that ω is closed and of type $(1,1)$. Our first step is to show that ω does not depend on the choice of Z. In fact any other choice is of the form ZF, where $F : U \to GL(r, \mathbb{C})$ is holomorphic. Then we have

$$\partial\bar{\partial}\ln\det(\bar{F}^T\bar{Z}^T ZF) = \partial\bar{\partial}\ln\det(\bar{Z}^T Z),$$

where we used the formula for differentiating the determinant (it can be found for example in [Mo] Lemma 12.3 on p.90) and the fact that F is holomorphic and $\bar{F}^T$ is anti-holomorphic. This shows that ω is indeed well defined.

As before, we denote by $[Z]$ the r-dimensional subspace spanned by $Z \in M^*$. Set $\lambda_A([Z]) = [AZ]$ for any $A \in GL(m, \mathbb{C})$. The group $GL(m, \mathbb{C})$ acts on M^* and $G_{r,m}$ holomorphically on the left

$$GL(m, \mathbb{C}) \times M^* \longrightarrow M^* \quad \text{by} \quad (A, F) \longmapsto AF$$

and

$$GL(m, \mathbb{C}) \times G_{r,m} \longrightarrow G_{r,m} \quad \text{by} \quad (A, F) \longmapsto [AF].$$

We now show that $\lambda_A^*\omega = \omega$ for any $A \in U(m)$. Let Z be a holomorphic section of π over some open set $U \in G_{r,m}$. Then $A^{-1}Z \circ \lambda_A$ is a section of π over $\lambda_A^{-1}(U)$ and

$$\begin{aligned} \lambda_A^*\omega = i\lambda_A^*(\partial\bar{\partial}\ln\det(\bar{Z}^T Z) = i\partial\bar{\partial}(\lambda_A^*(\ln\det(\bar{Z}^T Z))) \\ i\partial\bar{\partial}(\ln\det((\overline{A^{-1}Z})^T(A^{-1}Z) \circ \lambda_A) = \omega, \end{aligned}$$

where we used the facts that λ_A is holomorphic and A is unitary.

The last step is to prove that $g(X, Y) = \omega(X, JY)$ is a Kähler metric on $G_{r,m}$. We showed that g is invariant under the action of $U(m)$ and we know that $U(m)$ acts transitively on $G_{r,m}$. So it is sufficient to show that g is a Kähler metric only in some point p, which is the space spanned by first r unit vectors in $\mathbb{C}^m$. In the neighbourhood of p we have local holomorphic coordinates given by

$$\begin{pmatrix} 1 \\ C \end{pmatrix} \longmapsto w \in \mathbb{C}^{(m-r)\times r}.$$

We compute in some neighbourhood of p

$$\begin{aligned} \omega &= i\partial\bar{\partial}\ln\det(1 + \bar{C}^T C) = i\partial\mathrm{Tr}((\bar{\partial}\bar{C}^T)C(1 + \bar{C}^T C)^{-1}) \\ &= -i\mathrm{Tr}\bar{\partial}\bar{C}^T \wedge \partial C = -i\mathrm{Tr}d\bar{C}^T \wedge dC \\ &= i\sum_{j,k} dC_{jk} \wedge d\bar{C}_{jk}. \end{aligned}$$

It follows that g defined as above is a Kähler metric on $G_{r,m}$ with the Kähler form ω.

3. Classification of certain Kähler manifolds

In this section we would like to present an outline of a classification result by A. Derdziński and G. Maschler as appeared in [DM].

If not otherwise stated (M, g, J) will denote, throughout the whole section, a Kähler manifold with Kähler metric g, complex structure J and Levi-Civita connection ∇. We start with some results on so called Killing potentials.

3.1. **Killing potentials.** We start with a definition of the object of interest.

Definition 9. A real-valued and smooth function τ on (M, g, J) is called a **Killing potential** iff $J\nabla\tau$ is a Killing field on M.

