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Killing forms

Praca semestralna nr 3

(semestr letni 2011/12)

Opiekun pracy: Robert Wolak

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

KILLING FORMS

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1. Preliminaries

Let M be a smooth differential manifold of dimension n, $n \in \mathbb{N}$, and g be any Riemannian metric on M.

- For a vector bundle $\pi : E \to M$ we denote the space of sections of E by $\Gamma(E)$.
- For some vector space E we denote by $\Lambda^k E^*$ the space of k-forms on E, where $k \in \mathbb{N}$. By $\Omega^k(M)$ we denote the space of differential k-forms on M.
- For a differential k-form φ on M, we define its inner product with a vector field X as a $k-1$-form $X \lrcorner \phi$ defined by

$$X \lrcorner \phi(X_1, \dots, X_k) = \phi(X, X_1, \dots, X_k).$$

- The Lie derivative of a tensor T with respect to a vector field X is denoted by $L_X T$.
- We will use the same letter for a vector field X and the dual 1-form defined by $g(X, \cdot)$.
- The Hodge star operator is denoted by $*$.
- Let d denote the exterior differential operator. We define the codifferential δ to be the formal adjoint operator of d.

2. Algebraic preliminaries

Let E be a n-dimensional vector space with scalar product g and denote by $\{e_i\}_{i=1}^n$ some orthonormal basis of E. We can write the wedge product and the inner product as

$$\wedge : E \otimes \Lambda^k E^* \ni v \otimes \psi \mapsto v \wedge \psi \in \Lambda^{k+1} E^*, \tag{1}$$

$$\lrcorner : E \otimes \Lambda^k E^* \ni v \otimes \psi \mapsto v \lrcorner \psi \in \Lambda^{k-1} E^*. \tag{2}$$

Their adjoint operators with respect to the scalar product g are

$$\wedge^* : \Lambda^{k+1} E^* \ni \tau \mapsto \sum_{i=1}^n e_i \otimes e_i \lrcorner \tau \in E \otimes \Lambda^k E^*, \tag{3}$$

$$\lrcorner^* : \Lambda^{k-1} E^* \ni \tau \mapsto \sum_{i=1}^n e_i \otimes e_i \wedge \tau \in E \otimes \Lambda^k E^*. \tag{4}$$

We can now easily check that $\wedge \circ \lrcorner^* = \lrcorner \circ \wedge^* = 0$. Hence we get a decomposition

$$E \otimes \Lambda^k E^* = \mathrm{Im}(\lrcorner^*) \oplus \mathrm{Im}(\wedge^*) \oplus \Lambda^{k,1} E^*,$$

where $\Lambda^{k,1}E^*$ is the orthogonal complement of $\mathrm{Im}(\lrcorner^*) \oplus \mathrm{Im}(\wedge^*)$. We see that $\Lambda^{k,1}E^*$ is the intersection of kernels of the wedge product and the contraction map. It is easy to check that $\mathrm{Im}(\lrcorner^*)$ can be identified with $\Lambda^{k-1}E^*$ and $\mathrm{Im}(\wedge^*)$ with $\Lambda^{k+1}E^*$ embedded in $E \otimes \Lambda^k E^*$, hence we have the isomorphism

$$E \otimes \Lambda^k E^* \cong \Lambda^{k-1}E^* \oplus \Lambda^{k+1}E^* \oplus \Lambda^{k,1}E^*. \tag{5}$$

We would like to know how the projection on each summand looks like. Recall that we have $\sum_{i=1}^n e_i \lrcorner (e_i \wedge \tau) = (n-k)\tau$ and $\sum_{i=1}^n e_i \wedge (e_i \lrcorner \tau) = k\tau$ for any $\tau \in \Lambda^k E^*$. Hence, for $\alpha \in \Lambda^{k-1}E^*$ and $\beta \in \Lambda^{k+1}E^*$, we have

$$\lrcorner \circ \lrcorner^*(\alpha) = \sum_{i=1}^n e_i \lrcorner (e_i \wedge \alpha) = (n-k+1)\alpha, \tag{6}$$

$$\wedge \circ \wedge^*(\beta) = \sum_{i=1}^n e_i \wedge (e_i \lrcorner \beta) = (k+1)\beta. \tag{7}$$

We can write the projections of $\xi \in E \otimes \Lambda^k E^*$ on the elements of the decomposition (5) as

$$\mathrm{pr}_{\Lambda^{k-1}}(\xi) = \frac{1}{n-k+1} \lrcorner^* \circ \lrcorner \xi, \tag{8}$$

$$\mathrm{pr}_{\Lambda^{k+1}}(\xi) = \frac{1}{k+1} \wedge^* \circ \wedge \xi, \tag{9}$$

$$\mathrm{pr}_{\Lambda^{k,1}}(\xi) = \xi - \frac{1}{n-k+1} \lrcorner^* \circ \lrcorner \xi - \frac{1}{k+1} \wedge^* \circ \wedge \xi. \tag{10}$$

3. Definition and simple properties

The main reference for this section is [Sem]. If not otherwise stated all definitions, propositions, lemmas and theorems in this section are cited from this source.

