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Grupy i pierścienie w pewnych teoriomodelowych i motywowanych teorią modeli kontekstach

Praca doktorska

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Wrocław, 2014

University of Wrocław
Mathemtical Institute

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Groups and rings in some model-theoretic and model-theory-motivated contexts

Doctoral thesis

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Wrocław, 2014

Acknowledgments

I would like to express my deepest gratitude to my supervisor Krzysztof Krupiński for his guidance and support throughout my undergraduate and graduate studies, as well as for a very careful reading of the first version of this thesis and many helpful suggestions.

I gratefully acknowledge the support from the PhD programme ”Środowiskowe Studia Doktoranckie z Nauk Matematycznych” funded by the European Social Fund.

Streszczenie

Praca zawiera pewne nowe wyniki na temat grup i pierścieni w różnych kontekstach.

W Rozdziale 1 dowodzimy, że każda ω-kategoryczna grupa generycznie stabilna jest wirtualnie nilpotentna, oraz że każdy ω-kategoryczny pierścień generycznie stabilny jest wirtualnie nilpotentny. Wyniki te uogólniają twierdzenie Baura-Cherlina-Macintyre'a, Felgnera (w przypadku grup) oraz Baldwina-Rose (w przypadku pierścieni).

W Rozdziale 2 dowodzimy pewnych strukturalnych twierdzeń na temat (słabo) lokalnie skończonych pierścieni proskończonych. Główne wyniki to pełna klasyfikacja półprostych (słabo) lokalnie skończonych pierścieni proskończonych oraz twierdzenie mówiące, że radykał Jacobsona słabo lokalnie skończonego pierścienia proskończonego jest nil skończonego nilwykładnika. Wyniki te stosują się w szczególności do klasy małych zwartych G-pierścieni, dając uogólniania pewnych wyników uzyskanych przez Krupińskiego i Wagnera dla małych pierścieni proskończonych (w sensie Newelskiego).

W Rozdziale 3 prezentujemy pewne konstrukcje małych polskich struktur grupowych, dających odpowiedzi na dwa problemy sformułowane przez Krupińskiego. Pierwsza z nich to konstrukcja pierwszej znanej niezerowymiarowej małej polskiej G-grupy, natomiast druga konstrukcja dostarcza przykładów małych polskich struktur grupowych bez orbit nm-generycznych.

W Rozdziale 4 definiujemy pewne kanoniczne topologie indukowane przez działania grup topologicznych na grupach i pierścieniach przez automorfizmy. Dla grupy [pierścienia] H oraz działania grupy topologicznej G na H opisujemy najsilniejszą topologię grupy topologicznej [pierścienia topologicznego] na H, względem której działanie G na H jest ciągłe. Badamy również wprowadzone topologie w kontekście struktur polskich, dowodząc między innymi, że może nie istnieć żadna topologia Hausdorffa na grupie H względem której ustalone działanie grupy polskiej na H jest ciągłe.

Abstract

The thesis contains some new results about groups and rings in various contexts.

In Section 1, we prove that every ω-categorical, generically stable group is nilpotent-by-finite, and that every ω-categorical, generically stable ring is nilopotent-by-finite. These theorems generalize results of Baur-Cherlin-Macintyre and Felgner (in the case of groups) and of Baldwin-Rose (in the case of rings).

In Section 2, we prove some structural results about (weakly) locally finite profinite rings. The main results are: a complete classification of semisimple (weakly) locally finite profinite rings, and a theorem stating that the Jacobson radical of every locally finite profinite ring is nil of finite nilexponent. These results apply in particular to the class of small compact G-rings, yielding generalizations of certain results obtained by Krupiński and Wagner for small profinite rings (in the sense of Newelski).

In Section 3, we present certain constructions of examples of small Polish group structures which solve two problems stated by Krupiński. First of them is the construction of the first known non-zero dimensional small Polish G-group, and the second yields examples of small Polish group structures without nm-generic orbits.

In Section 4, we introduce some canonical topologies induced by actions of topological groups on groups and rings. For H being a group [or a ring] and G a topological group acting on H as automorphisms, we describe the finest group [ring] topology on H under which the action of G on H is continuous. We also study the topologies that we introduced in the context of Polish structures. In particular, we prove that there may be no Hausdorff topology on a group H under which a given action of a Polish group on H is continuous.

Contents

0 Introduction

Model theory as a subject has been existing since approximately the middle of the 20th century, and has been developing very rapidly in recent years. It is a deep branch of mathematics, which produces plenty of interesting mathematical problems itself, and also has various applications in other areas of mathematics. Among the most spectacular applications of model theory there are: the proof of Mordel-Lang conjecture ([19]) and the proof of Manin-Mumford conjecture ([20]).

One of the aims of model-theoretic algebra is to investigate the structure of the classical algebraic objects (such as groups and rings) satisfying some natural model-theoretic (i.e. concerning first-order logic) assumptions. The goal of this dissertation is to present some new results of this kind, as well as results about groups and rings in some contexts motivated by model theory.

The thesis splits naturally into two parts, first of them dealing with some problems in a first-order logic setting (Section 1), and the second one devoted to investigations motivated by model theory (Sections 2, 3 and 4).

In Section 1, we consider so-called ω-categorical structures satisfying some natural model-theoretic assumptions. We prove two main theorems, one of them dealing with groups definable in such structures, and the other dealing with rings. Both results are generalizations of now-classical theorems obtained in the 1970s for ω-categorical, stable groups and rings.

Main motivations for investigations contained in Sections 2, 3 and 4 are related to the notion of a Polish structure (see Definition 0.8) which was introduced by Krupiński in order to apply model-theoretic ideas to purely topological objects. The tools developed by Krupiński in this context provide a new insight into descriptive set theory, and lead to an innovative linkage of model-theoretic and descriptive-set-theoretic techniques. Some problems about groups and rings have been already considered in this context, as well as in the context of profinite structures (introduced by Newelski), which form a subclass of the class of Polish structures. In the thesis, we deal both with problems which are a natural continuation of the research that have been done in this area, and with much less explored issues. In Section 2, we prove some structural results about locally finite profinite rings that apply also in a certain context involving Polish structures. In Section 3, we present certain constructions of Polish group structures which solve two problems stated by Krupiński in [26]. In Section 4, we introduce some canonical topologies induced by actions of topological groups, and we study them in the contexts of Polish structures.

We shall try to emphasise the analogies between problems stated in different settings, and to enlighten the presence of model-theoretic ideas beyond the first-order logic. Particularly close similarities will be visible between the investigation of ω-categorical rings (with which we deal in Section 1) and small compact G-rings (with which we deal in Section 2).

Many of the results of Sections 2 and 4 are stated in fairly general (algebraic or topological) settings, which seems to make them interesting even regardless of the model-theoretic motivations.

0.1 Overview of the results of Section 1

It is natural to ask how precisely can a (first-order) theory describe the structure of its models. Since any theory with infinite models has models of arbitrarily large cardinalities, it never happens that all models of such a theory are isomorphic. It can, however, happen that all countably infinite models of a given theory (in a countable language) are isomorphic. We call such theories ω-categorical. A structure M for a language L is called ω-categorical if its theory (i.e. the collection of all L-sentences true in M) is such.

A general motivation for the considerations contained in Section 1 is to understand the structure of ω-categorical groups and rings satisfying various natural model-theoretic assumptions.

Let us recall some basic notions from ring theory. An element r of a ring R is nilpotent of nilexponent n if $r^n = 0$ and n is the smallest number with this property. The ring is nil [of nilexponent n] if every element is nilpotent [of nilexponent $\leqslant n$ and there is an element of nilexponent n]. The ring is nilpotent of class n if $r_1 \cdots r_n = 0$ for all $r_1, \ldots, r_n \in R$ and n is the smallest number with this property. An element r is null if $rR = Rr = \{0\}$. The ring is null if all its elements are.

The core of model theory is the stability theory, whose ideas often are applied also to non-stable theories, as well as to objects from classical mathematics. The fundamental theorem about ω-categorical groups, proved by Baur, Cherlin and Macintyre in [3] and by Felgner in [17] says that ω-categorical stable groups are nilpotent-by-finite. A long-standing conjecture states that they are even abelian-by-finite, which is known to be true under a stronger assumption of superstability. As to the ω-categorical, stable rings, they are nilpotent-by-finite [2], and it is conjectured that they are null-by-finite. As for groups, this conjecture is known to be true in the superstable case.

Among the most important generalizations of the notion of stability are: the negation of the independence property (NIP) and the negation of the strict order property (NSOP). It was proved by Shelah that stable theories are precisely those which satisfy both NIP and NSOP. There are many generalizations and variants of the results mentioned in the previous paragraph. For example, ω-categorical groups with NSOP (the negation of the strict order property) are nilpotent-by-finite [30], and ω-categorical rings with NSOP are nilpotent-by-finite [25], too.

More recently, in [24], an analysis of ω-categorical groups and rings in the NIP environment has been undertaken. It was proved there that ω-categorical rings with NIP are nilpotent-by-finite, and it was conjectured that ω-categorical groups with NIP are nilpotent-by-finite, too.

Another generalization of the notion of a stable group is a generically stable group [21, Definition 6.3]. Also, one can consider the notion of a generically stable ring which generalizes the notion of a stable ring. The main source of unstable examples of generically stable groups and rings is the theory of algebraically closed valued fields (ACVF). The following theorem was proved in [11] (that paper was based on my Master's thesis).

Fact 0.1 *Every ω-categorical, generically stable group is solvable-by-finite.*

In Section 1, using the above theorem, we obtain the following generalizations of theorems of Baur-Cherlin-Macintyre and Baldwin-Rose (to be more precise, in the proof of Theorem 0.3, we will use an appropriate variant of Fact 0.1).

Theorem 0.2 *Every ω-categorical, generically stable group is nilpotent-by-finite.*

Theorem 0.3 *Every ω-categorical, generically stable ring is nilpotent-by-finite.*

Theorem 0.2 proves [11, Conjecture 3.5], and Theorem 0.3 answers [11, Question 3.6] in the affirmative.

Moreover, Theorem 0.2 generalizes [24, Theorem 3.4], whose assumptions are equivalent to the assumptions of Theorem 0.2 together with the NIP assumption.

In contrast to the previous generalizations of Bauer-Cherlin-Macintyre and Baldwin-Rose theorems, the assumption of generic stability that we use is not a global property of a theory (it is not true that a group interpretable in a generically stable group is also generically stable). This forces us to use some new ideas and techniques in our proofs.

The results of Section 1 are also included in [12].

0.2 Small profinite structures and Polish structures

Since the considerations contained in Sections 2, 3 and 4 are motivated mainly by problems concerning the classes of small profinite structures and of Polish structures, we shall now introduce these concepts.

By a profinite topological space X we mean the inverse limit of a countable system of finite discrete topological spaces. Newelski considered structures consisting of a profinite topological space and a group of its automorphisms:

Definition 0.4 *A profinite structure is a pair $(X, Aut^*(X))$ consisting of a profinite topological space X and a closed subgroup $Aut^*(X)$ of the group of all homeomorphisms of X respecting a distinguished inverse system defining X (the topology is inherited from the product topology on X^X).*

His idea was to look at the pair $(X, Aut^*(X))$ like at a pair (M, G), where M is a first-order structure and G is the group of all its automorphisms. This correspondence allows to adopt many model-theoretic notions to the context of profinite structures. One of them is the assumption of smallness of a structure, which allowed to use various ideas from stability theory in studying profinite structures:

Definition 0.5 *We say that $(X, Aut^*(X))$ is small, if for every natural number n there are only countably many orbits on X^n under the action of $Aut^*(X)$.*

A fundamental concept for stability theory is the notion of forking independence. Newelski introduced an analogue (called m-independence) of this relation in the context of profinite structures.

Definition 0.6 *Let $a \subseteq X$ be a finite tuple, and $A, B \subseteq X$ be finite. We say that a is m-independent from B over A (written $a \overset{m}{\mathop{\perp}}_A B$) if $o(a/AB)$ (the orbit of a under the action of $Aut^*(X/AB)$) is open in $o(a/A)$. Otherwise we say that a is m-dependent on B over A, and write $a \overset{m}{\not\perp}_A B$ (in this case $o(a/AB)$ is nowhere dense in $o(a/A)$).*

In the case of stability theory, using the relation of forking independence, one defines the U-rank, which is one of the main tools in investigating superstable theories. Similarly, the notion of m-independence leads to the notion of $\mathcal{M}$-rank:

Definition 0.7 *$\mathcal{M}$-rank is the only function $\mathcal{M} : \{o(a/A) : a \in X, A \subseteq X, |A| < \omega\} \to Ord \cup \{\infty\}$ such that $\mathcal{M}(a/A) \geqslant \alpha + 1$ iff there is a finite $B \supseteq A$ such that $a \overset{m}{\not\perp}_A B$ and $\mathcal{M}(a/B) \geqslant \alpha$.*

These tools allowed to study small profinite structures by model-theoretic methods. Krupiński generalized these concepts to a much wider class of topological objects, which he called Polish structures.

Definition 0.8 *A Polish structure is a pair (X, G), where G is a Polish group acting faithfully on a set X so that the stabilizers of all singletons are closed subgroups of G. We say that (X, G) is small if for every $n < \omega$, there are only countably many orbits on X^n under the action of G.*

The class of Polish structures contains many more interesting objects from classical mathematics than the class of profinite structures. For a compact metric space X, by Homeo(X) we denote the group of all homeomorphisms of X equipped with the compact-open topology. Recall that Homeo(X) is a Polish group and $(X, \text{Homeo}(X))$ is a Polish structure. There are various examples of this kind which are small Polish structures. We list some examples of small Polish structures found by Krupiński.

1. $(S^n, \text{Homeo}(S^n))$, $n \in \omega$, where S^n is the n-dimensional sphere,
2. $(I^n, \text{Homeo}(I^n))$, $n \in \omega \cup \{\omega\}$, where I^n is the n-dimensional cube,
3. $((S^1)^n, \text{Homeo}((S^1)^n))$, $n \in \omega \cup \{\omega\}$,
4. $(P, \text{Homeo}(P))$, P - the pseudo-arc,
5. $(H, \text{Aut}(H))$, H - a profinite abelian group of finite exponent, Aut(H) - the group of all topological automorphisms of H,
6. $(H, \text{Aut}^0(H))$, H - as above, Aut$^0(H)$ - the group of all automorphisms of H preserving a distinguished inverse system indexed by ω.

Since, in general, we do not consider any topology on X (and even if we do, G-orbits need not to have good topological properties), Krupiński used the topology on G to define the notion of nm-independdce, which is a generalization of m-independence. Recall that a subset of a Polish space is called meager, if it is a union of countably many nowhere dense sets.

Definition 0.9 *Let (X, G) be a Polish structure. For any finite $C \subseteq X$, by G_C we denote the pointwise stabilizer of C in G. Let a be a finite tuple and A, B finite subsets of X. Let $\pi_A : G_A \to o(a/A)$ be defined by $\pi_A(g) = ga$. We say that a is nm-independent from B over A (written $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$) if $\pi_A^{-1}[o(a/AB)]$ is non-meager in $\pi_A^{-1}[o(a/A)]$. Otherwise, we say that a is nm-dependent on B over A (written $a \not\mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$).*

The notion of nm-dependence leads to the definition of $\mathcal{NM}$-rank, which generalizes $\mathcal{M}$-rank.

Definition 0.10 *The $\mathcal{NM}$-rank is the unique function from the collection of orbits over finite sets to the ordinals together with ∞, satisfying*
$\mathcal{NM}(a/A) \geqslant \alpha + 1$ iff there is a finite set $B \supseteq A$ such that $a \not\mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$ and $\mathcal{NM}(a/B) \geqslant \alpha$.
The $\mathcal{NM}$-rank of X is defined as the supremum of $\mathcal{NM}(x/\emptyset)$, $x \in X$.
We say that a Polish structure (X, G) is nm-stable, if $\mathcal{NM}(X) < \infty$.

It was proved in [26] that in small Polish structures, nm-independence has similar properties to those of forking independence in stable theories. More precisely, by Theorems 2.5 and 2.10 from [26], we have:

Fact 0.11 *In any Polish structure (X, G), nm-independence has the following properties:*
(0) (Invariance) $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B \iff g(a) \mathop{\smile\!\!\!\!\!\mid}^{nm}_{g[A]} g[B]$ whenever $g \in G$ and $a, A, B \subseteq X$ are finite.
(1) (Symmetry) $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_C b \iff b \mathop{\smile\!\!\!\!\!\mid}^{nm}_C a$ for every finite $a, b, C \subseteq X$.
(2) (Transitivity) $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_B C$ and $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$ iff $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_A C$ for every finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$.
(3) For every finite $A \subseteq X$, $a \in Acl(A)$ iff for all finite $B \subseteq X$ we have $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$.
If additionally (X, G) is small, then we also have:
(4) (Existence of nm-independent extensions) For all finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there is $b \in o(a/A)$ such that $b \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$.

In consequence, a counterpart of basic stability theory was developed for small Polish structures. The investigation of Polish structures has been undertaken in [5] and [27], and presently, in [9], [10] and [13].

Definition 0.12 *Let G be a Polish group.*
(i) A Polish group structure is a Polish structure (H, G) such that H is a group and G acts as a group of automorphisms of H.
(ii) A (topological) G-group is a Polish group structure (H, G) such that H is a topological group and the action of G on H is continuous.
(iii) A Polish [compact] G-group is a topological G-group (H, G), where H is a Polish [compact] group.

In the same way we define a Polish ring structure, G-rings, Polish G-rings and compact G-rings.

We wish to remark that, although small compact G-groups are profinite groups [26, Corollary 5.9], they form a much bigger class of structures than small profinite groups, since the group G need not to be compact.

0.3 Overview of the results of Section 2

By a locally finite ring we mean a ring whose every finitely generated subring is finite and by a profinite ring we mean the limit of any inverse system of finite rings.

In Section 2, our general motivation is to understand the structure of groups and rings in the context of small Polish structures. A particularly interesting and accessible situation is the case of small compact G-groups and G-rings. The initial motivation for the problems that we focused on was to describe the structure of small compact G-rings. Under the additional assumption of nm-stability, we know a lot by [25, 27]. For example, [25, Theorem 3.2] tells us that small, nm-stable compact G-rings are nilpotent-by-finite, and it is conjectured that they are null-by-finite, which was confirmed under the assumption that the $\mathcal{NM}$-rank of the ring is less than ω. However, without nm-stability, not much is known. We know for example that the Cartesian power R^ω of any finite ring R, considered together with the Polish group G of all permutations of coordinates, is a small compact G-ring which, of course, does not need to contain nilpotent elements.

Each small compact G-ring is a locally finite profinite ring (see Fact 2.6). So, our goal is to describe the structure of locally finite profinite rings admitting a structure of a small compact G-ring. It turns out, however, that our proofs work in the general context of locally finite profinite rings, or even more generally, in the context of profinite rings whose all 1-generated subrings are finite. So, the main structural results of Subsection 2.2 are stated in this general context, without using the notion of small compact G-rings, except for Corollary 2.17, where we additionally describe the action of G on the ring R.

Definition 0.13 *We say that a ring R is* weakly locally finite*, if every 1-generated subring of R is finite.*

In order to describe the structure of a ring R, it is important to understand the structure of the semisimple ring $R/J(R)$ and of the radical ring $J(R)$, where $J(R)$ is the Jacobson radical of R. The main results of this paper are: a complete classification of semisimple, weakly locally finite profinite rings established in Theorem 0.14, and important information on the structure of the Jacobson radical of weakly locally finite profinite rings obtained in Theorem 0.15.

Theorem 0.14 *Let R be a topological ring. Then, R is a semisimple, weakly locally finite profinite ring if and only if R is isomorphic (as a topological ring) to a direct product of complete matrix rings over finite fields with only finitely many non-isomorphic rings occurring as factors in this product.*

From this classification, one immediately concludes that semisimple, weakly locally finite profinite rings coincide with semisimple, locally finite profinite rings.

The above theorem not only yields a complete classification of the class of semisimple locally finite profinite rings, but also of the class of semisimple rings admitting a structure of a small compact G-ring (see Corollary 2.16); in Corollary 2.17, we also describe possible actions of G on the ring in question.

A problem which is "complementary" to the description of semisimple rings from a given class is the problem of describing the Jacobson radical of rings belonging to that class. Our second main result is the following (see also Corollary 2.3).

Theorem 0.15 *If R is a weakly locally finite profinite ring, then $J(R)$ is nil of finite nilexponent. More generally, each nil profinite ring has finite nilexponent (in the view of Corollary 2.3, this is indeed a more general statement).*

In particular, the Jacobson radical of each small compact G-ring is nil of finite nilexponent. This generalizes a similar result proved in [28] for small profinite rings. Recall that a small profinite ring [group] is the limit R of a countable inverse system of finite rings [groups] together with a closed subgroup $Aut^*(R)$ of the group of all automorphisms respecting the distinguished inverse system such that $Aut^*(R)$ has only countably many orbits on n-tuples for all $n < \omega$. In particular, every small profinite ring [group] is a small compact G-ring [G-group] (with $G := Aut^*(R)$).

It was proved in [28] that the Jacobson radical of a small profinite ring is open, and the following conjectures were formulated.

Conjecture 0.16 *The Jacobson radical of a small profinite ring R is nilpotent. In particular, R has an open nilpotent ideal.*

Conjecture 0.17 *A small profinite ring has an open null ideal.*

These conjectures are interesting in their own rights, but an additional motivation standing behind them comes from [28] and [25]. To explain this, recall the main conjecture concerning small profinite groups.

Conjecture 0.18 *A small profinite group has an open abelian subgroup.*

By [25, Corollary 2.4] (see also [25, Corollary 3.13]), we know that Conjecture 0.18 implies Conjecture 0.17 which, of course, implies Conjecture 0.16. On the other hand, [28, Theorem 3.5] tells us that Conjecture 0.16 for commutative rings implies an important intermediate conjecture towards the proof of Conjecture 0.18, namely that each small solvable profinite group has an open nilpotent subgroup; using [25, Theorem 2.10], we get that Conjecture 0.17 implies that each small solvable profinite group has an open abelian subgroup.

Several reductions of Conjecture 0.16 for commutative rings were obtained in [28]. In Subsection 2.3, we prove some further reductions of that conjecture. The main one (see Proposition 2.24) roughly says that if Conjecture 0.16 is false, then there is a counterexample which generically does not satisfy any polynomial identities which are not satisfied obvious reasons. At the end, notice that the counterpart of the second part of Conjecture 0.16 for small compact G-rings is false by Theorem 0.14. As to the first part, we do not know.

Question 0.19 *Is it true that the Jacobson radical of each small compact G-ring is nilpotent?*

It is clear, however, that the Jacobson radical of a small compact G-ring does not need to be null-by-finite. For this, take the Cartesian power R^ω of any non-null finite nil ring R and consider it together with the Polish group G of all permutations of coordinates. This is a nilpotent, small compact G-ring which is not null-by-finite.

Main results of Section 2 are also included in [13] .

0.4 Overview of the results of Section 3

Recall that all small compact G-groups are profinite, and hence zero-dimensional. It is natural to ask if we can obtain non-zero-dimensional examples if we drop the compactness assumption. The following problem was formulated in [26] (after Question 5.32):

Problem 0.20 *Find a non-zero-dimensional, small Polish G-group.*

In Subsection 3.1, we construct a small Polish G-group (H, G), such that H is homeomorphic to the complete Erdös space, which is known to be one-dimensional.

Another problem from [26] that we deal with is related to the notion of an *nm*-generic orbit (see Definition 3.9). An important motivation for considering this notion is the use of it in the proofs of the main results of [27]. The following is [26, Question 5.4]:

Question 0.21 *Let (H, G) be a small Polish group structure. Does H possess an nm-generic orbit?*

Proposition 5.5 from [26] gives us a positive answer to Question 0.21 in the class of small Polish G-groups. In Subsection 3.2, we construct a class of small Polish group structures for which the answer to Question 0.21 is negative.

The constructions from Subsections 3.1, 3.2 can also be found in [9].

0.5 Overview of the results of Section 4

In Subsection 4.1, we deal with the following general problem.

Problem 0.22 *Suppose G is a topological group acting on X, where X is a set, possibly equipped with some algebraic structure preserved by the action of G. When does there exist a "nice" topology on X, such that the action of G on X is continuous, and the topology is compatible with the structure on X?*

We notice that the situation is quite clear when X is a pure set, and, in the main results of the subsection, we describe the finest compatible topology on X in the case when it is equipped with a structure of a group or a ring.

In Subsection 4.2, we discuss the meaning of topologies introduced in 4.1 in the context of Polish structures. We also show that the finest group topology on a group X under which a given action of a Polish group on X is continuous may fail to be Hausdorff.

The results of Section 4 are included in [10].

