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Killing tensors on principal torus bundles

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KILLING TENSORS ON PRINCIPAL TORUS BUNDLES

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1. Introduction

In the paper [W-Z] authors proved the existence of Einstein metrics with positive salar curvature on principal torus bundles. In the similar fashion W. Jelonek in [Jel1] contructed a so-called $\mathcal{A}$-manifold, which can be viewed as a generalization of an Einstein manifold. Following both authors we will construct an $\mathcal{A}$-manifold on a principal torus bundle over a product of Kähler-Einstein manifolds. A similar construction was carried out in [PS] for Einstein-Weyl manifolds.

Let us recall some basic facts that will be used in this work. A Riemannian manifold M with metric g is called Einstein iff the Ricci tensor ρ satisfies

$$\rho = \lambda g$$

for some positive constant λ. We can generalize this condition and demand only that ρ satisfies

$$\nabla_X \rho(X, X) = 0.$$

Such a manifold will be called an $\mathcal{A}$-manifold.

Now we introduce the notational conventions used along the work.

- $A^k(M)$ - set of differential forms on M.
- If $p : P \to B$ is a fibre bundle we will refere to P as to this bundle for brievity.
- For a $(1,1)$-tensor field S denote by $\nabla S(X, Y)$ the ovariant derivative $\nabla_X S(Y)$.
- $[X, Y]$ denotes the Lie bracket of two vector fields.

In this work we consider only principal bundles with fibre a 2-dimensional torus. It is easy to see that what is written below can be generalized to principal fibre bundles with fibre a torus of any dimension. We use T^2 for clarity.

2. Preliminaries

In this section we prove a theorem which will help us to construct some $\mathcal{A}$-manifolds in the last section of this work.

Let (M, g) be a Riemannian manifold and $\mathfrak{t}^* = \{\xi^1, \ldots, \xi^k\}$ be a set of linearly independent Killing vector fields on M such that $g(\xi^i, \xi^j) = const$, $[\xi^i, \xi^j] = 0$ and $\mathfrak{t}^*$ gives a decompositon of TM into distribution spanned by $\mathfrak{t}^*$ and its orthogonal compliment. Define $T_i X = \nabla_X \xi^i$, $i = 1, \ldots, k$.

Lemma 1. *Tensor T_i satisfies, for $i, j = 1, \dots, k$,*

$$T_i \xi^j = 0.$$

Proof. For any $\xi^i, \xi^j \in \mathfrak{t}^*$ we have $g(\xi^i, \xi^j) = 0$, hence

$$0 = g(\nabla_{\xi^l} \xi^i, \xi^j) + g(\xi^i, \nabla_{\xi^l} \xi^j) = -g(\xi^l, \nabla_{\xi^j} \xi^i) - g(\nabla_{\xi^i} \xi^j, \xi^l),$$

for any $\xi^l \in \mathfrak{t}^*$. Since $[\xi^i, \xi^j] = 0$ we have $\nabla_{\xi^i} \xi^j = \nabla_{\xi^j} \xi^i$. This ends the proof. □

We can now prove a theorem which states under what conditions a Riemannian manifold is an $\mathcal{A}$-manifold, i.e. manifold with the Ricci tensor ρ satisfying

$$\nabla_X \rho(X, X) = 0,$$

for all vector fileds X. Recall that this condition is equivalent to the following condition (see [Jel1])

$$g(\nabla \operatorname{Ric}(X, Y), Z) + g(\nabla \operatorname{Ric}(Y, Z), X) + g(\nabla \operatorname{Ric}(Z, X), Y) = 0, \tag{1}$$

for all vector fields X, Y, Z, where the Ricci endomorphism Ric is defined by $g(\operatorname{Ric} X, Y) = \rho(X, Y)$.

Theorem 2. *Let (M, g) be as in the Lemma 1 and* Ric *the Ricci endomorphism. Assume that $\mu \in \mathbb{R}$ is an eigenvalue of* Ric *and define $\mathcal{H} = \ker(\operatorname{Ric} - \mu \mathrm{Id})$. Furthermore, assume that the orthogonal compliment $\mathcal{H}^\perp$ of $\mathcal{H}$ with respect to g is generated by $\mathfrak{t}^*$ and $\rho(\xi^i, \xi^j) = const.$ for every $i, j = 1, \dots, k$. Moreover, we want that $\rho(\xi^i, X) = \lambda_i g(\xi^i, X)$ for ξ^i and $X \in TM$. Then (M, g) is an $\mathcal{A}$-manifold.*