Let M be a smooth differential manifold of dimension n with a Riemannian metric g. The decomposition (5) on the bundle $TM \otimes \Lambda^k M$, where we identify T^*M and TM via the metric g, gives

$$TM \otimes \Lambda^k M \cong \Lambda^{k-1}M \oplus \Lambda^{k+1}M \oplus \Lambda^{k,1}M, \tag{11}$$

where $\Lambda^{k,1}M = \Lambda^{k,1}TM$ and an analogous decomposition of $TM \otimes \Omega^k(M)$. Let ω be a differential k-form on M and ∇ be the Levi-Civita connection on M. Then $\nabla\omega$ is a section of $TM \otimes \Lambda^k M$. The projection of $\nabla\omega$ onto $\Omega^{k+1}(M)$ gives $d\omega$ and onto $\Omega^{k-1}(M)$ the adjoint $\delta\omega$. Using the last projection we can define a differential operator $T : \Omega^k(M) \to \Gamma(\Lambda^{k,1}M)$, called the *twistor operator*, by

$$T(\omega) = \mathrm{pr}_{\Lambda^{k,1}}(\nabla\omega).$$

If we recall that $d\omega = \wedge(\nabla\omega)$ and $\delta\omega = -\lrcorner(\nabla\omega)$ we can write T explicitly

$$(T(\omega))(X) = \nabla_X\omega - \frac{1}{k+1}X \lrcorner d\omega + \frac{1}{n-k+1}X \wedge \delta\omega, \tag{12}$$

where $X \in TM$. The k-forms which are in the kernel of T are of special interest.

Definition 3.1. A differential k-form ϕ on M is called a *conformall Killing form* or a *twistor form* iff

$$\nabla_X\phi = \frac{1}{k+1}X \lrcorner d\phi - \frac{1}{n-k+1}X \wedge \delta\phi$$

for all vector fields X on M. If in addition ϕ is coclosed we call it a *Killing form.* This fact is equivalent to $X \lrcorner \nabla_X\phi = 0$. If ϕ is closed it is called a $*$*-Killing form.*

Using the decomposition (11) we can write the covariant derivative with respect to $X \in TM$ of any k-form ω as

$$\nabla_X\omega = \frac{1}{k+1}X \lrcorner d\omega - \frac{1}{n-k+1}X \wedge \delta\omega + (T(\omega))(X). \tag{13}$$

From this formula we can acquire the following norm estimate at each point of M ([G-M]).

Lemma 3.2. *Let (M,g) be a Riemannian manifold and let ω be a k-form. Then*

$$|\nabla\omega|^2 \geq \frac{1}{k+1}|d\omega|^2 + \frac{1}{n-p+1}|\delta\omega|^2, \tag{14}$$

with equality if and only if ω is a conformal Killing form.

This lemma can be used to prove that the Hodge star operator $*$ maps conformal Killing k-forms into conformal Killing $(n-k)$-forms. To see this we use the facts that ∇ commutes with $*$ and $*$ is metric, i.e. $g(*\psi, *\phi) = g(\psi, \phi)$ where ψ, ϕ are k-forms and g is the induced metric on $\Lambda^k M$. For a conformal Killing k-form ω we have $|\nabla * \omega|^2 = |\nabla\omega|^2$ and

$$|d\omega|^2 = |**d**\omega|^2 = |*\delta*\omega|^2 = |\delta*\omega|^2,$$
$$|\delta\omega|^2 = |**\delta**\omega|^2 = |*d*\omega|^2 = |d*\omega|^2.$$

It is immediate that $*$ interchanges closed and coclosed conformal Killing forms.

Next property of conformal Killing forms is the following.

Proposition 3.3. *Let (M,g) be a Riemannian manifold with a conformal Killing k-form ω. Then $\bar{\omega} = e^{(k+1)u}\omega$ is a conformal Killing k-form with respect to the conformally equivalent metric $\bar{g} = e^{2u}g$ for some smooth function u on M.*

Consider a symmetric bilinear form K_ω on a Riemannian manifold (M,g) defined by

$$K_\omega(X,Y) = g(X \lrcorner \omega, Y \lrcorner \omega), \tag{15}$$

where X, $Y \in TM$. We have

Lemma 3.4. *If ω is a Killing form, then the associated symmetric tensor K_ω is a Killing tensor i.e. K_ω satisfies*

$$(\nabla_X K_\omega)(X, X) = 0.$$

As conformal Killing forms were introduced as a generalization of conformal vector fields we have the following proposition.

