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Hardy inequalities and nonlinear PDEs

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Abstract

This paper is devoted to the study of connections between Hardy inequalities and other disciplines of mathematical analysis. In particular, we discuss importance of Hardy inequalities in nonlinear eigenvalue problems in PDEs.

Contents

1 Introduction

We give a general introduction to Hardy inequalities in Section 2. We provide therein classical statements with proofs. In Sections 3 and 4 we consider inequalities of the form

$$\int_\Omega |\xi(x)|^p \mu_1(dx) \le \int_\Omega |\nabla \xi(x)|^p \mu_2(dx), \tag{1}$$

where $1 \le p < \infty$, $\xi : \Omega \to \mathbb{R}$ belongs to general class of functions, Ω is a certain subset of $\mathbb{R}^n$, and the involved measures $\mu_1(dx), \mu_2(dx)$ have various forms. We are particularly interested in application of (1) to study PDEs. Section 5 gives several necessary and sufficient conditions for validity of a Hardy inequality. In particular, some results state equivalence between solvability of differential equations with validity of Hardy inequalities. In Section 6 we present sufficient conditions derived by the author.

2 The Classical Hardy Inequalities

In this section we recall the classical results. We refer to [51, 58, 61] for more information on the best constants in various classical Hardy–type inequalities.

The first inequalities of this type have been proven by Hardy [48] in 1920. In 1926, Landau [64] proved that the optimal constant equals

$$H_{\gamma,1,p} = \left(\frac{p}{|\gamma - p + 1|} \right)^p . \tag{2}$$

We give the statements from the book of Hardy, Littlewooda and Polya [51]. The first of them is a discrete version, while the second one is continuous. There are multiple various proofs of these classical results. We selected the ones indicating the best constants. We follow [51] and present the proof of a discrete inequality by Elliot [34]. The first proof of continuous case by Hardy can be found in [49].

Theorem 2.1 (Classical Hardy Inequality — the original discrete statement)**.** *If $p > 1$, $a_n \ge 0$, and $A = a_1 + a_2 + \dots + a_n$, then*

$$\sum \left(\frac{A_n}{n} \right)^p \le \left(\frac{p}{p-1} \right)^p \sum a_n^p \tag{3}$$

unless all the a's are zero. The constant is the best possible.

Elliot's proof. We may suppose that $a_1 > 0$. For if we suppose that $a_1 = 0$ and replace a_{n+1} by b_n, (3) becomes

$$\left(\frac{b_1}{2} \right)^p + \left(\frac{b_1 + b_2}{3} \right)^p + \dots < \left(\frac{p}{p-1} \right)^p \left({b_1}^p + b_2^p + \dots \right),$$

that is an inequality weaker than (3) itself.

Let us define $\alpha_n := A_n/n$ and $\alpha_0 = 0 = A_0$. We use Young inequality to observe that

$$\alpha_{n-1}\alpha_n^{p-1} \le \frac{1}{p}\alpha_{n-1}^p + \frac{p-1}{p}\alpha_n^p.$$

We have then

$$\begin{aligned}
\alpha_n^p - \frac{p}{p-1}a_n^{p-1}\alpha_n &= \alpha_n^p - \frac{p}{p-1}\left(n\alpha_n - (n-1)\alpha_{n-1}\right)\alpha_n^p = \\
&= \alpha_n^p\left(1 - \frac{np}{p-1}\right) + \frac{(n-1)p}{p-1}\alpha_n^{p-1}\alpha_{n-1} \le \\
&\le \alpha_n^p\left(1 - \frac{np}{p-1}\right) + \frac{n-1}{p-1}\left((p-1)\alpha_n^p + \alpha_{n-1}^p\right) = \\
&= \frac{1}{p-1}\left((n-1)\alpha_{n-1}^p - n\alpha_n^p\right).
\end{aligned}$$

Hence

$$\sum_{n=1}^N \alpha_n^p - \frac{p}{p-1}\sum_{n=1}^N \alpha_n^{p-1}a_n \le -\frac{N\alpha_N^p}{p-1} \le 0.$$

We apply Hölder inequality and obtain

$$\sum_{n=1}^N \alpha_n^p \le \frac{p}{p-1}\sum_{n=1}^N \alpha_n^{p-1}a_n \le \frac{p}{p-1}\left(\sum_{n=1}^N a_n^p\right)^{1/p}\left(\sum_{n=1}^N \alpha_n^p\right)^{1/p'}$$

Dividing by the last factor on the right (which is certainly positive), and raising the results to the p'th power, we obtain

$$\sum_{n=1}^N \alpha_n^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^N a_n^p$$

When we let $N \to \infty$ we obtain (3), except that we have '$\le$' instead of '$<$'. In particular we see that $\sum \alpha_n^p$ is finite. □

We give below the original statement with Hardy's proof and Pólya's improvements following [59].