1 ω-categorical structures

1.1 Preliminaries

By a type (in a first-order theory) we mean a consistent set of formulas in a fixed (possibly infinite) tuple of variables. By a complete type over a set of parameters A we mean a maximal type consisting of formulas with parameters in A. By $S(A)$ we denote the set of complete types (of a fixed arity) over A. A type is isolated if it contains a formula implying the type. If M is a model and $\Phi(x)$ is a formula or a type, then we set $\Phi(M) := \{a \in M^{|x|} : a \models \Phi\}$. We say that a model M is κ-saturated if all the types over subsets of M of a cardinality smaller that κ are realized in M. A M is strongly κ-homogeneous if every partial elementary map $M \to M$ with domain of cardinality smaller that κ extends to an automorphism of M. By a monster model we mean a κ-saturated and strongly κ-homogeneous model $\mathfrak{C}$, where κ is a cardinal big enough to allow us to work inside the model $\mathfrak{C}$ instead of working in all models of the theory.

Recall that a first order structure M in a countable language is said to be ω-categorical if, up to isomorphism, $\mathrm{Th}(M)$ has at most one model of cardinality $\aleph_0$. By Ryll-Nardzewski's theorem, this is equivalent to the condition that for every natural number n there are only finitely many complete n-types over $\emptyset$. Also, an equivalent condition is that every complete type over a finite set is isolated.

Assume M is ω-categorical. If M is countable or a monster model, two finite tuples have the same type over $\emptyset$ iff they lie in the same orbit under the action of the automorphism group of M, and hence for each natural number n the automorphism group of M has only finitely many orbits on n-tuples (which implies that M is locally finite). Moreover, for any finite subset A of such an M, a subset D of M is A-invariant iff D is A-definable. Indeed, since D is A-invariant, its set of realization is a union of (finitely many) sets of realizations of complete types over A, and every such set of realization is A-definable (by a formula isolating the type). The following fact was proved in [37]:

Fact 1.1 *For each infinite, countable, ω-categorical, characteristically simple group H, one of the following holds.*

(i) For some prime number p, H is an elementary abelian p-group (i.e. an abelian group in which every non-trivial element has order p).

(ii) $H \cong B(F)$ or $H \cong B^-(F)$ for some non-abelian, finite, simple group F, where $B(F)$ is the group of all continuous functions from the Cantor set $\mathcal{C}$ to F, and $B^-(F)$ is the subgroup of $B(F)$ consisting of the functions f such that $f(x_0) = e$ for a fixed element $x_0 \in \mathcal{C}$.

(iii) H is a perfect p-group (perfect means that H equals its commutator subgroup).

Let T be a first order theory. We work in a monster model $\mathfrak{C}$ of T. We say that a cardinal is bounded if it is smaller than κ. A subset $A \subseteq \mathfrak{C}$ is called small, if $|A|$

is bounded. Usually we consider only small sets of parameters, unless we consider global types, i.e. types over the monster model.

Let $p \in S(\mathfrak{C})$ be invariant over $A \subset \mathfrak{C}$. We say that $(a_i)_{i\in\omega}$ is a Morley sequence in p over A if $a_i \models p|Aa_{<i}$ for all i. Morley sequences in p over A are indiscernible over A and they have the same type over A (as ordered tuples). If $\mathfrak{C}' \succ \mathfrak{C}$ is a bigger monster model, then the generalized defining scheme of p determines a unique A-invariant extension $\widetilde{p} \in S(\mathfrak{C}')$ of p (by the generalized defining scheme of p we mean a family of sets $\{p_i^\varphi : i \in I_\varphi\}$ (with $\varphi(x,y)$ ranging over all formulas without parameters) of complete types over A such that $\varphi(x,c) \in p$ iff $c \in \bigcup_{i\in I_\varphi} p_i^\varphi(\mathfrak{C})$). By a Morley sequence in p we mean a Morley sequence in $\widetilde{p}$ over $\mathfrak{C}$. Finally, $p^{(k)}$ (where $k \in \omega \cup \{\omega\}$) denotes the type over $\mathfrak{C}$ of a Morley sequence in p of length k.

Definition 1.2 *For a small $A \subset \mathfrak{C}$, a global type $p \in S(\mathfrak{C})$ is said to be generically stable over A if it is A-invariant and for each formula $\varphi(x;y)$ there is a natural number m such that for any Morley sequence $(a_i : i < \omega)$ in p over A and any b from $\mathfrak{C}$ either less than m a_i's satisfy $\varphi(b;y)$ or less than m a_i's satisfy $\neg\varphi(b;y)$. A global type $p \in S(\mathfrak{C})$ is said to be generically stable if it is generically stable over some small $A \subset \mathfrak{C}$.*

It is well-known that in stable theories every type satisfies the above definition.

Let p be a global type invariant over A. Since the type over A of a Morley sequence in p over A does not depend on the choice of this Morley sequence, one can easily check that a generically stable type is in fact generically stable over any small set of parameters over which it is invariant.

Recently, Adler, Casanovas and Pillay have found an example of an ω-categorical theory and a generically stable type p for which $p^{(2)}$ is not generically stable (see [1, Example 1.7]). This shows that certain extra arguments that we use in this thesis in order to deal with the case when some powers of a generically stable type are not generically stable cannot be omitted.

Recall [33, Proposition 2.1].

Fact 1.3 *If p is generically stable over A, then any Morley sequence in p over A is an indiscernible set over A. In particular, a Morley sequence in p (over $\mathfrak{C}$) is an indiscernible set over $\mathfrak{C}$.*

The following observation was made in [11, Proposition 1.2].

Fact 1.4 *Let $p = tp(a/\mathfrak{C})$ be a type generically stable over A, and assume that $b \in dcl(A,a)$. Then $tp(b/\mathfrak{C})$ is also generically stable over A.*

The next lemma is a variant of a similar result for NIP groups (see [36, Theorem 1.0.5]), and its proof is very similar to the proof of [36, Theorem 1.0.5]. This is also a slight modification of [11, Lemma 2.1(i)]; in fact, it easily implies [11, Lemma 2.1(i)].

Lemma 1.5 *Let G be a group which is $\emptyset$-definable in $\mathfrak{C}$ by a formula $G(x)$. Assume that $p \in S(\mathfrak{C})$ is generically stable over A. Let $H(x, \bar{z}; y)$ be a formula defining a family of groups $H(G, c; g)$, $g \in (p|A)(\mathfrak{C})$, $c \in D$ (D is a definable set). Then, there is $N < \omega$ such that for any $c \in D$, $n \in \omega$ and $(g_1, \dots, g_n) \models p^{(n)}|A$ there are $i_1, \dots, i_N \in \{1, \dots, n\}$ for which*

$$\bigcap_{i=1}^{n} H(G, c; g_i) = \bigcap_{j=1}^{N} H(G, c; g_{i_j}).$$

Proof. Let $m > 0$ be such as in the definition of generic stability for p and $H(x, \bar{z}; y)$. We will show that $N := 2m$ satisfies our requirements. Suppose it is not the case. Then, there is $n > N$ (even $n = N + 1$ works) such that for some $c \in D$ and $(g_1, \dots, g_n) \models p^{(n)}|A$ the intersection $\bigcap_{i=1}^{n} H(G, c; g_i)$ is not an intersection of at most $n - 1$ groups among $H(G, c; g_i)$, $i = 1, \dots, n$. Hence, for each $j \in \{1, \dots, n\}$ there exists

$$a_j \in \bigcap_{i \neq j} H(G, c; g_i) \setminus \bigcap_{i=1}^{n} H(G, c; g_i).$$

Put $b = \prod_{j=1}^{m} a_j$. We see that

$$b \in H(G, c; g_i) \iff i \in \{m + 1, \dots, n\},$$

which contradicts the choice of m. $\square$

Recall that a subset of a group G is said to be left generic if finitely many left translates of this set cover G. Assuming that the group G is definable in a model M, a formula $\varphi(x)$ is called left generic if the set $\varphi(G)$ is left generic, and a type is said to be left generic (of G or in G) if every formula in it is left generic. The definitions of right generic sets, formulas and types are analogous. Finally, a subset, formula or type is called generic if it is both left generic and right generic.

Definition 1.6 *Let G be a group definable in $\mathfrak{C}$ by a formula $G(x)$. G has finitely satisfiable generics (fsg) if there is a global type p containing $G(x)$ and a model $M \prec \mathfrak{C}$, of cardinality less than the degree of saturation of $\mathfrak{C}$, such that for all $g \in G$, gp is finitely satisfiable in M.*

Let G be a group $\emptyset$-definable in $\mathfrak{C}$. By G^{00} we will denote the smallest type-definable (i.e. definable by a type over a small set of parameters) subgroup of bounded index (if it exists). We do not know whether G^{00} always exists when T is ω-categorical. Notice, however, that if it exists, then, being $\emptyset$-invariant, it must be $\emptyset$-definable, and hence of finite index in G. (To see this, let ϕ be a formula defining G^{00} and suppose G^{00} has infinite index in G. Then, the type $\{\neg\phi(x_i^{-1}x_j) : i \neq j, i, j < \lambda\}$ is consistent for any cardinal λ, so we can find unboundedly many elements of G lying in different G^{00}-cosets.)

The following fact was proved in [22, Section 4].

Fact 1.7 *Suppose G has fsg, witnessed by p. Then:*
(i) a formula is left generic iff it is right generic (iff it is generic),
(ii) p is generic,
(iii) the family of non-generic sets forms an ideal, so any partial generic type can be extended to a global generic type,
(iv) G^{00} exists, it is type-definable over $\emptyset$, and it is the stabilizer of any global generic type of G.

Recall [14, Proposition 0.26].

Fact 1.8 *Suppose G has fsg and G^{00} is definable. Then G^{00} has a unique global generic type.*

The next definition was introduced in [21, Section 6].

Definition 1.9 *(i) Let G be a group definable in $\mathfrak{C}$. We say that G is generically stable if it has fsg and some global generic type of G is generically stable.*
(ii) Let R be a ring definable in $\mathfrak{C}$. We say that R is generically stable if its additive group is generically stable.

We say that a group [or a ring] definable in a non-saturated model is generically stable if the group [or the ring] defined by the same formula in a monster model is such.

When we are talking about an ω-categorical, generically stable group [or ring], we mean a generically stable group G [or ring R] definable in a monster model $\mathfrak{C}$ of an ω-categorical theory. Replacing $\mathfrak{C}$ by G [or by R] equipped with the structure induced from $\mathfrak{C}$ (i.e. with predicates for all $\emptyset$-definable subsets of all finite Cartesian powers of G [or R]), neither ω-categoricity nor generic stability is lost. So, whenever we want to prove some algebraic properties of G [or R], we can assume that $\mathfrak{C} = G$ [or $\mathfrak{C} = R$] (possibly with some extra structure).

We say that G is connected if it does not have a proper, definable subgroup of finite index, and we will say that G is absolutely connected if it does not have a proper, type-definable subgroup of bounded index (i.e. $G = G^{00}$).

Throughout this thesis, rings are associative by definition, but they are not assumed to contain 1 or to be commutative. Let us recall fundamental issues concerning Jacobson radicals; for more details see [18]. The Jacobson radical of a ring R, denoted by $J(R)$, is the collection of all elements of R satisfying the formula $\phi(x) = \forall y \exists z (yx + z + zyx = 0)$ (that is, it is the set of all elements which generate quasi-regular left ideals, i.e. left ideals consisting of left quasi-regular elements which are defined as those elements $x \in R$ for which there is $z \in R$ such that $x+z+zx = 0$). Equivalently, $J(R)$ is the unique largest quasi-regular left [or right] ideal. Another equivalent definition says that $J(R)$ is the intersection of all the maximal regular left [or right] ideals, where a left ideal I is said to be regular if there is $a \in R$ such that $x - xa \in I$ for all $x \in R$ (notice that for rings with 1 all ideals are regular). For any ring R, $J(R)$ is a two-sided ideal. We say that R is semisimple if $J(R) = \{0\}$.

$R/J(R)$ is always a semisimple ring. Every nil left [or right] ideal is contained in $J(R)$; in particular, if R is nil, then $J(R) = R$.

Recall that a ring R is a subdirect product of rings $R_i, i \in I$, if there is a monomorphism of R into $\prod_{i\in I} R_i$ whose image projects onto each R_i. The following fact is [2, Corollary 1].

Fact 1.10 *If R is a semisimple, ω-categorical ring, then R is a subdirect product of complete matrix rings over finite fields. Moreover, only finitely many different matrix rings occur as subdirect factors.*

By [2, Lemma 1.3] and [6] we have:

Fact 1.11 *If R is an ω-categorical ring, then $J(R)$ is nilpotent.*

1.2 ω-categorical, generically stable rings

This subsection is devoted to the proof of Theorem 0.3 from the introduction. After a reduction to the situation when there is a unique global generic type, our proof splits into two cases depending on whether the generic type has non-nilpotent or nilpotent realizations. If they are non-nilpotent, the proof is a slight elaboration of the proof of [24, Theorem 2.1], which is based on Facts 1.10 and 1.11. For the reader's convenience, we include most of the details. The argument in the nilpotent case is completely different; in particular, it uses a variant of Fact 0.1 and some ideas from the proof of [25, Theorem 2.1(i)]. It will be noted in the course of the proof that if the ring in question is commutative, then it is enough to consider the non-nilpotent case.

Since [1, Example 1.7] shows that the generic stability of a type p does not imply the generic stability of all its powers $p^{(n)}$, $n \geqslant 1$, we will see that Fact 0.1 is too weak to complete the proof of Theorem 0.3 in the nilpotent case. Actually, we have to use a certain variant (in fact, strengthening) of Fact 0.1. Literally, it will be a strengthening of [11, Theorem 2.3], obtained by the same proof as in [11] modulo natural modifications and application of an appropriate strengthening of [11, Lemma 2.2] which we prove below.

For a subset A of a group G, we denote by $C_G(A)$ its centralizer in G.

Lemma 1.12 *Let G be a $\emptyset$-definable group in a monster model $\mathfrak{C}$ of an ω-categorical theory. Assume that $G_1 \trianglelefteq G$ is infinite, $\emptyset$-definable, and characteristically simple in $(G, \mathfrak{C})$, i.e. it has no non-trivial, proper subgroup which is invariant under conjugations by the elements of G and invariant under $Aut(\mathfrak{C})$. Let $p \in S(\mathfrak{C})$ be a type generically stable over $\emptyset$. Suppose that for some $\emptyset$-definable function f and a Morley sequence $(g_i)_{i<\omega}$ in p over $\emptyset$, $f(g_0, \dots, g_{k-1}) \in G_1 \setminus \{e\}$. Assume additionally that whenever $(h_i)_{i<k}$ is a Morley sequence in p over some $g \in G$, then the conjugate $f(h_0, \dots, h_{k-1})^g$ equals $f(h'_0, \dots, h'_{k-1})$ for some Morley sequence $(h'_i)_{i<k}$ in p over g.*

Then G_1 is abelian.

Proof. Define

$$H = \bigcap_{i_1 < \dots < i_k} C_{G_1}(f(g_{i_1}, \dots, g_{i_k})).$$

From [11, Lemma 2.1(ii)], we have that there is some $n < \omega$ such that for every $S \subseteq \omega$ of cardinality n,

$$H = \bigcap_{i_1 < \dots < i_k, i_1, \dots, i_k \in S} C_{G_1}(f(g_{i_1}, \dots, g_{i_k})).$$

It follows exactly as in the proof of [11, Lemma 2.2] that H is invariant under $Aut(\mathfrak{C})$, but we repeat the argument for the reader's convenience. Take any $h \in Aut(\mathfrak{C})$. Put $a_i = h(g_i)$, and choose a Morley sequence $(b_i)_{i<\omega}$ in p over $\{a_i, g_i : i < \omega\}$. Notice that the sequences $(g_i : i < \omega)^\frown(b_i : i < \omega)$ and $(a_i : i < \omega)^\frown(b_i : i < \omega)$ are Morley sequences in p over $\emptyset$, and thus they are indiscernible as sets (by Fact 1.3). Therefore,

$$\begin{aligned} H &= \textstyle\bigcap_{i_1<\dots<i_k<n} C_{G_1}(f(b_{i_1}, \dots, b_{i_k})) = \bigcap_{i_1<\dots<i_k<\omega} C_{G_1}(f(a_{i_1}, \dots, a_{i_k})) = \\ &= \textstyle\bigcap_{i_1<\dots<i_k<\omega} C_{G_1}(f(h(g_{i_1}), \dots, h(g_{i_k}))) = \bigcap_{i_1<\dots<i_k<\omega} C_{G_1}(h(f(g_{i_1}, \dots, g_{i_k}))) = \\ &= \textstyle\bigcap_{i_1<\dots<i_k<\omega} h[C_{G_1}(f(g_{i_1}, \dots, g_{i_k}))] = h[H], \end{aligned}$$

and so H is invariant under $Aut(\mathfrak{C})$.

Now, we will show that H is normal in G. Consider any $g \in G$. Choose a Morley sequence $(h_i)_{i<\omega}$ in p over g. By the uniqueness of a Morley sequence in p over $\emptyset$ up to the type (and by the strong κ-homogeneity of $\mathfrak{C}$), we can find $\alpha \in \mathrm{Aut}(\mathfrak{C})$ sending each h_i to g_i. Then, by the invariance of H under $Aut(\mathfrak{C})$, we have

$$H = \alpha[H] = \bigcap_{i_1<\dots<i_k} C_{G_1}(f(h_{i_1}, \dots, h_{i_k})).$$

Thus,

$$H^g = \bigcap_{i_1<\dots<i_k} C_{G_1}(f(h_{i_1}, \dots, h_{i_k})^g).$$

By assumption, for every $i_1 < \dots < i_k$ there are $h'_{i_1}, \dots, h'_{i_k}$ forming a Morley sequence in p over g for which $f(h_{i_1}, \dots, h_{i_k})^g = f(h'_{i_1}, \dots, h'_{i_k})$. Once again by the invariance of H under $Aut(\mathfrak{C})$ (and by the strong κ-homogeneity of $\mathfrak{C}$), we get that $H \leqslant C_{G_1}(f(h'_{i_1}, \dots, h'_{i_k}))$. Therefore, $H \leqslant H^g$. This shows that H is normal in G.

Knowing that H is invariant under $Aut(\mathfrak{C})$ and normal in G, the conclusion follows as in the proof of [11, Lemma 2.2], but, again, we repeat the argument.

We will show that $H \neq \{e\}$. It follows from the assumptions on G_1 that G_1 is a characteristically simple group. Take a countable $(M, \cdot) \prec (G_1, \cdot)$. Then, M is also a characteristically simple group (since, by ω-categoricity of $(M, \cdot)$, any characteristic subgroup of $(M, \cdot)$ is $\emptyset$-definable in $(M, \cdot)$). So, by Fact 1.1, M is either a p-group or it is isomorphic to a group of the form $B(F)$ or $B^-(F)$.

If M is a p-group, then G_1 is also a p-group, so $\langle\{f(g_{i_1}, \dots, g_{i_k}) : i_1 < \dots < i_k < n\}\rangle$ is a finite p-group, hence it has non-trivial center. As $Z(\langle\{f(g_{i_1}, \dots, g_{i_k}) : i_1 < \dots < i_k < n\}\rangle) \subseteq H$, H is also non-trivial.

Now, consider the case when M is of the form $B(F)$ (when $M = B^{-}(F)$, the argument is similar). Take any y_0 in the Cantor set. It is easy to see that if finitely many elements of $B(F)$ have the same value at a point from the Cantor set, then the intersection of their centralizers is non-trivial. By finite Ramsey theorem, there is a number $R < \omega$ such that for every $f_1, \dots, f_R \in B(F)$, there exist $1 \leqslant i_1 < \dots < i_n \leqslant R$ such that for all $1 \leqslant j_1 < \dots < j_k \leqslant n$ the value of the function $f(f_{i_{j_1}}, \dots, f_{i_{j_k}})$ at y_0 is the same. We conclude that M, and hence G_1, satisfies the following sentence

$$\forall x_1, \dots, x_R \bigvee_{1 \leqslant i_1 < \dots < i_n \leqslant R} \bigcap_{1 \leqslant j_1 < \dots < j_k \leqslant n} C(f(x_{i_{j_1}}, \dots, x_{i_{j_k}})) \neq \{e\}.$$

Thus, by the choice of n, H is non-trivial.

From these observations, and from the characteristic simplicity in $(G, \mathfrak{C})$ of G_1, we conclude that $H = G_1$. Thus, $Z(G_1) \neq \{e\}$. But $Z(G_1)$ is normal in G and invariant under $Aut(\mathfrak{C})$, so $Z(G_1) = G_1$. Hence, G_1 is abelian. □

Having Lemma 1.12, in order to get the next theorem (which strengthens [11, Theorem 2.3]), we modify slightly the proof of [11, Theorem 2.3].

Theorem 1.13 *Let G be a group $\emptyset$-definable in a monster model $\mathfrak{C}$ of an ω-categorical theory. Let $p \in S(\mathfrak{C})$ be a type generically stable over $\emptyset$. Suppose that for some $\emptyset$-definable function f and a Morley sequence $(g_i)_{i<\omega}$ in p over $\emptyset$, $tp(f(g_0, \dots, g_{k-1})/\emptyset)$ is a generic type of G. Assume additionally that whenever $(h_i)_{i<k}$ is a Morley sequence in p over A, g (for some $A \subseteq \mathfrak{C}$ and $g \in G$), then the conjugate $f(h_0, \dots, h_{k-1})^g$ equals $f(h'_0, \dots, h'_{k-1})$ for some Morley sequence $(h'_i)_{i<k}$ in p over A, g. Then G is solvable-by-finite.*

Proof. We will show that G has a $\emptyset$-definable, solvable subgroup of finite index. Of course, we can assume that G is infinite. The proof will be by induction on the greatest natural number n for which there is a series $\{e\} = G_0 < G_1 < \dots < G_n = G$ of $\emptyset$-definable (in $\mathfrak{C}$) normal subgroups of G (such a number exists by ω-categoricity). Notice that then G_k/G_{k-1} is characteristically simple in $(G, \mathfrak{C})$ for every $k \in \{1, \dots, n\}$.

If $n = 1$, then by Lemma 1.12, G is abelian (note that $f(g_0, \dots, g_{k-1})$ is distinct from e, as its type over $\emptyset$ is generic in G and G is infinite). We turn to the induction step, where we assume that $n > 1$. Let $\pi : G \to G/G_1$ be the quotient map. The function $f_1 := \pi \circ f$ witnesses (together with the type p) that G/G_1 satisfies the hypothesis of the theorem. To see this, take any $g \in G$, $A \subseteq \mathfrak{C}$ and $(h_0, \dots, h_{k-1}) \models p^{(k)}_{|A,g}$. Then,

$$f_1(h_0, \dots, h_{k-1})^{gG_1} = f(h_0, \dots, h_{k-1})^g G_1 = f(h'_0, \dots, h'_{k-1}) G_1 = f_1(h'_0, \dots, h'_{k-1}),$$

where $(h'_0, \dots, h'_{k-1})$ is a realization of $p^{(k)}_{|A,g}$, so also a realization of $p^{(k)}_{|A,gG_1}$ (as $gG_1 \in dcl(g)$). Also, it is clear that $f_1(p)$ is a generic type in G/G_1, so the assumptions are satisfied.

By the inductive hypothesis, there is a $\emptyset$-definable $H \trianglelefteq G$ such that $[G : H] < \omega$ and H/G_1 is solvable.

Let $(a_i)_{i<\omega}$ be a Morley sequence in p over $\emptyset$, and put $g_i = f(a_{ik}, a_{ik+1}, \dots, a_{ik+k-1})$ (notice that $(g_i)_{i<\omega}$ is a Morley sequence in $f(p^{(k)})$ over $\emptyset$). There exist $i < j < \omega$ such that $g_iH = g_jH$. Then, $[g_i, g_j] \in H$, so $[g_{i_1}, g_{i_2}] \in H$ for all $i_1, i_2 < \omega$. Hence, from the solvability of H/G_1, we get that there is a minimal $t < \omega$ such that $\delta_t(g_0, \dots, g_{2^t-1}) \in G_1$, where the iterated commutator δ_l is defined recursively as follows:

$$\delta_0(b_1) = b_1,$$
$$\delta_{l+1}(b_1, \dots, b_{2^{l+1}}) = [\delta_l(b_1, \dots, b_{2^l}), \delta_l(b_{2^l+1}, \dots, b_{2^{l+1}})].$$

Notice first that we can assume that $t > 0$. Indeed, if $t = 0$, then $f(p^{(k)}) \in S(\mathfrak{C}) \cap [G_1(x)]$, so G_1 is abelian by Lemma 1.12.

Case 1. $\delta_t(g_0, \dots, g_{2^t-1}) = e$.