Proposition 3.5. *Let (M, g) be a Riemannian manifold. A vector field ξ is dual to a conformal Killing 1-form iff it is a conformal vector field, i.e. there exists a smooth function f on M such that $L_\xi g = fg$. Moreover, ξ is dual to a Killing 1-form iff it is a Killing vector field.*

Proof. For ξ treated as a 1-form we have that it is a twistor 1-form iff for any $X, Y \in TM$ we have

$$\begin{aligned}
0 &= (\nabla_X \xi)(Y) - \frac{1}{2} d\xi(X, Y) + \frac{1}{n} g(X, Y)\delta\xi \\
&= (\nabla_X \xi)(Y) - \frac{1}{2}(\nabla_X \xi)(Y) + \frac{1}{2}(\nabla_Y \xi)(X) + \frac{1}{n} g(X, Y)\delta\xi \\
&= \frac{1}{2}(\nabla_X \xi)(Y) + \frac{1}{2}(\nabla_Y \xi)(X) + \frac{1}{n} g(X, Y)\delta\xi \\
&= \frac{1}{2}(L_\xi g)(X, Y) + \frac{1}{n} g(X, Y)\delta\xi.
\end{aligned}$$

We see that if ξ is a Killing form, i.e. it is coclosed, we have that the dual vector field ξ is Killing. □

4. Examples of conformal Killing forms

Once again the main source for this section is [Sem]. First example of a conformal Killing form on a Riemannian manifold (M, g) is a parallel form which is obviously in the kernel of the twistor operator. From this fact and Proposition 3.3 we deduce that from a parallel k-form ω we can create a new k-form $\bar{\omega} = e^{(k+1)u}\omega$ which is conformal Killing with respect to conformally equivalent metric $\bar{g} = e^{2u} g$. This new form is in general no longer parallel.

The simplest examples of manifolds with conformal Killing forms are those with constant scalar curvature.

Example 1. Let (S^n, g) be the standard sphere with scalar curvature $\tau = n(n-1)$. Recall that every eigenvalue of the Laplace operator on k-forms is of the form

$$\lambda_j^1 = (k + j)(n - k + j + 1) \quad \text{or} \quad \lambda_j^2 = (k + j + 1)(n - k + j),$$

for $j = 1, 2, \ldots$ The eigenvalues λ_j^1 and λ_j^2 correspond to closed and coclosed eigenforms respectively. For the minimal eigenvalues λ_0^1 and λ_0^2 the multiplicities are $\binom{n+1}{k}$ and $\binom{n+1}{k+1}$ respectively. We have

Proposition 4.1. *A k-form ω on the standard sphere (S^n, g) is a conformal Killing form iff it is a sum of eigenforms for the eigenvalue λ_0^1 or λ_0^2.*

The first wider class of manifolds admitting conformal Killing forms are Sasakian manifolds. We will use the following definition.

Definition 4.2. A Riemannian manifold (M, g) is called a *Sasakian manifold* if there exists a unit length Killing vector field satisfying

$$\nabla_X(d\xi) = -2X \wedge \xi \tag{16}$$

for any vector field $X \in TM$.

While the dual 1-form to a Killing vector field ξ is naturally a Killing form it is natural to ask whether the 2-form $d\xi$ is conformal Killing.

Theorem 4.3. *Let (M, g) be a Riemannian manifold of dimension n with a Sasakian structure defined by a unit length vector field ξ. Then the 2-form $d\xi$ is a conformal Killing form. Moreover, if (M, g) is an Einstein manifold with scalar curvature τ normalized to $\tau = n(n-1)$ and if ξ is a unit length Killing vector field such that $d\xi$ is a conformal Killing form, then ξ defines a Sasakian structure.*

The next proposition gives us some more examples of conformal Killing forms on Sasakian manifolds.

Proposition 4.4. *Let (M, g, ξ) be a Sasakian manifold of dimension $2n+1$ with Killing vector field ξ. Then*

$$\omega_k = \xi \wedge (d\xi)^k$$

is a Killing $2k+1$-form for $k = 0, \ldots, n$. Moreover, ω_k satisfies for any vector field X and any k the additional equation

$$\nabla_X(d\omega_k) = -2(k+1)X \wedge \omega_k.$$

In particular, ω_k is an eigenform of the Laplace operator corresponding to the eigenvalue $4(k+1)(n-k)$.

Recall the definition of a general vector cross product. Let V be a finite dimensional real vector space with a non-degenerate bilinear form g. An *r-fold vector cross product* on V is defined as a linear map $P : V \otimes \ldots \otimes V \to V$, where we take r copies of V, satisfying

- $g(P(v_1, \ldots, v_r), v_i) = 0$,
- $|P(v_1, \ldots, v_r)|^2 = \det(g(v_i, v_j))$,

where $v_i \in V$ for $i = 1, \ldots, r$ and $|\cdot| = g(\cdot, \cdot)$. There are only four types of possible vector cross products on a given vector space: 1-fold and $(n-1)$-fold products on n-dimensional vector spaces, 2-fold vector cross products on 7-dimensional spaces and 3-fold vector cross products on 8-dimensional manifolds.

