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Basics of Milnor-Thurston homology theory

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Basics of Milnor-Thurston homology theory

1 Introduction

The one of the most widely used constructions in algebraic topology is the singular homology theory. It derives from the simplicial homology, which encapsulates basic ideas of people like Henri Poincaré, Luitzen Brouwer and Emmy Noether (see historical notes in [12, chapter 4.6]). Yet it is more general and works for a wider class of topological spaces.

Unfortunately both theories have a limitation – they admit only chains composed of finite number of simplices – while in some cases it is convenient or even necessary to consider infinite chains. This happens, for instance, in the case of "wild" topological spaces.

One way to deal with infinite chains is to use methods of measure theory. This approach first appeared in Thurston's proof of the Gromov theorem, and was used to define fundamental class of hyperbolic manifold in a convenient way.

This theory was further developed by Zastrow [4] and Hansen [5] independently. Following Zastrow we refer to it as the *Milnor-Thurston homology theory*, while Hansen uses the name *measure homology theory*.

In this work we outline the basic knowledge in Milnor-Thurston homology theory. Starting with the original definition of Thurston we present main steps in proof of Gromov theorem. And in the next sections we give the definition for general topological spaces, and we prove some basic properties. Finally we outline proof of coincidence of Milnor-Thurston theory and singular theory in the case of simplicial complexes.

2 Preliminaries

Milnor-Thurston Homology is a homology theory in which chains are measures satisfying certain properties (one can also encounter term *measure homology*), therefore it is necessary to start with some basic notions connected with measures. Because we want our measures to form a vector space we need them to take also negative values. Such measures are called *signed measures*, however, for simplicity, in this paper by a *measure* we always mean a signed measure.

Definition 1 *(measurable space). Let Ω be a set and let $\mathcal{F} \subset 2^\Omega$ be a σ-algebra. An ordered pair $(\Omega, \mathcal{F})$ is called a measurable space.*

Definition 2 *(signed measure). Let $(\Omega, \mathcal{F})$ be a measurable space. A signed measure is a function $\mu : \mathcal{F} \to \mathbb{R} \cup \{-\infty, \infty\}$ with the following properties:*

(i) $\mu(\varnothing) = 0$,

(ii) $\mu\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n)$, *for any disjoint family of sets* $\{A_n\}_{n=1}^\infty \subset \mathcal{F}$.

Signed measures can be reduced to unsigned ones by the Hahn theorem:

Theorem 3 *(Hahn). Let μ be a signed measure on $(\Omega, \mathcal{F})$. Then there exist two disjoint sets Ω^+, $\Omega^- \in \mathcal{F}$ such that $\Omega = \Omega^+ \cup \Omega^-$ and such that for every $F \in \mathcal{F}$ it is $\mu(F \cap \Omega^+) \geq 0$, $\mu(F \cap \Omega^-) \leq 0$.*

The decomposition of our space Ω into sets Ω^+, Ω^- is not unique. Nevertheless for two distinct decompositions: Ω_i^+, Ω_i^-, $i = 1, 2$, we can prove that, given any $F \in \mathcal{F}$ it is $\mu(F \cap \Omega_1^+) = \mu(F \cap \Omega_2^+)$, $\mu(F \cap \Omega_1^-) = \mu(F \cap \Omega_2^-)$ [1, p. 122]. Therefore the signed measure μ can be uniquely decomposed into the sum of unsigned measures

$$\mu = \mu^+ - \mu^-,$$

where $\mu^+(\cdot) = \mu(\cdot \cap \Omega_+)$, $\mu^-(\cdot) = -\mu(\cdot \cap \Omega_-)$.

Let us introduce some further notions of measure theory.

Definition 4 *(norm of a measure). Let μ be a measure. We call*

$$\|\mu\| = \sup_{A \in \mathcal{F}} \mu(A) - \inf_{B \in \mathcal{F}} \mu(B).$$

the norm of μ.

