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# Hardy inequalities resulted from nonlinear problems dealing with $A$–Laplacian

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# Hardy inequalities resulted from nonlinear problems dealing with $A$–Laplacian

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**Abstract**

We consider the anticoercive partial differential inequality of elliptic type involving $A$–Laplacian: $-\Delta_A u = -\mathrm{div} A(\nabla u) \geq \Phi$, where $\Phi$ is a given locally integrable function and $u$ is defined on an open subset $\Omega \subseteq \mathbb{R}^n$. Knowing solutions, we derive Caccioppoli inequalities for $u$. As a direct consequence we obtain Hardy inequalities for compactly supported Lipschitz functions involving certain measures, having the form

$$\int_\Omega F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \int_\Omega \bar{A}(|\nabla \xi|)\mu_2(dx),$$

where $\bar{A}(t)$ is an $N$–function satisfying $\Delta'$–condition and $F_{\bar{A}}(t) = 1/(\bar{A}(1/t))$. Examples involving $\bar{A}(t) = t^p \log^\alpha(2+t)$, $p > 1$, $\alpha \geq 0$ are given. The work extends our previous work [44], where we dealt inequality $-\Delta_p u \geq \Phi$, leading to Hardy inequalities with the best constants.

**Key words and phrases:** Hardy–type estimates, $A$–harmonic PDEs, nonlinear eigenvalue problems, Orlicz—Sobolev spaces

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# 1 Introduction

In this paper we derive Hardy–Sobolev inequalities of the form

$$\int_\Omega F_{\bar{A}}(|\xi|)\mu_1(dx) \le \int_\Omega \bar{A}(|\nabla\xi|)\mu_2(dx), \tag{1.1}$$

where $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, $\Omega$ is an open subset of $\mathbb{R}^n$ not necessarily bounded, $\bar{A}(t)$ is an $N$–function satisfying $\Delta'$–condition and $F_{\bar{A}}(t) = 1/(\bar{A}(1/t))$. The involved measures $\mu_1(dx)$, $\mu_2(dx)$ depend on $u$ — a nonnegative weak solution to the anticoercive partial differential inequality of elliptic type involving $A$–Laplacian

$$-\Delta_A u = -\mathrm{div}A(\nabla u) = -\mathrm{div}\left(\frac{\bar{A}(|\nabla u|)}{|\nabla u|^2}\nabla u\right) =\ge \Phi \quad \text{in} \quad \Omega, \tag{1.2}$$

with locally integrable function $\Phi$. Quite a general function $\Phi$ is allowed. It can be even negative or sign changing if only there exists

$$\Phi + \sigma\,\frac{\bar{A}(|\nabla u|)}{g(u)}\chi_{\{\nabla u\neq 0\}} \ge 0 \quad \text{a.e.}$$

where nonnegative function $g$ satisfies some compatibility conditions (see Assumption ($\Psi$): (2.11) and (2.12)).

The idea of the proof is as follows. The main difficulty is to derive Caccioppoli–type inequality for weak solutions to the problem (1.2). The result is obtained by the certain substitution and application of some properties (resulting from $\Delta'$–condition) of the involved function $\bar{A}$.

Our method of construction of the inequalities is a handy tool. Not only is it easy to conduct, but also gives deep consequences. We retrieve all the results of [44, 45] preserving constants. They are among others classical Hardy inequality with optimal constant and Hardy–Poincaré inequalities

$$\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n} |\xi(x)|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)(\gamma-1)}dx \le \int_{\mathbb{R}^n} |\nabla\xi(x)|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)\gamma}\,dx, \quad \gamma > 1$$

with $\bar{C}_{\gamma,n,p}$ derived and proven to be optimal for sufficiently big $\gamma$'s in [45].

The motivation to consider Hardy–Sobolev–type inequalities (1.1) is clear. They are widely spread in various fieds of analysis playing significant role among others in functional analysis, harmonic analysis, probability theory, and PDEs. In theory of PDEs they are used to obtain a priori estimates, existence, regularity results and to study qualitative properties of solutions and their asymptotic behaviour [4, 7, 8, 21, 22, 39, 46]. Hardy inequalities are applied to derivation of embedding theorems, Gagliardo–Nirenberg interpolation inequalities and in real interpolation theory [12, 13, 14, 19, 26, 27, 25, 32]. Moreover, functions achieving the best constants in Hardy–Sobolev type inequalities satisfy some nonlinear eigenvalue problems [10, Chapter 5]. Our method is confirmed in some cases by the best constants in the obtained inequalities.

Hardy–Sobolev–type inequalities are also interesting on their own [34, 35, 36]. Many authors consider generalized versions of the inequalities with remainder terms [1, 2, 18] as well as those expressed in Orlicz setting [9, 31] or combing this both ideas [33].

Derivation of existence results and other properties via Hardy inequalities is a well known idea. We deal with the converse. Our purpose is to give the constructive method of derivation of Hardy–Sobolev inequalities. We find the idea of construction of Hardy inequalities via problems involving $p$–Laplacian in papers by D'Ambrosio [15, 16, 17]. The author derived inequality

$$\int |\xi(x)|^p W(x)dx \leq C \int |\nabla \xi(x)|^p V(x)dx, \quad \text{for every } \xi \in C_0^1(\Omega),$$

where the weights $V(x)$ and $W(x)$ depend on a function $u$, that is a nonnegative solution to $-\Delta_p(u^\alpha) \geq 0$, and on the constant $\alpha$. We generalize both: the right and the left hand side of the problem for u, which leads to far–reaching consequences.

Our cosiderations are based on the methods from [37] developed in [44]. The idea is as follows. In [37] the authors investigate nonexistence of nontrivial nonnegative weak solutions to the $A$–harmonic problem

$$-\Delta_A u \geq \Phi(u) \quad \text{on} \quad \mathbb{R}^n, \tag{1.3}$$

where $\Phi$ is a nonnegative function. The authors derive Caccioppoli–type estimate for nonnegative weak solutions to (1.3). Then, they obtain more specified a priori estimates involving general test functions and finally, choosing appropriate test functions, they obtain nonexistence.

In [44] the Caccioppoli–type estimate is considered in the case when (1.3) is

$$-\Delta_p u \geq \Phi \quad \text{in} \quad \Omega, \tag{1.4}$$

with a locally integrable function $\Phi$ (see Theorem 5.1), satisfying the following condition

$$\sigma_0 := \inf \{\sigma \in \mathbb{R} : \Phi \cdot u + \sigma |\nabla u|^p \geq 0 \quad \text{a.e. in } \Omega \cap \{u > 0\} \ \} \in \mathbb{R}.$$

The derived estimates violate some assumptions from [37]. The certain substitution in the derived Caccioppoli–type inequality for solutions implies family of Hardy–type inequalities having the form

$$\int_\Omega |\xi|^p \mu_{1,\beta}(dx) \leq \int_\Omega |\nabla \xi|^p \mu_{2,\beta}(dx),$$

where $1 < p < \infty$, $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, and $\Omega$ is an open subset of $\mathbb{R}^n$. The involved measures $\mu_{1,\beta}(dx), \mu_{2,\beta}(dx)$ depend on a certain parameter $\beta$ and on $u$ — a nonnegative weak solution to (1.4).

In Section 6 we supply a few new examples of weighted power–logarithm Hardy–Sobolev inequalities of a type

$$\int_\Omega |\xi|^p \log^{-\alpha}(2+1/|\xi|)\mu_1(dx) \le \widetilde{C} \int_\Omega |\nabla\xi|^p \log^{\alpha}(2+|\nabla\xi|)\mu_2(dx)$$

for compactly supported Lipschitz functions $\xi$.

# 2 Preliminaries

**Notation**

In the sequel we assume that $\Omega \subseteq \mathbb{R}^n$ is an open subset not necessarily bounded.

By $A$–harmonic problems we understand those, which involve $A$–Laplace operator $\Delta_A u = \mathrm{div}(A(\nabla u))$, understood in the weak sense, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$–function. Choosing $A(\lambda) = |\lambda|^{p-2}\lambda$ we deal with the usual $p$–Laplacian.

We restrict ourselves to $A$'s such that $A(\lambda) = B(|\lambda|)\lambda,\ \lambda \in \mathbb{R}^n$, and we set

$$\bar{A}(s) = B(s)s^2, \quad \text{where } s \in [0,\infty). \tag{2.1}$$

We assume that $\bar{A}$ is an $N$–function, i.e. it is convex and $\lim_{s\to 0} \frac{\bar{A}(s)}{s} = \lim_{s\to\infty} \frac{s}{\bar{A}(s)} = 0$. We refer to the monographs [38, 42] for basic properties of Orlicz spaces. By $\bar{A}^*$ we denote the Legendre transform of $\bar{A}$, e.i. $\bar{A}^* = \sup_{t>0}(st - \bar{A}(t))$.

