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Some properties of topological spaces concerning continuous actions of semigroups

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Abstract

We study some properties of topological spaces defined by means of continuous actions of semigroups. We examine some of them in case of certain spaces occuring in algebra.

0 Introduction

One of our motivations is to find some topological properties of spaces ocurring in valuation theory which would describe their "self-similarities". The following definition comes from [2].

Definition 0.1 *An iterated function system (IFS) is a finite set of contraction mappings on a complete metric space. By a contraction mapping on a metric space we mean a Lipschitz mapping with a Lipschitz constant smaller that 1.*

In 1981, Hutchinson proved that every such system $(f_1, \ldots, f_n)$ of functions on the euclidean metric space $\mathbb{R}^n$ has a unique non-empty compact fixed set, i. e., such a set $K \subset \mathbb{R}^n$ that $K = \bigcup_i f_i(K)$ (this is a consequence of the Banach fixed-point theorem). Such a space K is called an attractor of the IFS $(f_1, \ldots, f_n)$.

The above fact gives a method of constructing attractors of IFS's given on the euclidean spaces. However, the spaces coming from valuation theory that we want to deal with are usually not euclidean, and often they are even not metrizable. Thus, we will consider variant of the above approach allowing us to work with bigger classes of topological spaces.

In [4], a variant of the definition of IFS was generalized allowing it to consist of infinitely many functions. This allowed to generalize the above method of constructing IFS attractors. Another generalization of the classical approach was in [1], where the definition of a topological IFS attractor was introduced by replacing the contractivity assumption by a suitable topological "shirinking condition" (see Section 1 for details on notions mentioned in this paragraph).

Also, we propose in Section 1 one more possible variant of the definition of IFS attractor, which is a combination of the one mentioned in the preceeding paragraph. However, we will prove that even this definiton will substantially restrict the class of spaces on which such a system can exist.

A property of a topological space X which is in somewhat different spirit is smallness of the Polish structure $(X, Homeo(X))$ (although it also defined by means of a continuous action of a semigroup, by here this semigroup in fact assumed to be a group, and it acts as homeomorphisms onto the space X). The definition of a small Polish structure was introduced in [7] by Krupiński:

Definition 0.2 *A Polish structure is a pair (X, G), where G is a Polish group acting faithfully on a set X so that the stabilizers of all singletons are closed subgroups of G. We say that (X, G) is small if for every $n < \omega$, there are only countably many orbits on X^n under the action of G.*

For any compact metric space P, if we consider the group $Homeo(P)$ of all homeomorphisms of P equipped with the compact-open topology, then $(P, Homeo(P))$ is a Polish structure (examples of small Polish structures of this form were investigated in [7] and [3]). If P is not compact, then ussually we have to replace the full group $Homeo(P)$ by a smaller subgroup to obtain a Polish group G and, in consequence, a Polish structure (X, G). Also, we often consider smaller groups of homeomorphisms when there is some extra structure on X consider (for example, a structure of a topological group). We study some probems conerning small Polish structures in Sections 2 and 3.

1 IFS attractors

We start with the definition of a topological IFS-attractor given in [1].

Definition 1.1 *A compact topological space X is a topological IFS-attractor, if $X = \bigcup_i f_i[X]$ for some continuous functions $f_1, \dots, f_n : X \to X$ satisfying the following "shrinking condition":*
(SC) for any open cover $\mathcal{U}$ of X there is a natural number l such that for any $g_1, \dots, g_l \in \{f_1, \dots, f_n\}$, the image $g_1 \circ \dots \circ g_l[X]$ is contained in some $U \in \mathcal{U}$.

Notice that the existence of continuous functions $f_1, \dots, f_n$ on any topological space X, satysfying conditions (SC) and $X = \bigcup_i f_i[X]$ implies that X is quasi-compact. By the following example it can be seen that these conditions do not imply that X is Hausdorff:

Remark 1.2 *Let $X = [0, 1]$ be equipped with the topology in which the open sets are cofinite sets and $\emptyset$. Define $f_1, f_2 : X \to X$ by $f_1(x) = x/2$, $f_2(x) = 1/2 + x/2$. Then the system (f_1, f_2) consists of continuous functions, and satisfies conditions (SC) and $X = \bigcup_i f_i[X]$*

Now, let us recall the definion from [4].

Definition 1.3 *Let X be a compact metric space, and $\{f_i : i \in I\}$ any set of contractive mappings having a common contractivity factor $s < 1$. We define $\mathcal{F}(X, \{f_i : i \in I\})$ to be the unique compact subspace A of X such that $A = cl(\bigcup_{i \in I} f_i[A])$.*

Using the Banach fixed point theorem, it was proved in [4] that such a subspace $\mathcal{F}(X, \{f_i : i \in I\})$ indeed exists and is unique.

Connecting the property (SC) with the "covering property" from the above definition, we obtain the following definition of an attractor for possibly infinite systems of functions..