Theorem 2.2 (Classical Hardy Inequality)**.**
If $p \ge 1$, $f(x) \ge 0$, and $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)dt \tag{4}$$

unless $f \equiv 0$. The constant is the best possible.

Proof by Hardy and Pólya. By partial integration and the identity $\frac{d}{dx}(F^p(x)) = pF^{p-1}(x)f(x)$, which holds for almost all $x \in (0,\infty)$, we obtain for arbitrary α and A with $0 < \alpha < A < \infty$:

$$\begin{aligned} \int_\alpha^A \left(\frac{F(x)}{x}\right)^p dx &= -\frac{1}{p-1}\int_\alpha^A F^p(x)\frac{d}{dx}(x^{1-p})dx = \\ &= \frac{\alpha^{1-p}}{p-1}F^p(\alpha) - \frac{A^{1-p}}{p-1}F^p(A) + \frac{1}{p-1}\int_\alpha^A \frac{d}{dx}\left(F^p(x)\right)x^{p-1}dx \le \\ &= \frac{\alpha^{1-p}}{p-1}F^p(\alpha) + \frac{p}{p-1}\int_\alpha^A \left(\frac{F(x)}{x}\right)^{1-p} f(x)dx. \end{aligned}$$

Moreover, invoking Hölder inequality, we have

$$\int_\alpha^A \left(\frac{F(x)}{x}\right)^{1-p} f(x)dx \le \left(\int_\alpha^A f^p(x)dx\right)^{\frac{1}{p}} \left(\int_\alpha^A \left(\frac{F(x)}{x}\right)^p dx\right)^{\frac{p-1}{p}}.$$

Choosing β such that $\alpha \le \beta \le A$ and applying the preceding two inequalities to $F(x) - F(\alpha)$ instead of $F(x)$, we find that

$$\begin{aligned} \int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx &\le \frac{p}{p-1}\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^{p-1} f(x)dx \le \\ &\le \frac{p}{p-1}\left(\int_\alpha^A f^p(x)dx\right)^{\frac{1}{p}} \left(\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{\frac{p-1}{p}}. \end{aligned}$$

Hence

$$\left(\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{\frac{1}{p}} \le \frac{p}{p-1}\left(\int_\alpha^A f^p(x)dx\right)^{\frac{1}{p}}$$

and

$$\left(\int_\beta^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{\frac{1}{p}} \le \frac{p}{p-1}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}.$$

In this inequality we first let $\alpha \to 0^+$ and observe that $F(x) - F(\alpha)$ increases to $F(x)$. To finish the proof we let $A \to \infty$ and $\beta \to 0^+$. □

The following theorem presents the first weighted modification which was proved by Hardy [50] in 1927. It is probably the most famous version of inequalities adorned by the name of Hardy.

Theorem 2.3 (The first weighted Hardy inequality)**.** *Let $1 < p < \infty$ and denote $F(t) = \int_0^t f(x)dx$ for $\gamma < p-1$ and $F(t) = \int_t^\infty f(x)dx$ for $\gamma > p-1$, where f is a nonnegative measurable function defined on $(0,\infty)$. Then*

$$\int_0^\infty F^p(t)t^{\gamma-p}dt \le H_{\gamma,1,p}\int_0^\infty f^p(t)t^\gamma dt \tag{5}$$

with a constant $H_{\gamma,1,p} > 0$ independent of t.