Consider an r-fold vector cross product on a Riemannian manifold (M, g) defined as an $(r, 1)$-tensor field which in each fibre is an r-fold vector cross product. If an r-fold vector cross product P on (M, g) for any $Y, X_1, \ldots X_{r-1} \in TM$ satisfies

$$\nabla_Y P(Y, X_1 \ldots, X_{r-1}) = 0$$

we call it a *nearly parallel* vector cross product.

For any r-fold vector cross product P on (M, g) we can define an associated $(r+1)$-form ω by

$$\omega(X_1, \ldots, X_{r+1}) = g(P(X_1, \ldots, X_r), X_{r+1}).$$

The vector cross product P is nearly parallel iff $X \lrcorner \nabla_X \omega = 0$. We have

Lemma 4.5. *Let P be a nearly parallel vector cross product on (M, g) with associated form ω. Then ω is a Killing $r+1$-form.*

Proof. The vector cross product P is nearly parallel iff ω satisfies

$$X \lrcorner \nabla_X \omega = 0$$

for any vector field X. But this is equivalent to ω being a Killing $(r+1)$-form. □

Thanks to this lemma we can obtain four types of manifolds with conformal Killing forms.

- A 1-fold vector cross product is equivalent to an almost complex structure compatible with the given metric. The first class of manifolds are nearly Kähler manifolds, where the Kähler form is conformal Killing. If the manifold is Kähler then the Kähler form is parallel.
- A 2-fold vector cross product on a 7-dimensional Riemannian manifold exists iff the underlying manifold admits a topological G_2-structure, where $G_2 \subset O(7)$. Such a structure exists on any 7-dimensional 3-Sasakian manifold.
- The $(n-1)$-fold and 3-fold cross vector products yield only parallel associated forms by results of [Gr].

5. Special Killing forms

In this section we discuss, following [Sem], an additional condition imposed on some Killing form. It turns out that the existence of Killing forms satisfying this condition tells us much about the holonomy of the underlying manifold.

Definition 5.1. A *special Killing form* is a Killing form φ which for some constant c and any vector field X satisfies

$$\nabla_X (d\varphi) = cX \wedge \varphi.$$

From this definition and the Definition 4.2 we get the very first example of a special Killing form, namely the dual form of a Killing vector field defining a Sasakian structure. Moreover, it follows from Proposition 4.4 that the Killing forms $\xi \wedge (d\xi)^k$ are special for any $k \in \mathbb{N}$, where ξ is the Killing field defining a Sasakian structure.

It turns out that the question of existence of special Killing forms on a manifold M can be answered in terms of the metric cone $\hat{M}$ over M, i.e. a manifold defined by $\hat{M} = M \times \mathbb{R}_+$ with metric $\hat{g} = r^2 g + dr^2$, where g is some Riemannian metric on M.

One can show ([Sem]):

Lemma 5.2. *Let (M, g) be a Riemannian manifold and let φ be a k-form on M. Then the associated $(k+1)$-form $\hat{\varphi}$ on the metric cone $\hat{M}$ defined by*

$$\hat{\varphi} = r^k dr \wedge \varphi + \frac{r^{k+1}}{k+1} d\varphi$$

is parallel with respect to $\hat{\nabla}$ iff φ is a special Killing form on M with constant $c = -(k+1)$.

Together with the following lemma from the same work we get that the only parallel forms on the metric cone are those arising from special Killing forms on M.

Lemma 5.3. *Let ψ be a form on the metric cone $\hat{M}$. Then ψ is parallel with respect to $\hat{\nabla}$ iff there exists a special Killing form φ on M such that $\psi = \hat{\varphi}$.*

Let now M be an oriented, compact and simply connected m dimensional manifold. Then the metric cone $\hat{M}$ is either flat, and M is isometric to the standard sphere, or $\hat{M}$ is irreducible. In the latter case it can be showed that the holonomy of the manifold M is one of the following groups: $U(m)$,$SU(m)$,$Sp(m)$,G_2 or $Spin_7$. We have

Theorem 5.4. *Let (M, g) be a compact, simply connected manifold admitting a special Killing form. Then M is isometric to the standard sphere or M is Sasakian, 3-Sasakian, nearly Kähler or weak G_2-manifold.*

6. Killing forms and Kähler geometry

We will briefly present some results from [M-S]. We begin with some basic facts about differential forms on Kähler manifolds.

Recall that the complexified tangent space $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$ of any almost complex manifold (M, J) can be split into a direct sum $T^{1,0}M \oplus T^{0,1}M$ corresponding to the eigenvalue i and $-i$ of J.We have

$$T^{1,0}M = \{X - iJX : X \in TM\}, \quad T^{0,1}M = \{X + iJX : X \in TM\}.$$

With this we can define complex valued 1-forms of type $(1,0)$ and $(0,1)$. Moreover, we say that a complex valued differential k-form α is of type (p,q) iff it vanishes on $p+1$ vectors of type $(1,0)$ or $q+1$ vectors of type $(0,1)$.