As usual, $C^k(\Omega)$ (respectively $C_0^k(\Omega)$) denotes functions of class $C^k$ defined on an open set $\Omega \subset \mathbb{R}^n$ (respectively $C^k$-functions on $\Omega$ with compact support). If $f$ is defined on $\Omega$, by $f\chi_\Omega$ we understand function $f$ extended by 0 outside $\Omega$. When $V \subseteq \mathbb{R}^n$, by $|V|$ we denote its Lebesgue's measure.

We deal with $\Delta_2$ and $\Delta'$ conditions defined below.

**Definition 2.1.** *We say that the function $F : [0,\infty) \to [0,\infty)$ satisfies the $\Delta_2$–condition (denoted $F \in \Delta_2$), if there exists a constant $\bar{C}_F > 0$ such that for every $s > 0$ we have*

$$F(2s) \le \bar{C}_F F(s). \tag{2.2}$$

**Definition 2.2.** *We say that the function $F : [0,\infty) \to [0,\infty)$ satisfies the $\Delta'$–condition (denoted $F \in \Delta'$), if there exists a constant $C_F > 0$ such that for every $s_1, s_2 > 0$ we have*

$$F(s_1 s_2) \le C_F F(s_1)F(s_2). \tag{2.3}$$

**Remark 2.1.** Let us note that the $\Delta'$–condition is stronger than the $\Delta_2$–condition.

Typical examples of $N$–functions satisfying the $\Delta'$–condition can be found among Zygmund–type logarithmic functions. Their construction is based on the following easy observation.

**Fact 2.1** ([29])**.** *The family of functions satisfying $\Delta'$–condition is invariant under multiplications and compositions.*

**Example 2.1** ([29])**.** The following $N$–functions satisfy $\Delta'$–condition:
1. $F(s) = s^p$, $1 < p < \infty$,
2. $M_{p,\alpha}(s) = s^p(\ln(2+s))^\alpha$, $1 < p < \infty$, $\alpha \geq 0$,
3. $M^1_{p,\alpha}(s) = s^p(\ln(1+s))^\alpha$, $1 < p < \infty$, $\alpha \geq 0$,
4. $F(s) = M_{p_1,\alpha_1} \circ M_{p_2,\alpha_2} \circ \cdots \circ M_{p_k,\alpha_k}(s)$, $\alpha_1, \ldots, \alpha_k \geq 0$, $p_i > 1$ for $i = 1, \ldots, k$.

**Fact 2.2.** *Let $F_b(s) = s^p \log^\alpha(b+s)$, $b, p > 1$, $\alpha > 0$. Then, the constant from $\Delta'$–condition (see Definition 2.2), $C_F \leq \left(\frac{2}{\log b}\right)^\alpha$.*

*Proof.* Suppose $s_1 \leq s_2$. Then

$$\log(b + s_1 s_2) \leq \log(b + s_2^2) \leq \log(b + s_2)^2 = 2\log(b + s_2) \leq 2\log(b + s_2) \cdot \frac{\log(b + s_1)}{\log b},$$

and $F(s_1 s_2) = (s_1 s_2)^p \log^\alpha(b + s_1 s_2) \leq \left(\frac{2}{\log b}\right)^\alpha s_1^p s_2^p \log^\alpha(b + s_1) \log^\alpha(b + s_2) = C_F F(s_1) F(s_2)$. □

Let us state some useful facts and lemmas.

**Lemma 2.1** ([31], Lemma 4.2)**.** *Suppose that $F$ is a differentiable $N$–function satisfying $\Delta_2$–condition. Then there exists constants $1 < d_F \leq D_F$, such that for every $r > 0$*

$$d_F \frac{F(r)}{r} \leq F'(r) \leq D_F \frac{F(r)}{r}. \tag{2.4}$$

*Moreover, for every $r, s > 0$ the following estimate holds true*

$$\frac{F(r)}{r} s \leq \frac{D_F - 1}{d_F} F(r) + \frac{1}{d_F} F(s). \tag{2.5}$$

**Remark 2.2.** Let us comment above lemma.

1. When $F(r) = r^p$, $\frac{1}{p} + \frac{1}{p'} = 1$, we get $r^{p-1}s \leq \frac{1}{p'} r^p + \frac{1}{p} s^p$, equivalent to Young inequality $qs \leq \frac{q^{p'}}{p'} + \frac{s^p}{p}$.

2. For general convex function $F$ the latter inequality in (2.4) with finite constant $D_F$ is equivalent to $F \in \Delta_2$, while the condition $d_F > 1$ is equivalent to $F^* \in \Delta_2$ (see [38], Theorem 4.3 or [30], Proposition 4.1). If $d_F$ and $D_F$ are the best possible in (2.4), they are called Simonenko lower and upper index of $F$, respectively (see e.g. [6, 20, 24, 43]) for definition and discussion of properties.

**Fact 2.3.** *Let* $F(s) = s^p \log^\alpha(b+s)$, $b, p > 1$, $\alpha > 0$. *Then, the constants from* (2.4), *equals* $D_F = p + \frac{\alpha}{\log b}$ *and* $d_F = p$.

*Proof.* $F'(s) = (s^p \log^\alpha(b+s))' = ps^{p-1}\log^\alpha(b+s) + \alpha\frac{s^p}{b+s}\log^{\alpha-1}(b+s) =$
$s^{p-1}\log^\alpha(b+s)\left(p + \alpha\frac{s}{(b+s)\log(b+s)}\right) \le D_F\frac{F(s)}{s}$, with $D_F = \sup\left(p + \alpha\frac{s}{(b+s)\log(b+s)}\right)$.
$F'(s) \ge d_F\frac{F(s)}{s}$, with $d_F = \inf\left(p + \alpha\frac{s}{(b+s)\log(b+s)}\right)$. □

**Orlicz—Sobolev spaces**
By $W^{1,\bar{A}}(\Omega)$ we mean the completion of the set

$$\{u \in C^\infty(\Omega) : \|u\|_{W^{1,\bar{A}}(\Omega)} := \|u\|_{L^{\bar{A}}(\Omega)} + \|\nabla u\|_{L^{\bar{A}}(\Omega)} < \infty\},$$

under the Luxemburg norm

$$\|f\|_{L^{\bar{A}}(\Omega)} = \inf\left\{K > 0 : \int_\Omega \bar{A}\left(\frac{|f(x)|}{K}\right)dx \le 1\right\}$$

(in the sequel we assume that $\inf\emptyset = +\infty$). By $W^{1,\bar{A}}_{loc}(\Omega)$ we denote such functions $u : \Omega \to \mathbb{R}$ that $u\phi \in W^{1,\bar{A}}(\Omega)$ for every $\phi \in C_0^1(\Omega)$ (analogous notation is used for local Orlicz spaces $L^{\bar{A}}_{loc}(\Omega)$). Observe that we always have $W^{1,\bar{A}}_{loc}(\Omega) \subseteq W^{1,1}_{loc}(\Omega)$. By $W_0^{1,\bar{A}}(\Omega)$ we denote the completion of smooth compactly supported functions in $W^{1,\bar{A}}(\Omega)$.
The following fact holds true.

**Fact 2.4** ([37], Fact 2.3)**.** *If* $\bar{A}$ *is an N–function and* $u \in W^{1,\bar{A}}_{loc}(\Omega)$, *then*

$$B(|\nabla u|)\nabla u = \frac{\bar{A}(|\nabla u|)}{|\nabla u|}\chi_{\{|\nabla u|\neq 0\}} \in L^{\bar{A}^*}_{loc}(\Omega, \mathbb{R}^n),$$

*where* $B$ *and* $\bar{A}$ *are the same as in (2.1).*

Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$. For $w \in W^{1,\bar{A}}(\Omega)$ with compact support we define

$$\langle \Delta_A u, w\rangle := -\int_\Omega B(|\nabla u|)\langle \nabla u, \nabla w\rangle\, dx. \tag{2.6}$$

According to Fact 2.4 the right–hand side in (2.6) is well defined. Obviously when $A(\lambda) = |\lambda|^{p-2}\lambda$, then we retrieve the classical $p$–Laplacian, $\Delta_p u$.
Having an arbitrary $u \in L^1_{loc}(\Omega)$ it is possible to define its value at every point by the formula:

$$u(x) := \limsup_{r\to 0} \fint_{B(x,r)\cap\Omega} u(y)dy. \tag{2.7}$$

**Differential inequality**
The differential inequality we want to analyze is given by the following definition.

**Definition 2.3.** *Let $\Omega$ be any open subset of $\mathbb{R}^n$ and $\Phi$ be the locally integrable function defined in $\Omega$, such that for every nonnegative compactly supported $w \in W^{1,\bar{A}}(\Omega)$*

$$\left| \int_\Omega \Phi w \, dx \right| < \infty. \tag{2.8}$$

*Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$. We will say that*

$$-\Delta_A u \geq \Phi \tag{2.9}$$

*if for every nonnegative compactly supported $w \in W^{1,\bar{A}}(\Omega)$ we have*

$$\langle -\Delta_A u, w \rangle = \int_\Omega B(|\nabla u|) \langle \nabla u, \nabla w \rangle \, dx \geq \int_\Omega \Phi w \, dx. \tag{2.10}$$

**Remark 2.3.** We may choose $\Phi = \Phi(x, u, \nabla u)$.

**Set of assumptions.** In the sequel we will consider functions satisfying the following assumptions.

($\bar{A}$) $\bar{A}$ is an $N$–function satisfying $\Delta'$–condition;

($\Psi$) there exists a function $\Psi : [0, \infty) \to [0, \infty)$, which is nonnegative and belongs to $C^1((0, \infty))$ and satisfies the following conditions

**i)** inequality

$$g(t)\Psi^{'}(t) \leq -C\Psi(t) \tag{2.11}$$

holds for all $t > 0$ with $C > 0$ independent of $t$ and certain continuous function $g : (0, \infty) \to (0, \infty)$, such that $\Psi(t)/g(t)$ is nonincreasing.