Definition 1.4 *Let X be a compact metric space, and $\{f_i : i \in I\}$ any set of continuous mappings $X \to X$ satisfying (SC), i. e. for any finite open cover $\mathcal{U}$ of X there is a natural number l such that for any $g_1, \ldots, g_l \in \{f_i : i \in I\}$, the image $g_1 \circ \cdots \circ g_l[X]$ is contained in some $U \in \mathcal{U}$. We will say that X is a topological attractor for $\{f_i : i \in I\}$, if $X = cl(\bigcup_{i \in I} f_i[X])$. For any cardinal number κ, we will say that X is a topological κ-IFS-attractor, if X is an attractor for some set of continuous functions satisfying (SC) of cardinality at most κ.*

Although the above definiot seems to be the weakest reasonable generalization of the Definition 1.1 allowing inifinite systems of iterated functions, it turns out that the property of being a κ-IFS-attractor (where κ is fixed) cannot hold for spaces of arbitrary large weight (i. e., the minimal cardinality of a basis of open sets, denoted by $w(X)$):

Proposition 1.5 *Suppose X is a normal space which is a κ-IFS-attractor. Then, $w(X) \leq 2^\kappa + \aleph_0$.*

Proof. Choose a system of funtions $F = \{f_i : i \in I\}$ of cardinality at most κ satisfying (SC) such that X is an attractor for F, i. e., $X = cl(\bigcup_{i \in I} f_i[X])$.

Claim 1 *For any natural number l, we have that $S = cl(\bigcup_{g_1,\ldots,g_l \in F} g_1 \circ \cdots \circ g_l[X])$.*

Proof of the claim. We proceed by induction on l. Suppose that $S = cl(\bigcup_{g_1,\ldots,g_l \in F} g_1 \circ \cdots \circ g_l[X])$. Then, for any $i \in I$, we get by the continuity of f_i that

$$f_i[X] \subseteq cl(f_i[\bigcup_{g_1,\ldots,g_l \in F} g_1 \circ \cdots \circ g_l[X]]) =$$

$$cl(\bigcup_{g_1,\ldots,g_l \in F} f_i \circ g_1 \circ \cdots \circ g_l[X]) \subseteq cl(\bigcup_{g_1,\ldots,g_{l+1} \in F} g_1 \circ \cdots \circ g_{l+1}[X]).$$

Thus, be obtain that $X = cl(\bigcup_{i \in I} f_i[X]) \subseteq cl(\bigcup_{g_1,\ldots,g_{l+1} \in F} g_1 \circ \cdots \circ g_{l+1}[X])$. This completes the proof of the claim. □

Define

$$\mathcal{B} = \{X \backslash cl(\bigcup_{(g_1,\ldots,g_l) \in I} g_1 \circ \cdots \circ g_l[X]) : l < \omega, I \subseteq F^l\}.$$

Clearly, $|\mathcal{B}| \leq 2^\kappa + \aleph_0$. We will show that $\mathcal{B}$ is a basis of X. So take any open subset U of X and $x \in U$. Since X is normal, we can choose open sets V_1, V_2 such that

$$x \in V_1 \subseteq cl(V_1) \subseteq V_2 \subseteq cl(V_2) \subseteq U.$$

Let l be as in the condition (SC) for F and the covering $\{V_2, X\backslash cl(V_1)\}$ of X. Define $J = \{(g_1, \ldots, g_l) \in F^l : g_1 \circ \cdots \circ g_l[X] \subseteq X\backslash cl(V_1)\}$ and
$W = X\backslash cl(\bigcup_{(g_1,\ldots,g_l)\in J} g_1 \circ \cdots \circ g_l[X]) \in \mathcal{B}$. Since V_1 is disjoint from $\bigcup_{(g_1,\ldots,g_l)\in J} g_1 \circ \cdots \circ g_l[X]$, we get that $x \in W$. It remains to check that $W \subseteq U$. Take any $y \in X\backslash U$. We will show that $y \in cl(\bigcup_{(g_1,\ldots,g_l)\in J} g_1 \circ \cdots \circ g_l[X])$. Take any open neighbourhood Z of y. By the claim, $g_1 \circ \cdots \circ g_l[X]$ meets $Z \cap (X\backslash cl(V_2))$ for some $h_1, \ldots, h_l \in F$. Then, the image $g_1 \circ \cdots \circ g_l[X]$ is not contained in V_2, so it is contained in $X\backslash cl(V_1)$ and $(h_1, \ldots, h_l) \in J$. So, Z meets $\bigcup_{(g_1,\ldots,g_l)\in J} g_1 \circ \cdots \circ g_l[X]$, and we are done. $\square$

The above proposition applies in particular to caompact spaces (which are known to be normal). Hence, we obtain in particular that every topological IFS-attractor has a countable basis, and thus is metrizable (by the Urysohn metrization theorem).

Condition (SC) is not satisfied in some natural examples where the metric shrinking condition is satisfied (i. e. $\lim_l \sup_{i_1,\ldots,i_l} diam(f_{i_1} \circ \cdots \circ f_{i_l}[X]) = 0$):

Example 1.6 *Let $X = \omega^\omega$ be the Baire space (which is homeomorphic to $k((t))$ considered with the valuation topology, where k is any field of cardinality $\aleph_0$). For any $i < \omega$, define $f_i : X \to X$ as follows: $f_i(x)(0) = i$ and $f_i(x)(n) = x(n-1)$ for $n > 0$. Then (SC) is not satisfied for $f_i, i < \omega$, which is witnessed by the covering $\{U, X\backslash U\}$, where $U = \bigcup_{n<\omega}\{x \in X : x(0) = n, x(1) = \cdots = x(n) = 0\}$.* $\diamond$