In addition to standard differential operators d and δ on any Kähler manifold (M,J,ω) of real dimension $2m$ we can define the so-called "twisted differentials" d^c and δ^c by

$$d^c(\alpha) = \sum_{i=1}^{2m} Je_i \wedge \nabla_{e_i}\alpha, \quad \delta^c(\alpha) = -\sum_{i=1}^{2m} Je_i \lrcorner \nabla_{e_i}\alpha,$$

where α is a differential form.

In addition to the above, wedging and contraction with respect to the Kähler form ω gives two more natural operators on a Kähler manifold (M,J,ω). More precisely for any k-form α we can define:

$$L(\alpha) = \omega \wedge \alpha = \frac{1}{2}\sum_{i=1}^{2m} e_i \wedge Je_i \wedge \alpha, \quad \Lambda(\alpha) = \frac{1}{2}\sum_{i=1}^{2m} Je_i \lrcorner e_i \lrcorner \alpha.$$

Those operators commute with the complex structure tensor J and the Laplace operator. Moreover we have the so-called *Kähler identities*

$$[L,\delta] = d^c, \quad [L,d] = 0,$$
$$[\Lambda,d] = -\delta^c, \quad [\Lambda,\delta] = 0$$

and another important commutator, which on k-forms acts by

$$[L^r,\Lambda] = (r(k-m) + r(r-1))L^{r-1},$$

which in case $r=1$ gives $[L,\Lambda] = (k-m)\mathrm{Id}_{\Lambda^k}$ on k-forms.

Before we proceed further we give a theorem from [Sem] that gives answer to the question on which almost Hermitian manifolds the Kähler form is a twistor form.

Proposition 6.1. *Let (M,g,J) be an almost Hermitian manifold of dimension $2m$. The Kähler form ω is a conformal Killing form iff the manifold is nearly-Kähler.*

Proof. On an almost Hermitian manifold with Kähler form ω one has the following formula

$$\Lambda(d\omega) = J(\delta\omega).$$

Assume that ω is a twistor 2-form. We will show that it has to be coclosed. From the definition we have

$$(\nabla_X\omega)(Y,Z) = \frac{1}{3}d\omega(X,Y,Z) - \frac{1}{2m-1}\left(g(X,Y)\delta\omega(Z) - g(X,Z)\delta(Y)\right).$$

If we set $X = e_i$ and $Y = Je_i$ for some orthonormal basis $\{e_i\}_{i=1}^{2m}$ then summing over this basis the above equality yields

$$-\delta\omega(JZ) = \frac{1}{3}\sum_{i=1}^{2m} d\omega(e_i, Je_i, Z) + \frac{1}{2m-1}\sum_{i=1}^{2m} g(e_i, Z)\delta\omega(Je_i) \tag{17}$$

$$= \frac{2}{3}\Lambda(d\omega)(Z) + \frac{1}{2m-1}\delta\omega(JZ) \tag{18}$$

$$= \left(\frac{1}{2m-1} - \frac{2}{3}\right)\delta\omega(JZ). \tag{19}$$

From this it follows that $\delta\omega = 0$, which means exactly that ω is a Killing 2-form or equivalently $X \lrcorner \nabla_X\omega = 0$. From earlier discussion of nearly parallel vector cross products the fact that $X \lrcorner \nabla_X\omega = 0$ is equivalent to $(\nabla_X J)X = 0$ which in turn amounts to (M, g, J) being nearly-Kähler. □

On a Kähler manifold (M, J, ω) of real dimension $2m$ we can decompose any k-form α, $k \leq m$ as

$$\alpha = \alpha_0 + L\alpha_1 + \ldots + L^l\alpha_l,$$

where each α_i, $i = 0, \ldots, l$ is a *primitive form*, i.e. $\Lambda\alpha_i = 0$. We will call this decomposition the *Lefschetz decomposition*. This decomposition is valid also for differential forms of degree higher than m in sense that we decompose the Hodge-dual of a given form.

We can show that if α is a conformal Killing k-form such that $J\alpha$ is parallel then the Lefschetz decomposition of α is very simple.

Remark. On a compact Kähler manifold one can show that any conformal Killing k-form α with $k \neq m$ has $J\alpha$ parallel.

First, one can show that $d\alpha = L^k v$ and $\delta\alpha = L^{k-1}w$ for some 1-forms v, w. Second, the Lefschetz decomposition of α has exactly k summands with α_k being a function. Third, for any $X \in TM$ we have

$$\nabla_X\alpha = \nabla_X\alpha_0 + L(\nabla_X\alpha_1) + \ldots + L^k(\nabla_X\alpha_k)$$

and on the other hand

$$\begin{aligned}
\nabla_X\alpha &= \frac{1}{k+1}X \lrcorner L^k v - \frac{1}{m-k+1}X \wedge L^{k-1}w \\
&= L^{k-1}\left(\frac{k}{k+1}JX \wedge v - \frac{1}{m-k+1}X \wedge w\right) + L^k\left(\frac{1}{k+1}g(X, v)\right) \\
&= L^{k-1}\beta_1 + L^k\beta_2,
\end{aligned}$$

where β_1 and β_2 are primitive. Comparing both equalities we get that α_i are parallel for $i = 1, \ldots, k-2$. Without loss of generality we can assume that they are zero. This gives us a particularly simple expression for a conformal Killing k-form α with $J\alpha$ parallel, namely

$$\alpha = L^{k-1}\alpha_{k-1} + L^k\alpha_k,$$

where α_{k-1} is a primitive 2-form and α_k a function.