Hardy's proof. Suppose $\gamma < p-1$. Changing the variable ($t = xs$), using Minkowski inequality and again changing variable ($y = xs$) we obtain

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\gamma dx &= \left\| \int_0^1 f(xs)x^{\gamma/p} ds \right\|_{L^p(\mathbb{R}_+)} \\
&\le \int_0^1 \left\| f(xs)x^{\gamma/p} \right\|_{L^p(\mathbb{R}_+)} ds \\
&\le \int_0^1 \left(\int_0^\infty f^p(xs)x^\gamma ds\right)^{\frac{1}{p}} ds = \\
&= \int_0^1 \left(\int_0^\infty f^p(y)y^\gamma ds\right)^{\frac{1}{p}} s^{-\frac{\gamma+1}{p}} ds = \\
&= \frac{p}{p-\gamma-1}\left(\int_0^\infty f^p(y)y^\gamma ds\right)^{\frac{1}{p}}.
\end{aligned}$$

This implies the case $\gamma > p-1$ by the substitution. $\square$

By 'Classical Hardy Inequalities' we also understand the statement for the function and its derivative. Such statements hold are formulated for absolutely continuous functions or, sometimes, for C_0^∞.

Theorem 2.4 (Classical Hardy Inequalities)**.** *Let* $1 < p < \infty$.

1. *Assume further that* $\gamma \neq p-1$ *and that* $\xi = \xi(x)$ *is an absolutely continuous function in* $(0,\infty)$ *such that* $\int_0^\infty |\xi'(x)|^p x^\gamma \, dx < \infty$ *and let*

$$\xi^+(0) := \lim_{x\to 0} \xi(x) = 0 \quad \text{for } \gamma < p-1, \tag{6}$$

$$\xi(\infty) := \lim_{x\to\infty} \xi(x) = 0 \quad \text{for } \gamma > p-1. \tag{7}$$

Then

$$\int_0^\infty \left(\frac{|\xi|}{x}\right)^p x^\gamma \, dx \le H_{\gamma,1,p} \int_0^\infty |\xi'|^p x^\gamma \, dx, \tag{8}$$

where the constant $H_{\gamma,1,p} = \left(\frac{p}{|p-1-\gamma|}\right)^p$ *is optimal.*

2. *Assume further that* $\gamma \neq p-n$ *and* $\xi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. *Then*

$$\int_{\mathbb{R}^n\setminus\{0\}} |\xi|^p |x|^{\gamma-p} \, dx \le H_{\gamma,n,p} \int_{\mathbb{R}^n\setminus\{0\}} |\nabla \xi|^p |x|^\gamma \, dx, \tag{9}$$

where the constant $H_{\gamma,n,p} = \left(\frac{p}{|p-n-\gamma|}\right)^p$ *is optimal.*

3 Importance of Hardy inequalities

Hardy–type inequalities are important tools in functional analysis, harmonic analysis, probability theory, and PDEs. In theory of PDEs they are used to obtain a priori estimates, existence, and regularity ([7, 14, 15, 39, 43], Section 2.5 in [66]), as well as to study qualitative properties of solutions and their asymptotic behaviour [75]. Hardy inequalities are also applied in derivation of embedding theorems (Theorem 3.1 in [21], [47, 52]), Gagliardo–Nirenberg interpolation inequalities [25, 26, 46, 55] and in the real interpolation theory [37].

In the last three decades huge progress was made to understand Hardy–type inequalities, see e.g. books: [57, 58, 60, 59, 63, 66, 68] and their references. The applied tools are often expressed in the language of functional analysis, harmonic analysis, and probability. They seem to be rather abstract in general and the conditions for the validity of inequalities are often very hard to verify in practice.

Generally speaking, linking nonlinear eigenvalue problems of elliptic and parabolic type with Hardy inequalities is common in the literature. We observe this issue also in the articles [2, 3, 4, 13, 18, 19, 53, 54, 67]. For example it is well known that functions achieving best constants in Hardy–Sobolev type inequalities satisfy the nonlinear eigenvalue problems [16, Chapter 5]. Moreover, the best constants are investigated for proving existence of parabolic eigenvalue problems [7, 28, 39, 41]. What is less understood is the converse: that solutions or subsolutions to differential eigenvalue problems are helpful to construct Hardy–Sobolev inequalities.