Proof. Observe first that if $X \in TM$, $Y \in \mathcal{H}$ and $\xi_i \in \mathfrak{t}^*$ then for the Ricci endomorphism Ric we have

$$\begin{aligned} \operatorname{Ric} Y = \mu Y, &\quad \nabla \operatorname{Ric}(X, Y) = -(\operatorname{Ric} - \mu \mathrm{Id})(\nabla_X Y), \\ \operatorname{Ric} \xi^i = \lambda_i \xi^i, &\quad \nabla \operatorname{Ric}(X, \xi^i) = -(\operatorname{Ric} - \lambda_i \mathrm{Id})(\nabla_X, \xi^i), i = 1, \dots, k. \end{aligned}$$

Since Ric is symmetric, then so is $\nabla \operatorname{Ric}$.

We will now check the condition (1) for different choices of vector fields. First assume, that $X, Y, Z \in \mathcal{H}$. Then

$$g(\nabla \operatorname{Ric}(X, Y), Z) = g(-(\operatorname{Ric} - \mu \mathrm{Id})(\nabla_X Y), Z) = 0,$$

where we use the fact that $Z \in \mathcal{H}$.

Now assume, that $X, Y \in \mathcal{H}$ and $Z \in \mathcal{H}^\perp$. We can furthermore assume that $Z = \xi^i \in \mathfrak{t}^*$, hence it is Killing. We have $g(\nabla \operatorname{Ric}(Y, Z), X) = g(\nabla \operatorname{Ric}(Y, X), Z)$ and

$$g(\nabla \operatorname{Ric}(Z, X), Y) = g(-(\operatorname{Ric} - \mu \mathrm{Id})(\nabla_Z X), Y) = 0,$$

so the only components in the cyclic sum (1) are

$$\begin{aligned} & g(\nabla \operatorname{Ric}(X,Y),Z) + g(\nabla \operatorname{Ric}(Y,X),Z) \\ &= g(-(\operatorname{Ric} - \mu \mathrm{Id})(\nabla_X Y),Z) + g(-(\operatorname{Ric} - \mu \mathrm{Id})(\nabla_Y X),Z) \\ &= (\mu - \lambda_i)\,(g(\nabla_X Y,Z) + g(\nabla_Y X,Z)) \\ &= (\lambda_i - \mu)\,(g(Y,\nabla_X Z) + g(X,\nabla_Y Z)) = 0, \end{aligned}$$

because X, Y are perpendicular to Z and Z is Killing.

Now let $X = \xi^i, Y = \xi^j$ and $Z \in TM$. We have

$$\begin{aligned} g(\nabla \operatorname{Ric}(X,Y),Z) &= g(\nabla_X(\operatorname{Ric} Y),Z) - g(\operatorname{Ric}(\nabla_X Y),Z) \\ &= \lambda^j g(T_j X,Z) - g(T_j X, \operatorname{Ric} Z) = 0 \end{aligned}$$

by the Lemma 1. The same argument is valid for $g(\nabla \operatorname{Ric}(Y,Z),X)$ because of symmetry of $\nabla \operatorname{Ric}$. For the last summand of the cyclic sum we have

$$\begin{aligned} g(\nabla \operatorname{Ric}(Z,X),Y) &= g(\nabla_Z(\operatorname{Ric} X),Y) - g(\operatorname{Ric}(\nabla_Z X),Y) \\ &= \lambda_i g(\nabla_Z X,Y) - \lambda_j g(\nabla_Z X,Y) \\ &= -\lambda_i g(Z,T_i Y) + \lambda_j g(Z,T_i Y) = 0 \end{aligned}$$

again by Lemma 1. This proves the vanishing of the cyclic sum for $Z \in \mathcal{H}$ and $Z \in \mathfrak{t}^*$ and finishes the proof. □

3. Principal torus bundles

Let (B,h) be a Riemannian manifold. We will first recall some facts about principal S^1-bundles. If $p : S \to B$ is a principal circle bundle, then denote by $\theta \in A^1(S)$ its connection form and by ξ its fundametal vector field, i.e. $\theta(\xi) = 1$. Thanks to the fact that the Lie algebra of S^1 is the real line $\mathbb{R}$ this connection form is a differential form on S. Denote by $\Omega = d\theta$ the curvature form of S. Since Ω is projectable, there exists a closed 2-form ω on B, such that $\Omega = p^*\omega$.

For some positive function f on B we can define a metric on S by

$$g_f(X,Y) = f^2\theta(X)\theta(Y) + p^*h(X,Y),$$

where * dentotes a pullback of a tensor field.