Thus, we want to consider another topological shrinking condition, in which we are allowed to choose a basis from which the covering sets are taken. However, to make it possible to cover in this way the whole space (which is not assumed to be compact), we allow one of the covering sets to be not in the fixed basis. This leads to the following definition:

Definition 1.7 *A family of functions $(f_i)_{i\in I}$ on a topological space X satisfies $(SC*)$ if there is a basis $\mathcal{B}$ of X such that for every open covering $\mathcal{C}$ of X containing at most one set which is not in $\mathcal{B}$, there is some natural number k such that for every sequence $(i_1, \ldots, i_k) \in I^k$ there is $U \in \mathcal{C}$ with $f_{i_1} \circ \ldots \circ f_{i_k}(X) \subseteq U$.*

Every space is an attractor for the set of all constant functions from X to X (i. e., is covered by their images). We will say that X is a weak $*$-IFS attractor, if it is an attractor for a set of functions satisfying $(SC*)$ of a cardinality smaller than $|X|$. We will say that X is a $*$-IFS attarctor, if it is an attractor for a finite set of functions satisfying $(SC*)$.

Clearly, we have:

Remark 1.8 *If X is a compact space, then it is a $*$-IFS attractor if and only if it is a topological IFS attractor.*

By the following example it can be seen that being a $*$-IFS attarctor does not imply compactness:

Example 1.9 *Let $X = \omega$ be considered with the discrete topology. Define $f_0, f_1 : X \to X$ by $f_0(n) = 0$ and $f_1(n) = n + 1$. Then X is a $*$-IFS attractor for $\{f_0, f_1\}$, so X is a $*$-IFS attractor.*

Proof. We choose a basis $\mathcal{B}$ consisting of all singletons. Consider any covering of X of a form $\{U, \{n_1\}, \dots, \{n_l\}\}$. Then it is sufficient to take $k = max(n_1, \dots, n_l)+1$. □

Example 1.10 *Let $X = \omega^\omega$, and let $f_i : X \to X, i < \omega$ be as in Example 1.6. Then, $(f_i)_{i<\omega}$ satisfies $(SC*)$, so X is a weak $*$-IFS attractor. More generally, for any cardinal number κ, the space κ^ω is an attractor for a set of functions of cardinality κ, so it is a weak $*$-IFS attractor if $\kappa < \kappa^\omega$ (this holds for example for all cardinals with countable cofinality, so for unboundedly many cardinals)*

Proof. For any $\alpha \in \kappa$, define $f_\alpha : \kappa^\omega \to \kappa^\omega$ by $f_\alpha(x)(0) = \alpha$ and $f_\alpha(x)(n) = x(n-1)$ for $n > 0$. We choose the standard basis of κ^ω, i. e.,

$$\mathcal{B} = \{A_x : x \in \kappa^n, n < \omega\},$$

where $A_x = \{y \in \kappa^\omega : x \subseteq y\}$. Choose any open covering of κ^ω of the form $\{U, A_{x_1}, \dots, A_{x_n}\}$. Put $k = max(|x_1|, \dots, |x_n|)$. For any sequence $(\alpha_0, \dots, \alpha_{k-1}) \in \kappa^k$ we have that $f_{\alpha_0} \circ \dots \circ f_{\alpha_{k-1}}[\kappa^\omega] = A_y$, where $y(i) = \alpha_i$ for all $i < k$, so this image is either contained in one of the sets $A_{x_1}, \dots, A_{x_n}$, or disjoint from all of them and thus contained in U. □

Proposition 1.11 *Suppose A is a densely ordered abelian group and $|\cdot| : A \to \{a \in A : a > 0\}$ is the associated absolute value. Consider a collection of functions $f_i : A \to A, i \in I$. Suppose that there is a sequence $(a_i)_{i<\omega}$ of positive elements of A which convergers to 0, and that for every k and a sequence $(i_1, \dots, i_k) \in I^k$ we have $diam(f_{i_1} \circ \dots \circ f_{i_k}[A]) < a_i$. Then $f_i : A \to A, i \in I$ satisfies $SC*$ (where we consider A with the order topology).*

Proof. We choose a basis $\mathcal{B}$ of A consisting of all open intervals. Consider any covering of A of the form $\{U, (a_1, b_1), \dots, (a_n, b_n)\}$. For any i there are $c_i, d_i > 0$ such that each of the intervals $(a_i - c_i, a_i + c_i)$, $(b_i - d_i, b_i + d_i)$ is contained in one of the sets from the covering. Now it is sufficient to choose k such that for every sequence $(i_1, \dots, i_k) \in I^k$ we have $diam(f_{i_1} \circ \dots \circ f_{i_k}[A]) < min(c_1, d_1, \dots, c_n, d_n)$. □

Corollary 1.12 *$\mathbb{R}$ is a weak $*$-IFS attractor.*

Proof. Take a continuous bijection $f_0 : \mathbb{R} \to (-1, 1)$ which is Lipschitz with constant $1/2$. Define $f_n(x) = n + f_0(x)$ for any integer n. Then, clearly, the family $\{f_n : n \in \mathbb{Z}\}$ satisfies assumptions of Proposition 1.11, so we get that it satisfies $SC*$. Of course, $\mathbb{R}$ is an attractor for that family (and has a bigger cardinality).