4 Hardy inequalities and properties of solutions to PDEs

4.1 The best constant and existence

Analysis of the best constants $c_{n,\gamma,p}$ in Classical n–dimensional Hardy inequalities is crucial to decide existence. We refer to classical work of P. Baras and J. A. Goldstein [7], where existence, nonexistence of global solutions, and a blow–up for following parabolic problem is considered. For $x \in \mathbb{R}^n,\ n \geq 3$, and $t \in (0,T)$

$$\begin{cases} u_t - \Delta u = \lambda \frac{u}{|x^2|}, & \lambda \in \mathbb{R}, \\ u(x,0) = u_0(x) > 0, & u_0 \in L^2(\mathbb{R}^n), \end{cases} \tag{10}$$

has a solution if and only if $\lambda \leq \left(\frac{n-2}{2}\right)^2 = c_{n,0,2}$. See [7] for details and [43] for related generalized results. Nevertheless, the authors of [7] do not apply Hardy inequality in any version.

4.2 Asymptotic behaviour

In [75] J. L. Vazquez and E. Zuazua describe the asymptotic behaviour of the heat equation that reads

$$u_t = \Delta u + V(x)u \quad \text{and} \quad \Delta u + V(x)u + \mu u = 0,$$

where $V(x)$ is an inverse–square potential (e.g. $V(x) = \frac{\lambda}{|x|^2}$). The authors consider the Cauchy–Dirichlet problem in a bounded domain and for the Cauchy problem in $\mathbb{R}^n$ as well. The crucial tool is an improved form of Hardy–Poincaré inequality and its new weighted version. The main results show the decay rate of solutions. Well–posedness of the problem and problems with uniqueness are also considered. Furthermore, the authors of [75] explain and generalize the classical work of P. Baras and J. A. Goldstein [7] decribed in Subsection 4.1.

In [39] J. P. Garcia Azorero and I. Peral Alonso study elliptic and parabolic problems linking with Hardy inequality. The authors consider nonlinear critical p–heat equation (and the related stationary p–Laplacian equation) where $-\Delta_p u, f(x) \geq 0$, Ω is a bounded domain in $\mathbb{R}^n$, and $1 < p < N$. The analysis reveals that the behaviour depends on p. The results depend in general on the relation between existence and the best constant in Hardy inequality.

4.3 Radiality

Hardy inequality may play the key role to prove existence, nonexistence, as well as radiality of solutions. All the mentioned applications are studied in [40] by M. Garcia–Huidobro, A. Kufner, R. Manásevich, and C. S. Yarur.

The authors establish a critical exponent for the inclusion of a certain weighted Sobolev space into the weighted Lebesgue space. This result is applied in the proof of radiality of solutions for a quasilinear equation

$$\begin{cases} \operatorname{div}(a(|x|)|\nabla u|^{p-2}\nabla u) = b(|x|)|\nabla u|^{q-2}\nabla u & \text{in } B \subseteq \mathbb{R}^n, \\ u = 0 \quad \text{on } \partial B, \end{cases} \tag{11}$$

where $1 < p < q$, functions a, b are weight functions, and B is a ball.

5 Necessary and sufficient conditions for validity of Hardy inequalities

5.1 Maz'ya's methods

Let us mention that abstract conditions for existence of Hardy–type inequalities involving measures have been characterized completely in some cases. For example Theorem 2.4.1 in [66] (in case of $M(t) = |t|$) characterizes measures satisfying

Hardy inequalities

$$\int_\Omega |u|^p \mu(dx) \leq C \int_\Omega |\nabla u|^p dx, \quad 1 < p < \infty,$$

holding for smooth compactly supported functions u. The conditions, so–called isoperimetric inequalities, are expressed on compact sets and use capacities. Unfortunately, they are very hard to be verified in practice.

5.2 Muckenhoupt–type conditions

There are many conditions equivalent to validity of Hardy inequalities. They are usually associated with the name of Muckenhoupt, his work [69], and the following condition

$$A = \sup_{r>0} \left(\int_r^\infty u(x)dx \right)^{\frac{1}{p}} \left(\int_r^\infty v^{1-p'}(x)dx \right)^{\frac{1}{p'}} < \infty.$$

Muckenhoupt generalised it in the case when μ, ν are some Borel measures (and ν_{ac} is the absolutely continuous part of ν)

$$A = \sup_{r>0} [\mu(r,\infty)]^{\frac{1}{p}} \left(\int_r^\infty \left(\frac{d\nu_{ac}}{dx} \right)^{1-p'} (x)dx \right)^{\frac{1}{p'}} < \infty.$$

and proved that it is equivalent to inequality

$$\left(\int_0^\infty \left(\int_0^\infty f(t)dt \right)^p d\mu \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty f^p(t)d\mu \right)^{\frac{1}{p}}.$$