Definition 5 *(finite measure). Each measure μ where $\|\mu\| < \infty$ is called a finite measure.*

Supremum and infimum in the above definition of a norm can be changed into maximum and minimum respectively. In fact there exists the sequence of sets $A_n \subset \mathcal{F}$, such that $\lim_{n\to\infty} \mu(A_n) = \sup_{A\in\mathcal{F}} \mu(A)$. Let $B_n = \bigcup_{k=1}^{n} A_k$, then $\mu(A_n) \leq \mu(B_n)$, hence $\lim_{n\to\infty} \mu(B_n) = \sup_{A\in\mathcal{F}} \mu(A)$ and the sequence is monotonically increasing. Signed measures are continuous from above by the Hahn decomposition. Therefore $\lim_{n\to\infty} \mu(B_n) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right)$, so the supremum is reached for $\bigcup_{k=1}^{\infty} B_k \in \mathcal{F}$ and we can replace it with maximum. The analogous argument with $-\mu$ works for the infimum. Hence

$$\|\mu\| = \max_{A\in\mathcal{F}} \mu(A) - \min_{B\in\mathcal{F}} \mu(B).$$

It should be noted that by definition signed measures cannot take both ∞ and $-\infty$, otherwise there exists a set whose measure equals $\infty - \infty$. As a consequence for every measure norm is well defined.

Additionaly we define the absolute value of a signed measure

Definition 6 *(absolute value of measure). Let $\mu = \mu^+ - \mu^-$ be the Hahn decomposition of the signed measure μ. The positive measure*

$$|\mu| = \mu^+ + \mu^-$$

is called the absolute value of the measure μ.

There is a simple relation between norm and absolute value of measure. Namely $\|\mu\| = |\mu|(\Omega)$ for each measure μ on the space Ω.

There are some important types of measures

Definition 7 *(atomic measure). Let $\omega \in \Omega$. The measure of type*

$$\mu_\omega(A) = \begin{cases} 0, & \text{if } \omega \notin A \\ 1, & \text{if } \omega \in A \end{cases}$$

is called an atomic measure.

Definition 8 *(counting measure). Let $\omega_i \in \Omega$, $\alpha_i \in \mathbb{R}$ for $i = 1, 2, ..., n$. A linear combination*

$$\mu = \sum_{i=1}^{n} \alpha_i \cdot \mu_{\omega_i}$$

is called a counting measure.

We can easily compute the norm of a counting measure: $\|\mu\| = \sum_{k=1}^{n} |\alpha_k|$.

When Ω is a topological space and $\mathcal{F}$ contains all open sets, we can define the support of a measure.

Definition 9 *(support of a measure). Let μ be measure. The set*

$$\mathrm{supp}(\mu) := \{x \in X \mid \textit{for every open neighbourhood } U_x \textit{ of } x \textit{ we have } |\mu|(U_x) > 0\}.$$

is called the support of the measure μ.

We see, that $\mathrm{supp}(\mu)$ is a closed set. Therefore it is an element of $\mathcal{F}$.

Suppose that Ω satisfies the second axiom of countability. Then we can prove, that for every $\mathcal{F} \ni B \subset \Omega \setminus \mathrm{supp}(\mu)$ we have $|\mu|(B) = 0$.

3 First mention of Milnor-Thurston homology

Homology groups based on measures have appeared for the first time in Thurston's lecture notes *The Geometry and Topology of Three-Manifolds* and were used to prove the Gromov Theorem [2, Theorem 6.2] (Thurston proves two versions of Gromov theorem, namely Theorem 6.2 and Theorem 6.4, here we outline a proof of the weaker one).

The original definition of Milnor-Thurston Homology involved measures on the space $C^1(\Delta_k, X)$ of C^1-simplices equipped with the C^1-topology. The k-chains in this theory are finite measures with compact support. We will denote here chain, cycle, boundary and homology modules by $\mathcal{C}^d$, $\mathcal{Z}^d$, $\mathcal{B}^d$, $\mathcal{H}^d$.

As Thurston points out, the condition of a compact support is necessary, otherwise one can expect pathological cycles. For example let us consider the cycle $\mu = \sum_i (1/i^2)\mu_{\sigma_i}$, where σ_i is singular simplex which wraps the standard one-simplex i times around the circle S^1 and μ_{σ_i} denotes atomic measure. The measure is finite since $\sum_i 1/i^2 < \infty$. However σ_i covers each point of the circle i times, so it represents ith multiplicity of the fundamental class $[S^1]$. Hence $\mu \sim \sum_i (1/i)[S^1]$ seems to represent an infinite multiplicity of $[S^1]$.

The first application of this homology theory was Thurston's proof of the Gromov theorem which relates the volume of a hyperbolic manifold and the Gromov's norm of the fundamental class. According to Thurston [2, p. 126] "Gromov's norm measures the efficiency with which multiples of homology class can be represented by simplices. A complicated homology class needs many simplices". So intuitively the theorem says that manifold with bigger volume needs more simplices to cover it. Formally Thurston defines Gromov's norm in two different ways: one for singular homology and the other for measure homology (it has been proved that they are equivalent [3]).