Put

$$K = \bigcap_{i_1<\dots<i_{2^{t-1}k}<\omega} C(\delta_{t-1}(f(a_{i_1}, \dots, a_{i_{2^{t-1}}}), \dots, f(a_{i_{(k-1)2^{t-1}+1}}, \dots, a_{i_{(k-1)2^{t-1}+2^{t-1}}}))).$$

As in the proof of Lemma 1.12, using Lemma 2.1(ii) from [11], one can show that K is a $\emptyset$-invariant, normal subgroup of G. Let us show now that $Z(K)$ is non-trivial. By Lemma 2.1(ii) from [11], there is some $0 < m < \omega$ for which

$$K = \bigcap_{i_1<\dots<i_{2^{t-1}k}<m} C(\delta_{t-1}(f(a_{i_1}, \dots, a_{i_{2^{t-1}}}), \dots, f(a_{i_{(k-1)2^{t-1}+1}}, \dots, a_{i_{(k-1)2^{t-1}+2^{t-1}}}))).$$

Hence, by the assumption of Case 1, we get that $\delta_{t-1}(g_m, \dots, g_{m+2^{t-1}-1}) \in K$ (note that $mk \geqslant m$). On the other hand, it is clear that $K \subseteq C(\delta_{t-1}(g_m, \dots, g_{m+2^{t-1}-1}))$. Thus, $\delta_{t-1}(g_m, \dots, g_{m+2^{t-1}-1}) \in Z(K)$, and, by the choice of t, $\delta_{t-1}(g_m, \dots, g_{m+2^{t-1}-1}) \neq e$.

Summarizing, $Z(K)$ is a non-trivial, $\emptyset$-invariant, normal, abelian subgroup of G. So, by induction hypothesis, $G/Z(K)$ has a $\emptyset$-definable, solvable subgroup of finite index, hence so does G.

Case 2. $\delta_t(g_0, \dots, g_{2^t-1}) \neq e$.

Suppose first that G_1 is infinite. Define $f_2(x_0, \dots, x_{2^tk-1}) := \delta_t(f(x_0, \dots, x_{k-1}), \dots, f(x_{(2^t-1)k}, \dots, x_{2^tk-1}))$. Let us check that f_2 witnesses that G_1 satisfies the assumptions of Lemma 1.12. Take any $g \in G_1$. Notice that, by the uniqueness of the type over g of a Morley sequence in p over g, it is enough to check the condition for only one such a sequence. We choose inductively Morley sequences $(h_{ki}, \dots, h_{ki+k-1})$ and $(h'_{ki}, \dots, h'_{ki+k-1})$ $(i = 0, 1, \dots, 2^t - 1)$, in p over $g, h_{<ki}, h'_{<ki}$, in such a way that $f(h_{ki}, \dots, h_{ki+k-1})^g = f(h'_{ki}, \dots, h'_{ki+k-1})$ (we use here the assumption on f). Then, $h_{<2^tk}$ and $h'_{<2^tk}$ are both Morley sequences in p over g, and $f_2(h_{<2^tk})^g = f_2(h'_{<2^tk})$, whence the assumptions of the lemma are satisfied.

Thus, we get that G_1 is abelian. So, H is solvable, and we are done.
Now, assume that G_1 is finite. Then, $[G : C(G_1)] < \omega$, and hence $[g_{i_1}, g_{i_2}] \in C(G_1)$ for every $i_1, i_2 < \omega$. Since $t > 0$, we see that $\delta_t(g_0, \ldots, g_{2^t-1}) \in G_1 \cap C(G_1) = Z(G_1)$, and so $Z(G_1)$ is non-trivial. But, $Z(G_1)$ is a $\emptyset$-invariant, normal subgroup of G contained in G_1. Therefore, $G_1 = Z(G_1)$, i.e. G_1 is abelian, and we are done. □

Remark 1.14 *Suppose R is a ring definable in a monster model and that R^{00} in the additive sense exists. Then R^{00} is an ideal of R.*

Proof. Consider any $r \in R$. We will show that $rR^{00} \subseteq R^{00}$; the inclusion $R^{00}r \subseteq R^{00}$ can be proved analogously.

Let $f : R \to R$ be an additive homomorphism defined by $f(x) = rx$. Then $f[R^{00}]$ is a type-definable subgroup of R (if $\Phi(x)$ defines G^{00}, then $\Psi(x) := \{\exists y(\phi(y) \wedge x = ry) : \phi \in \Phi\}$) defines $f[R^{00}]$).

We need to show that $f[R^{00}] \subseteq R^{00}$. Suppose this is not the case. Then $A := R^{00} \cap f[R^{00}]$ is a proper, type-definable (by the union of the types defining R^{00} and $f[R^{00}]$) subgroup of $f[R^{00}]$ of bounded index. Hence, $R^{00} \cap f^{-1}[A]$ is a proper, type-definable, bounded index subgroup of R^{00}, which is not possible. □

Proof of Theorem 0.3. Let R be a generically stable ring which is definable in a monster model of an ω-categorical theory. By Fact 1.7(iv), R^{00} (in the additive sense) exists, and since it is $\emptyset$-invariant, it follows by ω-categoricity that it is $\emptyset$-definable. So, it has finite index in R, and, by Remark 1.14, we can assume that $R = R^{00}$ (because, being an additive translate of a generically stable generic type, the generic type of R^{00} is also generically stable by Fact 1.4). Fact 1.11 tells us that $J(R)$ is nilpotent, so we can assume that R is semisimple (replacing R by $R/J(R)$ and using Fact 1.4). If R is finite, we are done, so we can assume that R is infinite. By Fact 1.8, R has a unique (global) generic type $p \in S_1(R)$. Thus, p is invariant over $\emptyset$, and hence, generically stable over $\emptyset$. So, we can assume (by replacing $\mathfrak{C}$ by R with the induced structure) that the monster model is just the ring R (possibly with some extra structure).

As it was mentioned at the beginning of this subsection, the proof splits into two cases depending on whether the realizations of p are non-nilpotent or nilpotent. Notice, however, that in the special case of commutative rings, all non-zero elements of R are non-nilpotent (since, being semisimple, R does not have any non-trivial nil ideals). So, for commutative rings it is enough to consider only the first case.

Case 1. Realizations of p are not nilpotent.

By Fact 1.10, we can assume that R is a subring of $\prod_{i \in I} R_i$, where each R_i is finite, and $|\{R_i : i \in I\}| < \omega$. Let π_i be the projection onto the i-th coordinate. For $i_0, \ldots, i_n \in I$ and $r_0 \in R_{i_0}, \ldots, r_n \in R_{i_n}$, we define

$$R^{r_0,\ldots,r_n}_{i_0,\ldots,i_n} = \left\{ r \in R : \bigwedge_{j=0}^{n} \pi_{i_j}(r) = r_j \right\}.$$

Claim 1 *There are* $i_0, i_1, \dots \in I$, *non-nilpotent elements* $r_j \in R_{i_j}$ *and a Morley sequence* $(\eta_i)_{i<\omega}$ *in* p *over* $\emptyset$ *such that* $\eta_n \in R^{0,\dots,0,r_n}_{i_0,\dots,i_{n-1},i_n}$ *for every* $n < \omega$.

Proof of Claim 1. Assume that we have already constructed $i_0, \dots, i_n$, $r_0, \dots, r_n$ and $(\eta_i)_{i\leqslant n}$. Let $p_n = p|(\eta_i)_{i\leqslant n} \in S_1((\eta_i)_{i\leqslant n})$. If $R^{0,\dots,0}_{i_0,\dots,i_n} \cap p_n(R) = \emptyset$, then $R \setminus p_n(R)$ is generic in R (since $R^{0,\dots,0}_{i_0,\dots,i_n}$ has finite index in R), so, by the compactness theorem, there is $\phi \in p_n$ such that $\neg\phi$ is generic. (If R is covered by a finite union $\bigcup_i (a_i + (R \setminus p_n(R))$, then the definable sets $\bigcup_i (a_i + (R \setminus \phi(R)), \phi \in p_n$ cover R, so finitely many of them cover R).
From Fact 1.7(iii), we get that $\{\neg\phi\}$ extends to a global generic type, which contradicts the uniqueness of the generic type in R. So, $R^{0,\dots,0}_{i_0,\dots,i_n} \cap p_n(R) \neq \emptyset$. Take $\eta_{n+1} \in R^{0,\dots,0}_{i_0,\dots,i_n} \cap p_n(R)$. By the assumption of Case 1, η_{n+1} is non-nilpotent. Since $|\{R_i : i \in I\}| < \omega$, there is $i_{n+1} \in I$ such that $\pi_{i_{n+1}}(\eta_{n+1})$ is non-nilpotent. We put $r_{n+1} := \pi_{i_{n+1}}(\eta_{n+1})$. So, we have found i_{n+1}, r_{n+1} and η_{n+1} with the desired properties. □

For $r \in R$, let $RrR = \{a_1rb_1 + \dots + a_lrb_l : l < \omega, a_i, b_i \in R\}$ be the two-sided ideal of R generated by r. By ω-categoricity, for any $r \in R$, the ideal RrR consist of sums of a fixed number K of elements of the form r_1rr_2, where $r_1, r_2 \in R \cup \{1\}$ (otherwise there would be infinitely many types over r). Once again by ω-categoricity, there is a number K which works for every $r \in R$. Thus, there is a formula $H(x, z; y)$ expressing that $x \in R(z - y)R$. Let N be as in Lemma 1.5 for the type p, formula $H(x, z; y)$ and $D := R$.

Claim 2 *There are natural numbers* $n(0) < n(1) < \dots < n(N)$ *and a Morley sequence* $(a_i)_{i\leqslant N}$ *(of length* $N + 1$*) in* p *over* $\emptyset$ *such that*

$$a_0 \in R^{r_{n(0)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, a_1 \in R^{0,r_{n(1)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, \dots, a_N \in R^{0,\dots,0,r_{n(N)}}_{i_{n(0)},\dots,i_{n(N)}}. \qquad (*)$$

Proof of Claim 2. First, we will find natural numbers

$$n(0) < n'(0) < n(1) < n'(1) < \dots < n(N-1) < n'(N-1) < n(N)$$

such that for $a_k := \eta_{n(k)} - \eta_{n'(k)}$, $k = 0, \dots, N-1$, and $a_N := \eta_{n(N)}$ the condition $(*)$ is satisfied. This follows exactly as in the proof of Claim 2 in [24, Theorem 2.1], but we give some details for completeness.

Let $c = \max_{i\in I} |R_i|$. Define numbers $L_N, \dots, L_1, L_0$ recursively by:

$$\begin{aligned} L_N &= c + 1, \\ L_{N-k} &= c^{L_N+\dots+L_{N-k+1}+1} + 1 \ \text{ for } \ k = 1, \dots, N-1, \\ L_0 &= 0. \end{aligned}$$

Put $I_N = \{L_0 + \dots + L_N\}$, and define intervals $I_0, \dots, I_{N-1}$ as

$$I_k = [L_0 + \dots + L_k, L_0 + \dots + L_{k+1} - 1].$$

For each $k \in \{0, \dots, N-1\}$, by the pigeonhole principle, we can find two natural numbers $n(k) < n'(k)$ in I_k such that $\pi_{i_j}(\eta_{n'(k)}) = \pi_{i_j}(\eta_{n(k)})$ for every $j \in I_{k+1} \cup \dots \cup I_N$. Put additionally $n(N) = L_0 + \dots + L_N$. Now, it is easy to check that $(*)$ is satisfied for $a_k := \eta_{n(k)} - \eta_{n'(k)}$, $k = 0, \dots, N-1$, and $a_N := \eta_{n(N)}$.

It remains to show that $(a_i)_{i \leqslant N}$ is a Morley sequence in p over $\emptyset$. Fix any $k < N$. We have that $\eta_{n'(k)} \models p|(\eta_{n'(i)})_{i<k}(\eta_{n(i)})_{i \leqslant k}$. By the uniqueness of the generic type in R, we get that $\eta_{n(k)} - \eta_{n'(k)} \models p|(\eta_{n'(i)})_{i<k}(\eta_{n(i)})_{i<k}$ (since a translate of a generic type is generic), so $a_k \models p|(a_i)_{i<k}$. It is also clear that $a_N \models p|(a_i)_{i<N}$. This shows that $(a_i)_{i \leqslant N}$ is a Morley sequence in p over $\emptyset$. □

Let $c = \sum_{i \leqslant N} a_i$ and $b_j = \sum_{i \neq j} a_i = c - a_j$ for $j = 0, \dots, N$. Using Claim 2 and the choice of N, we reach a final contradiction in the same way as in the proof of [24, Theorem 2.1]. Namely,

$$\pi_{i_{n(j)}}[Rb_0R \cap \dots \cap Rb_NR] = \{0\} \text{ for } j = 0, \dots, N. \qquad (**)$$

On the other hand, $\prod_{k \neq j} b_k \in \bigcap_{k \neq j} Rb_kR$ for $j = 0, \dots, N$. We also have that $\pi_{i_{n(j)}}[\prod_{k \neq j} b_k] = r_{n(j)}^N \neq 0$ as $r_{n(j)}$ is non-nilpotent. So,

$$\pi_{i_{n(j)}}\left[\bigcap_{k \neq j} Rb_kR\right] \neq \{0\} \text{ for } j = 0, \dots, N. \qquad (***)$$

By $(**)$ and $(***)$, $Rb_0R \cap \dots \cap Rb_NR \neq \bigcap_{k \neq j} Rb_kR$ for all $j = 0, \dots, N$. This is a contradiction to the choice of N, because $Rb_iR = R(c - a_i)R = H(R, c; a_i)$ and $(a_i)_{i \leqslant N}$ is a Morley sequence in p over $\emptyset$.

Case 2. Realizations of p are nilpotent.

By ω-categoricity, R has a finite characteristic c. Put $R_1 = R \times \mathbb{Z}_c$, and define $+$ and $\cdot$ on R_1 by $(a, k) + (b, l) = (a + b, k +_c l)$ and $(a, k) \cdot (b, l) = (ab + l \times a + k \times b, k \cdot_c l)$, where $+_c$ and $\cdot_c$ are addition and multiplication modulo c, and $l \times a := a + \dots + a$ (l-many times). Then R_1 is a ring with 1 interpretable in R, and R is a two-sided ideal of finite index in R_1. Let G be the subgroup of $\mathrm{GL}_3(R_1)$ generated by $\{t_{ij}(\alpha) : \alpha \in R,\ i, j \in \{1, 2, 3\},\ i \neq j\} \cup \{t_j(\beta) : \beta \in (1 + R) \cap R_1^*,\ j \in \{1, 2, 3\}\}$, where $t_{ij}(\alpha)$ is the matrix with 1's on the diagonal, α on the (i, j)-th position and 0's elsewhere, and $t_j(\beta)$ is the matrix with β on the (j, j)-th position, 1's on the rest of the diagonal and 0's elsewhere. Since G is invariant over finitely many parameters (those over which R_1 is defined), it follows by ω-categoricity that it is definable. Let $(a_{ij})_{1 \leqslant i,j \leqslant 3} \models p^{(9)}$ (in a bigger monster model $\mathfrak{C} \succ R$); note that the order of a_{ij}'s is irrelevant, because p is generically stable and we have Fact 1.3. Define

$$A = \begin{pmatrix} 1 + a_{11} & a_{12} & a_{13} \\ a_{21} & 1 + a_{22} & a_{23} \\ a_{31} & a_{32} & 1 + a_{33} \end{pmatrix}.$$

Claim 3 *$A \in G(\mathfrak{C})$, where $G(\mathfrak{C})$ is the interpretation of G in $\mathfrak{C}$.*

Proof of Claim 3. The idea is to show that we can transform A to the identity matrix by a Gaussian elimination process in which all elementary matrices belong to $G(\mathfrak{C})$ (because then $BA = I$ for some $B \in G(\mathfrak{C})$, so $A = B^{-1} \in G(\mathfrak{C})$).

The following well-known remark is fundamental for our process: if $r \in R$ satisfies $r^n = 0$ for some n, then $(1+r)(1 - r + r^2 - \cdots \pm r^{n-1}) = 1$, so $(1+r)^{-1} \in (1 + R \cap dcl(r)) \cap R_1^*$.

Now, we describe the first step of the process. We have

$$t_{21}\left(-a_{21}(1+a_{11})^{-1}\right) A =$$
$$\begin{pmatrix} 1+a_{11} & a_{12} & a_{13} \\ 0 & 1+a_{22}-a_{21}(1+a_{11})^{-1}a_{12} & a_{23}-a_{21}(1+a_{11})^{-1}a_{13} \\ a_{31} & a_{32} & 1+a_{33} \end{pmatrix}.$$

But $a_{21}(1+a_{11})^{-1}a_{12} \in \mathfrak{C} \cap dcl((a_{ij})_{(i,j)\neq(2,2)})$, so, by the uniqueness of the generic type,

$$b_{22} := a_{22} - a_{21}(1+a_{11})^{-1}a_{12} \models p|R, (a_{ij})_{(i,j)\neq(2,2)}.$$

Similarly,

$$b_{23} := a_{23} - a_{21}(1+a_{11})^{-1}a_{13} \models p|R, (a_{ij})_{(i,j)\neq(2,3)}.$$

Therefore,

$$t_{21}\left(-a_{21}(1+a_{11})^{-1}\right) A = \begin{pmatrix} 1+a_{11} & a_{12} & a_{13} \\ 0 & 1+b_{22} & b_{23} \\ a_{31} & a_{32} & 1+a_{33} \end{pmatrix},$$

where $((a_{ij})_{(i,j)\neq(2,2),(2,3)}, b_{22}, b_{23}) \models p^{(8)}$. Continuing Gaussian elimination in this way, we obtain a matrix $C \in G(\mathfrak{C})$ such that

$$CA = \begin{pmatrix} 1+b_1 & 0 & 0 \\ 0 & 1+b_2 & 0 \\ 0 & 0 & 1+b_3 \end{pmatrix} = t_1(1+b_1)t_2(1+b_2)t_3(1+b_3)$$

for some $(b_1, b_2, b_3) \models p^{(3)}$. But $t_1(1+b_1), t_2(1+b_2), t_3(1+b_3) \in G(\mathfrak{C})$, so we conclude that $A \in G(\mathfrak{C})$. □

Claim 4 *All translates of the type $q := tp(A/R)$ by the elements of G are finitely satisfiable in some small model. Thus, G has fsg and q is a global generic type of G.*

Proof of Claim 4. We will show that every translate of A by an element of G belongs to the set

$$Z := \{(k_{ij} + b_{ij})_{i,j\in\{1,2,3\}} : k_{ij} \in \mathbb{Z}_c,\ (b_{ij})_{i,j\in\{1,2,3\}} \models p^{(9)}\}.$$

This will complete the proof of Claim 4, as every element of Z is in the definable closure of $\mathbb{Z}_c$ and some realization of the type $p^{(9)}$, and $p^{(9)}$ is finitely satisfiable in some small model. So, it suffices to show that Z is invariant under multiplication by the elements of the set $\{t_{ij}(\alpha) : \alpha \in R,\ i,j \in \{1,2,3\},\ i \neq j\} \cup \{t_j(\beta) : \beta \in (1+R) \cap R_1^*,\ j \in \{1,2,3\}\}$ (notice that this set is closed under the group inversion). Choose any $B = (k_{ij} + b_{ij})_{i,j \in \{1,2,3\}} \in Z$.

First, consider any element of Z of the form $t_j(\beta)$, where $\beta = 1 + r \in R_1^*$ for some $r \in R$. Denote the entries of the matrix $t_j(\beta)B$ by d_{im} $(i, m \in \{1,2,3\})$. Then $d_{im} = k_{im} + b_{im}$ for all m and $i \neq j$. Take any $m \in \{1,2,3\}$. Then $d_{jm} = \beta(k_{jm} + b_{jm}) = (1+r)(k_{jm} + b_{jm}) = k_{jm} + k_{jm} \times r + (1+r)b_{jm}$. Since multiplication by $(1+r)$ is a definable automorphism of $(R,+)$, it preserves generic types, so by the uniqueness of the generic type, we get that $(1+r)b_{jm} \models p|R, (b_{il})_{(i,l)\neq(j,m)}$, and hence $k_{jm} \times r + (1+r)b_{jm} \models p|R, (b_{il})_{(i,l)\neq(j,m)}$. This easily implies that $t_j(\beta)B \in Z$.

Now, consider any $t_{ij}(\alpha)$, where $i \neq j$ and $\alpha \in R$. Denote the entries of $t_{ij}(\alpha)B$ by f_{ij} $(i,j \in \{1,2,3\})$. Choose any $m \in \{1,2,3\}$. For all $l \neq i$ we have $f_{lm} = k_{lm} + b_{lm}$. Moreover, $f_{im} = k_{im} + b_{im} + \alpha(k_{jm} + b_{jm})$. But $\alpha(k_{jm} + b_{jm}) \in dcl(R, b_{jm})$, so $b_{im} + \alpha(k_{jm} + b_{jm}) \models p|R, (b_{pq})_{(p,q)\neq(i,m)}$. Hence, $t_{ij}(\alpha)B \in Z$. This completes the proof of Claim 4. □

If we knew that $p^{(9)}$ were generically stable (recall that we know that p is generically stable), then Claim 4 would imply that G is generically stable, so, by Fact 0.1, we would conclude that G is solvable-by-finite and we could turn to the last paragraph of the proof. Since, in general, we cannot conclude that $p^{(9)}$ is generically stable, we will prove Claim 5 below and then apply Theorem 1.13 in order to get that G is solvable-by-finite.

Adding to the language the appropriate parameters, we can assume that everything is definable over $\emptyset$.

Claim 5 *Let a $\emptyset$-definable function $f : M_{3\times3}(R) \to M_{3\times3}(R_1)$ be defined by*

$$f((x_{ij})_{1\leqslant i,j\leqslant 3}) = \begin{pmatrix} 1+x_{11} & x_{12} & x_{13} \\ x_{21} & 1+x_{22} & x_{23} \\ x_{31} & x_{32} & 1+x_{33} \end{pmatrix}.$$

Then, whenever $(h_{ij})_{1\leqslant i,j\leqslant 3}$ is a Morley sequence in p over A, g (for some $A \subseteq R$ and $g \in G$), then $f((h_{ij})_{1\leqslant i,j\leqslant 3}) \in G$ and $f((h_{ij})_{1\leqslant i,j\leqslant 3})^g = f((h'_{ij})_{1\leqslant i,j\leqslant 3})$ for some Morley sequence $(h'_{ij})_{1\leqslant i,j\leqslant 3}$ in p over A, g.

Proof of Claim 5. Let $(h_{ij})_{1\leqslant i,j\leqslant 3}$ be a Morley sequence in p over A, g. By the uniqueness of a Morley sequence in p over $\emptyset$, we can find $\alpha \in Aut(\mathfrak{C})$ sending each a_{ij} to h_{ij}. Then, $f((h_{ij})_{1\leqslant i,j\leqslant 3}) = f((\alpha(a_{ij}))_{1\leqslant i,j\leqslant 3}) = \alpha(f((a_{ij})_{1\leqslant i,j\leqslant 3})) \in \alpha(G)$ by Claim 3, but $\alpha(G) = G$ since G is invariant.

For the second part, first notice that it is enough to prove the statement for g of the form $t_{ij}(\alpha)$ (for $\alpha \in R$ and distinct $i,j \in \{1,2,3\}$) and $t_j(\beta)$ (for $\beta \in (1+R) \cap R_1^*$ and $j \in \{1,2,3\}$), which follows easily from the fact that G is generated by these elements and by the independence of the choice of a Morley sequence $(h_{ij})_{1\leqslant i,j\leqslant 3}$ in p over A, g. Next, apply a similar argument to the proof of Claim 4 in order to get that the conjugates by the elements of the form $t_{ij}(\alpha)$ or $t_j(\beta)$ have the desired property. □

From Claims 4 and 5 and Theorem 1.13, we conclude that G is solvable-by-finite.

The rest of the proof follows exactly as in [25, Theorem 2.1(i)], but we will repeat the argument for the reader's convenience. Let H be a normal subgroup of G of finite index, which is solvable. We have the following well-known formulas:

$$t_{ij}(\alpha)t_{ij}(\beta) = t_{ij}(\alpha+\beta) \text{ and } [t_{ik}(\alpha), t_{kj}(\beta)] = t_{ij}(\alpha\beta) \tag{†}$$

for pairwise distinct i, j, k. Define $I = \{r \in R : (\forall i \neq j) t_{ij}(r) \in H\}$. Using the normality of H in G and (†), we see that $I \lhd R$. If $|R/I| \geqslant \omega$, then, by Ramsey's theorem, for some distinct $i, j \in \{1, 2, 3\}$ there are r_k, $k < \omega$, such that $t_{ij}(r_n - r_m) \notin H$ for every $n < m < \omega$. But, by (†), $t_{ij}(r_n - r_m) = t_{ij}(r_n)t_{ij}(r_m)^{-1}$, which contradicts the finiteness of $[G : H]$. So, $|R/I| < \omega$. Since H is solvable, there exists n for which the n-th derived subgroup $H^{(n)}$ is trivial. Then (†) implies that for every $r_1, \ldots, r_{2^n} \in I$ and distinct $i, j \in \{1, 2, 3\}$ we have $t_{ij}(r_1 \ldots r_{2^n}) \in H^{(n)} = \{e\}$, so $r_1 \ldots r_{2^n} = 0$. This shows that I is a nilpotent ideal of R of finite index. □

1.3 ω-categorical, generically stable groups

The goal of this subsection is to prove Theorem 0.2 from the introduction. In the final part of the proof, we will apply Fact 0.1 and the argument from page 490 of [30]. However, in order to do that, first we need to prove a certain non-trivial lemma (a variant of [30, Corollary 3.5]), which uses some ideas from the proof of Theorem 0.3 in Case 1 and from the final part of the proof of [25, Corollary 3.17].