□

2 Small Polish structures

An initial motivation for the considerations done in this section was the following (purely topological) question.

Question 2.1 *Take a countable field F and define the action of $Homeo([0,1])$ on the field $L = F(x_t)_{t\in[0,1]}$ by $g \cdot f(x_{t_1},\dots,x_{t_n}) = f(x_{g(t_1)},\dots,x_{g(t_n)})$. Is there a Polish topology on the field L such that this action is continuous?*

The answer to this question is negative, but we were not able to find an easy elementary proof. Hence, we need to introduce some model theoretic tools developed in [7].

Throughout this paper, we follow the terminology from [7].

Definition 2.2 *Let G be a Polish group.*
(i) A Polish group structure is a Polish structure (H,G) such that H is a group and G acts as a group of automorphisms of H.
(ii) A (topological) G-group is a Polish group structure (H,G) such that H is a topological group and the action of G on H is continuous.
(iii) A Polish [compact] G-group is a topological G-group (H,G), where H is a Polish [compact] group.

Let (X,G) be a Polish structure. For any finite $C \subseteq X$, by G_C we denote the pointwise stabilizer of C in G, and for a finite tuple a of elements of X, by $o(a/C)$ we denote the orbit of a under the action of G_C (and we call it the orbit of a over C).

A fundamental concept for [7] is the relation of nm-independence in an arbitrary Polish structure.

Definition 2.3 *Let a be a finite tuple and A, B finite subsets of X. Let $\pi_A : G_A \to o(a/A)$ be defined by $\pi_A(g) = ga$. We say that a is nm-independent from B over A (written $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$) if $\pi_A^{-1}[o(a/AB)]$ is non-meager in $\pi_A^{-1}[o(a/A)]$. Otherwise, we say that a is nm-dependent on B over A (written $a \not\mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$).*

This is a generalization of m-independence, which was introduced by Newelski for profinite structures ([9, 10]). Under the assumption of smallness, nm-independence has similar properties to those of forking independence in stable theories, and hence it allows to transfer some ideas and techniques from stability theory to small Polish structures(which are purely topological objects). The investigation of Polish structures has been undertaken in [8] and [3]. For example, in [8], some structural theorems about compact G-groups were proved, and in [3], dendrites were considered as Polish structures, and some properties introduced in [7] were examined for them.

The class of Polish structures contains many more interesting examples from classical mathematics than the class of profinite structures. For example, for any compact metric space P, if we consider the group $Homeo(P)$ of all homeomorphisms of P equipped with the compact-open topology, then $(P, Homeo(P))$ is a Polish structure (examples of small Polish structures of this form were investigated in [7] and [3]). However, in the class of small Polish group structures, it is more difficult to construct interesting examples. In the present paper, we answer some questions from [7] by constructing suitable examples of small Polish group structures.

The following is [7, Question 5.4] (see Definition 2.9 for the notion of nm-generic orbit).

Question 2.4 *Let (H, G) be a small Polish group structure. Does H possess an nm-generic orbit?*

Proposition 5.5 from [7] gives us a positive answer to Question 2.4 in the class of small Polish G-groups. We construct a class of small Polish group structures for which the answer to Question 2.4 is negative.

If A is a finite subset of X (where (X, G) is a Polish structure), we define the algebraic closure of A (written $Acl(A)$) as the set of all elements of X with countable orbits over A. If A is infinite, we define $Acl(A) = \bigcup\{Acl(A_0) : A_0 \subseteq A \text{ is finite}\}$. By Theorems 2.5 and 2.10 from [7], we have:

Theorem 2.5 *In any Polish structure (X, G), nm-independence has the following properties:*
(0) (Invariance) $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B \iff g(a) \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_{g[A]} g[B]$ whenever $g \in G$ and $a, A, B \subseteq X$ are finite.
(1) (Symmetry) $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_C b \iff b \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_C a$ for every finite $a, b, C \subseteq X$.
(2) (Transitivity) $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_B C$ and $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$ iff $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A C$ for every finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$.
(3) For every finite $A \subseteq X$, $a \in Acl(A)$ iff for all finite $B \subseteq X$ we have $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$.
If additionally (X, G) is small, then we also have:
(4) (Existence of nm-independent extensions) For all finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there is $b \in o(a/A)$ such that $b \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$.

The notion of nm-independence leads to the definition of $\mathcal{NM}$-rank.

Definition 2.6 *The $\mathcal{NM}$-rank is the unique function from the collection of orbits over finite sets to the ordinals together with ∞, satisfying*
$\mathcal{NM}(a/A) \geq \alpha + 1$ iff there is a finite set $B \supseteq A$ such that $a \not\mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$ and $\mathcal{NM}(a/B) \geq \alpha$.
The $\mathcal{NM}$-rank of X is defined as the supremum of $\mathcal{NM}(x/\emptyset)$, $x \in X$.
We say that a Polish structure (X, G) is nm-stable, if $\mathcal{NM}(X) < \infty$.

The following fact is a part of [7, Proposition 2.3].