With this metric the map $p : (S, g_f) \to (B,h)$ becomes a Riemannian submersion with totally geodesic fibres, ξ is a Killing field and if f is a constant, then (S, g_f) becomes an $\mathcal{A}$-manifold (see [Jel1] for details).

A principal torus bundle $p : P \to B$ can be constructed as follows. Take principal S^1-bundles $p_i : P_i \to B$ for $i = 1, 2$. We can make them into one bundle, just by taking Cartesian product

$$p_1 \times p_2 : P_1 \times P_2 \longrightarrow B \times B.$$

This is also a principal bundle, with fiber a torus $T^2 = S^1 \times S^1$. Now we take the pullback of this bundle by the diagonal map $B \to B \times B$. The pullback bundle is, by definition, a Whitney sum of P_1 and P_2 – a principal bundle P over B with fiber T^2. The connection form of P is a differential form with values in the Lie algebra $\mathcal{L}(T^2)$ of T^2, namely $\mathbb{R}^2$. This gives us two differential forms on P, which are pullbacks of the connection forms on P_1 and P_2. We denote the respective pullbacks by θ^1 and θ^2. With this comes two fundamental fields ξ^1 and ξ^2 which are pullbacks of the fundamental fields on P_1 and P_2.

The curvature form Ω of P is just $\Omega_1 e_1 + \Omega_2 e_2 \in \mathcal{L}(T^2)$ where Ω_i is the curvature form of the S^1 bundle P^i and e_1 are basis vectors for $\mathbb{R}^2$. Hence there exists two closed 2-forms ω_i on B such that $\Omega_i = p_i^* \omega_i$. Pulling (ω_1, ω_2) back along the diagonal map, we get a single 2-form ω on B such that $\Omega = p^*\omega$.

We now introduce a metric on P which will make $p : P \to B$ a Riemannian submersion

$$g(X,Y) = \sum_{i,j=1}^{2} b_{ij} \theta^i(X)\theta^j(Y) + p^*h(X,Y),$$

where $[b_{ij}]_{i,j=1}^2$ is a positive definite symmetric matrix. This matrix induces a left-invariant metric on T^2.

Proposition 3. *Fields ξ^1 and ξ^2 are Killing fields with respect to the metric g.*

Proof. We will consider the field ξ^1 as for the other field the proof is the same. First observe, that the Lie derivative $L_{\xi^1} g$ depends only on the Lie derivative of θ^2 with respect to ξ^1

$$L_{\xi^1} g = \sum_{i,j=1}^{2} b_{ij} L_{\xi^1}(\theta^j \otimes \theta^i).$$

As $L_{\xi^1}\theta^1 = 0$, the only important part of the above derivative is

$$L_{\xi^1}\theta^2 = d(i_{\xi^1}\theta^2) + i_{\xi^1} d\theta^2.$$

Since $\theta^2(\xi^1) = 0$ we only have to check values of the 1-form $d\theta^2(\xi^1, X)$ for any X in TP. We have

$$d\theta^2(\xi^1, X) = \xi^1(\theta^2(X)) - X(\theta^2(\xi^1)) - \theta^2([\xi^1, X]).$$

As $\theta^2(\xi^1) = 0$ we have to check just the values of other two terms.

Assume first, that X belongs to the horizontal distribution of the connection on P. Then $\theta^2(X) = 0$ and $[\xi^1, X]$ is a horizontal vector field, as ξ^1 is a fundamental vector field. Hence the right hand side vanishes.

Next, assume that X is vertical. As $d\theta^2$ is $C^\infty(P)$-multilinear, we can assume, that X is just a linear combination of ξ^1 and ξ^2. Hence $\theta^2(X)$ is constant and the differential vanishes. The Poisson bracket $[\xi^1, X]$ also vanishes, as T^2 is abelian. This finishes the proof. □

We mark that the above facts can be proved in a more general setting, but we want to emphasise the case in which we are interested. Define T_i as in the Lemma 1.

Lemma 4. *Tensor T_i satisfies, for $i = 1, 2$,*

$$L_{\xi^i} T_j = 0.$$

Thus, the tensor T_i is horizontal for $i = 1, 2$, i.e. there exist a tensor $\tilde{T}_i$ on B such that $\tilde{T}_i \circ p_* = p_* \circ T_i$.