Below, we give two lemmas characterizing Killing potentials following [DM2]. In the proof of the first Lemma we use two well known facts.

- A vector field X is Killing iff for any vector fields Y, Z we have
$$g(\nabla_Y X, Z) = -g(\nabla_Z X, Y),$$
which means that ∇X is a skew-adjoint tensor given by $\nabla X(Y) = \nabla_Y X$.
- A vector field X on a Riemannian manifold is locally a gradient field iff ∇X is self-adjoint.

Lemma 19. *For a smooth function τ on a Kähler manifold M the following conditions are equivalent:*

(1) *τ is a Killing potential,*
(2) *the gradient $\nabla\tau$ is a holomorphic vector field,*
(3) *the Hessian H^τ is Hermitian, i.e. $H^\tau(X, JY) = -H^\tau(JX, Y)$ for all $X, Y \in TM$.*

Proof. (1) $\Leftrightarrow$ (2). Let τ be a Killing potential. For the Killing field $J\nabla\tau$ we have $\nabla(J\nabla\tau) = J(\nabla\nabla\tau)$ because $\nabla J = 0$. We compute
$$\begin{aligned} g(\nabla_Y J\nabla\tau, X) &= g(J\nabla_Y\nabla\tau, X) = -g(\nabla_Y\nabla\tau, JX) \\ &= -g(\nabla_{JX}\nabla\tau, Y). \end{aligned}$$
Hence we get
$$g(\nabla_X J\nabla\tau, Y) + g(\nabla_Y J\nabla\tau, X) = g(J\nabla_X\nabla\tau, Y) - g(\nabla_{JX}\nabla\tau, Y) = g([J, \nabla\nabla\tau]X, Y),$$
which completes the proof.

(1) $\Leftrightarrow$ (3). From the definition of Hessian we have $H^\tau(X, Y) = g(\nabla_X\nabla\tau, Y)$ for all $X, Y \in TM$. From this we compute
$$H^\tau(X, JY) = g(\nabla_X\nabla\tau, JY) = -g(J\nabla_X\nabla\tau, Y) = -g(\nabla_X J\nabla\tau, Y).$$
We can see that the Hessian of τ is skew-symmetric iff $J\nabla\tau$ is Killing. □

Lemma 20. *For every Killing potential τ on (M,g,J), the Killing vector field $J\nabla\tau$ is holomorphic. Conversely, if $H^1(M,\mathbb{R})$ is trivial, then every holomorphic Killing vector field on (M,g,J) has the form $J\nabla\tau$ for a Killing potential τ, which is unique up to an additive constant.*

Proof. The first assertion is clear from the previous lemma and the fact that a real vector field X is holomorphic iff JX is.

Let Y be a holomorphic Killing vector field. Define a vector field X by $X = -JY$. Taking covariant derivatives we get $\nabla X = -J(\nabla Y)$, since $\nabla J = 0$. From the fact that Y is holomorphic we have $[J, \nabla Y] = 0$. Both J and ∇Y are skew-adjoint, since Y is Killing and for J it follows from the definition of a Kähler metric. Because of that their composition ∇X is self-adjoint, i.e. locally $X = \nabla\tau$ for some function τ. ☐

3.2. **Special Kähler-Ricci potentials.** As in the previous section we start with a definition and then give some properties of the defined object. In the end we will give some examples.

Definition 10. A nonconstant Killing potential τ on (M,g,J) is called a **special Kähler-Ricci potential** iff at every point with $d\tau \neq 0$ all nonzero tangent vectors orthogonal to $\nabla\tau$ and $J\nabla\tau$ are eigenvectors of both the Hessian H^τ of τ and the Ricci tensor Ric of M.