Fact 2.7 *Let (X, G) be a Polish structure, a be a finite tuple and A, B be finite subsets of X. Then, TFAE:*
(1) $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$
(2) $G_{AB} G_{Aa} \subseteq_{nm} G_A$ (where $Y \subseteq_{nm} Z$ means that Y is a non-meager subset of Z)

By [7, 2.14], under some assumptions, nm-dependence in a G-group (H, G) can be expressed in terms of the topology on H:

Theorem 2.8 *Let (X, G) be a Polish structure such that G acts continuously on a Hausdorff space X. Let $a, A, B \subseteq X$ be finite. Assume that $o(a/A)$ is non-meager in its relative topology. Then, $a \mathop{\smile\!\!\!\!\!\!\,|}^{nm}_A B \iff o(a/AB) \subseteq_{nm} o(a/A)$.*

Counterparts of various notions from model theory were studied by Krupiński in the context of Polish structures. One of them is the notion of a generic orbit:

Definition 2.9 *Let (H, G) be a Polish group structure. We say that the orbit $o(a/A)$ is left nm-generic (or that a is left nm-generic over A) if for all $b \in H$ with $a \mathop{\smile\!\!\!\!\!\!\,|}^{nm}_A b$, one has that $b \cdot a \mathop{\smile\!\!\!\!\!\!\,|}^{nm} A, b$. We say that it is right nm-generic if, for b as above, we have $a \cdot b \mathop{\smile\!\!\!\!\!\!\,|}^{nm} A, b$. An orbit is nm-generic if it is both right and left nm-generic.*

It was noticed in [7] that *nm*-generics have similar properties to generics in simple theories, e.g. being right *nm*-generic is equivalent to being left *nm*-generic. We recall Proposition 5.5 from [7], which gives us a positive answer to Question 2.4 for the class of small G-groups (H, G) in which H in not meager in itself (this holds, for example, in all Polish G-groups).

Fact 2.10 *Suppose (H, G) is a small G-group. Assume H is not meager in itself (e.g. H is Polish or compact, or, more generally, Baire). Then, at least one nm-generic orbit in H exists, and an orbit is nm-generic in H iff it is non-meager in H.*

Now, we construct a class of small Polish group structures for which the answer to Question 2.4 is negative.

Suppose (X, G) is a Polish structure. Let H be an arbitrary group. For any $x \in X$ we consider an isomorphic copy $H_x = \{h_x : h \in H\}$ of H. By $H(X)$ we will denote the group $\bigoplus_{x\in X} H_x$. Although $H(X)$ is not necessarily commutative, we will denote its group action by $+$. For any $y \in H(X)$ there are $h_1, \dots, h_n \in H\backslash\{e\}$ and pairwaise distinct $x_1, \dots, x_n \in X$ such that $y = (h_1)_{x_1} + \dots + (h_n)_{x_n}$. Then, by $\tilde{y}$ we will denote the set $\{x_1, \dots, x_n\}$. We also put $\tilde{A} = \bigcup_{y\in A} \tilde{y}$ for any $A \subseteq H(X)$.

The group G acts as automorphisms on $H(X)$ by

$$g((h_1)_{x_1} + \dots + (h_n)_{x_n}) = (h_1)_{gx_1} + \dots + (h_n)_{gx_n}.$$

It is easy to see that if $h_1, \dots, h_k \in H\backslash\{e\}$ are pairwise distinct, and $x_{1,1}, \dots, x_{1,i_1}, x_{2,1}, \dots, x_{2,i_2}, \dots, x_{k,1}, \dots, x_{k,i_k} \in X$ are pairwise distinct as well, then the stabilizer of $(h_1)_{x_{1,1}} + \dots + (h_1)_{x_{1,i_1}} + \dots + (h_k)_{x_{k,1}} + \dots + (h_k)_{x_{k,i_k}}$ consists exactly of those elements of G which stabilise each of the finite sets $\{x_{ij} : j = 1, \dots, i_j\}$. Thus, we get that for every $a \in H(X)$, $G_{\tilde{a}}$ is a subgroup of finite index in G_a, and hence, for every finite $A \subseteq H(X)$, $G_{\tilde{A}}$ is a subgroup of finite index in G_A.

Proposition 2.11 *If (X, G) is a Polish structure, and H is a group, then $(H(X), G)$ is a Polish group structure. If, additionally, (X, G) is small and H is countable, then $(H(X), G)$ is small.*

Proof. For any $a \in H(X)$ we have that $G_{\tilde{a}}$ is closed in G and has finite index in G_a, so, G_a is also closed in G. Hence, $(H(X), G)$ is a Polish group structure.

Now, assume that (X, G) is small, and H is countable. Then, for every fixed $k < \omega$ and $i_1, \dots, i_n < \omega$, the orbit of a tuple $((h_{1,1})_{x_{1,1}} + \dots + (h_{1,i_1})_{x_{1,i_1}}, \dots, (h_{k,1})_{x_{k,1}} + \dots + (h_{k,i_k})_{x_{k,i_k}})$ depends only on $h_{1,1}, \dots, h_{1,i_1}, \dots, h_{k,1}, \dots, h_{k,i_k}$ and on the orbit of the tuple $(x_{1,1}, \dots, x_{1,i_1}, \dots, x_{k,1}, \dots, x_{k,i_k})$ in (X, G). So, there are only countably many k-orbits in $(H(X), G)$. □

Proposition 2.12 *Let (X, G) be a Polish structure, and H a countable group. Then, for any finite $A, B, C \subseteq H(X)$, we have:*
(1) $A \mathop{\downarrow}^{nm}_{C} B \iff \tilde{A} \mathop{\downarrow}^{nm}_{\tilde{C}} \tilde{B}$.
(2) If a is a finite tuple of elements of $H(X)$, and b is a tuple of elements of X enumerating $\tilde{a}$, then $\mathcal{NM}(a/A) = \mathcal{NM}(b/\tilde{A})$. In particular, $(H(X), G)$ is nm-stable iff (X, G) is.