Proof. As for the second equality, observe that for any $X \in TP$ we have

$$\theta^1(X) = \frac{1}{\det[b_{ij}]} \left(b_{22} g(\xi^1, X) - b_{12} g(\xi^2, X)\right) = b^{11} g(\xi^1, X) + b^{12} g(\xi^2, X),$$
$$\theta^2(X) = \frac{1}{\det[b_{ij}]} \left(b_{11} g(\xi^2, X) - b_{21} g(\xi^1, X)\right) = b^{21} g(\xi^1, X) + b^{22} g(\xi^2, X),$$

where b^{ij} is a coefficient of the inverse matrix of $[b_{ij}]$.

From this we easily get

$$d\theta^1(X, Y) = 2\left(b^{11} g(T_1 X, Y) + b^{12} g(T_2 X, Y)\right),$$
$$d\theta^2(X, Y) = 2\left(b^{21} g(T_1 X, Y) + b^{22} g(T_2 X, Y)\right).$$

Using the fact, that Lie derivative commutes with the exterior derivative we obtain

$$0 = d(L_{\xi^i} \theta^j)(X, Y) = L_{\xi^i}(d\theta^j)(X, Y), \quad i, j = 1, 2,$$

hence

$$b^{11} g((L_{\xi^i} T_1) X, Y) + b^{12} g((L_{\xi^i} T_2) X, Y) = 0,$$
$$b^{21} g((L_{\xi^i} T_1) X, Y) + b^{22} g((L_{\xi^i} T_2) X, Y) = 0.$$

Eventually we observe that as the matrix $[b^{ij}]$ is non-singular, this is equivalent to

$$L_{\xi^i} T_j = 0.$$

□

Lemma 5. *With the notation from above we have*

$$R(X, \xi^i) Y = \nabla T_i(X, Y), \quad \nabla T_i(X, \xi^j) = -T_i(T_j X),$$

where R is the curvature tensor and $X, Y \in TP$.

Proof. The proof consists in rewriting the analogous proof from [Jel1] for two Killing fields. Recall, that for any Killing vector field ξ we have

$$R(X, \xi) = \nabla_X(\nabla \xi).$$

Hence we have nothing to prove as

$$R(X, \xi^i) Y = (\nabla_X T_i) Y.$$

Since $T_i \xi^j = 0$ for $i, j = 1, 2$ we have $(\nabla_X T_i)\xi^j + T_i(\nabla_X \xi^j) = 0$ and there is also nothing to prove. □

Corollary 1. We have

$$\rho(\xi^i, X) = -g(X, \operatorname{tr}_g \nabla T_i),$$

for ξ^i from above and any $X \in TP$.

4. O'Neill fundamental tensors

We would like to compute the O'Neill tensor (see [ON]) A of the Riemannian submersion $p : P \to B$. Since the fibres are totally geodesic, the tensor T is zero.

Proposition 6. *The vertical distribution of the Riemannian submersion $p : P \to B$ is totally geodesic.*

Proof. We have to check that $g(\nabla_U V, X) = 0$ for all vertical vector fields U, V and all horizontal vector fields X. Since we are interested only in horizontal part of $\nabla_U V$ we can assume that $U = \xi^j$ since $\nabla(U, V) = \nabla_U V$ is tensorial in the first variable and $V = \xi^i$, $i, j = 1, 2$, since

$$\nabla_U(fV) = df(U)V + \nabla_U V$$

for any function f.

From Koszul formula, for any horizontal field X, we get

$$2g(\nabla_{\xi^i}\xi^j, X) = 0,$$

since the Poisson bracket of any fundamental and vertical vector fields is horizontal. □

Now we proceed to the fundamental tensor A.

Lemma 7. *We have*

$$A_E F = \sum_{i=1}^{2} \big(g(E, T_i F)\xi^i + g(\xi^i, F)T_i E\big),$$

for any E, F vector fields on P.

Proof. From [ON] we have

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F,$$

where $\mathcal{V}$ and $\mathcal{H}$ denote the vertical or horizontal part of a vector field respectively.

For any horizontal vector fields X, Y we have

$$g(\nabla_X Y, \xi^i) = Xg(Y, \xi^i) - g(Y, \nabla_X, \xi^i) = g(X, T_i Y),$$

for $i = 1, 2$.