We now give a definition of a special 2-form on a Kähler manifold.

Definition 6.2. A *special* 2*-form* on a Kähler manifold (M, J, ω) of real dimension $2m$ is a primitive 2-form φ of type $(1,1)$ satisfying the equation

$$\nabla_X \varphi = \gamma \wedge JX - J\gamma \wedge X - \frac{2}{m}\gamma(X)\omega$$

for some 1-form γ, which then equals $\frac{m}{2(m^2-1)}\delta^c\varphi$.

We can now cite a classification result ([M-S]) for conformal Killing forms on compact Kähler manifolds.

Theorem 6.3. *Let α be a differential k-form on a compact Kähler manifold (M, J, ω) of dimension $2m$. Suppose that $2m - 2 \geq k \geq 2$ and $k \neq m$. Then α is a conformal Killing form iff there exists a special form φ with generalized trace f and a positive integer l such that $k = 2l$ and*

$$\alpha = L^k\varphi - \frac{m-k}{k(m^2-1)}L^k f + \beta,$$

where β is any parallel k-form. The same is valid for $k = m$ with additional assumption that $J\alpha$ is parallel.

7. Killing vector fields with twistor derivative

In this section we would like to describe an application of the theory of Killing forms following [Mo]. All results come from this paper.

Let (M, g) be a Riemannian manifold of dimension m greater than 3 and let ξ be a Killing vector field. We denote by T a $(1,1)$-tensor defined by $TX = \nabla_X\xi$ for any $X \in TM$. By its definition T is skew-symmetric and we can identify it with $\frac{1}{2}d\xi$, i.e. $g(TX, Y) = \frac{1}{2}d\xi(X, Y)$ for any $X, Y \in TM$. The covariant derivative of T is given by

$$(\nabla_X T)Y = \nabla^2_{X,Y}\xi = R(X, \xi)Y$$

where the second equality is the Kostant formula for Killing fields. This shows that

$$\nabla_\xi T = 0. \tag{20}$$

From now on until the end of this section we assume that T viewed as a 2-form is a twistor form. Since it is closed, we have

$$\nabla_X T = \frac{1}{m-1}X \wedge \delta T,$$

and together with (20) this gives us

$$\xi \wedge \delta T = 0.$$

This means that ξ and δT are collinear, so we define a function f on the support of ξ by $\frac{1}{m-1}\delta T = f\xi$. Then on the support of ξ we have

$$\nabla_X T = fX \wedge \xi, \quad \forall X \in TM.$$

To proceed further we quote the following technical result. The proof can be found in [Mo].

Proposition 7.1. *Let α be a $*$-Killing 2-form, identified with a skew-symmetric endomorphism of TM. Then, for every other skew-symmetric endomorphism β of TM one has*

$$(m-2)\nabla^2_{X,Y}\alpha = -Y \wedge \alpha(\mathrm{Ric}(X)) - Y \wedge R_\alpha(X), \tag{21}$$

$$(m-2)(R_\beta \circ \alpha - \alpha \circ R_\beta) = (R_\alpha \circ \beta - \beta \circ R_\alpha) + (\alpha \circ \mathrm{Ric} \circ \beta - \beta \circ \mathrm{Ric} \circ \alpha), \tag{22}$$

where for a skew-symmetric endomorhpism u of TM we define another skew-symmetric endomorphism R_u by

$$R_u(X) = \frac{1}{2}R(e_j, u(e_j))X$$

for every X in TM.

From this follows

Corollary 1. If α is a $*$-Killing 2-form, the square of the endomorphism corresponding to α commutes with the Ricci tensor.