Proof. (1) By Fact 2.7, it is enough to show that $G_{CB}G_{CA} \subseteq_{nm} G_C \iff G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}} \subseteq_{nm} G_{\tilde{C}}$. First, suppose that $G_{CB}G_{CA} \subseteq_{nm} G_C$. Since $[G_{CB} : G_{\tilde{C}\tilde{B}}], [G_{CA} : G_{\tilde{C}\tilde{A}}] < \omega$, we get that $G_{CB}G_{CA}$ is a union of finitely many two-sided translates of $G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}}$ by elements of G_C. So, $G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}}$ is non-meager in G_C, and, hence, in $G_{\tilde{C}}$.

Now, suppose that $G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}} \subseteq_{nm} G_{\tilde{C}}$. Then $G_{CB}G_{CA} \cap G_{\tilde{C}}$ is non-meager in $G_{\tilde{C}}$, and hence, in G_C (because $[G_C : G_{\tilde{C}}] < \omega$). Thus, $G_{CB}G_{CA}$ is non-meager in G_C. This proves (1). Now, (2) follows by (1) and transfinite induction. □

The following corollary gives a negative answer to Question 2.4 in its full generality, i.e., in the class of all Polish group structures. Recall that Fact 2.10 tells us that the answer is positive for small Polish G-groups.

Corollary 2.13 *Let (X, G) be a Polish structure, where X is uncountable. If H is a countable group, then $(H(X), G)$ is a small Polish group structure, and it has no generic orbit (neither left nor right).*

Proof. Take any $a \in H(X)$ and a finite $A \subseteq H(X)$. We will show that $o(a/A)$ is not a generic orbit. Take any $h \in H \setminus \{e\}$ and $b \in X \setminus Acl(\emptyset)$ such that $b \mathop{\downarrow}^{nm} \tilde{A}, \tilde{a}$. Then, by Proposition 2.12, $h_b \mathop{\downarrow}^{nm}_{A} a$. Since $b \mathop{\downarrow}^{nm} \tilde{A}, \tilde{a}$ and $b \notin Acl(\emptyset)$, we see that $b \notin \tilde{a}$. Hence, $\widetilde{a + h_b} = \tilde{a} \cup \{b\}$. But $\tilde{a}, b \not\mathop{\downarrow}^{nm} b$, so, again by Proposition 2.12, we have that $a + h_b \not\mathop{\downarrow}^{nm} h_b$. Hence, $o(a/A)$ is not a generic orbit. □

By the above corollary and Fact 2.10, we get in particular that there is no Polish topology on $H(X)$ such that $(H(X), G)$ is a G-group, i.e., such that the action of G on $H(X)$ is continuous.

By [7] and [8], every nm-stable compact G-group is nilpotent-by-finite. When the assumption of compactness is dropped, the corresponding questions concern searching for a subgroup of countable index having some nice algebraic properties. The algebraic structure of nm-stable Polish G-groups remains unexplored. The following corollary shows that in general, not much can be said about the algebraic structure of small nm-stable Polish group structures.

Corollary 2.14 *Let (X, G) be an uncountable, small, nm-stable Polish structure, and H a non-solvable, countable group. Then $(H(X), G)$ is a small, nm-stable Polish group structure, which is not solvable-by-countable.*

Proof. By Proposition 2.12(2), $(H(X), G)$ is nm-stable. Now, take a subgroup A of countable index in $H(X)$. Then, there is some $x \in X$, such that $\pi_x[A] = H_x$, where $\pi_x : H(X) \to H_x$ is the projection on the x-th coordinate. Since H_x is not solvable, we get that A is not solvable. Thus, $H(X)$ is not solvable-by-countable. □

Now, we will give a variant of the above construction. Suppose R is a countable commutative ring, and (X, G) is a small Polish structure. Let $R(X) = R[(y_x)_{x \in X}]$ be the ring of polynomials in variables $(y_x)_{x \in X}$ with coefficients in R. Then G acts on $R(X)$ by $gw(y_{x_1}, \dots, y_{x_n}) = w(y_{gx_1}, \dots, y_{gx_n})$. If R is a countable field, we can additionally consider $R(X)_0 = R((y_x)_{x \in X})$, the field of rational functions in variables $(y_x)_{x \in X}$ with coefficients in R. Then, G acts on $R(X)$ by $gf(y_{x_1}, \dots, y_{x_n}) = f(y_{gx_1}, \dots, y_{gx_n})$. As for $H(X)$, one can check that $(R(X), G)$, $(R_0(X), G)$ are small Polish structures. Moreover, if we define $\tilde{w}$ as the set of all $x \in X$ such that y_x occurs in the reduced form of w, then we get the same description of nm-independence for $(R(X), G)$,$(R_0(X), G)$ as was done for $(H(X), G)$ in Proposition 2.12. Thus, we get that these structures (which we could call Polish ring structures and Polish field structures) have no generics (in the sense of the additive group), and hence, there is no Polish topology on $(R(X), G)$ or on $(R_0(X), G)$ such that the action of G is continuous. In particular, the answe to Question 2.1 is negative.