**ii)** function

$$s \mapsto \Theta(s) := \frac{\bar{A}\,(g(s))\, \Psi(s)}{g(s)} \tag{2.12}$$

is nonincreasing or bounded in certain neighbourhood of 0.

($u$) $u \in W^{1,\bar{A}}_{loc}(\Omega)$ is a given nonnegative solution to (2.9) which is nontrivial, i.e. $u \not\equiv const$, and there exists $\sigma \in \mathbb{R}$ such that

$$\Phi + \sigma \, \frac{\bar{A}(|\nabla u|)}{g(u)} \chi_{\{\nabla u \neq 0\}} \geq 0 \quad \text{a.e.} \tag{2.13}$$

We define

$$\sigma_0 = \inf\{\sigma \in \mathbb{R} : (2.13) \text{ is satisfied}\}, \tag{2.14}$$

where we set $\inf \emptyset = +\infty$.

| $\Psi(t)$ | g(t) | C | remarks |
|---|---|---|---|
| $t^{-\alpha}$ | $t$ | $\alpha$ | $\alpha > 0$ |
| $e^{-t}$ | bounded by $C$, $g' \geq -C$ | $C$ | $C > 0$ |
| $e^{-t}/t$ | $t/(1+t)$ | 1 | — |
| $e^{\frac{1}{2}\log^2(t)}$ | $t/\lvert\log t\rvert$ | 1 | considered on $(0,1)$ |

Table 1: Good pairs of $\Psi$ and $g$

**Remark 2.4.** Examples when those conditions are satisfied in the case when $\bar{A}(s) = s^p$, $g(s) = s,\ \Psi(s) = s^{-\beta}, \beta > 0$ can be found in [44, 45].

**Remark 2.5.** Let us discuss the assumption $(\Psi)$ **i)**. In particular, it implies that $\Psi$ is decreasing. Elementary calculation leads to following pairs of $\Psi$ and $g$ satisfying condition $g(t)\Psi'(t) \leq -C\Psi(t)$ a.e. To ensure that additionally $\Psi(t)/g(t)$ is nonincreasing we have assume that $g'(t) \geq -C$ with th same $C$. Indeed, $\Psi/g$ is nonincreasing if

$$\left(\frac{\Psi(t)}{g(t)}\right)' = \tfrac{\Psi'(t)g(t)-\Psi(t)g'(t)}{g^2(t)} \leq \frac{-C\Psi(t) - \Psi(t)g'(t)}{g^2(t)} = = -\tfrac{\Psi(t)}{g^2(t)}(C + g'(t)) \leq 0$$

I.e.: when $g'(t) \geq -C$.

The following pairs satisfy assumption $(\Psi)$ (see Table 1).

# 3 Caccioppoli estimates for solutions to $-\Delta_A u \geq \Phi$

Our main goal in this section is to obtain the following result.

**Theorem 3.1.** *Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 2.3, where $\Phi$ is locally integrable and assumptions $(\bar{A})$, $(\Psi)$, $(u)$ are satisfied satisfied with $C > 0$ and $\sigma \in [\sigma_0, C)$, where $\sigma_0$ is given by (2.14). Let $C_{\bar{A}} > 0$ be a constant coming from $\Delta'$–condition for $\bar{A}$ (see Definition 2.2) and $D_{\bar{A}} \geq d_{\bar{A}} > 1$ be constants coming from (2.4) applied to $\bar{A}$.*

*Then the inequality*

$$\int_\Omega \left(\Phi + \sigma\frac{\bar{A}(|\nabla u|)}{g(u)}\chi_{\{\nabla u \neq 0\}}\right)\Psi(u)\phi\,dx \leq \tag{3.1}$$

$$\leq K \int_{\Omega\cap\{\nabla u\neq 0\}} \frac{\bar{A}\left(g(u)\right)\Psi(u)}{g(u)} \cdot \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx,$$

*holds for every nonnegative Lipschitz function $\phi$ with compact support in $\Omega$, such that the integral $\int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0\}} \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx$ is finite and $K = (C-\sigma)\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)\frac{C^2_{\bar{A}}}{D_{\bar{A}}-1}$.*

We call (3.1) Caccioppoli inequality, because it involves $\nabla u$ on the left–hand side and only $u$ on the right–hand side (see e.g. [11, 28]).

The proof is based on careful analysis of the proof of Proposition 3.1 from [37]. However, here we are not restricted to $\Phi = \Phi(u)$, $\Phi \geq 0$ and integrals over $\mathbb{R}^n$.

**Remark 3.1.** We do not assume that right–hand side in (3.1) is finite.

***Proof of Theorem 3.1***. The proof follows by three steps.

STEP 1. DERIVATION OF LOCAL INEQUALITY.

We obtain the following lemma.

**Lemma 3.1.** *Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 2.3, where $\Phi$ is locally integrable and assumptions $(\bar{A})$, $(\Psi)$, $(u)$ are satisfied satisfied with $C > 0$ and $\sigma \in [\sigma_0, C)$, where $\sigma_0$ is given by (2.14). Let $K$ be the constant from Theorem 3.1.*

*Then for every $0 < \delta < R$ and every nonnegative Lipschitz function $\phi$ with compact support in $\Omega$, the inequality*

$$\int_{\{u\leq R-\delta\}} \left(\Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u+\delta)} \chi_{\{\nabla u \neq 0\}}\right) \Psi(u+\delta)\phi \; dx$$

$$\leq K \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \Theta(u+\delta)\cdot \bar{A}\left(\frac{|\nabla \phi|}{\phi}\right)\phi \; dx + \tilde{C}(\delta, R), \tag{3.2}$$

*holds with $\Theta(u)$ given by (2.12) and*

$$\tilde{C}(\delta, R) := \Psi(R)\left[\int_{\Omega\cap\{\nabla u\neq 0,\, u> R-\delta\}} B(|\nabla u|)\langle \nabla u, \nabla \phi\rangle \, dx - \int_{\Omega\cap\{u>R-\delta\}} \Phi\phi \, dx\right]. \tag{3.3}$$

Before we prove the theorem let us formulate the following facts.

**Fact 3.1** ([37])**.** *For $u, \phi$ as in the assumptions of Theorem 3.1 we fix $0 < \delta < R$ and denote*

$$u_{\delta,R}(x) := \min\left(u(x)+\delta, R\right), \qquad G(x) := \Psi(u_{\delta,R}(x))\phi(x). \tag{3.4}$$

*Then $u_{\delta,R} \in W^{1,\bar{A}}_{loc}(\Omega)$ and $G \in W^{1,\bar{A}}_0(\Omega) \subseteq W^{1,\bar{A}}(\Omega)$.*

**Fact 3.2** ([37])**.** *Let $u \in W^{1,1}_{loc}(\Omega)$ be defined everywhere by the formula (2.7) and let $t \in \mathbb{R}$. Then*

$$\{x \in \Omega : u(x) = t\} \subseteq \{x \in \Omega : \nabla u(x) = 0\} \cup N \tag{3.5}$$

*where $|N| = 0$.*

***Proof of Lemma 3.1.*** According to (2.8) integral $\int_\Omega \Phi\phi dx$ is finite. Before we start the proof of (3.2), let us introduce some notation, where $0 < \delta < R < \infty$:

$$\begin{aligned}
\tilde{A}(\delta, R) &= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\Psi'(u+\delta)\phi\, dx, \\
\tilde{A}_1(\delta, R) &= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\left(\frac{\Psi(u+\delta)}{g(u+\delta)}\right)\phi\, dx, \\
\tilde{B}(\delta, R) &= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} B(|\nabla u|)\langle\nabla u, \nabla\phi\rangle\Psi(u+\delta)\, dx, \\
\tilde{C}_1(\delta, R) &= \Psi(R)\int_{\Omega\cap\{u>R-\delta\}} \Phi\phi\, dx, \qquad (3.6)\\
\tilde{C}_2(\delta, R) &= \Psi(R)\int_{\Omega\cap\{\nabla u\neq 0,\, u>R-\delta\}} B(|\nabla u|)\langle\nabla u, \nabla\phi\rangle\, dx, \qquad (3.7)\\
\tilde{D}(\bar{\epsilon}, \delta, R) &= \bar{\epsilon}\bar{A}\left(\frac{1}{\bar{\epsilon}}\right)\frac{C_{\bar{A}}^2}{d_{\bar{A}}}\int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \Theta(u+\delta)\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx,
\end{aligned}$$

where $\Theta(u)$ is given by (2.12). Let us consider $u_{\delta,R}$ and $G$ defined by (3.4).