Nevertheless, we should not hesitate to refer to pioneer works of Tomaselli [74] and Talenti [73], who worked on this topic in the same time as Muckenhoupt. Independently, they obtained equivalence between

$$B = \sup_{r\in(0,b)} \left(\int_r^b u(x)dx \right) \left(\int_0^r v^{1-p'}(x)dx \right)^{p-1} < \infty,$$

where $f \geq 0,\ 0 < b \leq \infty,\ 0 < p < \infty$ and the inequality

$$\int_0^b \left(\int_0^x f(t)dt \right)^p u(x)dx \leq C \int_0^b f^p(t)v(x)dx.$$

Tomaselli in [74] derived two more equivalent conditions, namely

$$B^* = \sup_{r\in(0,b)} \left(\int_0^r u(x) \left(\int_0^x v^{1-p'}(t)dt \right)^p dx \right) \left(\int_0^r v^{1-p'}(x)dx \right)^{-1} < \infty,$$

$$B^{**} = \inf_f \sup_{r\in(0,b)} \frac{1}{f(x)} \int_0^x u(x)\left(f(t) + \int_0^t v^{1-p'}(s)ds\right)^p dt < \infty,$$

where the infimum is taken over all positive measurable functions f.

We give below the famous theorem, which summarizes efforts and ideas in this topic of wide range of great mathematicians such as Artola, Talenti [73], Tomaselli [74], Chisholm–Everitt [24], Muckenhoupt [69], Boyd–Erdos. The proof that we invoke follows [59] where, apart from this formulation, a lot of additional interesting historical information on the investigation of this problems can be found.

Theorem 5.1 (Talenti–Tomaselli–Muckenhoupt). *Let $1 \le p < \infty$. The inequality*

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^p u(x)dx\right)^{\frac{1}{p}} \le C\left(\int_0^b f^p(t)v(x)dx\right)^{\frac{1}{p}} \tag{12}$$

holds for all measurable functions $f(x) \ge 0$ on $(0,b)$, $0 < b \le \infty$ if and only if

$$A = \sup_{r>0}\left(\int_r^b u(x)dx\right)^{\frac{1}{p}}\left(\int_0^r v^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty. \tag{13}$$

Moreover, the best constant C in (12) *satifies $A \le C \le p^{\frac{1}{p}}p'^{\frac{1}{p'}}$ for $1 < p < \infty$ and $C = A$ for $p = 1$.*

Proof. Let $p = 1$. Then the estimate (12) is equivalent to the fact that the operator $Hf(x) = \int_0^x f(t)dt$ is bounded from $L^1(v)$ into $L^1(u)$ with the norm $\le C$, which is equivalent to boundedness of its dual $H^*f(x) = \int_x^b f(t)dt$ from $L^\infty(1/u)$ into $L^\infty(1/v)$ with the norm $\le C$. Thus, $\int_r^b u(t)dt \le Cv(r)$ for all $r \in (0,b)$.

Let $1 < p < \infty$.

Necessity. For any $f \in L^p(0,b;v)$ and $r \in (0,b)$ let $f_r = f\chi_{(0,r)}$. Then

$$\begin{aligned}
\left(\int_0^r f(t)dt\right)\left(\int_r^b u(x)dx\right)^{\frac{1}{p}} &= \left(\int_r^b \left(\int_0^x f_r(t)dt\right)^p u(x)dx\right)^{\frac{1}{p}} \le \\
&\le \left(\int_0^b \left(\int_0^x f_r(t)dt\right)^p u(x)dx\right)^{\frac{1}{p}} \le \\
&\le ||H||\left(\int_0^b f_r^p(t)v(x)dx\right)^{\frac{1}{p}} \le \\
&\le ||H||\left(\int_0^b f^p(t)v(x)dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Taking the supremum over all $f \in L^p(0,b;v)$ with the norm $\|f\|_{p,v} \le 1$ and observing that the dual space of $L^p(0,b;v)$ is $(L^p(0,b;v))^* = L^{p'}(0,b;v^{1-p'})$ we obtain

$$\|\chi_{(0,r)}\|_{p',v^{1-p'}} \|\chi_{(r,b)}\|_{p,u} \le \|H\|$$

for all $r \in (0,b)$ or

$$\sup_{r>0} \left(\int_r^b u(x)dx \right)^{\frac{1}{p}} \left(\int_0^r v^{1-p'}(x)dx \right)^{\frac{1}{p'}} < \|H\|.$$