Now take the inner product with Y in (21) for the 2-form T. From the well known formula $\sum_{i=1}^m e_i \lrcorner (e_i \wedge u) = (m-k)u$ for a k-form u we get

$$-(m-2)\nabla_X \delta T = -(m-1)(T(\mathrm{Ric}(X)) + R_T(X)).$$

If we take the scalar product with some vector Y in the above equation and symmetrize we get

$$-(m-2)\left(g(\nabla_X \delta T, Y) + g(\nabla_Y \delta T, X)\right) =$$
$$-(m-1)\left(g(T(\mathrm{Ric}(X)), Y) + g(T(\mathrm{Ric}(Y)), X) + g(R_T(X), Y) + g(R_T(Y), X)\right).$$

The last two terms on the right cancel out because of the skew-symmetry of the Riemannian curvature tensor in last two entries. Using the skew-symmetry of T and symmetry of Ric we can write this as

$$-\frac{m-2}{m-1}\left(g(\nabla_X \delta T, Y) + g(\nabla_Y \delta T, X)\right) = g(\mathrm{Ric}(TX), Y) + g(\mathrm{Ric}(TY), X)$$

We observe that thanks to Corollary 1 the expression

$$g(\nabla_X \delta T, TY) + g(\nabla_{TY}\delta T, X)$$

is symmetric in X and Y. Then, one can prove using $\nabla_X T = fX \wedge \xi$ and the above property, that

$$T\xi \wedge df + T(\nabla f) \wedge \xi = 0. \tag{23}$$

Since ξ is Killing we have the well-known formula $T\xi = \nabla_\xi \xi = -\frac{1}{2}d(|\xi|^2)$ and $X(|T|^2) = 2g(\nabla_X T, T) = -2fg(X, T\xi)$ for any vector field X. Hence

$$d(|T|^2) = fd(|\xi|^2),$$

where we treat T as a 2-form, so the norm $|T|$ differs from the norm of T treated as a tensor field by the factor $\sqrt{2}$. If we take the exterior derivative in the last equation we get

$$0 = df \wedge d(|\xi|^2) = -2df \wedge T\xi$$

and together with (23) we obtain

$$T(\nabla f) \wedge \xi = 0.$$

This leads to the following proposition.

Proposition 7.2. *Either f is constant, or T has rank 2 as an endomorphism of TM and $\xi \wedge T = 0$.*

In the case when f is constant we have a standard fact.

Theorem 7.3. *If the covariant derivative $T = \nabla\xi$ of a non-parallel Killing field ξ on M satisfies*

$$\nabla_X T = c\xi \wedge X$$

for some constant c, then either ξ defines a Sasakian structure on M, or M is a space form.

Remark. Moroianu defines a *Sasakian structure* on M as a Killing vector field ξ of constant length, such that

$$\nabla_X T = k\xi \wedge X, \quad k > 0,$$

where $T = \nabla\xi$.

In the remaining case when f is non-constant we have $\xi \wedge d\xi = 0$, where we identify T and $\frac{1}{2}d\xi$. This implies that the distribution orthogonal to ξ is integrable.

Proposition 7.4. *Around every point in the support of ξ the manifold M is locally isometric to a warped product $I \times_\lambda N$ where I is an open interval and N is a $(m-1)$-dimensional manifold such that $d\lambda$ is a twistor 1-form on N.*

Proof. Thanks to the Frobenius Theorem we can write M locally as a product $I \times N$, where $\xi = \frac{\partial}{\partial t}$ and N is a local leaf tangent to the distribution $\xi^\perp$. We can write the metric g on M as

$$g = \lambda^2(t)dt^2 + h_t$$

for some positive function λ on $I \times N$ and some family of Riemannian metrics h_t on N. From the fact that ξ is Killing we get that λ and h_t are independent of t. Concluding, we have that $(I \times_\lambda N, \lambda^2 dt^2 + h)$ is a warped product. Moreover we see that ξ is dual to the form $\lambda^2 dt$, so the covariant derivative T of ξ treated as a

2-form is $T = \lambda d\lambda \wedge dt$. Because M is a warped product the projection $M \to N$ is a Riemannian submersion and we can use the O'Neill formulas. We have

$$\nabla_\xi T = 0, \quad \nabla_X T = \lambda \nabla^N_X d\lambda \wedge dt, \quad \forall X \in TN,$$

where ∇^N is the Levi-Civita connection of (N, h). Now take the inner product with X in the second equation and sum over some orthonormal basis of N. We get that $\delta^M T = \lambda \Delta^N \lambda dt$. We have that T is a twistor 2-form iff

$$\nabla_X T = -\frac{1}{m-1} X \wedge \delta^M T = -\frac{1}{m-1} \lambda \Delta^N \lambda X \wedge dt.$$

Hence, T is a twistor 2-form iff

$$\nabla_X d\lambda = -\frac{1}{m-1} \Delta^N \lambda X, \quad \forall X \in TN,$$

which is equivalent to $d\lambda$ being a twistor 1-form. □

This leads to two more possibilities for a Riemmanian manifold M with a Killing vector field ξ whose covariant derivative T is a twistor 2-form which are complementary to the Theorem 7.3. Namely when ξ vanish on M or when ξ does not vanish.

We start with the second case, but first we give a definition.

Definition 7.5. Let N be a Riemannian manifold, $\lambda \in C^\infty(N)$ a positive function and let φ be an isometry of N preserving λ $(\lambda \circ \varphi = \lambda)$. The quotient of the warped product $\mathbb{R} \times_\lambda N$ by the free $\mathbb{Z}$-action generated by $(t, x) \mapsto (t+1, \varphi(x))$ is called the *warped mapping torus* of φ with respect to λ and is denoted by $N_{\lambda,\varphi}$.