The vertical part of any vector field is given by

$$\mathcal{V}F = \sum_{i=1}^{2} g(\xi^i, F)\xi^i.$$

Next, we have

$$\begin{aligned} g\left(\mathcal{H}E, T_i\mathcal{H}F\right) &= g\left(E-\sum_{i=1}^{2} g(\xi^i,F)\xi^i, T_i\left(F-\sum_{i=1}^{2} g(\xi^i,F)\xi^i\right)\right) \\ &= g(E,T_iF). \end{aligned}$$

Combining all of the above formulae we get

$$\mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F=\sum_{i=1}^{2} g(E,T_iF)\xi^i.$$

To compute the second summand of the O'Neill tensor A we observe, that for any function f and any $E\in TP$ we have $\mathcal{H}\nabla_E f\xi^i=\mathcal{H}f\nabla_E\xi^i$. Then

$$\begin{aligned} \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F &= \mathcal{H}\nabla_{E-\sum_{i=1}^{2} g(\xi^i,F)\xi^i}\sum_{i=1}^{2} g(\xi^i,F)\xi^i \\ &= \sum_{i=1}^{2} g(\xi^i,F)T_iE. \end{aligned}$$

□

Remark. Comparing with [Jel1] we see that the O'Neill tensor A on a principal T^2-bundle is just a sum of O'Neill tensors on principal S^1-bundles.

5. The structure of an $\mathcal{A}$-manifold on P.

We will check under what conditions a principal T^2-bundle P with some Riemannian metric g satisfies assumptions of Theorem 2. The distribution $\mathcal{H}$ from the theorem will be the horizontal distribution of the submersion.

The first condition we need to be satisfied is $\rho(X,V)=0$ for $X\in\mathcal{H}$ and V in its orthogonal compliment. From Corollary 1 we get

$$0=\rho(\xi^i,X)=-g(X,\operatorname{tr}_g\nabla T_i).$$

Recall, that for $i=1,2$ we have $\Omega^i=p^*\omega^i$, where ω^i is a 2-form on B, and

$$\begin{aligned} \Omega^1(X,Y) &= d\theta^1(X,Y)=2\left(b^{11}g(T_1X,Y)+b^{12}g(T_2X,Y)\right), \\ \Omega^2(X,Y) &= d\theta^2(X,Y)=2\left(b^{21}g(T_1X,Y)+b^{22}g(T_2X,Y)\right). \end{aligned}$$

As T_i are horizontal we get

$$\omega^1(p_*X,p_*Y)=2\left(b^{11}h(\tilde{T}_1(p_*X),p_*Y)+b^{12}h(\tilde{T}_2(p_*X),p_*Y)\right), \tag{2}$$

$$\omega^2(p_*X,p_*Y)=2\left(b^{21}h(\tilde{T}_1(p_*X),p_*Y)+b^{22}h(\tilde{T}_2(p_*X),p_*Y)\right), \tag{3}$$

where $\tilde{T}_i$ was defined in Lemma 4. The codifferential of ω^i, $i = 1, 2$ is given by $(\delta\omega^i)(p_*X) = -\sum_{k=1}^{m}(\nabla^h_{E_k}\omega^i)(E_k, p_*X)$, where $\{E_k\}_{k=1}^m$ is some orthonormal basis of B. Hence

$$(\delta\omega^1)(p_*X) = -2\left(b^{11}h(\mathrm{tr}_h\,\nabla^h\tilde{T}_1, p_*X) + b^{12}h(\mathrm{tr}_h\,\nabla^h\tilde{T}_2, p_*X)\right),$$
$$(\delta\omega^2)(p_*X) = -2\left(b^{21}h(\mathrm{tr}_h\,\nabla^h\tilde{T}_1, p_*X) + b^{22}h(\mathrm{tr}_h\,\nabla^h\tilde{T}_2, p_*X)\right).$$

By a straightforward computation for $i = 1, 2$ we get

$$\begin{aligned} h(p_*X, \mathrm{tr}_h\,\nabla^h\tilde{T}_i) &= \sum_{k=1}^{m}\left(h(p_*X, \nabla^h_{E_k}(\tilde{T}_iE_k)) - h(p_*X, \tilde{T}_i(\nabla^h_{E_k}E_k))\right) \\ &= \sum_{k=1}^{m}(g(X, \nabla_{X_k}(T_iX_k)) - g(X, T_i(\nabla_{X_k}X_k)) \\ &= g(X, \mathrm{tr}_g\,\nabla T_i). \end{aligned}$$

We easily see, that if the condition $\rho(V, X) = 0$ is satisfied, then ω^i is harmonic for $i = 1, 2$. Moreover, from equations for coddiferentials of ω^1 and ω^2 we see, they are both harmonic, then from non-singularity of the matrix $[b_{ij}]$ we get that $h(\mathrm{tr}_h\,\nabla^h\tilde{T}_1, p_*X) = h(\mathrm{tr}_h\,\nabla^h\tilde{T}_2, p_*X) = 0$. This means that $g(X, \mathrm{tr}_g\,\nabla T_i) = 0$, $i = 1, 2$. Hence the condition $\rho(V, X) = 0$ is satisfied iff ω^1 and ω^2 are harmonic.