We note that

$$\begin{aligned}
I &:= \int_\Omega \Phi G\, dx = \int_\Omega \Phi\Psi(u_{\delta,R})\phi\, dx = \\
&= \int_{\Omega\cap\{u\leq R-\delta\}} \Phi\Psi(u+\delta)\phi\, dx + \Psi(R)\int_{\Omega\cap\{u>R-\delta\}} \Phi\phi\, dx = \\
&= \int_{\Omega\cap\{u\leq R-\delta\}} \Phi\Psi(u+\delta)\phi\, dx + \tilde{C}_1(\delta, R), \qquad (3.8)
\end{aligned}$$

On the other hand, inequality (2.9) implies

$$\begin{aligned}
I &:= \int_\Omega \Phi G\, dx \leq \langle -\Delta_A u, G\rangle = \int_{\Omega\cap\{\nabla u\neq 0\}} B(|\nabla u|)\langle\nabla u, \nabla G\rangle\, dx = \\
&= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\Psi'(u+\delta)\phi\, dx + \\
&+ \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} B(|\nabla u|)\langle\nabla u, \nabla\phi\rangle\Psi(u+\delta)\, dx + \\
&+ \Psi(R)\int_{\Omega\cap\{\nabla u\neq 0,\, u>R-\delta\}} B(|\nabla u|)\langle\nabla u, \nabla\phi\rangle\, dx = \\
&= \tilde{A}(\delta, R) + \tilde{B}(\delta, R) + \tilde{C}_2(\delta, R). \qquad (3.9)
\end{aligned}$$

Note that all integrals above are finite, what follows from Fact 2.4 (for $0 \leq u \leq R-\delta$ we have $\delta \leq u+\delta \leq R$). Using assumption ($\Psi$) we get

$$\tilde{A}(\delta, R) \leq -C\int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\left(\frac{\Psi(u+\delta)}{g(u+\delta)}\right)\phi\, dx = -C\tilde{A}_1(\delta, R). \qquad (3.10)$$

Moreover, for an arbitrary $\bar\epsilon > 0$,

$$\begin{aligned}\tilde{B}(\delta,R) &\le \int_{\Omega\cap\{\nabla u\neq 0,\, u\le R-\delta\}} B(|\nabla u|)|\nabla u||\nabla\phi|\Psi(u+\delta)\,dx = \\ &= \bar\epsilon\int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\le R-\delta\}} (B(|\nabla u|)|\nabla u|)\cdot\left(\frac{|\nabla\phi|}{\phi}\frac{g(u+\delta)}{\bar\epsilon}\right)\left(\frac{\Psi(u+\delta)}{g(u+\delta)}\phi\right)dx.\end{aligned}$$

As $B(|\nabla u|)|\nabla u| = \frac{\bar{A}(|\nabla u|)}{|\nabla u|}$, we can apply (2.5) for the $N$–function $\bar{A}$ with $r = |\nabla u|$, $s = \left(\frac{|\nabla\phi|}{\phi}\frac{g(u+\delta)}{\bar\epsilon}\right)$ to get

$$\begin{aligned}\tilde{B}(\delta,R) \le\; & \bar\epsilon\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\le R-\delta\}}\bar{A}(|\nabla u|)\frac{\Psi(u+\delta)}{g(u+\delta)}\phi\,dx + \\ &+ \frac{\bar\epsilon}{d_{\bar{A}}}\int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\le R-\delta\}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\frac{g(u+\delta)}{\bar\epsilon}\right)\frac{\Psi(u+\delta)}{g(u+\delta)}\phi\,dx.\end{aligned}$$

Then, applying $\Delta'$–condition for $\bar{A}$ twice in the second expression above, we obtain

$$\tilde{B}(\delta,R) \le \bar\epsilon\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\tilde{A}_1(\delta,R) + \tilde{D}(\bar\epsilon,\delta,R). \tag{3.11}$$

Combining estimates (3.9), (3.10) and (3.11) we get

$$\begin{aligned} I &\le -C\tilde{A}_1(\delta,R) + \tilde{B}(\delta,R) + \tilde{C}_2(\delta,R) \le \\ &\le \left(-C + \bar\epsilon\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\right)\tilde{A}_1(\delta,R) + \tilde{D}(\bar\epsilon,\delta,R) + \tilde{C}_2(\delta,R).\end{aligned}$$

Moreover, $\tilde{C}_1(\delta,R)$ and $\tilde{A}_1(\delta,R)$ are finite (and $\tilde{D}(\epsilon,\delta,R)$ is finite as well). This and (3.8) imply

$$\begin{aligned}&\int_{\Omega\cap\{u\le R-\delta\}}\Phi\Psi(u+\delta)\phi\,dx + \left(C - \bar\epsilon\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\right)\tilde{A}_1(\delta,R) \le \\ &\qquad \le \tilde{D}(\bar\epsilon,\delta,R) + (\tilde{C}_2(\delta,R) - \tilde{C}_1(\delta,R)).\end{aligned}$$

This is (3.2). Indeed, we have $\tilde{C}(\delta,R) = \tilde{C}_2(\delta,R) - \tilde{C}_1(\delta,R)$. Moreover, when we substitute $\sigma := C - \bar\epsilon\frac{D_{\bar{A}}-1}{d_{\bar{A}}}$ we get

$$\bar\epsilon\bar{A}\left(\frac{1}{\bar\epsilon}\right)\frac{C_{\bar{A}}^2}{d_{\bar{A}}} = \frac{(C-\sigma)d_{\bar{A}}}{D_{\bar{A}}-1}\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)\frac{C_{\bar{A}}^2}{d_{\bar{A}}} = \frac{(C-\sigma)}{D_{\bar{A}}-1}\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)C_{\bar{A}}^2 = K.$$

We notice that $\bar\epsilon > 0$ is arbitrary and we may always choose $0 < \bar\epsilon \le \frac{(C-\sigma_0)d_{\bar{A}}}{D_{\bar{A}}-1}$, so that $\sigma_0 \le \sigma < C$. □

We have to introduce parameters $\delta$ and $R$ to make sure that some quantities in the estimates, which we move to opposite sides of inequalities, are finite.

STEP 2. PASSING TO THE LIMIT WITH $\delta \searrow 0$.

In this step we show that when assumptions $(\bar{A})$, $(\Psi)$ and $(\Phi)$ are satisfied with $\epsilon > 0$, $K$ is the constant from Theorem 3.1, then for any $R > 0$ inequality

$$\int_{\{u \le R\}} \left( \Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)} \chi_{\{\nabla u \neq 0\}} \right) \Psi(u) \phi \, dx \le$$

$$\le K \int_{\{\nabla u \neq 0,\, u \le R\}} \frac{\bar{A}\left(g(u)\right) \Psi(u)}{g(u)} \bar{A}\left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx + \tilde{C}(R), \tag{3.12}$$

where

$$\tilde{C}(R) = \Psi(R) \left[ \left| \int_{\Omega \cap \{u \ge \frac{R}{2}\}} B(|\nabla u|) |\nabla u| \cdot |\nabla \phi| \, dx \right| + \left| \int_{\Omega \cap \{u \ge \frac{R}{2}\}} \Phi \phi \, dx \right| \right] \tag{3.13}$$

holds for every nonnegative Lipschitz function $\phi$ with compact support in $\Omega$, such that the integral $\int_{\mathrm{supp}\,\phi \cap \nabla u \neq 0} \bar{A}\left(\frac{|\nabla \phi|}{\phi}\right) \phi \, dx$ is finite. Moreover, all quantities appearing in (3.12) are finite.

For this, we show first that under our assumptions, when $\delta \searrow 0$ we have

$$\int_{\Omega \cap \{\nabla u \neq 0,\, u + \delta \le R\}} \Theta(u + \delta) \cdot \bar{A}\left( \frac{|\nabla \phi|}{\phi} \right) \phi \; dx \to \int_{\Omega \cap \{\nabla u \neq 0,\, u \le R\}} \Theta(u) \cdot \bar{A}\left( \frac{|\nabla \phi|}{\phi} \right) \phi \; dx. \tag{3.14}$$

Note that $\Theta(u + \delta)\chi_{u+\delta \le R} \overset{\delta \to 0}{\to} \Theta(u)\chi_{u \le R}$, a.e. This follows from Lemma 3.2 (which gives that the sets $\{u = 0,\ |\nabla u| \neq 0\}$ and $\{u = R,\ |\nabla u| = 0\}$ are of measure zero) and the continuity outside zero of the involved functions.