Definition 10 *(Gromov's norm). Let h be some singular homology class. We define the Gromov norm $\|\cdot\|$ in the following way*

$$\|h\| := \inf\{\|z\| \mid z \textit{ is cycle which represents } h\},$$

where $\|\sum_i \alpha_i \cdot \sigma_i\| := \sum_i |\alpha_i|$.

Definition 11 *(Gromov's norm). Let h be some Milnor-Thurston homology class. We define the Gromov norm $\|\cdot\|$ in the following way*

$$\|h\| = \inf\{\|\mu\| \mid \mu \textit{ is a measure which represents } h\},$$

where $\|\mu\|$ denotes the norm of a measure.

Essentialy we will use the second definition, however some parts of the proof-outline for Gromov Theorem are done using the first one.

Theorem 12 *(Gromov). Let M be any closed oriented hyperbolic manifold of dimension n. Then*

$$\|[M]\| = \frac{v(M)}{v_n},$$

where $v(M)$ denotes the volume of the manifold M and v_n is the supremum to the n-dimensional volume of n-simplex in hyperbolic space $\mathbb{H}^n$.

Outline of proof. We will start with proving the inequality

$$\|[M]\| \geq \frac{v(M)}{v_n}. \tag{1}$$

Given the hyperbolic manifold M we can define the straightening process $\mathrm{straight}_M : C^1(\Delta_k, M) \to C^1(\Delta_k, M)$ (for details of the construction and some basic facts on hyperbolic manifolds see Appendix B). Straightening extends linearly to a chain homomorphism

$$\mathrm{straight}_M : C_k(M) \to C_k(M)$$

which is chain-homotopic to the identity, since $\mathrm{straight}_M(\sigma)$ and σ are homotopic (and the homotopy respects the chain complex structure). In addition, for any chain c we have

$$\|\mathrm{straight}(c)\| \leq \|c\|.$$

In fact it is true for any function $f : C(\Delta_k, X) \to C(\Delta_k, X)$, and is a simple consequence of a fact, that the chain c may contain simplices τ_1 and τ_2, for which $f(\tau_1) = f(\tau_2)$. As a consequence of the above inequality, for calculation of norms on the level of homology it is sufficient to consider only straight singular simplices.

Now we are ready to prove (1). Let Ω be the hyperbolic volume form of M. Let $z = \sum_i \alpha_i \sigma_i$ be any straight cycle representing $[M]$. Then

$$v(M) = \int_M \Omega = \sum_i \alpha_i \int_{\Delta_n} \sigma_i^* \Omega \leq \sum_i |\alpha_i| v_n,$$

where $\sigma_i^*\Omega$ denotes pullback form. Dividing the above inequality by v_n, we get (1).

Thurston arguments, that since the straightening process is uniform, this proof works also for measures. The formal proof of this fact requires a justification that the straightening process is continuous in the C^1-topology. However using recent results of measure homology we can prove, that both norms are equal, thus the inequality is true also for measure homology (see Appendix A.1). Consequently from now on we will use Definition 11 of Gromov's norm.

The next step is to prove the other inequality

$$\|[M]\| \leq \frac{v(M)}{v_n}. \tag{2}$$

The trick is to use measure homology in order to deal with infinite cycles that occur when one tries to define the fundamental cycle of a hyperbolic manifold.

We know that hyperbolic n-space is the universal covering of our manifold (see Appendix B), hence can we start with covering of $\mathbb{H}^n$ by n-simplices of nearly maximal volume. Then we can project this tesselation back to our manifold to get some multiplicity of the fundamental cycle. However it may consist of infinitely many simplices, so we need homology theory which can handle this problem.

Let $\sigma \in C^1(\Delta_n, \mathbb{H}^n)$. We will construct a measure $\mathrm{smear}_M(\sigma) \in \mathcal{C}_n^d(M)$ which we can see as the measure that is uniformly supported on simplices isometric to the projection of σ to M. Let $\mathrm{Isom}^+(\mathbb{H}^n)$ denote the group of orientation preserving isometries of $\mathbb{H}^n$. There exists the Haar measure h on $\mathrm{Isom}^+(\mathbb{H}^n)$ which can be normalised that the set of isometries which carry x into the region R has measure a $v(R)$.