3 A non-zero-dimensional small Polish G-group

The following problem was formulated in [7] (after Question 5.32):

Problem 3.1 *Find a non-zero-dimensional, small Polish G-group.*

In this section, we construct a small non-zero-dimensional Polish G-group.

First, we recall some results from [5]. Consider any $p \geq 1$ and the Banach space ℓ^p (over $\mathbb{R}$). We extend the p-norm from ℓ^p to $\mathbb{R}^\omega$ by putting $||z|| = \infty$ for every $z \in \mathbb{R}^\omega \setminus \ell^p$. The complete Erdös space is the intersection of ℓ^p with $(\mathbb{R} \setminus \mathbb{Q})^\omega$ (with the topology induced from ℓ^p).

Let $E_0, E_1, \dots$ be a fixed sequence of subsets of $\mathbb{R}$ and put

$$\mathcal{E} = \ell^p \cap \prod_{n<\omega} E_n.$$

The following is a part of [5, Theorem 1]:

Theorem 3.2 *Assume that $\mathcal{E}$ is not empty and that every E_n is zero-dimensional. For each $k \in \omega \setminus \{0\}$ we let $\eta(k) \in \mathbb{R}^\omega$ be given by*

$$\eta(k)_n = sup\{|a| : a \in E_n \cap [-1/k, 1/k]\},$$

where $\sup\emptyset = 0$. *The following statements are equivalent:*
(1) $||\eta(k)|| = \infty$ *for each* $k \in \omega\backslash\{0\}$
(2) $\dim\mathcal{E} > 0$

Under the assumptions of the above theorem, also the following theorem was proved there ([5, Theorem 3]):

Theorem 3.3 *If every* E_i *is closed in* $\mathbb{R}$*, then* $\mathcal{E}$ *is homeomorphic to the complete Erdös space if and only if* $\dim\mathcal{E} > 0$ *and every* E_n *is zero dimensional.*

We define a structure of a group on the complete Erdös space as in [6, Proposition 4.3] (with the only difference that we do not choose a particular p), which is done as follows. Fix any $p \in [1, \infty)$. We let $C \subseteq \mathbb{R}$ be the ternary Cantor set, and $X = C^\omega \cap \ell^p$. By Theorems 3.2 and 3.3, X (considered with the topology induced from ℓ^p) is homeomorphic to the complete Erdös space. Consider the standard bijection $\phi : 2^\omega \to C$ and the product map $\psi := \phi^\omega : (2^\omega)^\omega \to C^\omega$. It follows exactly as in [6, Proposition 4.3] that $H := \psi^{-1}[X]$ is a subgroup of $(2^\omega)^\omega$ (we will identify the latter group with $2^{\omega\times\omega}$ in the natural way), and becomes a Polish group with the topology induced from X by ψ (and is homeomorphic to the complete Erdös space). This topology is generated by the norm $||z|| := ||\psi(z)||_p$, $z \in H$. We also put $||z|| = \infty$ if $z \in 2^{\omega\times\omega}\backslash H$. For a subset A of $\omega \times \omega$, we define $||A|| := ||\chi_A||$, where χ_A is the characteristic function of A.

Now, we will define an action of a Polish group G on H. Let G_1 be the group of all permutations of $\omega \times \omega$. For any $g \in G_1$, we define the support of g to be $supp(g) = \{a \in \omega \times \omega : g(a) \neq a\}$. We put:

$$G = \{g \in G_1 : ||supp(g)|| < \infty\} < G_1.$$

It is clear that for any $g \in G$ and $h \in H$, the composition $h \circ g : \omega \times \omega \to 2$ is an element of H (since $||h \circ g|| \leq ||h|| + ||supp(g)||$). Hence, we can define an action of G on H by $gh = h \circ g^{-1}$. Then, G acts on H as automorphisms (both algebraic and topological). Notice, however, that if we consider G with the product topology, then this action is not continuous. Hence, we need another topology on G.

We define a metric d on G:

$$d(f, g) = ||supp(f^{-1}g)||.$$

We will consider G with the topology generated by d.