We assumed in $(\Theta)$ that $\Theta$ is nonincreasing or bounded in the neighbourhood of zero. Let we start with the case when there exists $\kappa > 0$ such that for $\lambda < \kappa$ the function $\Theta(\lambda)$ is nonincreasing. Without loss of generality we may consider $\kappa \le R$.

We divide the domain of integration

$$\int_{\Omega \cap \{\nabla u \neq 0,\, u + \delta \le R\}} \Theta(u + \delta) \cdot \bar{A}\left( \frac{|\nabla \phi|}{\phi} \right) \phi \; dx =$$

$$= \int_{E_\kappa} \Theta(u + \delta) \cdot \bar{A}\left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx + \int_{F_\kappa} \Theta(u + \delta) \chi_{\{u + \delta \le R\}} \cdot \bar{A}\left( \frac{|\nabla \phi|}{\phi} \right) \phi \, dx,$$

where

$$E_\kappa = \left\{ u < \frac{\kappa}{2},\ \nabla u \neq 0 \right\} \cap \mathrm{supp}\,\phi, \quad F_\kappa = \left\{ \frac{\kappa}{2} \le u,\ \nabla u \neq 0 \right\} \cap \mathrm{supp}\,\phi.$$

Let us begin with integral over $E_\kappa$. We consider $\delta \to 0$, so we may assume that $\delta < \kappa/2$. Then for $x \in E_\kappa$ we have $u + \delta < \kappa$. As function $\lambda \to \Theta(\lambda)$ is nonincreasing when $\lambda < \kappa$, thus for $\delta \searrow 0$ the function $\delta \to \Theta(u+\delta)$ is nondecreasing and so convergent monotonically almost everywhere to $\Theta(u)$. Therefore, due to The Lebesgue's Monotone Convergence Theorem

$$\lim_{\delta \to 0} \int_{E_\kappa} \Theta(u+\delta)\bar{A}\left(\frac{|\nabla \phi|}{\phi}\right)\phi\, dx = \int_{E_\kappa} \Theta(u)\bar{A}\left(\frac{|\nabla \phi|}{\phi}\right)\phi\, dx.$$

In the case of $F_\kappa$, we have $\kappa/2 \le u+\delta \le R$. Over this domain $\Theta$ is a bounded function, so in particular on $F_\kappa$:

$$\Theta(u+\delta)\chi_{\{u+\delta \le R\}}\bar{A}\left(\frac{|\nabla \phi|}{\phi}\right)\phi \le \sup_{t\in[\kappa/2,R]} \Theta(t)\cdot \bar{A}\left(\frac{|\nabla \phi|}{\phi}\right)\phi \in L^1(F_\kappa).$$

We apply The Lebesgue's Dominated Convergence Theorem to deduce that

$$\lim_{\delta \to 0} \int_{F_\kappa} \Theta(u+\delta)\chi_{\{u+\delta \le R\}}\bar{A}\left(\frac{|\nabla \phi|}{\phi}\right)\phi\, dx = \int_{F_\kappa \cap \{u \le R\}} \Theta(u)\bar{A}\left(\frac{|\nabla \phi|}{\phi}\right)\phi\, dx.$$

This completes the case of $\Theta$ nonincreasing in the neighbourhood of 0.
In the case when $\Theta$ is bounded in the neighbourhood of 0, we note that $\Theta$ is bounded on every interval $[0, R]$, where $R > 0$. Hence, we can use previous computations dealing with $F_\kappa$ in case $\kappa = 0$.

To finish the proof of this step we note that (3.14) says that when $\delta \searrow 0$ the first integral on the right–hand side of (3.2) is convergent to the first integral of right–hand side of (3.12). To deal with the second expression we note that for $\delta \le \frac{R}{2}$ :

$$|\tilde{C}(\delta, R)| \quad\le\quad |\tilde{C}_2(\delta, R)| + |\tilde{C}_1(\delta, R)| \le \tilde{C}(R),$$

where $\tilde{C}(\delta, R)$, $\tilde{C}_2(\delta, R)$, $\tilde{C}_1(\delta, R)$, $\tilde{C}(R)$ are given by (3.3), (3.6), (3.7), (3.13), respectively.

We can pass to the limit with $\delta \to 0$ on the left–hand side of (3.2) due to The Lebesgue's Monotone Convergence Theorem as an expression in brackets is nonnegative by (2.13) and the whole integrand therein is nonincreasing by assumption $(\Psi)$.

Step 3. We let $R \to \infty$ and finish the proof.

We are going to let $R \to \infty$ in (3.12). Without loss of generality we can assume that the integral in the right–hand side of (3.1) is finite, as otherwise the inequality follows trivially. Note that as $B(|\nabla u|)\langle \nabla u, \nabla \phi\rangle$ and $\Phi\phi$ are integrable, we have $\lim_{R\to\infty} \widetilde{C}(R) = 0$. Therefore (3.1) follows from (3.12) by the Lebesgue's Monotone Convergence Theorem. □

# 4 Hardy type inequalities

Our most general conclusion resulting from Theorem 3.1 reads as follows.

**Theorem 4.1.** *Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 2.3, where $\Phi$ is locally integrable and assumptions $(\bar{A})$, $(\Psi)$, $(u)$ are satisfied with $C > 0$ and $\sigma \in [\sigma_0, C)$, where $\sigma_0$ is given by (2.14). Set*

$$F_{\bar{A}}(\lambda) = \frac{1}{\bar{A}\,(1/\lambda)}, \text{ when } \lambda > 0 \text{ and } F_{\bar{A}}(0) = 0. \tag{4.1}$$

*Then for every Lipschitz function $\xi$ with compact support in $\Omega$, we have*

$$\int_\Omega F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \widetilde{C} \int_\Omega \bar{A}(|\nabla \xi|)\mu_2(dx). \tag{4.2}$$

*where*

$$\mu_1(dx) = \Psi(u)\left[\Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)}\right] \chi_{\{u>0\}}\, dx, \tag{4.3}$$

$$\mu_2(dx) = \frac{\bar{A}\,(g(u))\,\Psi(u)}{g(u)} \chi_{\{\nabla u \neq 0\}}\, dx, \tag{4.4}$$

$$\widetilde{C} = (C-\sigma)\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)\frac{\bar{A}(D_{\bar{A}})C^4_{\bar{A}}}{D_{\bar{A}}-1}. \tag{4.5}$$

*with constants $C_{\bar{A}} > 0$ coming from $\Delta'$–condition for $\bar{A}$ (see Definition 2.2) and $D_{\bar{A}} > d_{\bar{A}} \geq 1$ coming from (2.4) applied to $\bar{A}$.*

***Proof.*** Let $\xi$ be a compactly supported Lipschitz function. We define $\phi = F_{\bar{A}}(\xi)$ and apply Theorem 3.1. For this we have to verify that $\phi$ is compactly supported Lipschitz function and $\int_\Omega \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx < \infty$. We observe that $\phi$ is compactly supported, because $F_{\bar{A}}(t)$ is continuous at 0. Indeed,

$$\lim_{t\to 0} F_{\bar{A}}(t) = \lim_{t\to 0}\frac{1}{\bar{A}\,(1/t)} = \lim_{s\to\infty}\frac{1}{\bar{A}\,(s)} = 0,$$

which ensures that $\operatorname{supp}\phi = \operatorname{supp}\xi$. Furthermore, $F_{\bar{A}}(t)$ is a locally Lipschitz function. We obtain it from Lemma 2.1 which implies

$$F'_{\bar{A}}(t) = \left(\frac{1}{\bar{A}\,(1/t)}\right)' \sim \frac{1}{t\bar{A}(1/t)}.$$

Applying the condition $\lim_{s\to\infty}\frac{s}{\bar{A}(s)} = 0$ from definition of $N$–function, we get that $F'_{\bar{A}}(t)$ is a locally bounded function and bounded nearby 0. Therefore, $F_{\bar{A}}(t)$ is locally Lipshitz. The composition of locally Lipshitz function $F_{\bar{A}}(t)$ with Lipschitz and bounded $\xi$, i.e. $F_{\bar{A}}(\xi) = \phi$, is Lipschitz.

We verify that $\int_\Omega \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx < \infty$. Note that for every compactly supported Lipschitz function $\xi$ we have $\int_\Omega \bar{A}(|\nabla\xi|)\,dx < \infty$. Therefore, it suffices to prove that

$$\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \le C_{\bar{A}}^2\bar{A}(D_{\bar{A}})\bar{A}(|\nabla\xi|). \tag{4.6}$$

As $\bar{A} \in \Delta'$, we note that for each pair of $x, y \ge 0$ we have

$$\begin{aligned}\bar{A}(x)y &= \bar{A}\left(\frac{x}{\bar{A}^{-1}(\frac{1}{y})}\bar{A}^{-1}\left(\frac{1}{y}\right)\right)y \le \\ &\le C_{\bar{A}}\bar{A}\left(\frac{x}{\bar{A}^{-1}(\frac{1}{y})}\right)\bar{A}\left(\bar{A}^{-1}\left(\frac{1}{y}\right)\right)y = C_{\bar{A}}\bar{A}\left(\frac{x}{\bar{A}^{-1}(\frac{1}{y})}\right).\end{aligned} \tag{4.7}$$

Hence, taking $x = \frac{|\nabla\phi|}{\phi}$ and $y = \phi$, we obtain from (4.7)

$$\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \le C_{\bar{A}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\frac{1}{\bar{A}^{-1}(\frac{1}{\phi})}\right), \tag{4.8}$$

for any nonnegative $\phi$ at every $x$ where $\phi(x) > 0$.