Let τ be a special Kähler-Ricci potential on M and let $M' \subset M$ denote the subset of M where $d\tau \neq 0$. Define distributions $\mathcal{V}$ and $\mathcal{H}$ by $\mathcal{V} = \mathrm{Span}\{\nabla\tau, J\nabla\tau\}$ and $\mathcal{H} = \mathcal{V}^\perp$. Because of the condition on eigenvectors from the Definition 10 and Hermitian symmetry of the Ricci tensor Ric and the Hessian H^τ of τ there exist smooth functions $\phi, \psi, \lambda, \mu : M' \to \mathbb{R}$ such that

$$\begin{aligned} \mathrm{Ric} = \lambda g \quad &\text{and} \quad H^\tau = \phi g \quad \text{on} \quad \mathcal{H}, \\ \mathrm{Ric} = \mu g \quad &\text{and} \quad H^\tau = \psi g \quad \text{on} \quad \mathcal{V}, \\ \mathrm{Ric}(\mathcal{H},\mathcal{V}) &= H^\tau(\mathcal{H},\mathcal{V}) = \{0\}, \end{aligned}$$

and $\phi = \lambda = 0$ when $\dim_{\mathbb{C}} M = 1$. The last condition states that $\mathcal{H}$ and $\mathcal{V}$ are both Ric and H^τ-orthogonal.

Below we give some simple examples of special Kähler-Ricci potentials.

Example 11. In complex dimension 1 all non-constant Killing potentials are special Kähler-Ricci potentials. It follows easily from the eigenvector clause of the Definition 10.

Example 12. Let (N,h) be a Kähler manifold of complex dimension n, Einstein unless $n = 1$ and let (S,γ) be an oriented Riemannian surface with a non-constant Killing potential $\tau : S \to \mathbb{R}$. We construct a special Kähler-Ricci potential on $(M,g) = (N,h) \times (S,\gamma)$ treating τ as a function on M constant along the N factor. We can see that τ satisfies the three conditions above if we take $\psi = 0$, $\psi = \Delta\tau/2$

for the Laplacian of τ in S, λ such that λh is the Ricci tensor of (N, h) and μ the Gaussian curvature of (S, γ).

3.3. **Local classification results.** In this subsection we first present a model which locally describes, up to a biholomorphic isometry, all special Kähler-Ricci potentials τ on subsets of Kähler manifolds satisfying $d\tau \neq 0$. This is a classification result from [DM2].

Let us first mention all the data which will be needed in the later construction. Denote by I an open interval in $\mathbb{R}$, a variable $\tau \in I$ and a real, nonzero constant a. Let Q and r be positive diffeomorphisms of variable $\tau \in I$ satisfying $dr/d\tau = ar/Q$. Next, define constants ϵ and c by $\epsilon = 0$ and c left undefined, or $c \notin I$ and $\epsilon = \operatorname{sgn}(\tau - c) = \pm 1$ for all $\tau \in I$. We also set $r_- = \inf r$ and $r_+ = \sup r$ on I.

Let (N, h) be a Kähler manifold of complex dimension $m - 1$ for some $m \geq 1$ which we assume to be Einstein unless $m = 2$. Denote by $\pi : \mathcal{L} \to N$ a complex line bundle over N with Hermitian structure H which is parallel with respect to a fixed connection with horizontal distribution $\mathcal{H}$ and curvature form $\Omega = -2\epsilon a \omega_h$, where ω_h is the Kähler form of h.

Remark. When $m = 1$, then N is just a single point y and the total space $\mathcal{L}$ of the line bundle over N is equal to $\{y\} \times \mathcal{L}_y$ and may be identified with the fibre $\mathcal{L}_y$. Then, by definition, $\epsilon = 0$, h is the zero Einstein metric and $\Omega = 0$.

We now define a metric g on a subset U of $\mathcal{L} \setminus N$ given by $r \in (r_-, r_+)$, where $r : [0, +\infty) \to \mathcal{L}$ is treated as the norm function of H and N denote the image of the zero-section of $\mathcal{L}$. Let

(1) $g = f\pi^* h$ on $\mathcal{H}$,
(2) $g = (ar)^{-2} Q \operatorname{Re}(H)$ on $\mathcal{V}$,
(3) $g(\mathcal{H}, \mathcal{V}) = 0$,

where $f = 1$ if $\epsilon = 0$, or $f = 2|\tau - c|$ when $\epsilon = \pm 1$ and $\mathcal{V} = \mathcal{H}^\perp$.