Remark. Observe that this is the Yang-Mills condition for the principal connection θ of the bundle P.

Next we need that $\rho(U, V) = const.$ for any U, V in the vertical distribution. Again using Corollary 1 we have $\rho(\xi^i, \xi^j) = -g(\xi^j, \mathrm{tr}_g\,\nabla T_i)$. This means that

$$-g(\xi^j, \mathrm{tr}_g\,\nabla T_i) = const.$$

But for any orthonormal basis $\{X_k\}_{k=1}^m$ of the horizontal distribution $\mathcal{H}$ we have $\mathrm{tr}_g\,\nabla T_i = \sum_{k=1}^m(\nabla T_i)(X_k, X_k)$, so we get

$$g(\xi^j, \mathrm{tr}_g\,\nabla T_i) = -\sum_{k=1}^{m} g(\nabla T_i(X_k, \xi^j), X_k).$$

As $\nabla T_i(X_k, \xi^j) = -T_iT_jX_k$ we have

$$\sum_{k=1}^{m} g(T_jX_k, T_iX_k) = const.$$

and from the fact that T_i is horizontal of $i = 1, 2$ we get

$$\sum_{k=1}^{m} h(\tilde{T}_j(X_k)_*, \tilde{T}_i(X_k)_*) = const., \tag{4}$$

where X_* is the projection of the vector X on M to B. It follows that we can find a basis $\{\zeta^1, \zeta^2\}$ of the vertical distribution such that $\mathrm{Ric}\,\zeta^i = \lambda_i\zeta^i$, $i = 1, 2$.

Now we will check when the eigenvalue condition

$$\rho(X,Y) = \mu g(X,Y)$$

is satisfied for any $X, Y \in \mathcal{H}$.

First let us recall, that the tensor A for two horizontal vector fields X, Y is given by

$$A_X Y = \sum_{i=1}^{2} g(X, T_i Y)\xi^i.$$

We use the formula for ρ from [Be]:

$$\rho(X,Y) = \check{\rho}(X,Y) - 2\sum_{k=1}^{m} g(A_X X_k, A_Y X_k),$$

where $\check{\rho}$ is the horizontal symmetric 2-tensor given by

$$\check{\rho}(X,Y) = \rho_B(p_* X, p_* Y)$$

and ρ_B is the Ricci tensor of B. We can rewrite this as

$$\begin{aligned}
\rho(X,Y) &= \check{\rho}(X,Y) - 2\sum_{k=1}^{m} g(\sum_{i=1}^{2} g(X, T_i X_k)\xi^i, \sum_{i=j}^{2} g(Y, T_j X_k)\xi^j) \\
&= \check{\rho}(X,Y) - 2\sum_{k=1}^{m} g(\sum_{i=1}^{2} g(T_i X, X_k)\xi^i, \sum_{i=j}^{2} g(T_j Y, X_k)\xi^j) \\
&= \check{\rho}(X,Y) - 2\sum_{k=1}^{m} \sum_{i,j=1}^{2} b_{ij} g(T_i X, g(T_j Y, X_k) X_k) \\
&= \check{\rho}(X,Y) - 2\sum_{i,j=1}^{2} b_{ij} g(T_i X, T_j Y).
\end{aligned}$$

As ρ is symmetric it is determined by its values on the diagonal

$$\rho(X,X) = \check{\rho}(X,X) - 2\sum_{i,j=1}^{2} b_{ij} g(T_i X, T_j X). \tag{5}$$

Fortunately we can express this condition in terms of the metric on the base B as T_i and $\check{\rho}$ are horizontal for $i = 1, 2$.

6. Construction of an example over Kähler-Einstein base

In this section we will construct an $\mathcal{A}$-manifold on a torus bundle over a product of Kähler-Einstein manifolds. We will follow [Jel1] and [W-Z].

Let (B, g_B) be a compact Kähler manifold with positive definite Ricci tensor. By a theorem of S. Kobayashi ([Ko1]) it is simply connected. Moreover, by another theorem from the same work, if B has positive first Chern class, then $H_1(B; \mathbb{Z}) = 0$.

Hence if B is Kähler-Einstein with positive scalar curvature the above holds true. From the Universal Coefficient Theorem for Cohomology we get, that $H^2(B;\mathbb{Z})$ has no torsion. Then we can write the first Chern class $c_1(B)$ as an integer multiple of some indivisible class $\alpha \in H^2(B;\mathbb{Z})$, say $c_1(B) = q\alpha$, $q \in \mathbb{Z}_+ \setminus \{0\}$. We can normalize the Kähler metric g_B, so that the cohomology class of ω_B is the same as of $2\pi\alpha$, which is equivalent to choosing the Einstein constant to be equal q.