It is worth recalling that [25, Theorem 3.15] says that whenever each ring interpretable in a given ω-categorical structure (in which all definable groups have connected components) is nilpotent-by-finite, then each solvable group definable in this structure is also nilpotent-by-finite. This was used in [24] to conclude that ω-categorical groups with NIP and fsg are nilpotent-by-finite (using the fact that they are solvable-by-finite and that each ω-categorical ring with NIP is nilpotent-by-finite). The reason why, having Theorem 0.3, we cannot apply Fact 0.1 and [25, Theorem 3.15] in order to get Theorem 0.2 is that rings interpretable in a given ω-categorical, generically stable group need not to be generically stable. In the proof of Lemma 1.17 below, we undertake a detailed analysis of the relevant interpretable rings.

Recall a few basic definitions. Let H be a group. The commutator of $h_0, h_1 \in H$ is defined as $[h_0, h_1] = h_0^{-1}h_1^{-1}h_0h_1$. The iterated commutators γ_n on H are defined inductively as follows:

$$\gamma_1(h_0) = h_0 \text{ and } \gamma_{n+1}(h_0, \ldots, h_n) = [\gamma_n(h_0, \ldots, h_{n-1}), h_n].$$

For two subsets A and B of H, $[A, B]$ denotes the subgroup of H generated by all commutators $[a, b]$, where $a \in A$ and $b \in B$. The lower central series $H = \Gamma_1(H) \geqslant \Gamma_2(H) \geqslant \ldots$ of H is defined by:

$$\Gamma_1(H) = H \text{ and } \Gamma_{n+1}(H) = [\Gamma_n(H), H].$$

The group H is nilpotent of class n if $\Gamma_{n+1}(H) = \{e\}$ and n is the smallest number with this property. The following formulas for commutators will be very useful:

$$[x, zy] = [x, y][x, z]^y \text{ and } [xz, y] = [x, y]^z[z, y]. \qquad (1)$$

Recall that, in this thesis, a group H definable in a monster model is said to be absolutely connected if $H^{00} = H$.

Remark 1.15 *If H is an absolutely connected, nilpotent group definable in a monster model of an ω-categorical theory, then each group $\Gamma_k(H)$ is also definable and absolutely connected.*

Proof. The definability of $\Gamma_k(H)$ is a standard consequence of ω-categoricity: there is l such that each element of $\Gamma_k(H)$ is a product of at most l iterated commutators of length k (otherwise among the types of these products we would find infinitely many distinct ones). By induction on the nilpotency class n of H, we will show that each term of the lower central series is absolutely connected. For $n = 1$ it is clear. Assume H is of class $n + 1$ and the conclusion holds for groups of smaller class.

First, suppose for a contradiction that $\Gamma_{n+1}(H)$ is not absolutely connected, i.e., it has a proper, type-definable subgroup C of bounded index. Since $\Gamma_{n+1}(H)$ is central in H, C is a normal subgroup of H. For any $h/C \in \Gamma_n(H)/C$, $[h/C, H/C] \leqslant \Gamma_{n+1}(H)/C$ is finite, so $C_H(h/C)$ is of finite index in H, and hence, by the absolute connectedness of H, it is equal to H. Thus, $\Gamma_{n+1}(H) = [\Gamma_n(H), H] \leqslant C$, which implies $\Gamma_{n+1}(H) = C$, a contradiction to the properness of C. So, we have proved that $\Gamma_{n+1}(H)$ is absolutely connected. Since $H/\Gamma_{n+1}(H)$ is of nilpotency class n, by the induction hypothesis, each quotient $\Gamma_k(H)/\Gamma_{n+1}(H)$ is absolutely connected. To see that these two observations imply that each term $\Gamma_k(H)$ is also absolutely connected, consider any type-definable subgroup D of $\Gamma_k(H)$ of bounded index. Then, the intersection $D \cap \Gamma_{n+1}(H)$ it type definable, and has bounded index in $\Gamma_{n+1}(H)$, so, by the absolute connectedness of $\Gamma_{n+1}(H)$, we get $\Gamma_{n+1}(H) \subseteq D$. Now, the quotient $D/\Gamma_{n+1}(H)$ has a bounded index in $\Gamma_k(H)/\Gamma_{n+1}(H)$ (and is type-definable), so, we get that $D = \Gamma_k(H)$. □

Lemma 1.16 *Let H be an absolutely connected group with fsg (H is definable over A in a monster model). Then, for every $n \in \omega \setminus \{0\}$ each element of $\Gamma_n(H)$ is a product of conjugates of elements from the set $\{\gamma_n(g_0, \dots, g_{n-1}) : (g_0, \dots, g_{n-1}) \models p^{(n)}|A\}$, where p is the unique global generic type in H.*

Proof. Using (1), we easily get by induction that for every $x_0, \dots, x_k, y$ the commutator $[x_0 \dots x_k, y]$ is a product of conjugates of commutators $[x_i, y], i = 0, \dots, k$. To prove the lemma, we proceed by induction on n. The induction starts, since every element $a \in H$ is a product of two realizations of $p|A$ (if $b \models p|Aa$, then by the uniqueness of the generic type, also $b^{-1}a \models p|Aa$). Suppose that the conclusion of the lemma is satisfied for n. Take any $g \in \Gamma_{n+1}(H)$. Then $g = [a, b]$ for some $a \in \Gamma_n(H)$ and $b \in H$. By the inductive hypothesis, $a = \prod_{i<l} \gamma_n(\overline{g_i})^{c_i}$ for some $\overline{g_i} \models p^{(n)}|A$ and

$c_i \in H$ (for $i = 0, \ldots, l-1$). So, by (1), $[a,b] = \prod_{i<l}[\gamma_n(\overline{g_i})^{c_i}, b]^{d_i}$ for some $d_i \in H$, $i = 0, \ldots, l-1$. Choose $b_1, b_2 \models p|A, (\overline{g_i}, c_i)_{i<l}$ such that $b_1 b_2 = b$. Then, by (1), for every $i < l$ we have

$$\begin{aligned}
[\gamma_n(\overline{g_i})^{c_i}, b]^{d_i} &= [\gamma_n(\overline{g_i})^{c_i}, b_2]^{d_i}[\gamma_n(\overline{g_i})^{c_i}, b_1]^{b_2 d_i} \\
&= [\gamma_n(\overline{g_i}), b_2^{c_i^{-1}}]^{c_i d_i}[\gamma_n(\overline{g_i}), b_1^{c_i^{-1}}]^{c_i b_2 d_i} \\
&= \gamma_{n+1}(\overline{g_i}, b_2^{c_i^{-1}})^{c_i d_i}\gamma_{n+1}(\overline{g_i}, b_1^{c_i^{-1}})^{c_i b_2 d_i}.
\end{aligned}$$

By the uniqueness of the generic type, $b_1^{c_i^{-1}}, b_2^{c_i^{-1}} \models p|A, \overline{g_i}$. So, $(\overline{g_i}, b_1^{c_i^{-1}}), (\overline{g_i}, b_2^{c_i^{-1}}) \models p^{(n+1)}|A$. This completes the proof of the lemma. □

Lemma 1.17 *We work in a monster model $\mathfrak{C}$ of an ω-categorical theory. Let H be a nilpotent, absolutely connected, generically stable group definable in $\mathfrak{C}$ acting definably and by automorphisms on a definable vector space V over $F := GF(q^a)$ (q is a prime number), and assume that H has no elements of order q. Then* $\mathrm{Stab}_H(V) = H$.

Proof. First, we will prove the following claim.

Claim 1 *If H is a nilpotent group of class n which satisfies the assumptions of the lemma, then* $\mathrm{Stab}_{\Gamma_n(H)}(V) = \Gamma_n(H)$.

Proof of Claim 1. We can assume that everything is $\emptyset$-definable in $\mathfrak{C}$. Then, the unique global generic type p of H is generically stable over $\emptyset$. Put $A = \Gamma_n(H)$. As in [30], we define W as the sum of all finite dimensional FA-submodules of V. Then W is a subspace of V which is definable in $\mathfrak{C}$ and invariant under A. By [30, Proposition 3.4] (see also [25, Fact 3.16]) and Remark 1.15, it is enough to show that $W = V$. Suppose for a contradiction that $W \subsetneq V$, and put $\overline{V} = V/W$. Exactly as in in the proof of [30, Corollary 3.5], we get:

The FA-module $\overline{V}$ has no non-trivial, finite dimensional FA-submodules. $(*)$

Choose a non-trivial $v \in \overline{V}$, and put $V_0 = \mathrm{Lin}_F(Av)$. By ω-categoricity, every element of V_0 is an F-linear combination of a fixed number t of elements of Av, so we can identify V_0 with $(F \times Av)^{\times t}$ divided by the relation
$E((e_1, v_1, \ldots, e_t, v_t), (f_1, w_1, \ldots f_t, w_t)) \iff e_1 v_1 + \cdots + e_t v_t = f_1 w_1 + \cdots + f_t w_t$, so V_0 is interpretable. Let R be the ring of endomorphisms of V_0 generated by A. Since A is commutative (as $A = \Gamma_n(H)$ and H is nilpotent of class n), R is a commutative ring interpretable in $\mathfrak{C}$ (every element of R is a sum of a fixed number s of elements of A, and is determined by its value on v, so we can identify R with the set of s-tuples of elements of A divided by $E((a_1, \ldots, a_s), (b_1, \ldots, b_s)) \iff a_1(v) + \cdots + a_s(v) = b_1(v) + \cdots + b_s(v)$; the addition in R is given by $(a_1, \ldots, a_s)/E + (b_1, \ldots, b_s)/E = (c_1, \ldots c_s)/E \iff a_1(v) + \cdots + a_s(v) + b_1(v) + \cdots + b_s(v) = c_1(v) + \cdots + c_s(v)$ and the multiplication in R is given by $(a_1, \ldots, a_s)/E \cdot (b_1, \ldots, b_s)/E = (c_1, \ldots, c_s)/E \iff \sum_{i,j \leqslant s} a_i b_j(v) = \sum_{k \leqslant s} c_k(v)$, so the ring R is interpretable). Adding some parameters to the language, we can assume that R is interpretable in $\mathfrak{C}$ over $\emptyset$.

Let $(g_i)_{i<\omega} \models p^{(\omega)}|\emptyset$. We will show that

$$\gamma_n(g_0, \dots, g_{n-2}, g_n) - \gamma_n(g_0, \dots, g_{n-2}, g_{n-1}) \in J(R). \qquad (**)$$

Suppose for a contradiction that this is not the case. Put

$$a = \gamma_{n-1}(g_0, \dots, g_{n-2}).$$

By Fact 1.10, we can assume that $R/J(R)$ is a subring of $\prod_{i\in I} R_i$, where each R_i is finite and $|\{R_i : i \in I\}| < \omega$. Let $\pi_i : R \to R_i$ be the quotient map $R \to R/J(R)$ composed with the projection onto the i-th coordinate. For $i_0, \dots, i_m \in I$ and $r_j \in R_{i_j}$, we define

$$R^{r_0,\dots,r_m}_{i_0,\dots,i_m} = \left\{ r \in R : \bigwedge_{j=0}^{m} \pi_{i_j}(r) = r_j \right\}.$$

Subclaim 1 *There are $i_0, i_1, \dots \in I$, non-nilpotent elements $r_j \in R_{i_j}$ and a Morley sequence $(\eta_i)_{i<\omega}$ in p over a such that $[a, \eta_{2m+1}] - [a, \eta_{2m}] \in R^{0,\dots,0,r_m}_{i_0,\dots,i_{m-1},i_m}$ for every $m < \omega$.*

Proof of Subclaim 1. Suppose we have already constructed $i_0, \dots, i_{k-1}$, $r_0, \dots, r_{k-1}$ and $(\eta_i)_{i<2k}$. Let $(h_i)_{i<\omega} \models p^{(\omega)}|a, (\eta_i)_{i<2k}$. Choose $j < l < \omega$ such that $\pi_{i_m}([a, h_j]) = \pi_{i_m}([a, h_l])$ for every $m < k$. Put $\eta_{2k} = h_j$ and $\eta_{2k+1} = h_l$. Since $R/J(R)$ is a semi-simple, commutative ring, the only nilpotent element of $R/J(R)$ is zero. Hence, by our assumption that $(**)$ does not hold, $([a, \eta_{2k+1}] - [a, \eta_{2k}])/J(R)$ is non-nilpotent. Therefore, since $|\{R_i : i \in I\}| < \omega$, there is $i_k \in I$ such that $\pi_{i_k}([a, \eta_{2k+1}] - [a, \eta_{2k}])$ is non-nilpotent. Putting $r_k = \pi_{i_k}([a, \eta_{2k+1}] - [a, \eta_{2k}])$, the construction is completed.□

Now, using the assumption that H is nilpotent of class n and (1), we get

$$\begin{aligned}([a, \eta_{2k+1}] - [a, \eta_{2k}])[a, \eta_{2k}^{-1}] &= [a, \eta_{2k+1}][a, \eta_{2k}^{-1}] - [a, \eta_{2k}][a, \eta_{2k}^{-1}] \\ &= [a, \eta_{2k+1}][a, \eta_{2k}^{-1}]^{\eta_{2k+1}} - [a, \eta_{2k}][a, \eta_{2k}^{-1}]^{\eta_{2k}} \\ &= [a, \eta_{2k}^{-1}\eta_{2k+1}] - 1\end{aligned}$$

for any $k < \omega$. But $[a, \eta_{2k}^{-1}]$ is an invertible element of R, so putting $h_k = \eta_{2k}^{-1}\eta_{2k+1}$ for $k < \omega$, we obtain that $[a, h_k] - 1 \in R^{0,\dots,0,s_k}_{i_0,\dots,i_{k-1},i_k}$ for every $k < \omega$, where $s_k \in R_{i_k}$ are non-nilpotent. Also, by the uniqueness of the generic type, $(h_i)_{i<\omega}$ is a Morley sequence in p over a.

Let $H(x, \overline{z}; y)$ be a formula expressing that $x \in R(z_1 - [z_2, y] + 1)R$, where $\overline{z} = (z_1, z_2)$. Choose N as in Lemma 1.5 for the type p, formula $H(x, \overline{z}; y)$ and $D := R \times \{a\}$.

Now, exactly as in Claim 2 in the proof of Theorem 0.3, we find

$$n(0) < n'(0) < n(1) < n'(1) < \dots < n(N-1) < n'(N-1) < n(N) < n'(N)$$

such that for $a_k := ([a, h_{n(k)}] - 1) - ([a, h_{n'(k)}] - 1)$, $k = 0, \dots, N$, we have:

$$a_0 \in R^{s_{n(0)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}},\ a_1 \in R^{0,s_{n(1)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, \dots, a_N \in R^{0,\dots,0,s_{n(N)}}_{i_{n(0)},\dots,i_{n(N)}}.$$

By the fact that H is nilpotent of class n and (1), for all $k = 0, \dots, N$ we have

$$a'_k := a_k[a, h^{-1}_{n'(k)}] = [a, h_{n(k)}][a, h^{-1}_{n'(k)}] - [a, h_{n'(k)}][a, h^{-1}_{n'(k)}] = [a, h^{-1}_{n'(k)}h_{n(k)}] - 1.$$

Since each $[a, h^{-1}_{n'(k)}]$ is invertible in R, we obtain

$$a'_0 \in R^{s'_{n(0)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, a'_1 \in R^{0,s'_{n(1)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, \dots, a'_N \in R^{0,\dots,0,s'_{n(N)}}_{i_{n(0)},\dots,i_{n(N)}},$$

for some non-nilpotent elements $s'_{n(0)} \in R_{i_{n(0)}}, \dots, s'_{n(N)} \in R_{i_{n(N)}}$. Moreover, we see that $(h^{-1}_{n'(k)}h_{n(k)})_{k<\omega}$ is a Morley sequence in p over a. So, by the choice of N, the argument after Claim 2 in the proof of Theorem 0.3 leads to a contradiction, which completes the proof of $(**)$.

By $(**)$ together with the fact that H is nilpotent of class n and (1), we obtain

$$\begin{aligned} J(R) &\ni (\gamma_n(g_0, \dots, g_{n-2}, g_n) - \gamma_n(g_0, \dots, g_{n-2}, g_{n-1}))\gamma_n(g_0, \dots, g_{n-2}, g^{-1}_{n-1}) \\ &= \gamma_n(g_0, \dots, g_{n-2}, g^{-1}_{n-1}g_n) - 1, \end{aligned}$$

so $\gamma_n(h_0, \dots, h_{n-1}) - 1 \in J(R)$ for every $(h_0, \dots, h_{n-1}) \models p^{(n)}|\emptyset$. But, by Lemma 1.16, every element of A is a product of finitely many elements of the form $\gamma_n(h_0, \dots, h_{n-1})$, where $(h_0, \dots, h_{n-1}) \models p^{(n)}|\emptyset$. By ω-categoricity, every element of R is a sum of a fixed number of elements of the set $A \cup \{-h : h \in A\} \cup \{0\}$. So, we conclude that $J(R)$ is of finite index in R.

The rest of the proof follows the lines of the final part of the proof of [25, Corollary 3.17]. Namely, choose representatives $r_1, \dots, r_m$ of all cosets of $J(R)$ in R. Let $k \geqslant 1$ be the least number such that $J(R)^k = \{0\}$ (such a number exists by Fact 1.11). Take any non-trivial $i \in J(R)^{k-1}$ (where $J(R)^0 := R$). Then $i(v) \neq 0$ and $Ai(v) \subseteq Ri(v) = \{r_1 i(v), \dots, r_m i(v)\}$. Thus, $\mathrm{Lin}_F(Ai(v))$ is a non-trivial, finite-dimensional (over F) FA-submodule of $\overline{V}$. This is a contradiction to $(*)$, which completes the proof of Claim 1. □

Using Claim 1, the lemma follows easily by induction on the nilpotency class of H. □

Notice that the only places in the proof of Lemma 1.17 in which the assumption that H has no elements of order q is used are the proof of [30, Proposition 3.4] (an application of Maschke's theorem) and the proof of $(*)$.

Having Fact 0.1 and Lemma 1.17 in hand, the proof from page 490 of [30] goes through under the assumption of Theorem 0.2 after a slight modification (which is necessary, because the generic stability of a group is not inherited by definable subgroups).

Proof of Theorem 0.2. Let G be a generically stable group $\emptyset$-definable in a monster model of an ω-categorical theory. By fsg, G^{00} exists, and by ω-categoricity, it is $\emptyset$-definable. Hence, G^{00} has finite index in G, and we can assume that $G = G^{00}$. By Fact 0.1, G is solvable-by-finite, so it has a definable, solvable subgroup of finite index (the group generated by all normal, solvable subgroups of finite index is solvable and $\emptyset$-invariant, so by ω-categoricity it is also $\emptyset$-definable). Thus, G is solvable.

We will argue by induction on the maximal possible length of a chain of distinct, characteristic (in the group-theoretic sense) subgroups of G. Suppose $\{e\} = G_0 < G_1 < \cdots < G_t = G$ is a chain of maximal length of distinct, characteristic subgroups of G and that the theorem holds for absolutely connected, generically stable groups with smaller maximal length of such a chain. Notice that all groups G_i are invariant and so $\emptyset$-definable. They are also normal in G.

The group G_1 is characteristically simple and solvable, so it is abelian. The group G/G_1 is absolutely connected and generically stable, hence, by the induction hypothesis, it is nilpotent-by-finite and so nilpotent (by absolute connectedness). Put $N := G_1$

Since any nilpotent, ω-categorical group is a direct product of its Sylow subgroups, we may write

$$G/N = P_1 \times \cdots \times P_n, \tag{†}$$

where each P_i is a Sylow p_i-subgroup of G/N. By ω-categoricity, G/N has bounded exponent, so every P_i is definable. Hence, using (†), one can easily check that every P_i is a nilpotent, absolutely connected, generically stable group (if $tp(a/\mathfrak{C})$ is a generically stable generic type in G/N, and $a = (a_1, \dots, a_n)$ with $a_i \in P_i$, then each a_i is in $dcl(a)$, so $tp(a_i/\mathfrak{C})$ is generically stable; it is also a generic type by (†)). Thus, we can apply Lemma 1.17 to definable actions of these groups on the appropriate abelian groups. Having this, the rest of the proof from [30] goes through in our context, which we sketch below.

Let Q_i be the preimage of P_i, $i = 1, \dots, n$, under the quotient map $G \to G/N$. Applying Lemma 1.17 to the action of P_1 on the subgroup of N consisting of the elements whose order is co-prime to p_1, we get that the elements of Q_1 of co-prime orders commute, and so Q_1 is locally nilpotent. Since Q_1 is also solvable, ω-categoricity and [25, 4(8)] imply that Q_1 is nilpotent. Since the group Q_2Q_1/Q_1 is definably isomorphic with $P_2/P_2 \cap P_1$, we see that it is a nilpotent, absolutely connected and generically stable p_2-group. Thus, applying Lemma 1.17 to the action of Q_2Q_1/Q_1 on successive quotients of the lower central series of the normal Sylow r-subgroup of the nilpotent group Q_1 (for $r \neq p_2$), we conclude that the elements of co-prime orders in Q_2Q_1 commute. As before, this implies that Q_2Q_1 is nilpotent. Continuing in this way, we get that $G = Q_n \dots Q_1$ is nilpotent. □

2 Locally finite profinite rings

2.1 Preliminaries

Recall that we always assume rings to be associative, but we do not assume them to be commutative or unital. By an ideal we mean a two-sided one. In this section, by a topological ring we mean a ring equipped with a Hausdorff topology under which multiplication, addition and additive inversion are continuous functions.

If $r \in R$, then $\mathrm{Ann}(r) := \{a \in R : ar = ra = 0\}$ is the two-sided annihilator of r in R. Note that $\mathrm{Ann}(r)$ is always a subgroup of R^+, and if R is commutative, then $\mathrm{Ann}(r)$ is an ideal.

In this section, we will frequently use the facts about Jacobson radical mentioned at the end of Subsection 1.1.

For any left ideal I of R, define m_I to be the largest (two-sided) ideal of R contained in I. Then, for any maximal left regular ideal I, R/m_I is a (left) primitive ring (i.e. a ring having a left faithful irreducible module, namely the module R/I). The next fact follows from [18, Theorem 2.1.4].

Fact 2.1 *If I is a maximal left regular ideal of a ring R and the quotient R/m_I is finite, then R/m_I is the complete matrix ring $M_k(F)$ over a finite field F.*

Remark 2.2 *Let R be any ring. For every $x \in J(R)$ and $a \in R$ such that $xa+a = 0$, we have $a = 0$.*

Proof. Since $x \in J(R)$, we get that there is some z such that $zx + z + x = 0$. Then $0 = zxa + za + xa = -za + za - a = -a$. □

Corollary 2.3 *If R is a weakly locally finite ring, then $J(R)$ is nil.*

Proof. Take any $x \in J(R)$. By assumption, there are $n > m \geqslant 1$ such that $x^n = x^m$. Then $(-x^{n-m})x^m + x^m = 0$ and $-x^{n-m} \in J(R)$. Thus, $x^m = 0$ by Remark 2.2. □

The following remark follows easily from the definition of $J(R)$.

Remark 2.4 *If R is a compact topological ring, then $J(R)$ is a closed ideal.*

The following fact [35, Proposition 5.1.2] yields a characterization of when a topological unital ring is profinite.

Fact 2.5 *Let R be a topological unital ring (so, in particular, a Hausdorff topological space). Then the following conditions are equivalent:*

1. *R is profinite, i.e. the inverse limit of finite rings.*
2. *R is compact.*
3. *R is compact and totally disconnected.*

4. R is compact and there is a basis of open neighbourhoods of 0 *consisting of open ideals.*

Now, we will give some basic information about small compact G-rings. Let (R, G) be a compact G-ring. For any finite $C \subseteq R$, by G_C we denote the pointwise stabilizer of C in G, and for a finite tuple a of elements of R, by $o(a/C)$ we denote the orbit of a under the action of G_C (and we call it the orbit of a over C). Then we have that (R, G) is small iff for every finite $C \subseteq R$ there are only countably many orbits on R over C.

Fact 2.6 *Every small compact G-ring R is locally finite and profinite.*

Proof. Local finiteness of R follows as in [26, Proposition 5.7]. Namely, consider any finite subset S of R, and let $\overline{\langle S\rangle}$ be the closure of the subring generated by S. As each element of $\overline{\langle S\rangle}$ is fixed by G_S, smallness implies that $\overline{\langle S\rangle}$ is countable. But it is also a compact group, so it must be finite.