In [DM2] Derdziński and Maschler proved the following theorem.

Theorem 21. *The pair (U, g) is a Kähler manifold, and τ obtained as an inverse of r is a special Kähler-Ricci potential on (U, g). Moreover, the function $Q : U \to \mathbb{R}$ is defined by $Q = g(\nabla\tau, \nabla\tau)$.*

In what follows we will need the following facts.

Remark. Let $\mathcal{L}$ be a complex line bundle over a complex manifold N with a horizontal distribution $\mathcal{H}$ of a fixed smooth connection in $\mathcal{L}$. Moreover, let Ω be the curvature form of this connection which is skew-Hermitian. Such $\mathcal{L}$ admits a unique holomorphic structure such that $\mathcal{H}$ is invariant under the complex structure tensor J.

Remark. Let $\mathcal{L}$ be a holomorphic line bundle over a complex manifold N and let $\mathcal{L}^*$ be the dual bundle of $\mathcal{L}$. We identify N with the zero section N of $\mathcal{L}$ and N^* of $\mathcal{L}^*$. We define the **inversion biholomorphism** $\mathcal{L} \setminus N \to \mathcal{L}^* \setminus N^*$ by assigning to each $(y,z) \in \mathcal{L} \setminus N$ the pair (y, z^{-1}) where z^{-1} is the unique functional $\mathcal{L}_y \to \mathbb{C}$ that maps z to 1. One can prove that this biholomorphism maps the horizontal distribution of any smooth connection in $\mathcal{L}$ onto the horizontal distribution $\mathcal{H}^*$ of the corresponding dual connection in $\mathcal{L}^*$. Moreover, it sends any Hermitian structure H in $\mathcal{L}$ onto its dual structure H^* given by $H^*(z^{-1}, z^{-1}) = [H(z,z)]^{-1}$ for any $(y,z) \in \mathcal{L}_y \setminus \{0\}$.

Set $r^* = 1/r$, $a^* = -a$ and let $\mathcal{L}^*$ be the dual bundle of $\mathcal{L}$ with the dual Hermitian structure H^* and horizontal distribution $\mathcal{H}^*$. We have ([DM2] Lemma 4.4)

Lemma 22. *We use the notations from above.*

- *The inversion biholomorphism $\mathcal{L} \setminus N \to \mathcal{L}^* \setminus N^*$ sends U, g, τ from Theorem 21 to a Kähler manifold (U^*, g^*) with a special Kähler-Ricci potential τ^* obtained by applying the above construction with $r, a, \mathcal{L}, H$ and $\mathcal{H}$ replaced by $r^*, a^*, \mathcal{L}^*, H^*$ and $\mathcal{H}^*$.*
- *If I has a finite endpoint τ_0 such that Q admits a smooth extension to $I \cup \{\tau_0\}$ with $Q = 0$ and $dQ/d\tau = 2a$ at τ_0, while either $\epsilon = \pm 1$ and $\tau_0 \neq c$, or $\epsilon = 0$ and c is left undefined, then g and τ have smooth extensions to a metric and a function on the open set $U^o \subset \mathcal{L}$ given by $0 \leq r \leq r_+$ – the bundle of radius r_+ open disks on $\mathcal{L}$.*

We now state the main classification result of this section ([DM2] Theorem 18.1). In this theorem we treat τ as an inverse function of r.