The fundamental group $\pi_1(M)$ can be seen as a subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$ (see Theorem 37 in Appendix B), so there exists the quotient space $P(M) = \pi_1(M)\backslash\mathrm{Isom}^+(\mathbb{H}^n)$. Because the Haar measure h is invariant from both sides, it induces the a measure on $P(M)$ also denoted by h.

Let us look at the continuous map $P(M) \to C^1(\Delta_n, M)$ which assigns $p \circ \phi \circ \sigma$ to a coset $\pi_1(M)\phi$, where $p : \mathbb{H}^n \to M$ is the covering projection. The measure h pushes forward with respect to this map to give $\mathrm{smear}_M(\sigma) \in \mathcal{C}_n(M)$.

The smearing operation $\mathrm{smear}_M(\cdot)$ extends to $C_n(\mathbb{H}^n)$, so we can consider the chain $z = (1/2)\mathrm{smear}_M(\sigma - \sigma^-)$, where σ^- is a reflected copy of the straight simplex σ. This chain is a cycle since faces of σ and σ^- cancel up to isometry.

First of all, notice that

$$\|z\| = v(M), \tag{3}$$

by normalisation of the Haar measure h.

Secondly we have to determine the homology class of z. We will use coincidence of original Milnor-Thurston homology and singular homology which was proved by Zastrow [4, Theorem 3.4]. Hence for an n-dimensional manifold M we have $\mathcal{H}^d_n(M) = \mathbb{R}$. So a homology class of z must be the multiplicity of fundamental class $[M]$.

Intuitively the fundamental class covers the whole manifold, so if $Z = [M]$, then (see appendix Appendix A.2 for the discussion on the volume covered by a chain)

$$\int_{C^1(\Delta_n,M)} \left(\int_{\Delta_n} \tau^*\Omega \right) dZ(\tau) = v(M).$$

The measure z is supported on simplices isometric to the projection of σ to M. Therefore the integral $\int_{\Delta_n} \tau^*\Omega$ equals $v(\sigma)$ on the support of $z^+ = (1/2)\mathrm{smear}_M(\sigma)$ and it equals $-v(\sigma)$ on the support of $z^- = (1/2)\mathrm{smear}_M(\sigma^-)$. Therefore using (3) we have

$$\int_{C^1(\Delta_n,M)} \left(\int_{\Delta_n} \tau^*\Omega \right) dz(\tau) = v(\sigma) \int_{C^1(\Delta_n,M)} d|z|(\tau) = v(\sigma)\|z\| = v(\sigma)v(M),$$

so finally

$$[z] = v(\sigma)[M],$$

therefore by (3) we obtain

$$\|v(\sigma)[M]\| \le v(M).$$

Taking supremum with respect to $v(\sigma)$ we get the inequality (2).

□

4 The Milnor-Thurston homology

The definition of Milnor-Thurston homology for general topological spaces was given independently by Zastrow [4] and Hansen [5]. Lack of the notion of differentiability implies the necessity for considering only continuous simplices instead of differentiable ones. Consequently chains in this theory are measures on $C^0(\Delta_k, X)$ satisfying certain properties discussed in the preceding sections.

4.1 Chain modules

Let us introduce some auxiliary notions.

Definition 13 *(quasicompact space). We say that a topological space is quasicompact when for any open covering there exists an open subcovering.*

Notice, that a quasicompact space may not be Hausdorff. When X is quasicompact and Hausdorff, we say that X is compact. Notions of compactness and quasicompactness are equivalent in the case of metric spaces.

Definition 14 *(determination set). Let μ be a measure on $(\Omega, \mathcal{F})$. We say, that $D \subset \Omega$ is the determination set for μ if all sets $\mathcal{F} \ni A \subset \Omega \setminus D$ satisfy $|\mu|(A) = 0$.*

Definition 15 *(quasicompactly determined measure). Let μ be a measure on a topological space Ω. We say, that μ is quasicompactly determined if there exists a quasicompact determination set for μ.*

Now we are ready to define chain modules for a topological space X. We will use the compact-open topology on $C^0(\Delta_k, X)$ and we will restrict to considering only Borel measures (i.e. we assume that our σ-algebra are Borel sets). We will denote chain, cycle, boundary modules and homology modules of Milnor-Thurston theory with calligraphic letters $\mathcal{C}, \mathcal{Z}, \mathcal{B}, \mathcal{H}$. Respective symbols for singular theory are: C, Z, B, H.