By Fact 2.5, we see that R is profinite when R has a unit, and we will use this information to show that R is always profinite. By the Baire category theorem and local finiteness of R, we get that for some non-zero $n < \omega$ the set $\{r \in R : nr = 0\}$ has non-empty interior. Hence, we can cover R with finitely many translates of this set, which yields that R has a finite characteristic c. As in Case 2 in the proof of Theorem 0.3 from the previous section, we define $+$ and $\cdot$ on $R_1 := R \times \mathbb{Z}_c$ by $(a, k) + (b, l) = (a + b, k +_c l)$ and $(a, k) \cdot (b, l) = (ab + l \times a + k \times b, k \cdot_c l)$, where $+_c$ and $\cdot_c$ are addition and multiplication modulo c, and $l \times a := a + \cdots + a$ (l-many times). Then, R_1 is an unital compact ring, and we can treat R as a clopen ideal of R in the natural way. By Fact 2.5, R_1 is profinite, so also R is profinite. □

So, we see that local finiteness is a property shared by ω-categorical rings and small compact G-rings. At the end of Subsection 2.2, we discuss some correspondences between problems about ω-categorical rings and small compact G-rings.

The following remarks provide some examples of small compact G-rings, and they will be used later. The proof of the first of them is the same as that in Example A(ii) from [26] (formulated after Conjecture 5.15), and the proof of the second one is straightforward.

Remark 2.7 *Let R be a finite ring. Consider the action of the permutation group $G = S_\omega$ on R^ω given by $(\sigma \cdot f)(i) = f(\sigma^{-1}(i))$. Then, (R^ω, G) is a small compact G-ring.*

Remark 2.8 *Let (R_1, G_1), (R_2, G_2) be two small compact G-rings. Then also $(R_1 \times R_2, G_1 \times G_2)$ is a small compact G-ring, where the action is given by $(g_1, g_2) \cdot (r_1, r_2) = (g_1 \cdot r_1, g_2 \cdot r_2)$.*

Suppose we have a profinite ring R which is the inverse limit of a distinguished countable inverse system, or equivalently, R is a compact topological ring with a distinguished countable basis of open neighbourhoods of 0 consisting of clopen ideals.

Definition 2.9 *A* profinite ring regarded as profinite structure *is a pair of the form* $(R, Aut^*(R))$*, where* $Aut^*(R)$ *is a closed subgroup of the group* $Aut^0(R)$ *of all automorphisms of* R *respecting the inverse system defining* R *(equivalently,* $Aut^*(R)$ *is a closed subgroup of the group of all automorphisms of* R *fixing setwise the clopen ideals from the distinguished basis of open neighbourhoods of 0). The group* $Aut^*(R)$ *is called the* structural group *of* R *and* $Aut^0(R)$ *is the* standard structural group *of* R*.*

The general definition of a profinite structure is given in Definition 0.4.

Definition 2.10 *A profinite ring* $(R, Aut^*(R))$ *is* small *if there are only countably many orbits under* $Aut^*(R)$ *on finite tuples over* $\emptyset$ *(equivalently, if it is small regarded as a compact G-ring).*

Until the end of this subsection, by a profinite ring we mean a profinite ring regarded a profinite structure. Let $(R, Aut^*(R))$ be a profinite ring. We will write R having in mind either the topological ring R or the pair $(R, Aut^*(R))$.

By an A-invariant subset of $R^{\times n}$ we mean a subset invariant under the pointwise stabilizer of A (denoted by $Aut^*(R/A)$), and by an A-closed subset we mean a closed and A-invariant subset of $R^{\times n}$; if we do not want to specify A, we write $*$-invariant or $*$-closed. For $a \in R^{\times n}$ and $A \subseteq R$ we define $o(a/A) = \{f(a) : f \in Aut^*(R/A)\}$, the orbit of a over A.

The following remark follows directly from the definition.

Remark 2.11 *A profinite ring* R *has a descending chain* $(I_n : n < \omega)$ *of open* $\emptyset$*-invariant ideals forming a basis of open neighbourhoods of* 0 *(hence with trivial intersection).*

It is clear that whenever I is an A-closed ideal of R, then R/I can be treated as a profinite ring (regarded as profinite structure) with the structural group induced by $Aut^*(R/A)$.

Since every orbit is a closed subset of R (as a continuous image of a compact space $Aut^*(R/A)$), it follows, by the Baire category theorem, that if R is small, then over any finite subset A there exists an open orbit.

By $acl(A)$ we denote the set of elements of R which have finite orbits over A. We say that $g \in R$ is generic over A if $o(g/A)$ is open; g being generic means that it is generic over $\emptyset$ or over a set of parameters which is obvious from the context.

Recall that, in the case of profinite structures, m-independence coincides with nm-independence, so, in particular, it satisfies properties listed in Fact 0.11 (for profinite structures this was proved by Newelski).

The following fact was proved in [31].

Fact 2.12 *Let* $(G, Aut^*(G))$ *be a small profinite group, and* A *a finite subset. An* A*-invariant subgroup of* G *is* A*-closed. In particular, all characteristic subgroups of* G *are* $\emptyset$*-closed. The group generated by any family of* A*-invariant sets is* A*-closed, and generated in finitely many steps from finitely many sets. There is no infinite increasing chain of* A*-invariant subgroups of* G*.*

It was observed in [28] that in order to obtain Conjecture 0.16 for commutative rings, it is enough to prove it only for rings with the characteristic and nilexponent equal to the same prime number (this relies on the Nagata-Higman theorem which yields that nil rings of nilexponent smaller than the characteristic are nilpotent). Also, the following reduction was obtained there (see [28, Proposition 4.4]).

Fact 2.13 *To show Conjecture 0.16 for commutative rings, one can assume that for each non-zero $a \in R$, the quotient ring $R/\operatorname{Ann}(a)$ is not nilpotent.*

2.2 Main Results

This subsection is devoted mainly to the proofs of our main results about weakly locally finite profinite rings. We start from a certain characterization of Jacobson radicals in profinite rings.

Proposition 2.14 *Let R be a profinite ring, and let $\mathcal{I}$ be the family of all maximal regular left open (so clopen) ideals of R. Recall that for any left ideal I, m_I denotes the unique largest (two-sided) ideal of R contained in I.*

1. *Each $I \in \mathcal{I}$ is a maximal regular left ideal of R.*
2. *For each $I \in \mathcal{I}$, m_I is the unique largest (two-sided) clopen ideal of R contained in I; in particular, R/m_I is finite.*
3. *$J(R) = \bigcap_{I \in \mathcal{I}} m_I = \bigcap \mathcal{I}$.*

Proof. (1) Take $I \in \mathcal{I}$. Then, any left ideal extending I is also clopen.
(2) Take any $I \in \mathcal{I}$. Since it is clopen, there is a clopen (two-sided) ideal m of R contained in I. Then $m_I + m$ is a clopen ideal of R contained in I, so, by the maximality of m_I, we get that $m_I = m_I + m$ is clopen.
(3) Put $J_1(R) = \bigcap \mathcal{I}$ and $J_2(R) = \bigcap_{I \in \mathcal{I}} m_I$. By [18, Theorem 1.2.1], we know that $J(R) = \bigcap_{J \in \mathcal{J}} m_J$, where $\mathcal{J}$ is the collection of all maximal regular left ideals. So, by (1), we get get $J(R) \subseteq J_2(R) \subseteq J_1(R)$. It remains to show that $J_1(R) \subseteq J(R)$. By the proof of [18, Theorem 1.2.2], we see that it is enough to show that for any $x \in J_1(R)$ the set $A := \{yx + y : y \in R\}$ is all of R. Suppose for a contradiction that A is proper. Since R is compact, A is a regular left closed ideal of R. Thus, there exists a clopen ideal I such that $A + I$ is a proper regular left clopen ideal of R. Hence, A is contained in a maximal regular left clopen ideal J. Then, for every $y \in R$, we have $x \in J_1(R) \subseteq J$ and $yx + y \in J$, so $y \in J$. Thus, $J = R$, a contradiction. □

In the proof of Theorem 0.14, we will use the above proposition only for weakly locally finite unital rings. In this case, one can give a direct proof (not referring to the proof of [18, Theorem 1.2.2]), which we do below for the reader's convenience.

Proof of Proposition 2.14(3) for unital rings.

As before, it is enough to show that $J_1(R) \subseteq J(R)$. Since R is unital and $J_1(R)$ is a left ideal, we will be done if we show that $1+x$ is left invertible in R for all $x \in J_1(R)$. So, take any $x \in J_1(R)$ and suppose for a contradiction that $1 \notin R(1+x)$. By compactness, $R(1+x)$ is closed in R, so there is an open ideal I of R such that $1 \notin R(1+x)+I$. We can extend $R(1+x)+I$ to a maximal left open ideal J of R. Then $x \in J_1(R) \subseteq J$ and $1+x \in J$, so $1 \in J$. Thus, $J=R$, a contradiction. □

Recall that for a filed F, $M_k(F)$ denotes the ring of all matrices of size $k \times k$ with entries from F.

Lemma 2.15 *Let R be a weakly locally finite profinite ring. Then there exists $n<\omega$ such that for every finite field F and a number $k<\omega$ if there exists an epimorphism $f: R \to M_k(F)$, then $k<n$ and $|F|<n$.*

Proof. For any $1<m<\omega$, consider the polynomial

$$w_m = \prod_{0<i<j\leqslant m} (x^i - x^j).$$

Since R is weakly locally finite, we have that $R=\bigcup_{m<\omega} Z_R(w_m)$, where $Z_R(p)$ denotes the set of all zeros in R of a polynomial p. Since each $Z_R(w_m)$ is closed in R, we get, by the Baire category theorem, that for some $m<\omega$, $Z_R(w_m)$ has a non-empty interior. Take $a \in R$ and an open ideal I of R of index s such that $a+I \subseteq Z_R(w_m)$. Since $f[I]$ is an ideal of $M_k(F)$, we get that it is either trivial or equal to $M_k(F)$.

Now, $[M_k(F) : f[I]] \leqslant s$, and if $f[I]=\{0\}$, then $[M_k(F) : f[I]] \geqslant 2^{k^2}$. So, either $2^{k^2} \leqslant s$, or $f[I]=M_k(F)$ and then $f[a+I]=f(a)+f[I]=M_k(F)$. In the latter case, since $(\forall x \in a+I)(w_m(x)=0)$, we get that $(\forall x \in M_k(F))(w_m(x)=0)$. Using the matrix with 1's right above the diagonal and 0's elsewhere, one easily gets that $k \leqslant \sum_{i=1}^{m-1} i(m-i) = \frac{m(m-1)(m+1)}{6}$. So, we obtain the following bound on k:

$$k \leqslant \max\left(\sqrt{\log_2(s)}, \frac{m(m-1)(m+1)}{6}\right).$$

Now, if $f[I]=M_k(F)$, then $(\forall x \in F)(w_m(x)=0)$ (as F embeds into $M_k(F)$), so $|F| \leqslant \deg(w_m)$. On the other hand, if $f[I]=\{0\}$, then $|M_k(F)| \leqslant s$, so $|F| \leqslant s$. Thus, in any case,

$$|F| \leqslant \max(s, \deg(w_m)).$$

□

Now, we proceed to the proof of Theorem 0.14.

Proof of Theorem 0.14. The implication $(\leftarrow)$ is easy. Let R be topologically isomorphic to the product of complete matrix rings over finite fields with only finitely many factors up to isomorphism. Clearly R is profinite and locally finite. To see semisimplicity, is is enough to use a classical fact that the complete matrix ring over any field is semisimple (see e.g. [18, Theorem 1.2.6]) and to show that products of semisimple rings are semisimple, which is a very easy exercise.

Now, we turn to the proof of ($\rightarrow$). Let R be our semisimple, weakly locally finite profinite ring.

Claim 1 *We can assume that R is unital.*

Proof of Claim 1. Suppose that the theorem is true for unital rings. Define R_1 as in the proof of Fact 2.6. Since R is semisimple, we get that $R \cap J(R_1) = J(R) = \{0\}$. Thus, R is isomorphic (via the quotient map) to a closed ideal of $R_1/J(R_1)$. Since $R_1/J(R_1)$ is semisimple, we get, by our assumption, that it is of the form $\prod_{i=1}^m M_{n_i}(F_i)^{\kappa_i}$ for some cardinal numbers κ_i. Every closed ideal I of this ring consists precisely of elements with zeros on a fixed set of coordinates. Indeed, consider the set A of coordinates on which the projection of I is nontrivial (so, the projection has to be equal to the corresponding complete matrix ring). Since I is an ideal, every element of $\prod_{i=1}^m M_{n_i}(F_i)^{\kappa_i}$ with a finite support contained in A belongs to I. By the closedness of I, we conclude that I consists exactly of elements vanishing outside of A. So, every such I is also isomorphic to a ring of the form $\prod_{i=1}^m M_{n'_i}(F'_i)^{\kappa'_i}$. In particular, R can be presented in this form. □

Now, assume that R is unital. Let $\mathcal{I}$ be the family of all maximal left open ideals of R. Let $\mathcal{I}_1$ be the subfamily of those $I \in \mathcal{I}$ for which m_I is minimal in the family of all m_I's, $I \in \mathcal{I}$. Finally, for each $m \in \{m_I : I \in \mathcal{I}_1\}$ choose exactly one ideal I from the set $\{I \in \mathcal{I}_1 : m_I = m\}$, and denote the collection of all ideals obtained in this way by $\mathcal{I}_2$.

By Proposition 2.14(2), for each $I \in \mathcal{I}$, R/m_I is finite, so Proposition 2.14(1) and Fact 2.1 imply that R/m_I is isomorphic to a complete matrix ring over a finite field. Thus, using Lemma 2.15, we get that the family $\{m_I : I \in \mathcal{I}\}$ is well-founded with respect to inclusion. Therefore, $\bigcap_{I\in\mathcal{I}} m_I = \bigcap_{I\in\mathcal{I}_2} m_I$. By Proposition 2.14(3) and the assumption that R is semisimple, we end up with

$$\bigcap_{I\in\mathcal{I}_2} m_I = J(R) = \{0\}. \qquad (*)$$

Define a homomorphism $f : R \to \prod_{I\in\mathcal{I}_2} R/m_I$ as the diagonal of the quotient homomorphisms. By $(*)$, we get that f is injective. Also, f is continuous, so $\mathrm{Im}(f)$ is closed.

Now, consider any $I, J \in \mathcal{I}_2$ with $I \neq J$. Since m_J is not contained in m_I, we get that $(m_I + m_J)/m_I$ is a non-trivial ideal of R/m_I. Hence, as we know that R/m_I is a complete matrix ring over a finite field, we get that $(m_I + m_J)/m_I$ is equal to R/m_I, so $m_I + m_J = R$. Therefore, since R is unital, it follows from the Chinese Remainder Theorem that $\mathrm{Im}(f)$ is dense in $\prod_{I\in\mathcal{I}_2} R/m_I$. So, $\mathrm{Im}(f) = \prod_{I\in\mathcal{I}_2} R/m_I$, and f is an isomorphism. Moreover, by Lemma 2.15, the complete matrix rings R/m_I have bounded size. This completes the proof of Theorem 0.14. □

Corollary 2.16 *Let $\mathcal{A}$ denote the class all topological rings isomorphic to a product of complete matrix rings of bounded size over finite fields, and let $\mathcal{A}_0$ be its subclass consisting of products of only countably many matrix rings.*

1. *$\mathcal{A}$ is the class of all semisimple weakly locally finite profinite rings.*
2. *$\mathcal{A}$ is the class of all semisimple locally finite profinite rings.*
3. *$\mathcal{A}_0$ is the class of all semisimple topological rings admitting a structure of a small compact G-ring.*

Proof. (1) is a restatement of Theorem 0.14.
(2) follows from (1) and the observation that each ring from $\mathcal{A}$ is locally finite.
(3) By Remarks 2.7 and 2.8, any member of $\mathcal{A}_0$ admits a structure of a small compact G-ring. By Fact 2.6 and (2), we get that each semisimple topological ring R admitting a structure of a small compact G-ring belongs to $\mathcal{A}$. However, [26, Proposition 3.9] tells us that R is second countable. Thus, we conclude that $R \in \mathcal{A}_0$. □

Using this classification, we can also describe possible actions of G on R for semisimple small compact G-rings (R, G).

Corollary 2.17 *Let $\mathcal{A}$ be defined as above. Take any $R \in \mathcal{A}$. Present R in the form $\prod_{i=1}^{m} M_{n_i}(F_i)^{\kappa_i}$ for some cardinal numbers κ_i so that the rings $M_{n_i}(F_i)$ are non-isomorphic for distinct i's. Then the group of all topological automorphisms of R is equal to*

$$\{\prod_{i=1}^{m} f_{\sigma_i} \circ g : \sigma_i \in Sym(\kappa_i), g \in \prod_i Aut(M_{n_i}(F_i))^{\kappa_i}\},$$

where $f_\sigma(x)(\alpha) = x(\sigma^{-1}(\alpha))$. Hence, it is isomorphic to the semidirect product of $\prod_i Sym(\kappa_i)$ and $\prod_i Aut(M_{n_i}(F_i))^{\kappa_i}$. Thus, if (R, G) is a small compact G-ring, then we can treat G as a subgroup of the above group, acting on R in the natural way.

Proof. Consider an arbitrary topological automorphism f of R. Fix any i and consider any $\alpha \in \kappa_i$.

We have that $f[\{r \in R : (\forall (j, \beta) \neq (i, \alpha))(r(j, \beta) = 0)\}]$ is a closed ideal of R isomorphic to $M_{n_i}(F_i)$, and since it is not a product of two non-trivial rings, we get that it is equal to $\{r \in R : (\forall (j, \beta) \neq (i, \gamma))(r(j, \beta) = 0)\}$ for some $\gamma \in \kappa_i$. Define

$$\sigma_i(\alpha) = \gamma.$$

By composing f with canonical isomorphisms $M_{n_i}(F_i) \to \{r \in R : (\forall (j, \beta) \neq (i, \alpha))(r(j, \beta) = 0)\}$ and $\{r \in R : (\forall (j, \beta) \neq (i, \gamma))(r(j, \beta) = 0)\} \to M_{n_i}(F_i)$, we obtain an automorphism $g_{i,\alpha}$ of $M_{n_i}(F_i)$. Put

$$h = \prod_{i=1}^{m} (f_{\sigma_i} \circ \prod_{\alpha \in \kappa_i} g_{i,\alpha}).$$

By the choice of h, we get that h agrees with f on elements of R having one-element supports. Since f and g are algebraic isomorphisms, we get that they agree on all elements with finite supports. Finally, we conclude, by continuity of f and h, that $f = h$. This completes the proof. $\square$

Question 2.18 *Let (R, G) be a semisimple small compact G-ring. By Corollary 2.16, we know that $R \in \mathcal{A}_0$, and we have that the group G, treated as a permutation group of R, is a subgroup of the concrete Polish group $Aut(R)$ of all topological automorphisms of R described in Corollary 2.17. Consider $Aut(R)$ equipped with the compact-open topology. Is it the case that G is also a topological subgroup of $Aut(R)$, i.e. is the topology on G inherited from $Aut(R)$? Equivalently, is G a closed subgroup of $Aut(R)$? The equivalence of these questions follows from the fact that a topological subgroup of a Polish group is Polish iff it is closed, and from the fact that each two comparable Polish group topologies on a given group are equal, see for example [23, Theorem 9.10]. (Notice that it follows from the latter that if the compact-open topology on $Aut(R)$ is Polish, then it is the unique Polish topology on it under which the action is continuous, as every such a topology is finer than the compact-open topology).*

Notice that any $R \in \mathcal{A}_0$ has a countable dense set invariant under all topological automorphisms of R (namely, the set of elements with finite supports). Hence, by the following remark, there can be at most one Polish topology on a group G acting faithfully on R, under which the action is continuous.

Remark 2.19 *If G is a Polish group acting continuously and faithfully on a T_2-topological space X, and $D \subseteq X$ is a countable dense subset which is invariant under the action of G, then there is no other Polish group topology on G under which the action of G on X is continuous.*

Proof. Consider the family F of all subsets of G of the form $\{g \in G : g(c) = d\}$, where $c, d \in D$. Then F is a countable family separating points of G (since D is dense). Now, if τ_1, τ_2 are Polish group topologies on G under which the action of G on X is continuous, then all sets from F are Borel (even closed) in τ_1, τ_2, so F generates the families of Borel subsets of $(G, \tau_1), (G, \tau_2)$ as sigma-algebras (by [23, Exercise 15.4 ii)]). Hence, the map $id : (G, \tau_1) \to (G, \tau_2)$ is a Borel isomorphism, and by [23, Theorem 9.10], the topologies are equal. $\square$

Now, we turn to the proof Theorem 0.15.

Proof of Theorem 0.15. If R is a weakly locally finite profinite ring, then $J(R)$ is nil by Corollary 2.3 and closed by Remark 2.4, so $J(R)$ is a nil profinite ring. Thus, the second part of the theorem is indeed more general than the first part. So, from now on, assume that R is a nil profinite ring. Then $J(R) = R$.

Since R is nil, we get, by the Baire category theorem, that there is a non-zero $n < \omega$ such that the set $\{r \in R : r^n = 0\}$ has a non-empty interior. Take an open ideal I of R and a coset $e = a + I$ such that $e \subseteq \{r \in R : r^n = 0\}$.

We put $Z_{-1} = \{0\}$, and define inductively

$$Z_{i+1} = \{r \in R : IrI/(Z_i \cap IrI) \text{ is nilpotent}\}.$$

It is easy to see that Z_i is an ideal of R for each i (if $r_1 \in Z_{i+1}$ yields the quotient $Ir_1I/(Z_i \cap Ir_1I)$ of nilpotency class c_1 and r_2 yields the quotient of nilpotency class c_2, then $r_1 + r_2$ yields the quotient of nilpotency class at most $c_1 + c_2$).

Claim 1 *For all $k \in \{0, \dots, n-1\}$, $y \in I$ and $x \in e$ we have $x^{n-k}yx^{n-k} \in Z_{k-1}$.*

Proof of Claim 1. We proceed by induction on k. The case $k = 0$ is trivial. Suppose the claim is true for a certain $k < n-1$. Then $w^{n-k} \in Z_k$ for every $w \in e$. So, in R/Z_k, we have

$$\begin{aligned}
&0 + Z_k = (x + yx^{n-k-1})^{n-k} + Z_k = \\
&= x^{n-k-1}yx^{n-k-1} + x^{n-k-2}(yx^{n-k-1})^2 + \dots + (yx^{n-k-1})^{n-k} + Z_k = \\
&= x^{n-k-1}yx^{n-k-1} + zx^{n-k-1}yx^{n-k-1} + Z_k,
\end{aligned}$$

where $z = x^{n-k-2}y + x^{n-k-3}yx^{n-k-1}y + \dots + (yx^{n-k-1})^{n-k-2}y$. But $z + Z_k \in R/Z_k = J(R/Z_k)$, because R/Z_k is nil. Hence, by Remark 2.2, $x^{n-k-1}yx^{n-k-1} + Z_k = 0 + Z_k$, which shows that the claim is true for $k+1$. □

Applying Claim 1 for $k = n-1$, we easily get that $e \subseteq Z_{n-1}$, so Z_{n-1} is clopen. For $x \in R$ and $\bar{i} = (i_1, \dots, i_{2m})$ (where each i_j is from R), we define

$$\bar{i} * x = i_1 x i_2 i_3 x i_4 \dots i_{2m-1} x i_{2m}.$$

Claim 2 *The following statement is true for all $k \in \omega$.*

$$\begin{gathered}
(\forall x \in Z_{k-1})(\exists m_1 < \omega)(\forall \overline{i_1} \in I^{\times 2m_1})(\exists m_2 < \omega)(\forall \overline{i_2} \in I^{\times 2m_2}) \dots \\
(\exists m_k < \omega)(\forall \overline{i_k} \in I^{\times 2m_k})(\overline{i_k} * (\overline{i_{k-1}} * (\dots (\overline{i_1} * x) \dots)) = 0).
\end{gathered}$$

Proof of Claim 2. The claim follows from the definition of the Z_j's by induction on k. The case $k = 0$ is trivial. For the induction step, consider any $x \in Z_{k-1}$. Let m_1 be the nilpotency class of $IxI/(Z_{k-2} \cap IxI)$. Then, for any $\overline{i_1} \in I^{\times 2m_1}$ we have that $\overline{i_1} * x \in Z_{k-2}$, so the assertion follows from the inductive hypothesis. □

In the next claim, we will show that in the statement from Claim 2 one can move all the existential quantifiers to the left, obtaining a statement of the form $(\forall x \in Z_{k-1})\exists\forall$.