Now we show that at every $x$, where $\phi(x) > 0$ we have

$$\frac{|\nabla\phi(x)|}{\phi(x)}\frac{1}{\bar{A}^{-1}\left(\frac{1}{\phi(x)}\right)} \le D_{\bar{A}}|\nabla\xi(x)|. \tag{4.9}$$

Indeed, we have $\phi = \frac{1}{\bar{A}(\frac{1}{\xi})}$, so that

$$\nabla\phi = F_{\bar{A}}'(\xi) = -\frac{1}{\bar{A}^2\left(\frac{1}{\xi}\right)}\bar{A}'\left(\frac{1}{\xi}\right)\left(-\frac{1}{\xi^2}\right)\nabla\xi.$$

Applying (2.4) to $\bar{A} \in \Delta_2$ we have $\bar{A}'(\lambda) \le D_{\bar{A}}\frac{\bar{A}(\lambda)}{\lambda}$, with the constant $D_{\bar{A}}$. Therefore

$$|\nabla\phi| \le \frac{1}{\bar{A}^2\left(\frac{1}{\xi}\right)}D_{\bar{A}}\bar{A}\left(\frac{1}{\xi}\right)\frac{|\nabla\xi|}{\xi} = D_{\bar{A}}\phi\frac{|\nabla\xi|}{\xi}.$$

Hence, we have $\frac{|\nabla\phi|}{\phi}\xi \le D_{\bar{A}}|\nabla\xi|$, which is exactly (4.9).

Summing up the estimates (4.8) and (4.9) we obtain (4.6)

$$\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \le C_{\bar{A}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\frac{1}{\bar{A}^{-1}(\frac{1}{\phi})}\right) \le C_{\bar{A}}\bar{A}\left(D_{\bar{A}}|\nabla\xi|\right) \le C_{\bar{A}}^2\bar{A}(D_{\bar{A}})\bar{A}(|\nabla\xi|).$$

Thus the assumptions of Theorem 3.1 are satisfied. We obtain (3.1). The substitution $\phi = F_{\bar{A}}(\xi)$, equivalently taking

$$\xi(x) = \begin{cases} \frac{1}{\bar{A}^{-1}\left(\frac{1}{\phi(x)}\right)}, & \text{when } \phi(x) \neq 0, \\ 0, & \text{when } \phi(x) = 0, \end{cases}$$

where $\bar{A}^{-1}$ is the inverse function of $\bar{A}$, transforms the left–hand side of (3.1) into the left–hand side of (4.2). What remains to show is that the right–hand side in (3.1) is estimated as follows

$$\int_{\{\nabla u\neq 0\}} \frac{\bar{A}\left(g(u)\right)\Psi(u)}{g(u)}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx \leq C_{\bar{A}}^2\bar{A}(D_{\bar{A}})\int_{\{\nabla u\neq 0\}} \frac{\bar{A}\left(g(u)\right)\Psi(u)}{g(u)}\bar{A}(|\nabla\xi|)\, dx.$$

This is a direct consequence of (4.6). The proof is complete. □

Examples dealing with various $F_{\bar{A}}$ and $g$ are given in the following sections.

# 5 Retrieving our previous results

When we consider $\Delta_A = \Delta_p$ (i.e. we take $\bar{A}(t) = t^p$), the method becomes much simpler and the obtained inequality (4.2) involves $F_{\bar{A}}(t) = \frac{1}{(1/t)^p} = t^p = \bar{A}(t)$. In this case we have

$$\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla\xi|^p \mu_2(dx)$$

with certain measures.

We concentrate on retrieving our previous results from [44, 45]. Theorem 4.1 imply the following theorem obtained in [44]. It leads among others to Hardy and Hardy–Poincaré inequalities with optimal constants (see [44, 45]).

**Theorem 5.1** ([44], Theorem 4.1)**.** *Assume that $1 < p < \infty$ and $u \in W^{1,p}_{loc}(\Omega)$ is a nonnegative solution to PDI $-\Delta_p u \geq \Phi$, in the sense of Definition 2.3, where $\Phi$ is locally integrable and*

$$\sigma_0 := \inf\{\sigma \in \mathbb{R} : \Phi\cdot u + \sigma|\nabla u|^p \geq 0 \quad a.e.\ in\ \Omega\} \in \mathbb{R}, \tag{5.1}$$

*where we set $\inf\emptyset = \infty$. Assume further that $\beta$ and $\sigma$ are arbitrary numbers such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0$.*

*Then, for every Lipschitz function $\xi$ with compact support in $\Omega$, we have*

$$\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla\xi|^p \mu_2(dx), \tag{5.2}$$

*where*

$$\begin{aligned} \mu_1(dx) &= \left(\frac{\beta-\sigma}{p-1}\right)^{p-1}\left[\Phi\cdot u + \sigma|\nabla u|^p\right]\cdot u^{-\beta-1}\chi_{\{u>0\}}\, dx, \\ \mu_2(dx) &= u^{p-\beta-1}\chi_{\{|\nabla u|\neq 0\}}\, dx. \end{aligned}$$

***Sketch of the proof of Theorem 5.1.*** We apply Theorem 4.1, respectively, with $\bar{A}(t) = t^p = F_{\bar{A}}(t)$, $g(t) = t$, $\Psi(t) = t^{-\beta}$, $C = \beta > 0$ (then $C_{\bar{A}} = 1$, $d_{\bar{A}} = D_{\bar{A}} = p$). We note that the assumption (5.1) matches with the assumption $(u)$. Inequality (5.2) follows from (4.2). Involved measures and constants are the same. □

In particular as a consequence of Theorem 4.1, we may obtain all examples from [44] and [45], such as classical Hardy inequalities with the best constants, Hardy inequalities with radial weights and others. As an implication of the above result we present Hardy–Poincaré inequalities from [45]. To derive them we consider Barrenblatt profiles $u_\alpha(x) = (1+|x|^{\frac{p}{p-1}})^{-\alpha}$, where $\alpha > 0$. We obtained the family of inequalities for function from weighted Sobolev space defined as follows.

By $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where nonnegative measurable functions $v_1, v_2$ are given, we mean the completion of $\varphi \in C^\infty(\mathbb{R}^n)$, such that $\int_{\mathbb{R}^n} |\varphi|^p \, v_1 dx < \infty$ and $\int_{\mathbb{R}^n} |\nabla\varphi|^p \, v_2 dx < \infty$, under the norm $\|\varphi\|_{W^{1,p}_{v_1,v_2}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\varphi|^p \, v_1 dx + \int_{\mathbb{R}^n} |\nabla\varphi|^p \, v_2 dx\right)^{\frac{1}{p}}$.

**Theorem 5.2** (Hardy–Poincaré inequalities, [45])**.** *Suppose $p > 1$ and $\gamma > 1$. Then, for every function $\xi \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where $v_1(x) = \left(1+|x|^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}$, $v_2(x) = \left(1+|x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma}$, we have*

$$\bar{C}_{\gamma,n,p} \int_{\mathbb{R}^n} |\xi|^p \left[(1+|x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dx \leq \int_{\mathbb{R}^n} |\nabla\xi|^p \left[(1+|x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dx, \tag{5.3}$$

*with $\bar{C}_{\gamma,n,p} = n\left(\frac{p(\gamma-1)}{p-1}\right)^{p-1}$. Moreover, for $\gamma \geq n+1-\frac{n}{p}$, the constant $\bar{C}_{\gamma,n,p}$ is optimal.*

Related results can be found in [3, 23].

**Remark 5.1.** Theorem 4.1 enables us to derive various measures in (5.2). In the above examples we apply $\Psi(t) = t^{-\beta}$, $g(t) = t$. When we check the other pairs e.g. $\Psi(t) = e^{-t}$, $g(t) \equiv 1$, or $\Psi(t) = \frac{e^{-t}}{t}$, $g(t) = 1/(1+t)$, we obtain comparable inequalities.

# 6 Hardy–Sobolev inequalities dealing with Orlicz functions of power–logarythmic type

Now we deal with the case $\bar{A}(t) = t^p \log^\alpha(2+t)$, $p > 1$, $\alpha > 0$.