By another theorem of Kobayashi ([Ko2]) every principal S^1-bundle S over a Kähler manifold (B, g_B) is classified by a cohomology class in $H^2(B;\mathbb{Z})$. In fact this class is just the Euler class $e(S)$ of S. Hence every principal T^2-bundle $p : P \to B$ is classified by two cohomology classes β_1, β_2 in $H^2(B;\mathbb{Z})$. Those classes can be described as the Euler classes of the quotient principal S^1-bundle P/S_i over B, where S_i is the i-th S^1 factor in the canonical decomposition $T^2 = S^1 \times S^1$.

Theorem 8. *Let (B_i, g_i) be compact Kähler-Einstein manifolds with $c_1(B_i) > 0$ for $i = 1, 2$ and define a Kähler manifold $B = B_1 \times B_2$, with metric $h = x_1 g_1 + x_2 g_2$ where x_1, x_2 are some positive constants. Let $p : P \to B$ be a principal T^2 bundle characterised by $\beta_i = a_{i1} pr_1^* \alpha_1 + a_{i2} pr_2^* \alpha_2$, $i = 1, 2$, where $pr_i : B \to B_i$ is the projection on the i-th factor, α_i is the indivisible class in $H^2(B_i;\mathbb{Z})$ such that $c_1(B_i) = q_i \alpha_i$ for $q_i \in \mathbb{Z}_+ \setminus \{0\}$ and $[a_{ij}]$ is a 2×2 matrix with integer coefficients. Then P with a metric defined by*

$$g(X, Y) = \sum_{i,j=1}^{2} b_{ij} \theta^i(X) \theta^j(Y) + p^* h(X, Y),$$

where θ^i is as before, is an $\mathcal{A}$-manifold for some positive, non-singular matrix $[b_{ij}]$.

Proof. We assume that $i = 1, 2$ along the proof. Moreover, we normalize the metric g_i so that the Kähler form η_i of g_i is in the cohomology class of $2\pi\alpha_i$, i.e. we just divide the metric by q_i.

Let P_i be the principal S^1-bundle over B determined by β_i. Denote by η_i the Kähler form of g_i. From Theorem 1.4 [W-Z] we know, that there exist a connection form θ_i on P_i such that

$$d\theta_i = a_{i1} p^* \eta_1^* + a_{i2} p^* \eta_2^*,$$

where $\eta_i^* = pr_i^* \eta_i$. Let P be the principal T^2 bundle obtained as the Whitney sum of P_1 and P_2. We know that θ_1 and θ_2 are the components of the principal connection on P and $\Omega_i = d\theta_i$ is the i-th component of the curvature form on P. Recal that we have $\Omega_i = p^* \omega_i$ for some 2-form on B. Comparing the above formula and (2) - (3) we have

$$a_{11} \eta_1^*(X, Y) + a_{12} \eta_2^*(X, Y) = 2 \left(b^{11} h(\tilde{T}_1(X), Y) + b^{12} h(\tilde{T}_2(X), Y) \right),$$

$$a_{21} \eta_1^*(X, Y) + a_{22} \eta_2^*(X, Y) = 2 \left(b^{21} h(\tilde{T}_1(X), Y) + b^{22} h(\tilde{T}_2(X), Y) \right),$$

where $X, Y \in TB$. Because h is the product metric, the complex structure tensor J_i is an endomorphism of TB_i and $TB = TB_1 \oplus TB_2$ the pullback J_i^* of J_i by pr_i

preserves TB_i. Hence we can write $\eta_i^*(X,Y) = \frac{1}{x_i}h(J_i^*X,Y)$. We can now define $\tilde{T}_i$ with the following formulae

$$\tilde{T}_1 = \frac{1}{2}\left(\frac{1}{x_1}(b_{11}a_{11}+b_{12}a_{21})J_1^* + \frac{1}{x_2}(b_{11}a_{12}+b_{12}a_{22})J_2^*\right),$$
$$\tilde{T}_2 = \frac{1}{2}\left(\frac{1}{x_1}(b_{22}a_{21}+b_{21}a_{11})J_1^* + \frac{1}{x_2}(b_{22}a_{22}+b_{21}a_{12})J_2^*\right).$$