Claim 3 *The following statement is true for all $k \in \omega$.*

$$\begin{gathered}
(\forall x \in Z_{k-1})(\exists m_1, \dots, m_k < \omega)(\forall \overline{i_1} \in I^{\times 2m_1}, \dots, \overline{i_k} \in I^{\times 2m_k}) \\
(\overline{i_k} * (\overline{i_{k-1}} * (\dots (\overline{i_1} * x) \dots)) = 0).
\end{gathered}$$

Proof of Claim 3. By induction on k, we will show that whenever $x \in R$ is such that

$$(\exists m_1 < \omega)(\forall \overline{i_1} \in I^{\times 2m_1})(\exists m_2 < \omega)(\forall \overline{i_2} \in I^{\times 2m_2}) \dots (\exists m_k < \omega)(\forall \overline{i_k} \in I^{\times 2m_k}) \\ (\overline{i_k} * (\overline{i_{k-1}} * (\dots (\overline{i_1} * x) \dots)) = 0),$$

then

$$(\exists m_1, \dots, m_k < \omega)(\forall \overline{i_1} \in I^{\times 2m_1}, \dots, \overline{i_k} \in I^{\times 2m_k}) \\ (\overline{i_k} * (\overline{i_{k-1}} * (\dots (\overline{i_1} * x) \dots)) = 0).$$

This together with Claim 2 will finish the proof.

The cases $k = 0$ and $k = 1$ are trivial, as there are no quantifiers to switch. Now, take $k > 1$ and assume that the statement is true for numbers less than k. Consider any $x \in R$ satisfying the assumption of our statement. By the inductive hypothesis, we get

$$(\exists m_1 < \omega)(\forall \overline{i_1} \in I^{\times 2m_1})(\exists m_2, \dots, m_k < \omega)(\forall \overline{i_2} \in I^{\times 2m_2}, \dots, \overline{i_k} \in I^{\times 2m_k}) \\ (\overline{i_k} * (\overline{i_{k-1}} * (\dots (\overline{i_1} * x) \dots)) = 0). \qquad (*)$$

Now, the goal is to switch $(\forall \overline{i_1} \in I^{\times 2m_1})$ with $(\exists m_2, \dots, m_k < \omega)$. We will do this in $2m_1$ steps, switching at every step all existential quantifiers $\exists m_2, \dots, \exists m_k$ with one universal quantifier corresponding to one of the variables in the sequence $\overline{i_1}$. We will only show how to switch all these existential quantifiers with the universal quantifier corresponding to the last variable in $\overline{i_1}$, as the other steps can be done in a similar fashion.

Denote $\overline{i_1} = (s_1, \dots, s_{2m_1})$, and fix $s_1, \dots, s_{2m_1-1} \in I$. Put

$$t = s_1 x s_2 s_3 x s_4 \dots s_{2m_1-1} x.$$

For any $\overline{m} = (m_2, \dots, m_k) \in (\omega \setminus \{0\})^{k-1}$ define

$$D_{\overline{m}} = \{i \in I : (\forall \overline{i_2} \in I^{\times 2m_2}, \dots, \overline{i_k} \in I^{\times 2m_k})(\overline{i_k} * (\overline{i_{k-1}} * (\dots * (ti) \dots)) = 0)\}.$$

By $(*)$, we have $I = \bigcup_{\overline{m}} D_{\overline{m}}$. It also follows that each $D_{\overline{m}}$ is a closed subset of I. Hence, by the Baire category theorem, there is some $\overline{m_0}$ such that $D_{\overline{m_0}}$ has a non-empty interior in I. Thus, for some $a_1, \dots, a_w \in I$ we have that

$$I = (a_1 + D_{\overline{m_0}}) \cup \dots \cup (a_w + D_{\overline{m_0}}).$$

We also know that there are $\overline{m_1}, \dots, \overline{m_w}$ such that $a_1 \in D_{\overline{m_1}}, \dots, a_w \in D_{\overline{m_w}}$. So, in order to finish the proof, it is enough to show the following subclaim (in which $\overline{m_i}$'s and a_i's are NOT the particular tuples or elements chosen above).

Subclaim 1 *For any $l \geqslant 1$ and $\overline{m} = (m_1, \dots, m_l) \in (\omega \setminus \{0\})^l$ define*

$$D'_{\overline{m}} = \{a \in I : (\forall \overline{i_1} \in I^{\times 2m_1}, \dots, \overline{i_l} \in I^{\times 2m_l})(\overline{i_l} * (\overline{i_{l-1}} * (\dots * (\overline{i_1} * a) \dots)) = 0)\}.$$

Let $r \geqslant 1$. Then, for every $\overline{m_1}, \dots, \overline{m_r}$ there is $\overline{m}$ such that for any $a_1 \in D'_{\overline{m_1}}, \dots, a_r \in D'_{\overline{m_r}}$ one has $a_1 + \dots + a_r \in D'_{\overline{m}}$.

Proof of Subclaim 1. The proof is by induction on l. Consider the base step $l = 1$. Take any $\overline{m_1}, \dots, \overline{m_r} < \omega$. Let $m = \overline{m_1} + \dots + \overline{m_r}$. Consider any $a_1 \in D'_{\overline{m_1}}, \dots, a_r \in D'_{\overline{m_r}}$. Then, for every $\overline{i_1} \in I^{\times 2m}$ the element $\overline{i_1} * (a_1 + \dots + a_r)$ is a sum of elements from the sets $(Ia_1I)^{\overline{m_1}}, \dots, (Ia_rI)^{\overline{m_r}}$ which are all equal to $\{0\}$, so $\overline{i_1} * (a_1 + \dots + a_r) = 0$ and $a_1 + \dots + a_r \in D'_{\overline{m}}$.

The induction step is similar. Take any $\overline{m_1} = (m_1^1, \dots, m_l^1), \dots, \overline{m_r} = (m_1^r, \dots, m_l^r)$. Let $m' = m_1^1 + \dots + m_1^r$. Consider any $a_i \in D'_{\overline{m_i}}$ for $i = 1, \dots, r$. Then, the elements from the set

$$\{\overline{i_1} * (a_1 + \dots + a_r) : \overline{i_1} \in I^{\times 2m'}\}$$

are sums of a bounded number of elements from the sets

$$\{\overline{i_1} * a_1 : \overline{i_1} \in I^{\times 2m_1^1}\}, \dots, \{\overline{i_1} * a_r : \overline{i_1} \in I^{\times 2m_1^r}\},$$

and we finish using the inductive hypothesis. □

So, we have proved the statement formulated at the beginning of the proof of Claim 3 which together with Claim 2 completes the proof of Claim 3. □

Consider the statement from the last claim for $k = n$. Using the fact that Z_{n-1} is compact, we can apply the same trick as in the proof of Claim 3 to switch all the existential quantifiers with the quantifier $\forall x \in Z_{n-1}$, and so we get that there are $m_1, \dots, m_n < \omega$ such that

$$(\forall x \in Z_{n-1})(\forall \overline{i_1} \in I^{\times 2m_1}, \dots, \overline{i_k} \in I^{\times 2m_n})(\overline{i_n} * (\overline{i_{n-1}} * (\dots (\overline{i_1} * x) \dots)) = 0).$$

Hence, $I \cap Z_{n-1}$ has finite nilexponent. Since $R/(I \cap Z_{n-1})$ is finite, it has also finite nilexponent. Thus, we conclude that R has finite nilexponent, which completes the proof of Theorem 0.15. □

Summarizing, we have the following corollary of Theorems 0.14 and 0.15.

Corollary 2.20 *Every weakly locally finite profinite ring is (nil of finite nilexponent)-by-(product of complete matrix rings over finite fields with only finitely many factors up to isomorphism).*

Having Theorem 0.15, one could ask if we can strengthen it by showing that the Jacobson radical of a locally finite profinite ring is nilpotent. The following easy example shows that this is not always true.

Example 2.21 *Let p be a prime number and let F_0 be the free commutative nil ring of nilexponent p and of characteristic p on generators $(x_i : i < \omega)$. For $n < \omega$ let I_n be the ideal generated by $\{x_i : i \geqslant n\}$. Then each quotient ring F_0/I_n is finite and nilpotent; their inverse limit, say F, is the free commutative profinite ring which is nil of nilexponent p and of characteristic p with free topological generators $\{y_n : n < \omega\}$, where $y_n = (x_n + I_k : k \in \omega)$. We see that $F = J(F)$ is locally finite, but it is not nilpotent.*

However, for the class of small compact G-rings that question (see Question 0.19) is open. In particular, we do not know whether the ring F from Example 2.21 considered together with G being the group of all topological automorphisms of F is small. We should remark here that by an easy counting argument (see [28, Example 1 in Section 5]), one can check that F cannot be a counterexample to Conjecture 0.16, i.e. it does not admit a structure of a small profinite ring.

Let us observe the following.

Proposition 2.22 *1. Suppose* $(\prod_{i \in I} R_i, G)$ *is a small profinite ring. Then only finitely many* R_i*'s are not null rings.*

2. Suppose $(\prod_{i \in I} H_i, G)$ *is a small profinite group. Then only finitely many* G_i*'s are non-abelian groups.*

Proof. (1) Put $J = \{i \in I : R_i \text{ is not null}\}$. For any $i \in J$ choose $s_i \in R_i$ such that the two-sided annihilator of s_i in R_i is not equal to R_i. Let o be an open orbit in $\prod_{i \in I} R_i$. Then, for some finite $I_0 \subseteq I$ and elements $r_i \in R_i$, $i \in I_0$, we have that $\{r \in \prod_{i \in I} R_i : (\forall i \in I_0)(r(i) = r_i)\} \subseteq o$. Define $x, y \in \prod_{i \in I} R_i$ by: $x(i) = r_i$ if $i \in I_0$ and $x(i) = 0$ otherwise; $y(i) = r_i$ if $i \in I_0$, $y(i) = 0$ if $i \in I \setminus (I_0 \cup J)$ and $y(i) = s_i$ if $i \in J \setminus I_0$. Since $x, y \in o$ and the two-sided annihilator of x is open in R, we get that the same is true about y. This clearly implies that J is finite.
(2) is similar, using centralizers instead of annihilators. □

Part (2) of the above proposition slightly strengthens Remark 4.3 from [31], where it is additionally assumed that the structural group consists of automorphisms respecting the inverse system $H_1 \leftarrow H_1 \times H_2 \leftarrow \dots$. Part (1) is a ring counterpart of (2).

Notice that Theorem 3.1 from [28] follows easily from Theorems 0.14, 0.15 and Proposition 2.22(1).

Let us finish this subsection with noting some correspondences between the problems we dealt with and some issues concerning ω-categorical structures. In [29], all ω-categorical reduced (i.e. having no non-trivial nilpotent elements) unital rings were classified up to theory. A corresponding problem was to classify reduced rings admitting a structure of a small compact G-ring. We see by Corollary 2.16 that this class of rings consists precisely of products of countably many finite fields, among which there are only finitely many non-isomorphic ones. This is, in fact, much easier to prove than the full classification of semisimple compact G-rings that we have obtained, but it was a starting point for our work.

As to Theorem 0.15, recall that a stronger conclusion (namely, nilpotency) was obtained for the Jacobson radical of an ω-categorical ring by Cherlin in Fact 1.11. As we have already mentioned, such a conclusion cannot be obtained for the Jacobson radical of a locally finite profinite ring (Example 2.21).

2.3 Remarks on Conjecture 0.16

In this subsection, we prove some reductions for Conjecture 0.16. It has already been recalled at the end of Subsection 2.1 that, in order to prove Conjecture 0.16 for commutative rings, one can assume that the ring in question is nil of nilexponent p and of characteristic p for some prime number p, which justifies this assumption in the results below.

Throughout this subsection, we will skip the structural group $Aut^*(R)$ in the notation.

Lemma 2.23 *Fix any prime number p. Let R be a non-nilpotent, commutative small profinite ring of characteristic p which is nil of nilexponent p. Then, for every $g \in G$ generic over $\emptyset$ and for every $j \in \{1, 2, \ldots, p-1\}$, we have $g^j \neq 0$ (recall than an element is called generic over A if its orbit over A is open).*

Proof. We proceed by induction on j. The conclusion is clear for $j = 1$, so suppose $j > 1$ and that the lemma holds for smaller numbers. Suppose for a contradiction that $g^j = 0$ for some generic $g \in R$. Then there is an open ideal I of R such that $(g+i)^j = 0$ for all $i \in I$. Therefore, $i^j g^{j-1} = g^{j-1}(g+i)^j = 0$ for $i \in I$. Take any $i \in I$ which is m-independent from g. If $R/\operatorname{Ann}(i^j)$ is not nilpotent, then it is a commutative small profinite ring of characteristic p and nilexponent p (by the Nagata-Higman theorem), and since $g+\operatorname{Ann}(i^j)$ is its generic satisfying $(g+\operatorname{Ann}(i^j))^{j-1} = 0+\operatorname{Ann}(i^j)$, we would get a contradiction with the inductive hypothesis. So, $R/\operatorname{Ann}(i^j)$ is nilpotent. Thus, taking $I' := \{i \in I : i \overset{m}{\perp} g\}$ and $K := \{b \in R : R/\operatorname{Ann}(b) \text{ is nilpotent}\}$, we get that $\{i^j : i \in I'\} \subseteq K$. By the claim in the proof of Proposition 4.4 in [28], we have that K is a nilpotent ideal of R. It follows from Fact 2.12 that K is also closed. Since I' is a dense subset of I, we get, by the continuity of the mapping $x \mapsto x^j$, that $\{i^j : i \in I\} \subseteq K$. Thus, $I/(I \cap K)$ is a nil ring of nilexponent not greater than $j < p$, whence we get, by the Nagata-Higman theorem, that it is nilpotent. Since also K is nilpotent, we get that I is nilpotent. Hence, R is nilpotent, a contradiction. □

Now, we will make the main observation of this subsection. We will assume that $R/\operatorname{Ann}(a)$ is non-nilpotent for every $a \in R\backslash\{0\}$. This assumption is justified by Fact 2.13.

Proposition 2.24 *Let R be as in the lemma, and assume additionally that $R/\operatorname{Ann}(a)$ is non-nilpotent for every $a \in R\backslash\{0\}$. Then, for every polynomial $f(x_1, \ldots, x_n) \in F_p[x_1, \ldots, x_n]\backslash\{0\}$ with $\deg_{x_1}(f), \ldots, \deg_{x_n}(f) < p$ and for all independent (i.e. such that every element in it is m-independent from the other elements) tuples $(g_1, \ldots, g_n)$ of generics of R we have that $f(g_1, \ldots, g_n) \neq 0$.*

Proof. We proceed by induction on n. Suppose that $n \geqslant 1$ and that the proposition is true for smaller positive natural numbers (the argument will also cover the base induction step). We can present any f satisfying the assumptions in the form

$$f = h_0(x_1, \ldots, x_{n-1}) + h_1(x_1, \ldots, x_{n-1})x_n + \cdots + h_{p-1}(x_1, \ldots, x_{n-1})x_n^{p-1}$$

(where $h_0, \dots, h_{p-1}$ are constants in the case when $n = 1$). We will prove the conclusion by induction on the number k of non-zero polynomials among $h_0, \dots, h_{p-1}$.

Suppose first that $k = 1$ and $f(x_1, \dots, x_n) = h_i(x_1, \dots, x_{n-1})x_n^i$ for some $i > 0$ (we can assume that $\deg_{x_n} f > 0$ by the inductive hypothesis). Take any tuple $(g_1, \dots, g_n)$ of independent generics of R. Put $a = h_i(g_1, \dots, g_{n-1})$. By the inductive hypothesis of the first induction, $a \neq 0$. By assumptions on R, we have that $R/\operatorname{Ann}(a)$ is non-nilpotent, so it has characteristic and nilexponent equal to p. The coset $g_n + \operatorname{Ann}(a)$ is a generic element of this ring, so, by Lemma 2.23, $g_n^i \notin \operatorname{Ann}(a)$. Hence, $f(g_1, \dots, g_n) = ag_n^i \neq 0$.

Now, we turn to the inductive step, where we assume that $k > 1$. Then,

$$f(x_1, \dots, x_n) = h_{i_1}(x_1, \dots, x_{n-1})x_n^{i_1} + h_{i_2}(x_1, \dots, x_{n-1})x_n^{i_2} + \dots + h_{i_k}(x_1, \dots, x_{n-1})x_n^{i_k},$$

where $i_1 < i_2 < \dots < i_k$ are all indices i for which h_i is non-zero (if $i_1 = 0$, then by $h_{i_1}(x_1, \dots, x_{n-1})x_n^{i_1}$ we mean just $h_{i_1}(x_1, \dots, x_{n-1})$). By the inductive hypothesis of the second induction, we get that the element

$$g_n^{p-i_k} f(g_1, \dots, g_n) = h_{i_1}(g_1, \dots, g_{n-1})g_n^{i_1+p-i_k} + h_{i_2}(g_1, \dots, g_{n-1})g_n^{i_2+p-i_k} + \dots + \\ + h_{i_{k-1}}(g_1, \dots, g_{n-1})g_n^{i_{k-1}+p-i_k} + h_{i_k}(g_1, \dots, g_{n-1})g_n^{i_k+p-i_k} = h_{i_1}(g_1, \dots, g_{n-1})g_n^{i_1+p-i_k} + \\ + h_{i_2}(g_1, \dots, g_{n-1})g_n^{i_2+p-i_k} + \dots + h_{i_{k-1}}(g_1, \dots, g_{n-1})g_n^{i_{k-1}+p-i_k}$$

is non-zero, so $f(g_1, \dots, g_n) \neq 0$. □

Corollary 2.25 *If Conjecture 0.16 is not true in the class of commutative rings, then there is a counterexample, say R, to it such that the topological ring F defined in Example 2.21 (for some prime p) is topologically isomorphic to a closed subring of R.*

Proof. As it has already been explained, by results of [28, Section 4], we can take R to be a counterexample to Conjecture 0.16 which satisfies the assumptions (and hence, the conclusion) of Proposition 2.24. Let $(I_n)_{n<\omega}$ be a decresing chain of open ideals of R with trivial intersection. We choose inductively a sequence of independent generics $g_0, g_1, \dots$ of R such that $g_{n+1} \in I_{k_n}$, where k_n is a natural number such that for any polynomial $f(x_0, \dots, x_n) \in F_p[x_0, \dots, x_n] \setminus \{0\}$ with $\deg_{x_0}(f), \dots, \deg_{x_n}(f) < p$, we have that $f(g_0, \dots, g_n) \notin I_{k_n}$. Let S be the closure of the subring of R generated by $\{g_i : i < \omega\}$.

We define $\phi : F \to S$ as follows. Consider any $y \in F$. Choose polynomials $p_i(t_0, \dots, t_i)$, $i < \omega$, such that $y = \lim_i p_i(y_0, \dots, y_i)$ for y_i's defined in Example 2.21. Then, by the choice of the sequence (g_i), the sequence $p_i(g_0, \dots, g_i)$ is convergent, and we define $\phi(y)$ to be its limit. It is easy to check that $\phi : F \to S$ is a well-defined isomorphism of topological rings. □

Note that Conjecture 0.16 would be proved if in the above proof we were able to embed topologically F into R as an invariant (under all topological automorphisms, or only under the ones coming from the structural group of R) ring. Indeed, if there was such an embedding, then F with the structural group induced by the structural group of R would be a small profinite ring, which is impossible by the comment in the paragraph below Example 2.21.

3 New examples of small Polish group structures

3.1 A non-zero-dimensional small Polish G-group

In this subsection, we construct a first example of a small non-zero-dimensional Polish G-group.

First, let us recall some results from [7]. Consider any $p \geqslant 1$ and the Banach space ℓ^p (over $\mathbb{R}$). We extend the p-norm from ℓ^p to $\mathbb{R}^\omega$ by putting $||z|| = \infty$ for every $z \in \mathbb{R}^\omega \backslash \ell^p$. The complete Erdös space is the intersection of ℓ^p with $(\mathbb{R}\backslash\mathbb{Q})^\omega$ (with the topology induced from ℓ^p).

Let $E_0, E_1, \dots$ be a fixed sequence of subsets of $\mathbb{R}$ and put

$$\mathcal{E} = \ell^p \cap \prod_{n<\omega} E_n.$$

The following is a part of [7, Theorem 1]:

Theorem 3.1 *Assume that $\mathcal{E}$ is not empty and that every E_n is zero-dimensional. For each $k \in \omega\backslash\{0\}$ we let $\eta(k) \in \mathbb{R}^\omega$ be given by*

$$\eta(k)_n = sup\{|a| : a \in E_n \cap [-1/k, 1/k]\},$$

where $\sup\emptyset = 0$*. The following statements are equivalent:*
(1) $||\eta(k)|| = \infty$ for each $k \in \omega\backslash\{0\}$,
(2) $\dim\mathcal{E} > 0$ *(where $\mathcal{E}$ is considered with the topology inherited from ℓ^p).*

Also the following theorem was proved there ([7, Theorem 3]):

Theorem 3.2 *If every E_i is closed in $\mathbb{R}$, then $\mathcal{E}$ is homeomorphic to the complete Erdös space if and only if* $\dim\mathcal{E} > 0$ *and every E_n is zero-dimensional.*

We define a structure of a group on the complete Erdös space as in [8, Proposition 4.3] (with the only difference that we do not choose a particular p), which is done as follows. Fix any $p \in [1, \infty)$. We let $C \subseteq \mathbb{R}$ be the ternary Cantor set, and $X = C^\omega \cap \ell^p$. By Theorems 3.1 and 3.2, X (considered with the topology induced from ℓ^p) is homeomorphic to the complete Erdös space. Consider the standard bijection $\phi : 2^\omega \to C$ and the product map $\psi := \phi^\omega : (2^\omega)^\omega \to C^\omega$. It follows exactly as in [8, Proposition 4.3] that $H := \psi^{-1}[X]$ is a subgroup of $(2^\omega)^\omega$ (we will identify the latter group with $2^{\omega\times\omega}$ in the natural way), and becomes a Polish group with the topology induced from X by ψ (and is homeomorphic to the complete Erdös space, which is known to be one-dimensional, see [15]). This topology is generated by the norm $||z|| := ||\psi(z)||_p$, $z \in H$. We also put $||z|| = \infty$ if $z \in 2^{\omega\times\omega}\backslash H$. For a subset A of $\omega \times \omega$, we define $||A|| := ||\chi_A||$, where χ_A is the characteristic function of A.

Now, we will define an action of a Polish group G on H. Let G_1 be the group of all permutations of $\omega \times \omega$. For any $g \in G_1$, we define the support of g to be $supp(g) = \{a \in \omega \times \omega : g(a) \neq a\}$. We put:

$$G = \{g \in G_1 : ||supp(g)|| < \infty\} < G_1.$$

It is clear that for any $g \in G$ and $h \in H$, the composition $h \circ g : \omega \times \omega \to 2$ is an element of H (since $||h \circ g|| \leqslant ||h|| + ||supp(g)||$). Hence, we can define an action of G on H by $gh = h \circ g^{-1}$. Then, G acts on H as automorphisms (both algebraic and topological). Notice, however, that if we consider G with the product topology, then this action is not continuous. Hence, we need another topology on G.

We define a metric d on G:

$$d(f, g) = ||supp(f^{-1}g)||.$$

We will consider G with the topology generated by d.

Proposition 3.3 *G is a Polish group.*

Proof. It is easy to check that d is a complete metric on G. Also, the set of elements of G with finite support is a countable, dense subset of G. Now, we will check that the composition $\circ : G \times G \to G$ is continuous. For any $(f, g), (f_1, g_1) \in G \times G$ we have $d(fg, f_1g_1) = ||supp((fg)^{-1}f_1g_1)|| = ||supp(g^{-1}(f^{-1}f_1g_1g^{-1})g)||$. Clearly, the composition is continuous at $(e, e) \in G \times G$ (since $supp(fg) \subseteq supp(f) \cup supp(g)$), so it is enough to check that conjugating by g is continuous at $e \in G$. We will check that for every $f \in G$ conjugating by f^{-1} is continuous at $e \in G$, which is of course sufficient. Notice that $supp(fhf^{-1}) = f[supp(h)]$. For any $\epsilon > 0$ there is $n < \omega$ such that $||supp(f)\backslash n \times \omega|| < \epsilon$ (where n denotes the set $\{0, 1, \dots, n-1\}$), and since $supp(f) \cap n \times \omega$ is finite, we can choose $m < \omega$ such that $f[\omega \times \omega\backslash m \times \omega] \subseteq \omega \times \omega\backslash n \times \omega$. So, for h so close to e that $supp(h) \subseteq \omega \times \omega\backslash m \times \omega$, we have that $f[supp(h)] \subseteq supp(h) \cup (f[supp(h)] \cap supp(f)) \subseteq supp(h) \cup (supp(f)\backslash n \times \omega)$. This shows that the conjugation by f is continuous, and hence, so is the group composition. Similarly, one checks that the group inversion on G is continuous. $\square$

The next proposition shows that we have constructed (the first known) example of a small, non-zero-dimensional Polish G-group.