Then we have

Proposition 7.6. *A compact Riemannian manifold (M, g) carries a nowhere vanishing Killing vector field with twistor covariant derivative and $\xi \wedge d\xi = 0$ iff it is isometric to a warped mapping torus $N_{\lambda,\varphi}$, where (N, h) is a compact Riemannian manifold carrying a conformal vector field which is a gradient vector field of the function λ and φ is an isometry preserving λ.*

In the case when ξ has a zero we see that the distribution orthogonal to ξ is no longer globally defined. But it turns out that the orbits of ξ are always closed which follows from the next proposition.

Proposition 7.7. *Let M be a compact Riemannian manifold and ξ be a Killing vector field on M. If the covariant derivative of ξ, treated as a skew-symmetric tensor, has rank 2 at some point $x \in M$ where ξ vanishes, then ξ is induced by an isometric action of the circle S^1 on M and its orbits are closed.*

Denote by M_0 the set of points where ξ is not zero. Along this set the integrable distribution $\xi^\perp$ is well-defined and the circle S^1 acts freely and transitively on its maximal integral leaves. We set (N, h) to be such a maximal leaf. Then by the Proposition 7.4 we have that M_0 is isometric to the warped product $S^1 \times N$ with metric g defined by $g = \lambda^2 d\theta^2 + h$, where λ is a positive function on N whose gradient $\nabla\lambda$ is a conformal vector field (gradient conformal vector field for short).

Now we can apply the following proposition.

Proposition 7.8. *Let X be a gradient conformal vector field on a Riemannian manifold (M, g) vanishing at some point $x \in M$. Then there exists an open neighbourhood of x in M on which the metric can be expressed in polar coordinates*

$$g = ds^2 + \gamma^2(s) g_{S^{m-1}},$$

where $g_{S^{m-1}}$ denotes the canonical metric on the sphere S^{m-1} and γ is some positive function $\gamma : (0, \varepsilon) \to \mathbb{R}_+$, for $\varepsilon > 0$. The norm of X in these coordinates is expressed by $|X| = c\gamma$ for some real constant c.

From this follows that we can write the metric on N as $h = ds^2 + \gamma^2(s) g_{S^{m-2}}$ on some neighbourhood of x and the length of $\nabla\lambda$ depends only on the distance from x. Assume that $\nabla\lambda$ vanishes in some other point $y = \exp_x tV$, where V is some unit vector in $T_x N$. This implies that it vanishes on the whole geodesic sphere of radius t. But $\nabla\lambda$ only has isolated zeros, so the above geodesic sphere is just the point y. This in turn implies that N is compact and homeomorphic to S^{m-1} giving that M_0 is compact. Because M_0 is open and connected it follows that $M_0 = M$ which is a contradiction with the assumption that ξ has zeros on M, hence x is the unique zero of ξ.

To proceed further we need the following definition.

Definition 7.9. Let $l > 0$ be a real number and let $\gamma, \lambda : (0, l) \to \mathbb{R}_+$ be two smooth functions satisfying

$$\lim_{s\to 0} \gamma(s) = 0, \lim_{s\to l} \gamma(s) > 0,$$
$$\lim_{s\to 0} \lambda(s) > 0, \lim_{s\to l} \lambda(s) = 0.$$

We view the sphere S^{m-2} as the *topological join* of S^{m-2} and S^1, obtained from $[0, l] \times S^{m-2} \times S^1$ by shrinking $\{0\} \times S^{m-2} \times S^1$ to $\{point\} \times S^1$ and by shrinking $\{l\} \times S^{m-2} \times S^1$ to $\{point\} \times S^{m-2}$.

Then S^m with the Riemannian metric

$$g = ds^2 + \gamma^2(s) g_{S^{m-2}} + \lambda^2(s) d\theta^2$$

defined on its open submanifold $(0, l) \times S^{m-2} \times S^1$ is called the *Riemannian join* of S^{m-2} and S^1 with respect to γ and λ and is denoted by $S^{m-2} *_{\gamma,\lambda} S^1$.

The metric g from the definition extends to a continuous metric on S^m. Under some additional conditions this Riemannian join metric extends smoothly to the whole manifold.

Now we can give the result which finishes the classification.

Theorem 7.10. *Let M and ξ be as above. Then if f in $\nabla_X T = f X \wedge \xi$ is not constant, then*

- *M is a warped mapping torus $N_{\lambda,\varphi}$ where N is compact $(m-1)$-dimensional Riemannian manifold such that $\nabla\lambda$ is a conformal vector field and φ is an isometry of N preserving λ and ξ is the fundamental vector field of the action of $\mathbb{R}$ on M.*
- *M is a Riemannian join $S^{m-2} *_{\gamma,\lambda} S^1$ with λ and γ satisfying certain boundary conditions which makes the metric defined in 7.9 smooth.*

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