Sufficiency. Let $h(x) = \left(\int_0^x v^{1-p'}(t)dt \right)^{\frac{1}{pp'}}$. By Hölder inequality and the Fubini theorem (with some modifications when $v(t)h(t)$ is 0 or ∞ on a set of positive measure) we obtain

$$\int_0^b \left(\int_0^x f(t)dt \right)^p u(x)dx \le$$

$$\le \int_0^b \left[\int_0^x \left(f(t)v^{\frac{1}{p}}(t)h(t) \right)^p dt \right] \left[\int_0^x \left(v^{\frac{1}{p}}(s)h(s) \right)^{-p'} ds \right]^{\frac{p}{p'}} u(x)dx =$$

$$= \int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t) \right)^p \left(\int_t^b \left[\int_0^x \left(v^{\frac{1}{p}}(s)h(s) \right)^{-p'} ds \right]^{p-1} u(x)dx \right) dt.$$

Moreover,

$$\frac{d}{ds} \left(\int_0^s v^{1-p'}(t)dt \right)^{1-\frac{1}{p}} = \left(1 - \frac{1}{p} \right) \left(\int_0^s v^{1-p'}(t)dt \right)^{-\frac{1}{p}} v^{1-p'}(s),$$

which gives

$$\int_0^x v^{-\frac{p'}{p}}(s)h^{-p'}(s)ds = p' \left(\int_0^x v^{1-p'}(t)dt \right)^{\frac{1}{p'}}$$

and, hence, the last integral in the estimate above equals

$$(p')^{p-1} \int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t) \right)^p \left[\int_t^b \left(\int_0^x v^{1-p'}(s)ds \right)^{\frac{p-1}{p'}} u(x)dx \right] dt.$$

By using the definition of A twice we find that

$$(p')^{p-1} \int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t) \right)^p \left[\int_t^b \left(\int_0^x v^{1-p'}(s)ds \right)^{\frac{p-1}{p'}} u(x)dx \right] dt \le$$

$$\le (Ap')^{p-1} \int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t) \right)^p \left[\int_t^b \left(\int_x^b u(s)ds \right)^{\frac{p-1}{p}} u(x)dx \right] dt =$$

$$\leq (Ap')^{p-1}\int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t)\right)^p \left[\int_t^b p\frac{d}{dx}\left\{-\left(\int_x^b u(s)ds\right)^{\frac{1}{p}}\right\}\right] dt =$$

$$\leq p(Ap')^{p-1}\int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t)\right)^p \left(\int_t^b u(s)ds\right)^{\frac{1}{p}} dt \leq$$

$$\leq p(Ap')^{p-1}A\int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t)\right)^p \left(\int_0^t v^{1-p'}(s)ds\right)^{-\frac{1}{p'}} dt =$$

$$\leq pA^p(p')^{p-1}\int_0^b \left(f(t)v^{\frac{1}{p}}(t)h(t)\right)^p h^{-p}dt = pA^p(p')^{p-1}\int_0^b f^p(t)v(t)dt.$$

We conclude that (12) holds with a constant $C \leq Ap^{\frac{1}{p}}(p')^{\frac{1}{p'}}$ and the theorem is proven. □

5.3 Solvability of ODEs equivalent to existence of Hardy inequality

Constructing Hardy–type inequalities on the basis of differential problems is a known way. In paper [45] Gurka investigated the existence of one–dimensional Hardy–type inequality between L^q and L^p (allowing the case $p = q$) that reads

$$\left(\int_0^a s(x)|u(x)|^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^a r(x)|u'(x)|^p dx\right)^{\frac{1}{p}} \tag{14}$$

and found necessary and sufficient conditions for the existence of (14) in a certain class of admitted functions. The work [45] generalises previous results by Beesack [8], Kufner and Triebel [62], Muckenhoupt [69], and Tomaselli [74]. Some of them are already summarised in Subsection 5.2. The main result of [45] reads

Theorem 5.2 ([45], Theorem 1.3)**.** *Assume* $0 < a \leq \infty$, $1 < p \leq q < \infty$. *Let* $r(x) > 0$, $s(x) \geq 0$ *be functions measurable on* $[0, a]$.