Proposition 3.4 *(H, G) is a small, Polish G-group.*

Proof. To check that the action of G on H is continuous at every $(g, h) \in G \times H$, consider any $(g_1, h_1) \in G \times H$. Then, the functions $gh = h \circ g^{-1}$ and $g_1h_1 = h_1 \circ g_1^{-1}$ agree on the set $\{a \in \omega \times \omega : g^{-1}(a) = g_1^{-1}(a)\} \cap \{a \in \omega \times \omega : h(g^{-1}(a)) = h_1(g^{-1}(a))\}$. The complement of this set in H is the union of $supp(g_1g^{-1})$ and $g[\{a \in \omega \times \omega : h(a) \neq h_1(a)\}]$. For (g_1, h_1) sufficiently close to (g, h) these sets are arbitrary small in the sense of $|| \cdot ||$ (by a similar argument to the one in the proof of Proposition 3.3). So, the action of G on H is continuous.

It remains to check that for every finite $A \subseteq H$, there are countably many G_A-orbits in H. Fix such an A. For any $h \in H$ we put $h_0 := h^{-1}[\{0\}], h_1 := h^{-1}[\{1\}]$ (then $||h_1|| < \infty$ and $||h_0|| = \infty$). Let B be the Boolean algebra generated by the family of sets: $\{a_0, a_1 : a \in A\}$, and let $b_0, b_1, \dots, b_n$ be all its atoms. For exactly one $i \leqslant n$ we have $||b_i|| = \infty$ (this happens for i such that $b_i = \bigcap_{a \in A} a_0$), and we assume that this is the case for $i = 0$. We will show that the G_A-orbit of an element x of H depends only on the cardinalities of the sets $x_0 \cap b_0, x_0 \cap b_1, \ldots, x_0 \cap b_n$ and $x_1 \cap b_0, x_1 \cap b_1, \ldots, x_1 \cap b_n$. Suppose that for two elements $x, y \in H$ these cardinalities are the same. For all $0 \leqslant i \leqslant n$ let g_i be a permutation of b_i such that $g_i[b_i \cap x_0] = b_i \cap y_0$ and $g_i[b_i \cap x_1] = b_i \cap y_1$. We can choose g_0 in such a way that $||supp(g_0)|| < \infty$ (to see this, choose any $c \subseteq b_0$ such that $b_0 \cap x_1, b_0 \cap y_1 \subseteq c$, $c \backslash (x_1 \cup y_1)$ is infinite and $||c|| < \infty$, and notice that we can choose g_0 such that $supp(g_0) \subseteq c$). Then, $\bigcup_{i \leqslant n} g_i$ is an element of G_A, and $gx = y$. This completes the proof. □

Proposition 3.5 *(H, G) is not nm-stable.*

Proof. For $c \in 2^{\omega \times \omega}$, we will write c_{ij} instead of $c(i, j)$. Consider $o = o(a/\emptyset)$, where $a_{ij} = 1$ if $j = 0$ and $a_{ij} = 0$ if $j > 0$. For any $n < \omega$, let $b_n \in H$ be given by $(b_n)_{ij} = 1$ if $j = n + 1$, and $(b_n)_{ij} = 0$ if $j \neq n + 1$. Then, by the description of orbits obtained in the proof of Proposition 3.4, for every $n < \omega$, we have

$$o(a/b_{<n}) = \{x \in H : |(i, j) \in \omega \times \omega : x_{ij} = 1| = \omega \wedge \forall j \in \{1, 2, \ldots n\} x_{ij} = 0\}.$$

So, $o(a/b_{<n})$ is a G_δ subset of H, and hence, it is non-meager in itself. Morever, for every $n < \omega$, $o(a/b_{<n+1})$ is nowhere dense in $o(a/b_{<n})$. Thus, by Theorem 4.13, $\mathcal{NM}(o) = \infty$. So, (H, G) is not *nm*-stable. □

Similarly, one can check that $\mathcal{NM}$-rank of every uncountable 1-orbit in (H, G) is equal to ∞.

We end this subsection by stating some questions about the existence of some further interesting examples of Polish G-groups.

Question 3.6 *Is there an nm-stable, non-zero-dimensional small Polish G-group?*

Notice that since the product $H \times H$ is homeomorphic to H, we cannot obtain examples of higher dimensions just by taking finite Cartesian powers of H.

Question 3.7 *Is there a small Polish G-group of dimension greater than one?*

3.2 Small Polish group structures without generic elements

In this subsection, we construct a class of small Polish group structures for which the answer to Question 0.21 is negative.

The following fact is a part of [3, Proposition 2.3].

Fact 3.8 *Let (X, G) be a Polish structure, a be a finite tuple and A, B be finite subsets of X. Then, TFAE:*
(1) $a \mathop{\smile\!\!\!\!\!\!\mid}^{nm}_A B$
(2) $G_{AB} G_{Aa} \subseteq_{nm} G_A$ (where $Y \subseteq_{nm} Z$ means that Y is a non-meager subset of Z)

Now, let us the notion of an *nm*-generic orbit.

Definition 3.9 *Let (H, G) be a Polish group structure. We say that the orbit $o(a/A)$ is left nm-generic (or that a is left nm-generic over A) if for all $b \in H$ with $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A b$, one has that $b \cdot a \mathop{\smile\!\!\!\!\!\!\;|}^{nm} A, b$. We say that it is right nm-generic if, for b as above, we have $a \cdot b \mathop{\smile\!\!\!\!\!\!\;|}^{nm} A, b$. An orbit is nm-generic if it is both right and left nm-generic.*

It was observed in [26] that nm-generics have some nice properties (similar to those of generics in so-called simple theories), e.g. being right nm-generic is equivalent to being left nm-generic. Let us recall Proposition 5.5 from [26], which gives us a positive answer to Question 0.21 for the class of small G-groups (H, G) in which H in not meager in itself (this holds, for example, in all Polish G-groups).

Fact 3.10 *Suppose (H, G) is a small G-group. Assume H is not meager in itself (e.g. H is Polish or compact, or, more generally, Baire). Then, at least one nm-generic orbit in H exists, and an orbit is nm-generic in H iff it is non-meager in H.*

Suppose (X, G) is a Polish structure. Let H be an arbitrary group. For any $x \in X$ we consider an isomorphic copy $H_x = \{h_x : h \in H\}$ of H. By $H(X)$ we will denote the group $\bigoplus_{x \in X} H_x$. Although $H(X)$ is not necessarily commutative, we will denote its group action by $+$. For any $y \in H(X)$ there are $h_1, \ldots, h_n \in H \backslash \{e\}$ and pairwise distinct $x_1, \ldots, x_n \in X$ such that $y = (h_1)_{x_1} + \cdots + (h_n)_{x_n}$. Then, by $\tilde{y}$ we will denote the set $\{x_1, \ldots, x_n\}$. We also put $\tilde{A} = \bigcup_{y \in A} \tilde{y}$ for any $A \subseteq H(X)$.

The group G acts as automorphisms on $H(X)$ by

$$g((h_1)_{x_1} + \cdots + (h_n)_{x_n}) = (h_1)_{gx_1} + \cdots + (h_n)_{gx_n}.$$

It is easy to see that if $h_1, \ldots, h_k \in H \backslash \{e\}$ are pairwise distinct, and $x_{1,1}, \ldots, x_{1,i_1}, x_{2,1}, \ldots, x_{2,i_2}, \ldots, x_{k,1}, \ldots, x_{k,i_k} \in X$ are pairwise distinct as well, then the stabilizer of $(h_1)_{x_{1,1}} + \cdots + (h_1)_{x_{1,i_1}} + \cdots + (h_k)_{x_{k,1}} + \cdots + (h_k)_{x_{k,i_k}}$ consists exactly of those elements of G which stabilise each of the finite sets $\{x_{ij} : j = 1, \ldots, i_j\}$. Thus, we get that for every $a \in H(X)$, $G_{\tilde{a}}$ is a subgroup of finite index in G_a, and hence, for every finite $A \subseteq H(X)$, $G_{\tilde{A}}$ is a subgroup of finite index in G_A.

Proposition 3.11 *If (X, G) is a Polish structure, and H is a group, then $(H(X), G)$ is a Polish group structure. If, additionally, (X, G) is small and H is countable, then $(H(X), G)$ is small.*

Proof. For any $a \in H(X)$ we have that $G_{\tilde{a}}$ is closed in G and has finite index in G_a, so, G_a is also closed in G. Hence, $(H(X), G)$ is a Polish group structure.

Now, assume that (X, G) is small, and H is countable. Then, for every fixed $k < \omega$ and $i_1, \ldots, i_n < \omega$, the orbit of a tuple $((h_{1,1})_{x_{1,1}} + \cdots + (h_{1,i_1})_{x_{1,i_1}}, \ldots, (h_{k,1})_{x_{k,1}} + \cdots + (h_{k,i_k})_{x_{k,i_k}})$ depends only on $h_{1,1}, \ldots, h_{1,i_1}, \ldots, h_{k,1}, \ldots, h_{k,i_k}$ and on the orbit of the tuple $(x_{1,1}, \ldots, x_{1,i_1}, \ldots, x_{k,1}, \ldots, x_{k,i_k})$ in (X, G). So, there are only countably many k-orbits in $(H(X), G)$. □

Proposition 3.12 *Let (X, G) be a Polish structure, and H a countable group. Then, for any finite $A, B, C \subseteq H(X)$, we have:*
(1) $A \mathop{\smile\!\!\!\!\!\!\;\mid}^{nm}_C B \iff \tilde{A} \mathop{\smile\!\!\!\!\!\!\;\mid}^{nm}_{\tilde{C}} \tilde{B}$.
(2) If a is a finite tuple of elements of $H(X)$, and b is a tuple of elements of X enumerating $\tilde{a}$, then $\mathcal{NM}(a/A) = \mathcal{NM}(b/\tilde{A})$. In particular, $(H(X), G)$ is nm-stable iff (X, G) is.

Proof. (1) By Fact 3.8, it is enough to show that $G_{CB}G_{CA} \subseteq_{nm} G_C \iff G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}} \subseteq_{nm} G_{\tilde{C}}$. First, suppose that $G_{CB}G_{CA} \subseteq_{nm} G_C$. Since $[G_{CB} : G_{\tilde{C}\tilde{B}}], [G_{CA} : G_{\tilde{C}\tilde{A}}] < \omega$, we get that $G_{CB}G_{CA}$ is a union of finitely many two-sided translates of $G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}}$ by elements of G_C. So, $G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}}$ is non-meager in G_C, and, hence, in $G_{\tilde{C}}$.

Now, suppose that $G_{\tilde{C}\tilde{B}}G_{\tilde{C}\tilde{A}} \subseteq_{nm} G_{\tilde{C}}$. Then $G_{CB}G_{CA} \cap G_{\tilde{C}}$ is non-meager in $G_{\tilde{C}}$, and hence, in G_C (because $[G_C : G_{\tilde{C}}] < \omega$). Thus, $G_{CB}G_{CA}$ is non-meager in G_C. This proves (1). Now, (2) follows by (1) and transfinite induction. □

The following corollary gives us a negative answer to Question 0.21 in its full generality, i.e., in the class of all Polish group structures. Recall that Fact 3.10 tells us that the answer is positive for small Polish G-groups.

Corollary 3.13 *Let (X, G) be a Polish structure, where X is uncountable. If H is a countable group, then $(H(X), G)$ is a small Polish group structure, and it has no nm-generic orbit (neither left nor right).*

Proof. Take any $a \in H(X)$ and a finite $A \subseteq H(X)$. We will show that $o(a/A)$ is not an nm-generic orbit. Take any $h \in H \backslash \{e\}$ and $b \in X \backslash Acl(\emptyset)$ such that $b \mathop{\smile\!\!\!\!\!\!\;\mid}^{nm} \tilde{A}, \tilde{a}$. Then, by Proposition 3.12, $h_b \mathop{\smile\!\!\!\!\!\!\;\mid}^{nm}_A a$. Since $b \mathop{\smile\!\!\!\!\!\!\;\mid}^{nm} \tilde{A}, \tilde{a}$ and $b \notin Acl(\emptyset)$, we see that $b \notin \tilde{a}$. Hence, $\widetilde{a + h_b} = \tilde{a} \cup \{b\}$. But $\tilde{a}, b \not\mathop{\smile\!\!\!\!\!\!\;\mid}^{nm} b$, so, again by Proposition 3.12, we have that $a + h_b \not\mathop{\smile\!\!\!\!\!\!\;\mid}^{nm} h_b$. Hence, $o(a/A)$ is not an nm-generic orbit. □

By the above corollary and Fact 3.10, we get in particular that there is no Polish (or even non-meager) topology on $H(X)$ such that $(H(X), G)$ is a G-group, i.e., such that the action of G on $H(X)$ is continuous.

By [26] and [27], every nm-stable compact G-group is nilpotent-by-finite. When the assumption of compactness is dropped, the corresponding questions concern searching for a subgroup of countable index having some nice algebraic properties (this is because any countable group has a structure of a small Polish G-group). The algebraic structure of nm-stable Polish G-groups remains unexplored. The following corollary shows that in general, not much can be said about the algebraic structure of small nm-stable Polish group structures.

Corollary 3.14 *Let (X, G) be an uncountable, small, nm-stable Polish structure, and H a non-solvable, countable group. Then $(H(X), G)$ is a small, nm-stable Polish group structure, which is not solvable-by-countable.*

Proof. By Proposition 3.12(2), $(H(X), G)$ is nm-stable. Now, take a subgroup A of countable index in $H(X)$. Then, there is some $x \in X$, such that $\pi_x[A] = H_x$, where

$\pi_x : H(X) \to H_x$ is the projection on the x-th coordinate. Since H_x is not solvable, we get that A is not solvable. Thus, $H(X)$ is not solvable-by-countable. □

Now, we will give a variant of the above construction. Suppose R is a countable commutative ring, and (X, G) is a small Polish structure. Let $R(X) = R[(y_x)_{x\in X}]$ be the ring of polynomials in variables $(y_x)_{x\in X}$ with coefficients in R. Then G acts on $R(X)$ by $gw(y_{x_1}, \dots, y_{x_n}) = w(y_{gx_1}, \dots, y_{gx_n})$. If R is a countable field, we can additionally consider $R(X)_0 = R((y_x)_{x\in X})$, the field of rational functions in variables $(y_x)_{x\in X}$ with coefficients in R. Then, G acts on $R(X)$ by $gf(y_{x_1}, \dots, y_{x_n}) = f(y_{gx_1}, \dots, y_{gx_n})$. As for $H(X)$, one can check that $(R(X), G)$, $(R_0(X), G)$ are small Polish structures. Moreover, if we define $\tilde{w}$ as the set of all $x \in X$ such that y_x occurs in the reduced form of w, then we get the same description of nm-independence for $(R(X), G)$,$(R_0(X), G)$ as was done for $(H(X), G)$ in Proposition 3.12. Thus, we get that these structures (which we could call Polish ring structures and Polish field structures) have no nm-generics (in the sense of the additive group), and hence, there is no Polish topology on $(R(X), G)$ or on $(R_0(X), G)$ such that the action of G is continuous. This suggests the following question.

Question 3.15 *Is there an uncountable small Polish G-field?*

4 Topologies induced by group actions

4.1 A general setting

Suppose G is a group equipped with a group topology σ, and G acts on a set X, which can be possibly equipped with an algebraic structure.

As mentioned in the introduction, we shall consider the question: When does there exist a "nice" topology on X, such that the action of G on X is continuous, and the topology is compatible with the structure on X? Clearly, if there is such a topology which is at least T_1, then, for every element $x \in X$, its stabilizer G_x is closed in G. On the other hand, if the latter is satisfied, then, by Proposition 4.1 below, we can equip X with an appropriate topology, which inherits many properties of the given topology on G.

Now, suppose that X is equipped with a group structure. Then, the topology τ defined below usually fails to be a group topology. The main result of this subsection is Theorem 4.3, which yields a description of the finest group topology on X, under which the action of G on X is continuous.

In this section, by a topological group we mean a group equipped with a topology, such that the multiplication and the inversion are continuous functions (we do not assume that the topology is Hausdorff).

Proposition 4.1 *Suppose G is a Hausdorff topological group acting on a set X. Define $\mathcal{U} := \{U \cdot x : U \subseteq G, U$ is open, $x \in X\}$. Then, we have:*
(1) The family $\mathcal{U}$ is a basis of a topology on X. Denote this topology by τ.
(2) The action of G on (X, τ) is continuous.
(3) All G-orbits in X are clopen in τ; moreover, for every $x \in X$, $G/G_x \approx G \cdot x$ (where the orbit $G \cdot x$ is equipped with the topology τ).
(4) $(\forall x \in X)(G_x$ is closed in $G) \iff \tau$ is $T_1 \iff \tau$ is Hausdorff.

Proof.
(1) Take any $x_1, x_2 \in X$, open sets $U_1, U_2 \subseteq G$ and a point $y \in U_1x_1 \cap U_2x_2$. Then, $y = u_1x_1 = u_2x_2$ for some $u_1 \in U_1$ and $u_2 \in U_2$. For $i = 1, 2$, let W_i be an open neighbourhood of e in G such that $W_iu_i \subseteq U_i$. Put $W = W_1 \cap W_2$. Then, $Wy = Wu_ix_i \subseteq U_ix_i$ for $i = 1, 2$, so we are done.
(2) Take any $g \in G$, $x, y \in X$ and an open set $U \subseteq G$ such that $gx \in Uy$. Choose $u \in U$ so that $gx = uy$. Let U_1 be an open neighbourhood of e in G such that $U_1u \subseteq U$, and let $V_1, V_2 \subseteq G$ be open neighbourhoods of e such that $V_1gV_2 \subseteq U_1g$. Then, $(V_1g)(V_2x) \subseteq U_1gx = U_1uy \subseteq Uy$, which yields the continuity of the action.
(3) By the definition of τ, every G-orbit is open, so also clopen. Let $f_x : G/G_x \to Gx$ given by $aG_x \mapsto ax$. We check that f_x is a homeomorphism. Let $i : G \to G/G_x$ be the quotient map, and take an open set $U \subseteq Gx$. Then, $i^{-1}[f^{-1}[U]] = \{g \in G : gx \in U\}$ is an open set, so $f^{-1}[U]$ is open in G/G_x. On the other hand, if V is open in G/G_x, then $i^{-1}[V]$ is open in G, so $f[V] = i^{-1}[V]x$ is open.
(4) First condition implies third by (3) and the fact that the quotient of a Hausdorff group by a closed subgroup is Hausdorff. Third condition implies second trivially,

and second implies first by (2). $\square$

Denote the topology τ from the above proposition by $\tau(X,G)$.

We will now focus on the case where G acts on a group H as automorphisms. First, let us recall a recent result of Bergman, which we use in our construction. Let G be a group. We will denote the neutral element of G by e, and for $S \subseteq G$ we will write $S^* = S \cup \{e\} \cup S^{-1}$. Following the notation from [4], given any $\mathbb{Q}$-tuple $(S_q)_{q\in\mathbb{Q}}$, put

$$U((S_q)_{q\in\mathbb{Q}}) = \bigcup_{n<\omega} \bigcup_{q_1<\cdots<q_n\in\mathbb{Q}} S^*_{q_1}\dots S^*_{q_n}.$$

Below, we will omit the symbol $\bigcup_{n<\omega}$ in similar expressions. For a family F of subsets of G, F^G will denote the collection of all subsets of G of the form $\bigcup_{g\in G} gS_g g^{-1}$, for G-tuples $(S_g)_{g\in G}$ of members of F. We say that a filter on G converges to e in a given topology, if every neighbourhood of e contains a member of the filter. By Lemma 14 and Proposition 15 from [4], we have:

Fact 4.2 *Let F be a downward directed family of nonempty subsets of G. Then the sets $U((S_q)_{q\in\mathbb{Q}})$, where $(S_q)_{q\in\mathbb{Q}}$ ranges over all $\mathbb{Q}$-tuples of members of F^G, form a basis of open neighbourhoods of e in a group topology $\mathcal{T}_F$, which is the finest group topology on G under which F converges to e.*

When ρ is a topology on G, we will denote by ρ^* the topology $\mathcal{T}_F$, where F consists of ρ-open neighbourhoods of e. In particular, if ρ is a group topology, then $\rho^* = \rho$.

Let G be a topological group equipped with a topology σ.

Theorem 4.3 *Suppose G acts on a group H as automorphisms. We identify G and H with $\{e\}\times G < H \rtimes G$ and $H\times\{e\} < H \rtimes G$, respectively. Put $T = (D\times\sigma)^*$, where D is the discrete topology on H. Denote by T_H and T_G the topologies induced by T on the subgroups identified with H and G, respectively. Then:*
(1) $T_G = \sigma$.
(2) T_H is a group topology on H under which the action of G on H is continuous.
(3) If ρ is another group topology on H under which the action of G on H is continuous, then T_H is finer than ρ.
We will denote the topology T_H by $T(H,G)$.

Proof.
(1) We will show that $T_G = \sigma^*$ (using the description of $T = (D\times\sigma)^*$ and σ^* given by Fact 4.2), which suffices since $\sigma = \sigma^*$. It is easy to see that T_G is a group topology on G. Hence, it is enough to show that the neighbourhoods of e in T_G are the same as in σ^*.

Take a T_G-open neighbourhood of e of the form

$$V = G\cap \bigcup_{q_1<\cdots<q_n\in\mathbb{Q}} \Big(\bigcup_{(h,g)\in H\rtimes G} (\{e\}\times U^{q_1}_{(h,g)})^{(h,g)}\Big)\dots\Big(\bigcup_{(h,g)\in H\rtimes G} (\{e\}\times U^{q_n}_{(h,g)})^{(h,g)}\Big) =$$

$$= G \cap \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \Big(\bigcup_{(h,g) \in H \rtimes G} \{(h(u^g h^{-1}), u^g) : u \in U^{q_1}_{(h,g)}\} \Big) \dots$$

$$\dots \Big(\bigcup_{(h,g) \in H \rtimes G} \{(h(u^g h^{-1}), u^g) : u \in U^{q_n}_{(h,g)}\} \Big),$$

where each $U^q_{(h,g)}$ is a symmetric σ-open neighbourhood of e in G. Then,

$$\bigcup_{q_1 < \dots < q_n} \Big(\bigcup_{g \in G} g U^{q_1}_{(e,g)} g^{-1} \Big) \dots \Big(\bigcup_{g \in G} g U^{q_n}_{(e,g)} g^{-1} \Big)$$

is a σ^*-open neighbourhood of e contained in V.

Conversely, take a σ^*-open neighbourhood of e of the form

$$W = \bigcup_{q_1 < \dots < q_n} \Big(\bigcup_{g \in G} g U^{q_1}_g g^{-1} \Big) \dots \Big(\bigcup_{g \in G} g U^{q_n}_g g^{-1} \Big),$$

where each U^q_g is a symmetric neighbourhood of e in G. For any $(h, g) \in G$ and $q \in \mathbb{Q}$, find a σ-open, symmetric neighbourhood $U^q_{(h,g)}$ of e, whose conjugate by g is contained in U_g (it can be chosen independently from h). Then,

$$G \cap \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \Big(\bigcup_{(h,g) \in H \rtimes G} (\{e\} \times U^{q_1}_{(h,g)})^{(h,g)} \Big) \dots \Big(\bigcup_{(h,g) \in H \rtimes G} (\{e\} \times U^{q_n}_{(h,g)})^{(h,g)} \Big)$$

is a T_G-open neighbourhood of e contained in W.
(2) T_H is a group topology since H is a subgroup of $H \rtimes G$, and T is a group topology on $H \rtimes G$ by Fact 4.2. For the continuity of the action, take any $g \in G$, $h \in H$ and a T-open set U, such that $gh \in U \cap H$. This means that, in $H \rtimes G$, $(e, g)(h, e)(e, g)^{-1} \in U$, so we can choose T-open sets U_1 and U_2, such that $(e, g) \in U_1$, $(h, e) \in U_2$ and $U_1 U_2 U_1^{-1} \subseteq U$. Then, for any $g_1 \in U_1 \cap G$ and h_1 in $U_2 \cap H$, we have that $(e, g_1)(h_1, e)(e, g_1^{-1}) = (g_1 h_1, e)$ belongs to U, so $g_1 h_1 \in U \cap H$. This proves the continuity of the action.
(3) Suppose ρ is a group topology on H under which the action of G on H is continuous. Then, the product topology $\rho \times \sigma$ is a group topology on $H \rtimes G$, which is coarser than $D \times \sigma$, so, by the choice of T, we have that T is finer than $\rho \times \sigma$. In particular, T_H is finer than ρ. □

Using the above theorem we obtain an explicit formula describing the topology $T(H, G)$:

Corollary 4.4 *With the notation from the above theorem, $T(H, G)$ has a basis of open neighbourhoods of e consisting of the sets:*

$$\bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(u_1 h_1^{-1}) u_1 (h_2(u_2 h_2^{-1})) u_1 u_2 (h_3(u_3 h_3^{-1})) \dots u_1 u_2 \dots u_{n-1} (h_n(u_n h_n^{-1})) : h_i \in H,$$

$$u_i \in U^{q_i}_{h_i}, u_1 \dots u_n = e\},$$

where $(U^q_h)_{h \in H, q \in \mathbb{Q}}$ *range over all* $H \times \mathbb{Q}$*-tuples of* σ*-open symmetric neighbourhoods of* e *in* G*.*

Proof. By the description of the topology T_H given in Fact 4.2, we get that it has a basis of open neighbourhoods of e consisting of the sets:

$$\bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(v_1^{g_1} h_1^{-1}) v_1^{g_1} (h_2(v_2^{g_2} h_2^{-1})) \dots (v_1^{g_1} \dots v_{n-1}^{g_n})(h_n(v_n^{g_n} h_n^{-1})) : h_i \in H,$$

$$g_i \in G, v_i \in U^{q_i}_{(h_i,g_i)}, v_1^{g_1} \dots v_n^{g_n} = e\} =$$

$$= \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(u_1 h_1^{-1}) u_1 (h_2(u_2 h_2^{-1})) \dots (u_1 \dots u_{n-1})(h_n(u_n h_n^{-1})) : h_i \in H,$$

$$g_i \in G, u_i \in (U^{q_i}_{(h_i,g_i)})^{g_i}, u_1 \dots u_n = e\},$$

where $(U^q_{(h,g)})_{(h,g) \in H \rtimes G, q \in \mathbb{Q}}$ range over all $(H \rtimes G) \times \mathbb{Q}$-tuples of σ-open symmetric neighbourhoods of e in G. Since the tuples $(U^q_{(h,g)})^g_{(h,g) \in H \rtimes G, q \in \mathbb{Q}}$ range over the same set, we can omit the conjugations in the formula. Now, if we replace each $U^q_{(h,g)}$ by $U^q_{(h,e)}$ in a tuple $(U^q_{(h,g)})_{(h,g) \in H \rtimes G, q \in \mathbb{Q}}$, then the corresponding neighbourhood of $e \in H$ will be contained in the original one. Thus, we obtain the same topology when we restrict ourselves to tuples in which $U^q_{(h,g)} = U^q_h$ does not depend on g. This gives the conclusion. □

In Subsection 4.2, we will use the description of $T(H,G)$ that we have obtained to prove the absence of a compatible Hausdorff topology for some classes of Polish group structures (see Proposition 4.15).