Theorem 23. *Let τ be a special Kähler-Ricci potential on a Kähler manifold (M, g) of complex dimension $m \geq 1$. Let $M' \subset M$ be the open set where $d\tau \neq 0$. Then M' is dense in M and every point of M' has a neighbourhood on which, up to biholomorphic isometry, g and τ are obtained as above.*

3.4. **Class 1.** Suppose that $[\tau_{min}, \tau_{max}]$ is a nontrivial, closed interval and $Q : [\tau_{min}, \tau_{max}] \to \mathbb{R}$ is a smooth function of variable τ, which is positive in the interior of $[\tau_{min}, \tau_{max}]$ and vanishes on the endpoints τ_{min} and τ_{max}. Moreover, suppose that the values of $dQ/d\tau$ at those endpoints are mutually opposite and nonzero. Now select an endpoint τ_0 of $[\tau_{min}, \tau_{max}]$ and a diffeomorphism $r : (\tau_{min}, \tau_{max}) \to (0, +\infty)$ of the variable τ satisfying $dr/d\tau = ar/Q$, where a is defined by $2a = dQ/d\tau$ at τ_0. Let ϵ and c be real numbers defined by either $\epsilon = 0$ and c is left undefined, or $\epsilon = \operatorname{sgn}(\tau - c) = \pm 1$ for all $\tau \in [\tau_{min}, \tau_{max}]$.

Next, let (N, h) be a compact Kähler manifold of complex dimension $m-1$ for some $m \geq 1$, which is Einstein unless $m = 2$. Denote by $\mathcal{L}$ the total space of a complex line bundle over N with a $U(1)$-connection having the curvature form $\Omega = -2\epsilon a \omega_h$, where ω_h is the Kähler form of (N, h).

The formula from the previous subsection defines a Kähler metric on $U = \mathcal{L} \setminus N$, where N is treated as a subset of $\mathcal{L}$ by the zero section. The function τ, treated as the inverse of the norm function $r : U \to [0, +\infty)$ becomes a special Kähler-Ricci potential on U.

Denote by M a manifold obtained from $\mathcal{L} \cup \mathcal{L}^*$ by identifying the supplements of zero sections in $\mathcal{L}$ and $\mathcal{L}^*$ by the inversion biholomorphism.

First class of compact Kähler manifolds with special Kähler-Ricci potential is obtained from the above construction, since both g and τ have smooth extensions to the whole of M by Lemma 22.

To see how it works in our setting we first note that by our assumptions the second part of Lemma 22 holds both for $r, a, \mathcal{L}, \tau_0$ and $r^*, a^*, \mathcal{L}^*$ with the other endpoint of I. Moreover, we replace the connection and the Hermitian structure in $\mathcal{L}$ by their duals in $\mathcal{L}^*$. Now by the first part of Lemma 22 the inversion biholomorphism sends g and τ to the dual objects in $\mathcal{L}^* \setminus N^*$ obtained by the same construction on the new data. Using the second part of the Lemma we obtain extensions of the metrics and potentials to the whole of $\mathcal{L}$ and $\mathcal{L}^*$.

3.5. **Class 2.** All notations are the same as before. In addition let V be a complex vector space of dimension m and with a Hermitian inner product $<,>$.

We define a Riemannian metric g on $V \setminus \{0\}$ by

- $|a|r^2 g = 2|\tau - c|\mathrm{Re} <,>$ on $\mathcal{H}$,
- $g = (ar)^{-2} Q \mathrm{Re} <,>$ on $\mathcal{V}$,
- $g(\mathcal{H}, \mathcal{V}) = 0$,

where $\mathcal{V}$ is the distribution on $V \setminus \{0\}$ given by $\mathcal{V}_x = \mathbb{C}x$ nad $\mathcal{H}$ is its orthogonal complement relative to the metric Re $<,>$. As before, the diffeomorphism r is the norm function on V, and we treat τ as an inverse diffeomorphism of r.

Let M be the projective space of $V \times \mathbb{C}$, biholomorphic to $\mathbb{C}P^m$. We treat V as an open subspace of M by the holomorphic embedding $V \ni x \mapsto \mathrm{Span}\{[x : 1]\} \in M$. Both g and τ have then extensions to M, denoted again by g, τ, such that g is Kähler and τ is a special Kähler-Ricci potential on M.