**Lemma 6.1.** *Suppose $p > 1$, $\alpha > 0$, $\bar{A}(t) = t^p \log^\alpha(2+t)$ and $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 2.3, where $\Phi$ is locally integrable and assumptions $(\Psi)$, $(u)$ are satisfied with $\sigma \in \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$.*

*Then there exists a constant $\widetilde{C} > 0$, such that for every Lipschitz function $\xi$ with compact support in $\Omega$, we have*

$$\int_{\Omega\cap\{\xi\neq 0\}} |\xi|^p \log^{-\alpha}(2+1/|\xi|)\mu_1(dx) \le \widetilde{C}\int_\Omega |\nabla\xi|^p \log^{\alpha}(2+|\nabla\xi|)\mu_2(dx),$$

*where*

$$\mu_1(dx) \;=\; \Psi(u)\left(\Phi + \frac{\sigma}{g(u)}|\nabla u|^p \log^\alpha(2+|\nabla u|)\right)\chi_{\{u>0\}}\, dx, \tag{6.1}$$

$$\mu_2(dx) \;=\; g^{p-1}(u)\log^\alpha(2+g(u))\Psi(u)\chi_{\{\nabla u\neq 0\}}\, dx, \tag{6.2}$$

***Proof.*** We apply Theorem 4.1. We remark first that assumption $(\bar{A})$ is satisfied as, according to Example 2.1, $\bar{A}\in\Delta'$ if $p>1,\ \alpha>0$. We notice, that

$$F_{\bar{A}}(t) = \frac{1}{\bar{A}(1/t)} = \frac{1}{(1/t)^p\log^\alpha(2+1/t)} = t^p\log^{-\alpha}(2+1/t), \quad F_{\bar{A}}(0)=0. \tag{6.3}$$

□

As a direct consequence of Lemma 6.1 we obtain the following corollary.

**Corollary 6.1.** *Suppose $p>1,\ \alpha>0$, $\bar{A}(t)=t^p\log^\alpha(1+t)$ and $\Omega\subseteq\mathbb{R}^n$, $n\ge 1$. Let $u\in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \ge \Phi$, in the sense of Definition 2.3, where $\Phi$ is locally integrable and assumptions $(\Psi)$, $(u)$ are satisfied satisfied with $\sigma\in\mathbb{R}$ and $g:\mathbb{R}_+\to\mathbb{R}_+$.*

*Then there exists $\widetilde{C}>0$, such that for every Lipschitz function $\xi$ with compact support in $\Omega$, we have*

$$\int_\Omega |\xi|^{p+\alpha}\mu_1(dx) \le \widetilde{C}\int_\Omega \bar{A}(|\nabla\xi|)\mu_2(dx),$$

*where $\mu_1(dx),\mu_2(dx),\widetilde{C}$ comes from Theorem 6.1.*

*Proof.* We note, that $t^\alpha < \log^{-\alpha}(2+1/t)$. Indeed, $\log(2+1/t) = \log\left(\frac{2t+1}{t}\right) = \frac{\log(2t+1)-\log(t)}{2t+1-t} = \log'(t_1) = \frac{1}{t_1}$, for some $t_1\in(t,2t+1)$.

This implies

$$\int_\Omega |\xi|^{p+\alpha}\mu_1(dx) < \int_\Omega |\xi|^p\log^{-\alpha}(1+2/|\xi|)\mu_1(dx)$$

and the result follows from estimate proven in Theorem 6.1. □

We give two examples of application Theorem 4.1 to power–logarithm function $\bar{A}$ and $u$ being a power function defined on a halfline. We start with a lemma confirming common assumptions.

**Lemma 6.2.** *Suppose $p > 1,\ \alpha > 0,\ \beta \in (0, 1)$ and $\Omega \subseteq \mathbb{R}_+$. Assume further that assumption $(\Psi)$ is satisfied with functions $\Psi, g$ and $(u)$ is satisfied with*

$$\sigma > -(1/\beta - 1)(p - 1) \inf_{x>0} g(x^\beta)x^{-\beta} =: \sigma_0.$$

*Then there exists a constant $\widetilde{C} > 0$, such that for every Lipschitz function $\xi$ with compact support in $\Omega$, we have*

$$\int_\Omega |\xi|^p \log^{-\alpha}(2 + 1/|\xi|)\mu_1(dx) \leq \widetilde{C} \int_\Omega |\xi'|^p \log^\alpha(2 + |\xi'|)\mu_2(dx),$$

*where*

$$\mu_1(dx) = \frac{\Psi(x^\beta)}{g(x^\beta)} x^{p(\beta-1)} \log^\alpha \left(2 + \beta x^{\beta-1}\right) dx, \tag{6.4}$$

$$\mu_2(dx) = \frac{\Psi(x^\beta)}{g(x^\beta)} g^p(x^\beta) \log^\alpha(2 + g(x^\beta))\, dx. \tag{6.5}$$

*Moreover*

$$\widetilde{C} \leq \frac{\beta^{1-p}}{(1-\beta)(p-1) + \sigma\beta}(C - \sigma)\bar{A}\left(\frac{D_{\bar{A}} - 1}{(C - \sigma)d_{\bar{A}}}\right)\frac{\bar{A}(D_{\bar{A}})C_{\bar{A}}^4}{D_{\bar{A}} - 1}, \tag{6.6}$$

*where $C_{\bar{A}} = (\frac{2}{\log 2})^\alpha,\ d_{\bar{A}} = p,\ D_{\bar{A}} = p + \frac{\alpha}{\log 2}$.*

*Proof.* We are to apply Theorem 4.1. We consider $\bar{A}(t) = t^p \log^\alpha(1+t)$. The assumption $(\bar{A})$ is satisfied as, according to Example 2.1, $\bar{A} \in \Delta'$ for $p > 1,\ \alpha > 0$. We notice, that (as in (6.3)) $F_{\bar{A}}(t) = t^p \log^{-\alpha}(2 + 1/t)$, when $t > 0$ and $F_{\bar{A}}(0) = 0$. We note that $u = u_\beta(x) = x^\beta$, with $\beta \in (0, 1)$, is the solution to PDI $-\Delta_A u \geq \Phi$, where

$$\Phi = -(\beta - 1)\beta^{p-1}(p - 1)x^{p\beta-\beta-p} \log^\alpha \left(2 + \beta x^{\beta-1}\right). \tag{6.7}$$

Indeed, we have $\nabla u = \beta x^{\beta-1}$, $|\nabla u| = |\beta| x^{\beta-1}$ and we compute the function $\Phi$

$$-\Delta_A u = -\mathrm{div}\left(\frac{\bar{A}(|\nabla u|)}{|\nabla u|^2}\nabla u\right) = -\beta|\beta|^{p-2}\left(x^{(p-1)(\beta-1)} \log^\alpha \left(2 + |\beta| x^{\beta-1}\right)\right)' =$$

$$= -\beta|\beta|^{p-2}(\beta - 1)x^{(p-1)(\beta-1)-1} \log^{\alpha-1} \left(2 + |\beta| x^{\beta-1}\right) \cdot$$

$$\cdot \left((p - 1) \log \left(2 + |\beta| x^{\beta-1}\right) + \alpha \frac{|\beta| x^{\beta-1}}{2 + |\beta| x^{\beta-1}}\right) \geq$$

$$\geq -\beta|\beta|^{p-2}(\beta - 1)(p - 1)x^{p\beta-p-\beta} \log^\alpha \left(2 + |\beta| x^{\beta-1}\right) =$$

$$= |\beta|^p(1/\beta - 1)(p - 1)x^{p\beta-p-\beta} \log^\alpha \left(2 + |\beta| x^{\beta-1}\right) = \Phi,$$

where the inequality holds for $\beta \in (0,1)$, thus we remove the absolute value of $\beta$ and write (6.7).

Now let us verify assumption $(u)$.
We note first that $\bar{A}(|\nabla u|) = \beta^p x^{p(\beta-1)} \log^\alpha \left(2 + \beta x^{\beta-1}\right)$. Therefore

$$g(u)\Phi + \sigma \bar{A}(|\nabla u|) = \beta^p x^{p(\beta-1)} \log^\alpha \left(2 + \beta x^{\beta-1}\right) \left[(1/\beta - 1)(p-1)g(x^\beta)x^{-\beta} + \sigma\right]$$

is positive for $\sigma > -(1/\beta - 1)(p-1)\inf_{x>0} g(x^\beta)x^{-\beta} = \sigma_0$.

We reach the goal by computing the weights according to Theorem 4.1 and dividing both sides by the constant.

We notice that, due to the above method, we can estimate the constant $\widetilde{C}$ as in (6.6). We have to note that, according to Facts 2.2 and 2.3, $C_{\bar{A}} = (\frac{2}{\log 2})^\alpha$, $d_{\bar{A}} = p$, $D_{\bar{A}} = p + \frac{\alpha}{\log 2}$. □

## 6.1 Inequalities on $(0, \infty)$

Applying $\Psi(t) = t^{-C}$, $g(t) = t$ in Lemma 6.2, we obtain the following result.