Proposition 3.4 *G is a Polish group.*

Proof. It is easy to check that d is a complete metric on G. Also, the set of elements of G with finite support is a countable, dense subset of G. Now, we will check that the composition $\circ : G \times G \to G$ is continuous. For any $(f, g), (f_1, g_1) \in G \times G$ we have $d(fg, f_1g_1) = ||supp((fg)^{-1}f_1g_1)|| = ||supp(g^{-1}(f^{-1}f_1g_1g^{-1})g)||$. Clearly, the composition is continuous at $(e, e) \in G \times G$ (since $supp(fg) \subseteq supp(f) \cup supp(g)$), so it is enough to check that conjugating by g is continuous at $e \in G$. We will check that for every $f \in G$ conjugating by f^{-1} is continuous at $e \in G$, which is of course sufficient. Notice that $supp(fhf^{-1}) = f[supp(h)]$. For any $\epsilon > 0$ there is $n < \omega$ such that $||supp(f) \backslash n \times \omega|| < \epsilon$ (where n denotes the set $\{0, 1, \ldots, n-1\}$), and since $supp(f) \cap n \times \omega$ is finite, we can choose $m < \omega$ such that $f[\omega \times \omega \backslash m \times \omega] \subseteq \omega \times \omega \backslash n \times \omega$. So, for h so close to e that $supp(h) \subseteq \omega \times \omega \backslash m \times \omega$, we have that $f[supp(h)] \subseteq supp(h) \cup (f[supp(h)] \cap supp(f)) \subseteq supp(h) \cup (supp(f) \backslash n \times \omega)$. This shows that the conjugation by f is continuous, and hence, so is the group composition. Similarly, one checks that the group inversion on G is continuous. □

The next proposition shows that we have constructed (the first known) example of a small, non-zero-dimensional Polish G-group.

Proposition 3.5 *(H, G) is a small, Polish G-group.*

Proof. To check that the action of G on H is continuous at every $(g, h) \in G \times H$, consider any $(g_1, h_1) \in G \times H$. Then, the functions $gh = h \circ g^{-1}$ and $g_1 h_1 = h_1 \circ g_1^{-1}$ agree on the set $\{a \in \omega \times \omega : g^{-1}(a) = g_1^{-1}(a)\} \cap \{a \in \omega \times \omega : h(g^{-1}(a)) = h_1(g^{-1}(a))\}$. The complement of this set in H is the union of $supp(g_1 g^{-1})$ and $g[\{a \in \omega \times \omega : h(a) \neq h_1(a)\}]$. For (g_1, h_1) sufficiently close to (g, h) these sets are arbitrary small in the sense of $|| \cdot ||$ (by a similar argument to the one in the proof of Proposition 3.4). So, the action of G on H is continuous.

It remains to check that for every finite $A \subseteq H$, there are countably many G_A-orbits in H. Fix such an A. For any $h \in H$ we put $h_0 := h^{-1}[\{0\}], h_1 := h^{-1}[\{1\}]$ (then $||h_1|| < \infty$ and $||h_0|| = \infty$). Let B be the Boolean algebra generated by the family of sets: $\{a_0, a_1 : a \in A\}$, and let $b_0, b_1, \ldots, b_n$ be all its atoms. For exactly one $i \leq n$ we have $||b_i|| = \infty$ (this happens for i such that $b_i = \bigcap_{a \in A} a_0$), and we assume that this is the case for $i = 0$. We will show that the G_A-orbit of an element x of H depends only on the cardinalities of the sets $x_0 \cap b_0, x_0 \cap b_1, \ldots, x_0 \cap b_n$ and $x_1 \cap b_0, x_1 \cap b_1, \ldots, x_1 \cap b_n$. Suppose that for two elements $x, y \in H$ these cardinalities are the same. For all $0 \leq i \leq n$ let g_i be a permutation of b_i such that $g_i[b_i \cap x_0] = b_i \cap y_0$ and $g_i[b_i \cap x_1] = b_i \cap y_1$. We can choose g_0 in such a way that $||supp(g_0)|| < \infty$ (to see this, choose any $c \subseteq b_0$ such that $b_0 \cap x_1, b_0 \cap y_1 \subseteq c$, $c \backslash (x_1 \cup y_1)$ is infinite and $||c|| < \infty$, and notice that we can choose g_0 such that $supp(g_0) \subseteq c$). Then, $\bigcup_{i \leq n} g_i$ is an element of G_A, and $gx = y$. This completes the proof. □

Proposition 3.6 *(H, G) is not nm-stable.*

Proof. For $c \in 2^{\omega \times \omega}$, we will write c_{ij} instead of $c(i, j)$. Consider $o = o(a/\emptyset)$, where $a_{ij} = 1$ if $j = 0$ and $a_{ij} = 0$ if $j > 0$. For any $n < \omega$, let $b_n \in H$ be given by $(b_n)_{ij} = 1$ if $j = n + 1$, and $(b_n)_{ij} = 0$ if $j \neq n + 1$. Then, by the proof of Proposition 3.5, for every $n < \omega$, we have

$$o(a/b_{<n}) = \{x \in H : |(i, j) \in \omega \times \omega : x_{ij} = 1| = \omega \wedge \forall j \in \{1, 2, \ldots n\} x_{ij} = 0\}.$$

So, $o(a/b_{<n})$ is a G_δ subset of H, and hence, it is non-meager in itself. Morever, for every $n < \omega$, $o(a/b_{<n+1})$ is nowhere dense in $o(a/b_{<n})$. Thus, by Theorem 2.8, $\mathcal{NM}(o) = \infty$. So, (H, G) is not *nm*-stable. □

Similarly, one can check that $\mathcal{NM}$-rank of every uncountable 1-orbit in (H, G) is equal to ∞.

Question 3.7 *Is there an nm-stable, non-zero-dimensional small Polish G-group?*

Notice that since the product $H \times H$ is homeomorphic to H, we cannot obtain examples of higher dimensions just by taking finite cartesian powers of H.

Question 3.8 *Is there a small Polish G-group of dimension greater than one?*

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