Let us keep the notation from above and define $\lambda(H,G)$ to be the topology on H in which a set U is open if for each $h_1, h_2 \in H$, the sets $h_1 U h_2, h_1 U^{-1} h_2$ are open in the topology $\tau(H,G)$ (defined after Proposition 4.1). It is easy to see that if we equip H with $\lambda(H,G)$, then the action of G on H is continuous, the inversion on H is continuous and the multiplication on H is separately continuous. Moreover, $\lambda(H,G)$ is the finest topology on H with these properties. Indeed, let ξ be any other such topology. Take any ξ-open set V. Then, for any $h_1, h_2 \in H$, $h_1 U h_2, h_1 U^{-1} h_2$ are ξ-open, so also τ-open. Hence, V is $\lambda(H,G)$-open.

Remark 4.5 *In Theorem 4.3, we can replace the discrete topology* D *by any topology on* H *which is finer than all group topologies under which the action of* G *on* H *is continuous. Examples of such topologies are* $\tau(H,G)$ *and* $\lambda(H,G)$*. However, the simplest description of* $T(H,G)$ *we obtain starting from the discrete topology on* H*.*

Let us formulate a remark about the topology $\lambda(H, G)$ defined above.

Remark 4.6 *If the topology of G and $\lambda(H, G)$ are metrizable and Baire, then $\lambda(H, G) = T(H, G)$*

Proof. Let us equip H with the topology $\lambda(H, G)$. Since the multiplication on H is separately conitnuous, the inversion on H is continuous and the action of G on H is separately continuous, we get by Theorem 9.14 from [23] that H is a topological group with the topology $\lambda(H, G)$, and the action of G on H is continuous. So, $\lambda(H, G)$ is coarser than $T(H, G)$. But $\lambda(H, G)$ is always finer than $T(H, G)$ (see the discussion preceeding Remark 4.5), so these topologies are equal. □

The above remark can by illustrated by the following example.

Example 4.7 *Let $G = S_\omega$ be the group of permutations of ω, considered with the product topology (which is Polish, so, in particular, metrizable and Baire). Consider the action of G on $H := 2^\omega$, given by $g \cdot h = h \circ g^{-1}$. Then, $\lambda(H, G)$ is the product topology on 2^ω, so it coincides with $T(H, G)$.*

Proof. Clearly $\lambda(H, G)$ is finer than the product topology. For the converse, let U be any $\lambda(H, G)$-open neighbourhood of $0 \in H$. We will show that it contains an open neighbourhood of $0 \in H$ in the sense of the product topology. Let $\omega = A \dot\cup B \dot\cup C$ be a partition of ω into three infinite sets. For $i, j, k \in \{0, 1\}$ define $h_{i,j,k} \in H$ to be equal to i on A, equal to j on B, and equal to k on C. For $i, j, k \in \{0, 1\}$, $h_{i,j,k} + U$ contains a $\tau(H, G)$-open neighbourhood of $h_{i,j,k}$ of the form $[\alpha_{i,j,k}] \cdot h_{i,j,k}$, where $\alpha_{i,j,k} : \omega \to \omega$ is a partial function with a finite domain, and $[\alpha] = \{\eta \in S_\omega : \alpha \subseteq \eta\}$. We finish by the following claim:

Claim 1 *Put $I = \bigcup_{i,j,k\in\{0,1\}}(dom(\alpha_{i,j,k}) \cup rng(\alpha_{i,j,k}))$. Then, $\{x \in H : x_{|I} = 0\} \subseteq U$.*

Proof of Claim 4.1. Take any $x \in H$ such that $x_{|I} = 0$. Notice that we can choose $i, j, k \in \{0, 1\}$ so that $(h_{i,j,k} + x)^{-1}[\{0\}]$ and $(h_{i,j,k} + x)^{-1}[\{1\}]$ are both infinite, and i, j, k are not all equal. Then, since $h^{-1}_{i,j,k}[\{0\}]$ and $h^{-1}_{i,j,k}[\{1\}]$ are also both infinite, and $h_{i,j,k} + x$ agrees with $h_{i,j,k}$ on I, we can find a permutation $\eta \in [\alpha_{i,j,k}]$, such that $\eta \cdot h_{i,j,k} = h_{i,j,k} + x$. Thus, $h_{i,j,k} + x \in h_{i,j,k} + U$, so $x \in U$. □

Now, we aim towards a description of the finest ring topology on R under which a given action of a topological group on R is continuous. First, we give a variant of Fact 4.2, in which we are interested in semigroup topologies (i.e. topologies under which the multiplication is continuous) rather than group topologies on G, but still we assume that G is a group. For a subset S of G, we will write $S^\# = S \cup \{e\}$. We define

$$U'((S_q)_{q\in\mathbb{Q}}) = \bigcup_{q_1<\dots<q_n\in\mathbb{Q}} S^\#_{q_1} \dots S^\#_{q_n}.$$

Then, by a straightforward modification (which is just replacing expressions of the form S^* by $S^\#$) of the proof of Lemma 14 and Proposition 15 from [4], we obtain:

Fact 4.8 *Let F be a downward directed family of nonempty subsets of G. Then, the sets $U'((S_q)_{q\in\mathbb{Q}})$, where $(S_q)_{q\in\mathbb{Q}}$ ranges over all $\mathbb{Q}$-tuples of members of F^G, form a basis of open neighbourhoods of e in a semigroup topology $\mathcal{T}'_F$, which is the finest semigroup topology on G under which F converges to e.*

When ρ is a topology on H, we will denote by $\rho^{\#}$ the topology $\mathcal{T}'_F$, where F consist of ρ-open neighbourhoods of e.

Recall that σ is the fixed topology on the group G. Repeating the proof of Theorem 4.3, we obtain:

Proposition 4.9 *Suppose G acts on a group H as automorphisms. We identify G and H with $\{e\}\times G < H \rtimes G$ and $H\times\{e\} < H\rtimes G$, respectively. Put $T' = (D\times\sigma)^{\#}$, where D is the discrete topology on H. Denote by T'_H and T'_G the topologies induced by T' on the subgroups identified with H and G, respectively. Then, $T'_G = \sigma$ and T'_H is the finest semigroup topology on H under which the action of G on H is continuous. We will denote it by $T'(H,G)$.*

Now, we are in a position to give a description of the finest topology in the ring case.

Theorem 4.10 *Suppose G is a group equipped with a group topology σ, acting as automorphisms on a ring R. Put $R_1 = R\times\mathbb{Z}$, and define $+$ and $\cdot$ on R_1 by $(a,k)+(b,l) = (a+b, k+l)$ and $(a,k)\cdot(b,l) = (ab + l\times a + k\times b, k\cdot l)$. Clearly, G acts on R_1 as automorphisms by $g(a,k) := (g(a),k)$. Consider the induced action of G on $GL_3(R_1)$. We identify R with a subset $\left\{\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} : x\in R\right\}$ of $GL_3(R_1)$. Denote by $T^r(R,G)$ the topology induced on R by $T'(GL_3(R_1),G)$. Then, $T^r(R,G)$ is the finest ring topology on R such that the action of G on R is continuous.*

Proof. Let T_1 be the topology induced on R_1 by $T'(GL_3(R_1),G)$ (we identify R_1 with a subset of $GL_3(R_1)$ in the same manner as we do with R).

Claim 1 *T_1 is the finest ring topology on R_1 under which the action of G on R_1 is continuous.*

First, suppose the claim is proved and let us see that the conclusion of the theorem follows.

By the claim, $T^r(R,G)$ is a ring topology on R, and the action of G on R equipped with $T^r(R,G)$ is continuous (as the restriction of the action on R_1).

Suppose ρ is another topology on R such that the action of G on R is continuous. Consider R_1 equipped with the product of ρ and the discrete topology E on $\mathbb{Z}$. Then, the action G on R_1 is also continuous and R_1 is a topological ring, so, by the claim, T_1 is finer than $\rho\times E$. Hence, $T^r(R,G)$ (which is equal to the topology induced on R by T_1) is finer than ρ, so we are done.

Proof of Claim 1. First, we will check that T_1 is finer than every ring topology on R_1 under which the action of G on R_1 is continuous. Let χ be any such topology.

Let us equip $GL_3(R_1)$ with the topology Z induced from the product topology χ^9 on R_1^9. Then, $GL_3(R_1)$ becomes a topological semigroup, and the action of G on it is continuous. Thus, $T'(GL_3(R_1), G)$ is finer that Z, so T_1 is finer than the topology induced by Z on R_1, i.e. T_1 is finer than χ.

Now, consider R_1 equipped with the topology T_1. The action of G on R_1 is continuous (as a restriction of the action on $GL_3(R_1)$) and the addition in R is continuous (as a restriction of the multiplication in $GL_3(R_1)$). Moreover, the additive inversion in R is continuous, as it is given by the map

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is continuous with respect to $T'(GL_3(R_1), G)$.

It remains to show that the multiplication on R_1 is continuous. So, we will be done if we show that the map

$$\left(\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \mapsto \begin{pmatrix} 1 & 0 & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is continuous with respect to $T'(GL_3(R_1), G)$. The latter follows, since

$$\begin{pmatrix} 1 & 0 & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix},$$

and maps

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are continuous.

□

The proof of the theorem has been completed. □

Remark 4.11 *In the context of Theorem 4.10, we obtain the same topology on R if we identify R with*

$$\{\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in R\},$$

Proof. This follows from the fact that

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x & x \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \left(\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right)^2$$

is a continuous function (the continuity of the inverse to that map follows from the calculations made at the end of the proof of Theorem 4.10). □

4.2 Topologies on Polish structures

In this subsection, we will study topologies considered in the previous subsection in the context of Polish structures.

By Proposition 4.1, we have:

Corollary 4.12 *(1) If G is a Polish group acting on a set X, then (X, G) is a Polish structure iff $\tau(X, G)$ is T_1 iff $\tau(X, G)$ is completely metrizable.*
(2) If (X, G) is a small Polish structure, then $\tau(X, G)$ is a Polish topology. In particular, (X, G) is a Polish G-space if we equip X with the topology $\tau(X, G)$.

By [26, 2.14], under some assumptions, nm-dependence in a G-group (H, G) can be expressed in terms of the topology on H:

Fact 4.13 *Let (X, G) be a Polish structure such that G acts continuously on a Hausdorff space X. Let $a, A, B \subseteq X$ be finite. Assume that $o(a/A)$ is non-meager in its relative topology. Then, $a \mathop{\smile\!\!\!\!\!\!\mid}^{nm}_A B \iff o(a/AB) \subseteq_{nm} o(a/A)$.*

Using the above fact, we will now express the relation of nm-independece in terms of a family of topologies on X, without assuming anything about the Polish structure (X, G).

Remark 4.14 *Let (X, G) be a Polish structure and let $a, A, B \subseteq X$ be finite. Then $a \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B \iff o(a/B) \subseteq_{nm} o(a/A)$, where X is equipped with the topology $\tau(G_A, X)$ (and the action of G_A on X is the restriction of the action of G on X).*

Proof. The conclusion follows from Fact 4.13 and Corollary 4.12(2). □

Using Remark 4.14, we can slightly simplify some of the arguments from [26]. For example, we reprove the existence of non-forking extensions in small Polish structures (point 4 of Fact 0.11):
Let $a \subseteq X$ and $A \subseteq B \subseteq X$ be all finite. Since $\tau(X, G_A)$ is Polish, and there are countably many orbits over B, we can find, by the Baire category theorem, an element $b \in o(a/A)$, such that $o(b/B)$ is non-meager in $\tau(X, G_A)$. Then, by Remark 4.14, we get that $b \mathop{\smile\!\!\!\!\!\mid}^{nm}_A B$.

We will now apply Corollary 4.4 to some of the structures of the form $H(X)$ constructed in Subsection 3.2.

Recall that if (X, G) is an uncountable small Polish structure, and H is a non-trivial countable group, then $(H(X), G)$ has no nm-generic orbits, and hence (by Fact 3.10) there is no group topology on $H(X)$ under which the action of G on it is continuous.

We strengthen this observation in some cases:

Proposition 4.15 *Let H be any non-trivial group and let X be a compact Hausdorff space containing an open subset homeomorphic to $(0, 1)^n$ for some $n > 0$ (notice that the classes of examples mentioned in points 1, 2, 3 in the list after Definition 0.8 satisfy this assumption). Then, there is no Hausdorff group topology on $H(X)$ under which the action of* $\mathrm{Homeo}(X)$ *on $H(X)$ is continuous (where* $\mathrm{Homeo}(X)$ *is considered with the compact-open topology).*

Proof. Suppose first that $X = [0, 1]$. It is enough to show that the topology $\rho := T(H([0, 1]), \mathrm{Homeo}([0, 1]))$ is not Hausdorff. Take any $a \in H \backslash \{e\}$. We will show that any ρ-open neighbourhood of $e \in H([0, 1])$ contains the element $a_{1/3} - a_{2/3}$. Let W be any such neighbourhood and choose (by Corollary 4.4) a ρ-open set

$$V = \bigcup_{q_1 < \dots < q_n \in \mathbb{Q}} \{h_1(u_1 h_1^{-1}) u_1(h_2(u_2 h_2^{-1})) u_1 u_2 (h_3(u_3 h_3^{-1})) \dots u_1 u_2 \dots u_{n-1}(h_n(u_n h_n^{-1})) :$$

$$h_i \in H, u_i \in U^{q_i}_{h_i}, u_1 \dots u_n = e\},$$

such that $V + V \subseteq W$. Let $B_\epsilon(id) \subseteq \mathrm{Homeo}([0, 1])$ be a ball (in the supremum metric) contained in $U^0_0 \cap U^3_0$, and choose $n < \omega$ such that $1/3n < \epsilon$. Put

$$h = a_{n/3n} + a_{(n+1)/3n} + \dots + a_{(2n-1)/3n}, h' = -a_{(n+1)/3n} - a_{(n+2)/3n} - \dots - a_{2n/3n}$$

and $U = U^1_h \cap U^1_{h'} \cap U^2_0$. Notice that $\{u_0(h - u_1 h), u_0(h' - u_1 h') : u_0 \in B_\epsilon(id), u_1 \in U\} \subseteq V$ (to see this, choose $q_j = j$ for $j = 0, 1, 2, 3$, $u_2 = u_1^{-1}$, $u_3 = u_0^{-1}$, $h_0 =$

$h_2 = h_3 = 0$ and h_1 equal to either h or h'). Since U is open, we can find $u_1 \in U$ such that $u_1(k/3n) \in (k/3n, (k+1)/3n)$ for $k = n, n+1, \ldots, 2n-1$. Then, there is some $u_0 \in B_\epsilon(id)$ such that $u_0(k/3n) = k/3n$ and $u_0 u_1(k/3n) = (2k+1)/6n$ for $k = n, n+1, \ldots, 2n-1$. So, we get that $\sum_{k=n}^{2n-1}(a_{k/3n} - a_{(2k+1)/6n}) \in V$. Similarly, we obtain using h' that $\sum_{k=n+1}^{2n}(-a_{k/3n} + a_{(2k-1)/6n}) \in V$. Thus, $a_{1/3} - a_{2/3} \in V + V \subseteq W$.

Now, suppose X is any space as in the statement. Then, we can find a copy F of $[0,1]^n$ contained in $(0,1)^n \subseteq X$, and a copy I of $[0,1]$ contained in F, such that every homeomorphism of I preserving its endpoints can be extended to a homeomorphism of F having the same distance from the identity (with respect to the supremum metric induced by the Euclidean metric on $[0,1]^n$) and equal to the identity on the border of F in $(0,1)^n$. Futhermore, since $(0,1)^n$ is open in X, we can extend such a homeomorphism of F to a homeomorphism of X equal to the identity on $X \backslash F$. Notice that any open neighbourhood of $id \in \mathrm{Homeo}(X)$ contains $\{f \in \mathrm{Homeo}(X) : f_{|X\backslash int(F)} = id, d(id_F, f_{|F}) < \epsilon\}$ for some $\epsilon > 0$, where d is the supremum metric. Indeed, by the definition of the compact-open topology, such a neighbourhood is of the form $\{f \in \mathrm{Homeo}(X) : f[K_1] \subseteq W_1, \ldots, f[K_l] \subseteq W_l\}$ where W_i's are open, and each K_i is a compact subset of W_i. Then, it is enough to choose ϵ such that for each i and $x \in K_i \cap F$, $B_F(x, \epsilon) \subseteq F \cap U_i$. Now, we can repeat the proof that we gave in the case of $X = [0,1]$. Namely, choosing V as above (but for an arbitrary X), we define ϵ to be such that $\{f \in \mathrm{Homeo}(X) : f_{|X\backslash int(F)} = id, d(id_F, f_{|F}) < \epsilon\} \subseteq U_0^0 \cap U_0^3$ and define h, h' in the same way as above (identyfing $[0,1]$ with a subset of F). Since u_0 and u_1 can be chosen to preserve endpoints of $[0,1]$, the choice of F and of the copy of $[0,1]$ inside it allows us to repeat the argument. □

The only known examples of small Polish group structures without nm-generic orbits are of the form $H(X)$. For those counterexamples for which we were able to compute the finest compatible topology, it turned out that it is not Hausdorff. This may suggest that there could be a topological property of a group H other than being non-meager in itself, which guarantees the existence of nm-generic orbits in a structure (H, G).

Problem 4.16 *Characterize the existence of nm-generic orbits in a Polish group structure (H, G) in terms of topological properties of H.*

In particular, we can ask:

Question 4.17 *Does the existence of a Hausdorff group topology on a group H such that the action of a Polish group G on H is continuous imply that the structure (H, G) has an nm-generic orbit?*

Also, we do not know whether the converse is true.

Question 4.18 *Does the existence of nm-generic orbits in a Polish group structure (H, G) imply the existence of a compatible Hausdorff topology on H?*

Literatura

[1] H. Adler, E. Casanovas, A. Pillay, *Generic stability and stability*, Journal of Symbolic Logic (79), 179-185, 2014.

[2] J. Baldwin, B. Rose, $\aleph_0$*-categoricity and stability of rings*, Journal of Algebra (45), 1-16, 1977.

[3] W. Baur, G. Cherlin, A. Macintyre, *Totally categorical groups and rings*, Journal of Algebra (57), 407-440, 1979.

[4] G. M. Bergman, *On group topologies determined by families of sets*, preprint, 2013.

[5] R. Camerlo, *Dendrites as Polish structures*,Proceedings of the American Mathematical Society (139), 2217-2225, 2011.

[6] G. Cherlin, *On* $\aleph_0$*-categorical nilrings. II*, Journal of Symbolic Logic (45), 291-301, 1980.

[7] J. J. Dijkstra, *A criterion for Erdös spaces*, Proceedings of the Edinburgh Mathematical Society (48), 595601, 2005.

[8] J. J. Dijkstra, J. V. Mill, J. Steprāns, *Complete Erdös space is unstable*, Mathematical Proceedings of the Cambridge Philosophical Society (137), 465-473, 2004.

[9] J. Dobrowolski, *New examples of small Polish structures*, Journal of Symbolic Logic (78), 969-976, 2013.

[10] J. Dobrowolski, *Topologies induced by group actions*, in preparation.

[11] J. Dobrowolski, K. Krupiński, *On* ω*-categorical, generically stable groups*, Journal of Symbolic Logic (77), 1047-1056, 2012.

[12] J. Dobrowolski, K. Krupiński, *On* ω*-categorical, generically stable groups and rings*, Annals of Pure and Applied Logic (164), 802-812, 2013.

[13] J. Dobrowolski, K. Krupiński, *Locally finite profinite rings*, Journal of Algebra (401), 161-178, 2014.

[14] C. Ealy, K. Krupiński, A. Pillay, *Superrosy dependent groups having finitely satisfiable generics*, Annals of Pure and Applied Logic (151), 1-21, 2008.

[15] P. Erdös, *The dimension of the rational points in Hilbert space*, Annals of Mathematics (41), 734-736, 1940.

[16] D. Evans, F. Wagner, *Supersimple* ω*-categorical groups and theories*, Journal of Symbolic Logic (65), 767-776, 2000.

[17] U. Felgner, $\aleph_0$-*categorical stable groups*, Mathematische Zeitschrift (160), 27-49, 1978.

[18] I. N. Herstein, *Noncommutative Rings*, Carus Mathematical Monographs (15), Mathematical Association of America, 1968.

[19] E. Hrushovski, *The Mordell-Lang conjecture for function fields*, Journal of the American Mathematical Society (9), 667-690, 3499-3533, 2010.

[20] E. Hrushovski, *The Manin-Mumford conjecture and the model theory of difference fields*, Annals of Pure and Applied Logic (112), no. 1, 43115, 2001.

[21] E. Hrushovski, A. Pillay, *On NIP and invariant measures*, Journal of the European Mathematical Society (13), 1005-1061, 2011.

[22] E. Hrushovski, A. Pillay, Y. Peterzil, *Groups, measures, and the NIP*, Journal of the American Mathematical Society (21), 563-595, 2008.

[23] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1995.

[24] K. Krupiński, *On* ω-*categorical groups and rings with NIP*, Proceedings of the American Mathematical Society (140), 2501-2512, 2012.

[25] K. Krupiński, *On relationships between algebraic properties of groups and rings in some model-theoretic contexts*, Journal of Symbolic Logic (76), 1403-1417, 2011.

[26] K. Krupiński, *Some model theory of Polish structures*, Transactions of the American Mathematica Society (362), 3499-3533, 2010.

[27] K. Krupiński, F. Wagner, *Small, nm-stable compact G-groups*, Israel Journal of Mathematics (194), 907-933, 2013.

[28] K. Krupiński, F. Wagner, *Small profinite groups and rings*, Journal of Algebra (306), 494-506, 2006.

[29] A. Macintyre, J. G. Rosenstein $\aleph_0$-*categoricity for rings without nilpotent elements and forBoolean structures*, Journal of Algebra (43), 129-154, 1976.

[30] H. D. Macpherson, *Absolutely ubiquitous structures and* $\aleph_0$-*categorical groups*, Quarterly Journal of Mathematics Oxford (2) 39, 483-500, 1988.

[31] L. Newelski, *Small profinite groups*, Journal of Symbolic Logic (66), 859872, 2001.

[32] L. Newelski, *Small profinite structures*, Transactions of the American Mathematica Society (354), 925943, 2002.

[33] A. Pillay, P. Tanović, *Generic stability, regularity, and quasi-minimality*, CRM Proceedings and Lecture Notes 53, 189-211, 2011.

[34] B. Poizat, *Stable groups*, American Mathematical Society, Providence, 2001.

[35] L. Ribes, P. Zalesskii, *Profinite groups*, Ergeb. Math. Grenzgeb. 40, Springer, Berlin, 2000.

[36] F. Wagner, *Stable groups*, London Mathematical Society Lecture Notes Series 240, Cambridge University Press, UK, 1997.

[37] J. Wilson, *The algebraic structure of ω-categorical groups*, in: Groups-St. Andrews, Ed. C. M. Campbell, E. F. Robertson, London Mathematica Society Lecture Notes 71, Cambridge, 345-358, 1981.

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