We first show that g is a Kähler metric and τ a special Kähler-Ricci potential on $V \setminus \{0\}$. Let I, τ, Q, a, c, m be as before and $\epsilon a > 0$ to ensure that $\epsilon(\tau - c) > 0$ for all $\tau \in I$. Moreover, let N be the projective space of V, biholomorphic to $\mathbb{C}P^{m-1}$, the metric h on N equal $1/|a| g_{FS}$, where g_{FS} is the Fubini-Study metric on $\mathbb{C}P^{m-1}$ and $\mathcal{L}$ the tautological bundle over N with distributions $\mathcal{H}$ and $\mathcal{V}$ defined above.

Remark. By the tautological bundle of $\mathbb{C}P^m$ we mean a complex line bundle with fibre over $[z]$ consisting of all the points which lie in the complex line in $\mathbb{C}^{m+1}$ to which z belongs.

We can treat $\mathcal{H}$ as a distribution and $<,>$ as a Hermitian structure in $\mathcal{L}$ because of the standard biholomorphic identification of $\mathcal{L} \setminus N$ with $V \setminus \{0\}$ given by $([y], z) \mapsto z$. The restriction to the fibres of $\mathcal{L}$ of the Hermitian metric $<,>$ in V gives a Hermitian structure on $\mathcal{L}$. We obtain the distribution $\mathcal{H}$ as the horizontal distribution of the connection on $\mathcal{L}$, which is a projection of the flat connection on the product bundle $\mathcal{P} = N \times V$ on to the $\mathcal{L}$ summand in the decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{L}^\perp$. The last condition to apply the construction from 3.3 is to check that $\Omega = -a\epsilon\omega_h$. It follows from the fact that ω_h is equal up to a constant to ω_{FS} – the fundamental form of the Fubini-Study metric.

We can extend the metric g from $V \setminus \{0\}$ to $M \setminus \{0\}$. Take the dual $\mathcal{L}^*$ of the tautological bundle $\mathcal{L}$ over N. Define a mapping from $\mathcal{L}^*$ to $M \setminus \{0\}$ by $([y], \zeta) \mapsto \text{Graph}(\zeta)$ where $\zeta \in [y]^*$. As a line in $[y] \times \mathbb{C}$ this graph is an element of M. Asa map it is a biholomorphism and, restricted to $\mathcal{L}^* \setminus N^*$ it becomes the inverse of the inversion biholomorphism under the identification of $V \setminus \{0\}$ with $\mathcal{L} \setminus N$. The existence of the extension is immediate from the second part of Lemma 22.

The last step is to extend the metric to V and M. Let $V(x) = ax$ and $U(x) = iax$, for a chosen above, be vector fields on V and define 1-forms ξ, ξ' by formulae $\xi(X) =< V, X >$ and $\xi'(X) =< U, X >$. The existence of the extension follows from the fact that we can describe the metric g on $V \setminus \{0\}$ as

$$\frac{[Q - 2a(\tau - c)]}{(ar)^4}(\xi \otimes \xi + \xi' \otimes \xi') + \frac{2\tau - c}{(ar)^2}\text{Re} <,> .$$

To see that the formula above has the same values as g it suffices to check it on pairs of vectors from $\mathcal{H}$ and $\mathcal{V}$. The smoothness of the coefficients is checked in [DM] Section 6 and Remarks 4.2 ad 4.3 from that paper.

3.6. **Global classification.** In the end of this paper we want to cite the main result from [DM] – Theorem 16.3.

Theorem 24. *Let (M, g) be a compact Kähler manifold with a special Kähler-Ricci potential τ. Then, up to a biholomorphic isometry, (M, g) and τ belongs to one of Class 1 or 2.*

The proof consists in checking that all the data needed in the construction of the respective class satisfies the positivity and boundary conditions stated in the beginning of above sections.

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