**Theorem 6.1** (Power–logarithm Hardy–Sobolev inequality on $(0,\infty)$)**.** *Let $p > 1$, $\alpha > 0$, $\beta \in (0,1)$, $C > 0$, $C > \sigma > -(1/\beta - 1)(p-1)$.*

*Then there exists $c > 0$, such that for every compactly supported Lipschitz function $\xi$, we have*

$$\int_0^\infty |\xi|^p \log^{-\alpha}(2 + 1/|\xi|)\mu_1(dx) \le c \int_0^\infty |\xi'|^p \log^\alpha(2 + |\xi'|)\mu_2(dx),$$

*where*

$$\begin{aligned} \mu_1(dx) &= x^{\gamma-p} \log^\alpha \left(2 + \beta x^{\beta-1}\right) dx \sim x^{\gamma-p} \log^\alpha (2+x)\, dx, \\ \mu_2(dx) &= x^\gamma \log(2 + x^\beta)\, dx \sim x^\gamma \log(2+x)\, dx, \end{aligned}$$

*with $\gamma = -\beta(C + 1 - p)$ and the constant $c$ depends on $\bar{A}$, $p$, $C$, $\beta$, $\sigma$.*

*Proof.* We apply Lemma 6.2. It suffices now to check that the pair $\Psi(t) = t^{-C}$, $g(t) = t$ with $C > 0$ satisfies the assumption $(\Psi)$ **i)** and **ii)** and finally we compute the weights.

**i)** The mentioned $\Psi, g$ are positive functions. $\Psi$ is locally Lipschitz, $\Psi/g$ is decreasing, moreover

$$\Psi'(t)g(t) = -Ct^{-C-1}g(t) = -Ct^{-C-1}t = -Ct^{-C-1+1} = -C\Psi(t).$$

**ii)** The function $\Theta = t^{p-1-C}\log^\alpha(2+t)$ (see (2.12)) is bounded in the neighbourhood of 0 when $p - 1 - C \ge 0$ and decreasing when $p - 1 - C < 0$.

We note that

$$\begin{aligned}\sigma &> -(1/\beta-1)(p-1)\inf_{0<x} g(x^\beta)x^{-\beta} = \\ &= -(1/\beta-1)(p-1)\inf_{0<x} x^\beta x^{-\beta} = -(1/\beta-1)(p-1) = \sigma_0.\end{aligned}$$

Thus there exists $\sigma \in [\sigma_0, C)$ for any $C > 0$.

We apply Lemma 6.2 and obtain the following measures in inequality (6.4)

$$\begin{aligned}\mu_1(dx) &= (x^\beta)^{-C-1}\beta^p x^{p(\beta-1)}\log^\alpha\left(2+\beta x^{\beta-1}\right)[(1/\beta-1)(p-1)+\sigma]\,dx = \\ &= x^{-\beta(C+1-p)-p}\log^\alpha\left(2+\beta x^{\beta-1}\right)\beta^p[(1/\beta-1)(p-1)+\sigma]\,dx, \\ \mu_2(dx) &= \widetilde{C}x^{-\beta(C+1-p)}\log(2+x^\beta)\,dx.\end{aligned}$$

Now it suffices to take $\gamma = -\beta(C+1-p)$. □

**Remark 6.1.** We may estimate $c$ due to (6.6).

## 6.2 Inequalities on $(0,1)$

We present application with $g(\lambda)$ different from identity. For this, it is convenient to consider the extension of previous results where we consider the restriction of $\Psi$ to the codomain of $u$. We need Theorems 3.1 and 4.1, and Lemma 6.2, where instead of Assumption $(\Psi)$ we suppose $(\Psi)_2$ (see below). Their proofs in this case are easy modifications of the proofs from previous sections.

$(\Psi)_2$ for a given nonnegative $u \in W^{1,\bar{A}}_{loc}(\Omega)$, there exists a function $\Psi : [0,\infty) \to [0,\infty)$, which is nonnegative and belongs to $C^1(u(\Omega\setminus\{0\})$, where $u(\Omega) = \{u(x) : x \in \Omega\}$. Furthermore, the following conditions are satisfied

- **i)** inequality
$$g(t)\Psi'(t) \le -C\Psi(t),$$
holds for all $t \in u(\Omega)\setminus\{0\}$ with $C > 0$ independent of $t$ and certain continuous function $g : (0,\infty) \to (0,\infty)$, such that $\Psi(t)/g(t)$ is nonincreasing for $t \in u(\Omega)$. Moreover, we set $\Psi(t) \equiv 0$ for $t \notin u(\Omega)$.
- **ii)** function $\Theta(t)$ given by (2.12) is nonincreasing or bounded in the neighbourhood of 0.

When we restict ourselves to $(0,1)$ and apply $\Psi(t) = e^{\frac{1}{2}\log^2(t)}$, $g(t) = t/|\log t|$. They do not satisfy assumption $(\Psi)$, but only $(\Psi)_2$. In particular assumption $(\Psi)$ **i)** requires $\Psi$ to be a decreasing function, but it does not hold outside $(0,1)$. This choice in Lemma 6.2 leads to the following result.

**Theorem 6.2** (Hardy–Sobolev inequality on $(0,1)$)**.** *Let $p>1,\ \alpha>0,\ \beta\in(0,1)$ and $\bar{A}(t)=t^p\log^{\alpha}(2+t)$.*

*Then there exists a constant $c>0$, such that for every Lipschitz function $\xi$ with compact support in $(0,1)$, we have*

$$\int_0^1 |\xi|^p \log^{-\alpha}(2+1/|\xi|)\mu_1(dx) \le c\int_0^1 \bar{A}(|\xi'|)\mu_2(dx),$$

*where*

$$\mu_1(dx) = e^{\frac{\beta}{2}\log^2(x)}|\log x|\frac{x^{(p-1)\beta}}{x^p}\log^{\alpha}\left(2+\beta x^{\beta-1}\right)\,dx, \tag{6.8}$$

$$\mu_2(dx) = e^{\frac{\beta}{2}\log^2(x)}|\log x|\frac{x^{(p-1)\beta}}{|\log x|^p}\log^{\alpha}(2+\frac{x^{\beta}}{|\log x^{\beta}|})\,dx. \tag{6.9}$$

*Proof.* We apply Lemma 6.2, where $u=u_\beta(x)=x^\beta$ is considered, with Assumption $(\Psi)_2$ instead of $(\Psi)$. It suffices now to check that the pair $\Psi(t)=e^{\frac{1}{2}\log^2(t)}$, $g(t)=t/|\log t|$, with $C=1$ (for $t\in(0,1)$) satisfies the assumption $(\Psi)_2$ **i)** and **ii)**.

**i)** The functions $\Psi,g$ are positive. $\Psi$ is locally Lipschitz. Moreover

$$\begin{aligned}\Psi'(t)g(t) &= -\frac{t}{\log t}\cdot\frac{1}{2}(\log^2 t)'e^{\frac{1}{2}\log^2(t)} = -\frac{t}{\log t}\cdot\frac{1}{2}2\frac{\log t}{t}e^{\frac{1}{2}\log^2(t)} = \\ &= -e^{\frac{1}{2}\log^2(t)} = -\Psi(t).\end{aligned}$$

As $t\in(0,1)$, we have $\log t<0$. Therefore

$$\begin{aligned}g'(t) &= \left(-\frac{t}{\log t}\right)' = -\frac{t'\log t - t\log' t}{\log^2 t} = -\frac{\log t-1}{\log^2 t} = \\ &= \frac{1+|\log t|}{\log^2 t} \ \ge\ 0\ >\ -1.\end{aligned}$$

According to Remark 2.5 it is enough to ensure that $\Psi/g$ is nonincreasing.

**ii)** The function $\Theta(s)=\frac{\bar{A}(g(s))\Psi(s)}{g(s)}=\left(\frac{s}{|\log s|}\right)^{p-1}\log^{\alpha}\left(2+\frac{s}{|\log s|}\right)e^{\frac{1}{2}\log^2(s)}$ is decreasing in the neighbourhood of 0. Indeed, it is easy to show that for sufficiently small positive $s$ we have $\Theta'(s)<0$.

We note that there exists $\sigma\in[\sigma_0,C)=[0,1)$. Indeed, the only condition for $\sigma$ is the following

$$\begin{aligned}\sigma>\sigma_0 &= -(1/\beta-1)(p-1)\inf_{0<x<1} g(x^\beta)x^{-\beta} = -(1/\beta-1)(p-1)\inf_{0<x<1} x^\beta|\log x^\beta|x^{-\beta} = \\ &= -(1/\beta-1)(p-1)\inf_{0<x<1}|\log x^\beta| = 0.\end{aligned}$$

We apply Lemma 6.2 and obtain the following measures in inequality (6.4)

$$\begin{aligned}\mu_1(dx) &= e^{\frac{1}{2}\log^2(x^\beta)}|\log(x^\beta)|x^{p\beta-\beta-p}\log^\alpha\left(2+\beta x^{\beta-1}\right)dx,\\ \mu_2(dx) &= e^{\frac{1}{2}\log^2(x^\beta)}|\log(x^\beta)|^{-p+1}x^{p\beta-\beta}\log^\alpha\left(2+\frac{x^\beta}{|\log x^\beta|}\right)dx.\end{aligned}$$

We compute the final measures by removing unnecessary constants from logarithm terms. □

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