Moreover, let us suppose that the first derivative $r'(x)$ *exists for all* $x \in (0, a)$. *Then the equation*

$$\lambda\frac{d}{dx}\left(r^{\frac{q}{p}}(x)(y'(x))^{\frac{q}{p'}}\right) + s(x)y^{\frac{q}{p'}}(x) = 0 \tag{15}$$

(with a certain $\lambda > 0$*) has a solution* $y(x)$ *(with a locally absolutely continuous first derivative) such that*

$$y(x) > 0,\ y'(x) > 0, \quad (x \in (0, a))$$

if and only if there exists a constant $C_0 > 0$ *such that the inequality*

$$\left(\int_0^a s(x)|u(x)|^q dx\right)^{\frac{1}{q}} \leq C_0\left(\int_0^a r(x)|u'(x)|^p dx\right)^{\frac{1}{p}}$$

holds for every function $u(x)$ absolutely continuous on $[0,a]$ such that $u(0) = \lim_{t\to\infty} u(t) = 0$.

We find connections between validity of Hardy inequality in $\mathbb{R}^n$ with radial measures in the recent paper by Ghoussoub and Moradifam [42]. The authors proved that the existence of inequalities

$$c\int_B W(x)u^2 dx \leq \int_B V(x)|\nabla u(x)|^2 dx \text{ for all } u \in C_0^\infty(B),$$

with radially symmetric functions V and W (so–called Bessel pairs), where B is a ball with center at zero, is equivalent to the existence of solutions to the one dimensional nonlinear eigenvalue problem

$$y''(r) + \left(\frac{n-1}{r} + \frac{V'(r)}{V(r)}\right) y'(r) + \frac{cW(r)}{V(r)} y(r) = 0, \quad y > 0.$$

This is in the spirit of Gurka's inequality from (15).

We find this method also in books of Kufner e.g. [60].

5.4 Solvability of PDEs equivalent to existence of Hardy inequality

We find connections between p–harmonic problems and Hardy inequalities in papers [9, 10] by Barbatis, Filippas and Tertikas. Their method is based on the following geometric observation: if $K \subseteq \bar{\Omega} \subseteq \mathbb{R}^n$ is a smooth surface of codimension k $(1 \leq k \leq n)$ and $d(x) = dist(x, \partial\Omega)$, then the following inequality holds in the weak sense for $p \neq k$

$$-\Delta_p \left(d^{\frac{p-k}{p-1}}\right) \geq 0 \quad \text{in} \quad \Omega \setminus K.$$

Using this observation the authors obtain improved Hardy inequalities involving function d.

More general approach is presented in several papers by D' Ambrosio [29, 30, 31]. We find there an alternative method of construction of Hardy inequalities from problems of a type $-\Delta_p\phi \geq 0$ and similar ones involving analysis on Heisenberg groups $\mathbb{H}^n$. Among other inequalities the author obtained the following one described in Heisenberg setting

$$\int_\Omega |u|^p \left(\frac{|\nabla_L \phi|}{\phi}\right)^p d\xi \leq C \int_\Omega |\nabla_L u|^p \, d\xi, \quad \text{for every } u \in C_0^1(\Omega),$$

where $\phi : \Omega \to \mathbb{R}^n$ satisfies $-\Delta_p\phi \geq 0$. The author deals also with Hardy inequalities involving a term with distance from the boundary.

6 Works by the author

6.1 Nonexistence

Our methods are inspired by the techniques from paper [56], where nonexistence of nontrivial nonnegative weak solutions to A–harmonic problem

$$-\Delta_A u \geq \Phi(u) \quad \text{on} \quad \mathbb{R}^n, \tag{16}$$

where Φ is a nonnegative function, is investigated. The authors derive Caccioppoli–type estimate for nonnegative weak solutions to (16). Then they obtain more specified a priori estimates involving general test functions and finally, choosing appropriate test functions, they obtain nonexistence.

6.2 Hardy inequalities derived from p–harmonic problems

We consider the anti–coercive partial differential inequality of elliptic type involving p–Laplacian

$$-\Delta_p u \geq \Phi,$$

where Φ is a given locally integrable function and u is defined on an open subset $\Omega \subseteq \mathbb{R}^n$. Knowing solutions, we derive Caccioppoli inequalities for u. As a direct consequence we obtain Hardy inequalities involving certain measures for compactly supported Lipschitz functions. Our methods allow to retrieve classical Hardy inequalities with optimal constants. We present several applications leading to various weighted Hardy inequalities.