For brievity we will denote by c_{ij} the coefficient of T_i standing before $\frac{1}{x_j}J_j^*$. We have to determine what conditions have to be imposed on x_i for (P,g) to be an $\mathcal{A}$-manifold. Let us denote by $\{X_k\}_{k=1}^m$ some orthonormal basis of B such that $\{X_k\}_{k=1}^{n_1}$ and $\{X_k\}_{k=1}^{n_2}$ are orthogonal basis of B_1 and B_2 respectively, where n_1 and n_2 are their real dimensions. We compute

$$\begin{aligned}
&\sum_{k=1}^m h(\tilde{T}_iX_k,\tilde{T}_jX_k)\\
&= \frac{1}{4}\sum_{k=1}^m h((\frac{1}{x_1}c_{i1}J_1^* + \frac{1}{x_2}c_{i2}J_2^*)X_k, (\frac{1}{x_1}c_{j1}J_1^* + \frac{1}{x_2}c_{j2}J_2^*)X_k)\\
&= \frac{1}{4}\sum_{k=1}^m \left(\frac{1}{x_1^2}c_{i1}c_{j1}h(J_1^*X_k,J_1^*X_k) + \frac{1}{x_2^2}c_{i2}c_{j2}h(J_2^*X_k,J_2^*X_k)\right),
\end{aligned}$$

where the last equality follows from the above discussion of tensor J_i. We can descend furhter to metrics on B_1 and B_2:

$$\begin{aligned}
&\sum_{k=1}^m h(\tilde{T}_iX_k,\tilde{T}_jX_k)\\
&= \frac{1}{4}\left(\sum_{k=1}^{n_1}\frac{1}{x_1}c_{i1}c_{j1}g_1(J_1X_k,J_1X_k) + \sum_{k=1}^{n_2}\frac{1}{x_2}c_{i2}c_{j2}g_2(J_2X_k,J_2X_k)\right)\\
&= \frac{1}{4}\left(\frac{1}{x_1^2}c_{i1}c_{j1}n_1 + \frac{1}{x_2^2}c_{i2}c_{j2}n_2\right)
\end{aligned}$$

and we see that the condition (4) is satisfied.

Let us look at the equation defining the other eigenvalue.

$$\begin{aligned}
\rho(X^*, X^*) &= \check{\rho}(X^*, X^*) - 2 \sum_{i,j=1}^{2} b_{ij} g(T_i X^*, T_j X^*) \\
&= \rho_B(X, X) - 2 \sum_{i,j=1}^{2} b_{ij} h(\tilde{T}_i X, \tilde{T}_j X) \\
&= \rho_B(X, X) - \frac{1}{2} \sum_{i,j=1}^{2} b_{ij} \left(\frac{1}{x_1^2} c_{i1} c_{j1} h(J_1 X, J_1 X) + \frac{1}{x_2^2} c_{i2} c_{j2} h(J_2 X, J_2 X) \right).
\end{aligned}$$

Assuming, that X is an element of the local orthonormal frame on B_1 or B_2 we get two equations

$$\mu = \frac{q_1}{x_1} - \frac{1}{2} \sum_{i,j=1}^{2} \frac{b_{ij} c_{j1} c_{i1}}{x_1^2}$$

$$\mu = \frac{q_2}{x_2} - \frac{1}{2} \sum_{i,j=1}^{2} \frac{b_{ij} c_{j2} c_{i2}}{x_2^2}.$$

It is now easy to see that we can choose the coefficients x_1 and x_2 so that the above equations are satisfied for any a_{ij} and b_{ij}. One solution is for example when $x_1 = \alpha x_2$ for some $\alpha > 0$. In this case we have

$$\frac{\alpha q_1 - q_2}{\alpha x_1} = \frac{1}{2} \sum_{i,j=1}^{2} \frac{\alpha^2 b_{ij} c_{j1} c_{i1} - b_{ij} c_{j2} c_{i2}}{\alpha^2 x_1^2}.$$

This equation is equivalent to

$$\alpha x_1 = \frac{1}{2} \sum_{i,j=1}^{2} \frac{\alpha^2 b_{ij} c_{j1} c_{i1} - b_{ij} c_{j2} c_{i2}}{\alpha q_1 - q_2}.$$

Hence the positive solution exists iff the right hand side is greater then zero and this is the case iff

$$\alpha^2 > \frac{b_{ij} c_{j2} c_{i2}}{b_{ij} c_{j1} c_{i1}}, \quad \alpha > \frac{q_2}{q_1}$$

or when the inequalities are opposite. If we take α big or small enough so those inequalities are satisfied and we get a solution.

We also see that the assumptions of the Theorem 2 are satisfied, hence (P, g) is an $\mathcal{A}$-manifold. $\square$

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