Let us summarize the main goals reached by the author in [70, 71].

Theorem 6.1 ([70], Theorem 4.1)**.** *Assume that* $1 < p < \infty$ *and that* $u \in W^{1,p}_{loc}(\Omega)$ *is a nonnegative solution to PDI* $-\Delta_p u \geq \Phi$*, in the sense of distributions, where* Φ *is locally integrable and satisfies the condition*

$$(\mathbf{\Phi}, \mathbf{p}) \qquad \sigma_0 := -\inf\{\sigma \in \mathbb{R} : \Phi \cdot u + \sigma|\nabla u|^p \geq 0 \quad a.e. \ in \ \{u > 0\} \cap \Omega\} \in \mathbb{R},$$

where we set $\inf \emptyset = -\infty$*. Assume further that* β *and* σ *are arbitrary numbers such that* $\beta > 0$ *and* $\beta > \sigma \geq \sigma_0$*.*

Then, for every Lipschitz function ξ *with compact support in* Ω*, we have*

$$\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla \xi|^p \mu_2(dx), \tag{17}$$

where

$$\mu_1(dx) = \left(\frac{\beta - \sigma}{p - 1}\right)^{p-1} [\Phi \cdot u + \sigma|\nabla u|^p] \cdot u^{-\beta-1} \chi_{\{u>0\}} \, dx, \tag{18}$$

$$\mu_2(dx) = u^{p-\beta-1} \chi_{\{|\nabla u| \neq 0\}} \, dx. \tag{19}$$

Theorem 6.2 (Classical Hardy Inequality)**.** *Let $1 < p < \infty$ and $\gamma \neq p-1$. Then, for every nonnegative Lipschitz function ξ with compact support, we have*

$$\int_0^\infty \left(\frac{|\xi|}{x}\right)^p x^\gamma\, dx \le C_{min} \int_0^\infty |\xi'|^p x^\gamma\, dx, \tag{20}$$

where the constant $C_{min} = \left(\frac{p}{|p-1-\gamma|}\right)^p$ is optimal.

Corollary 6.1 (Hardy Inequality on $\mathbb{R}^n \setminus \{0\}$)**.** *Suppose $p > 1$, $\gamma < p - n$. Then, for every nonnegative Lipschitz function ξ with compact support, we have*

$$\int_{\mathbb{R}^n\setminus\{0\}} |\xi|^p |x|^{\gamma-p}\, dx \le \widetilde{C}_{min} \int_{\mathbb{R}^n\setminus\{0\}} |\nabla\xi|^p |x|^\gamma\, dx.$$

where constant $\widetilde{C}_{min} = \left(\frac{p}{p-n-\gamma}\right)^p$ is optimal.

In [71] we prove the result for ξ's from the weighted Sobolev space defined as follows.

Definition 6.1 (Weighted Sobolev space)**.** *By $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where nonnegative measurable functions v_1, v_2 are given, we mean the completion of the set of functions $u \in C^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |u|^p\, v_1 dx < \infty$ and $\int_{\mathbb{R}^n} |\nabla u|^p\, v_2 dx < \infty$, under the norm*

$$\|u\|_{W^{1,p}_{v_1,v_2}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u|^p\, v_1 dx + \int_{\mathbb{R}^n} |\nabla u|^p\, v_2 dx\right)^{\frac{1}{p}}.$$

Theorem 6.3 (Hardy–Poincaré inequalities, [71])**.** *Suppose $p > 1$ and $\gamma > 1$. Then, for every compactly supported function $\xi \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where $v_1(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}$, $v_2(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma}$, we have*

$$\bar{C}_{\gamma,n,p} \int_{\mathbb{R}^n} |\xi|^p \left[(1 + |x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dx \le \int_{\mathbb{R}^n} |\nabla\xi|^p \left[(1 + |x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dx, \tag{21}$$

with $\bar{C}_{\gamma,n,p} = n\left(\frac{p(\gamma-1)}{p-1}\right)^{p-1}$. Moreover, for $\gamma \ge n + 1 - \frac{n}{p}$, the constant $\bar{C}_{\gamma,n,p}$ is optimal.

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