Definition 16 *(chain module). Let $k \in \mathbb{N}$. We define the k-chain module for a topological space X in the following way*

$$\mathcal{C}_k(X) = \{\mu \mid \textit{where } \mu \textit{ is quasicompactly determined measure} \\ \textit{on } C^0(\Delta_k, X) \textit{ with } \|\mu\| < \infty\}$$

We can easily see that $\mathcal{C}_k(X)$ is a vector space over $\mathbb{R}$ with natural addition and multiplication. Indeed, the determination set of $a \cdot \mu$ is the same as for μ, and determination set of $\mu_1 + \mu_2$ is the union of respective determination sets.

We can also define relative chains in X with respect to some subspace W. But first notice some useful fact [4, Proposition 1.10]

Lemma 17 *Let Ω be a topological space and let Ψ be its subspace, then*

$$\mathcal{B}(\Psi) = \Psi \cap \mathcal{B}(\Omega),$$

where $\mathcal{B}(\cdot)$ denotes Borel sets and $B \cap \mathcal{A} := \{B \cap A | A \in \mathcal{A}\}$ for a set B and a family of sets $\mathcal{A}$.

Proof. It is easy to see, that $\Psi \cap \mathcal{B}(\Omega)$ is a σ-algebra on Ψ, which contains all open sets. Thus from the minimality property: $\mathcal{B}(\Psi) \subset \Psi \cap \mathcal{B}(\Omega)$.

To prove the other inclusion we will use the fact, that if $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras in Ψ and $\Omega \setminus \Psi$ respectively, then $\mathcal{F} \cup \mathcal{G} := \{A \cup B | A \in \mathcal{F}, B \in \mathcal{G}\}$ is a σ-algebra in Ω. We can easily check, that $(\Omega \setminus \Psi) \cap \mathcal{B}(\Omega)$ is a σ-algebra in $\Omega \setminus \Psi$. Hence $\mathcal{B}(\Psi) \cup [(\Omega \setminus \Psi) \cap \mathcal{B}(\Omega)]$ is a σ-algebra in Ω.

Because Ψ has its topology induced from Ω, the above σ-algebra contains all open sets. Thus from the minimality property we have: $\mathcal{B}(\Omega) \subset \mathcal{B}(\Psi) \cup [(\Omega \setminus \Psi) \cap \mathcal{B}(\Omega)]$. Applying operation $\Psi \cap$ on both sides we have: $\Psi \cap \mathcal{B}(\Omega) \subset \mathcal{B}(\Psi)$.

□

The next lemma allows us to identify measures on the subspace Ψ with measures on Ω.

Lemma 18 *Let ν be a finite quasicompactly determined Borel measure on $\Psi \subset \Omega$, then*

$$\mu(\cdot) := \nu(\cdot \cap \Psi)$$

is finite quasicompactly determined Borel measure on Ω.

Proof. The measure μ is a well-defined Borel measure by Lemma 17.

$$\|\mu\| = (\mu^+ + \mu^-)(\Omega) = (\nu^+ + \nu^-)(\Psi) = \|\nu\| < \infty,$$

hence μ is finite. Apart from that the determination set for ν is also the determination set for μ. Thus μ is a finite quasicompactly determined Borel measure.

□

At first if W is a subspace of the topological space X, then $C^0(\Delta_k, W) \subset C^0(\Delta_k, X)$. Secondly we can see, that the topology of $C^0(\Delta_k, W)$ that is induced from $C^0(\Delta_k, X)$ (which is compact-open topology) coincides with compact-open topology on $C^0(\Delta_k, W)$. Hence by Lemma 18 we can treat $\mathcal{C}_k(W)$ as a submodule of $\mathcal{C}_k(X)$ and consequently we can define relative chains.

Definition 19 *(relative chain module). Let W be a subspace of the topological space X. We define a relative chain module $\mathcal{C}_k(X, W)$ as the following quotient module*

$$\mathcal{C}_k(X, W) := \mathcal{C}_k(X)/\mathcal{C}_k(W).$$

4.2 The boundary operator

In this section we will outline a general technique which allows us to extend the boundary operator from singular homology to Milnor-Thurston homology [4, Theorem 2.1].

Theorem 20 *Let $f_{i,j} : C^0(\Delta_{k_{i-1}}, X_{i-1}) \to C^0(\Delta_{k_i}, X_i)$ be continuous functions for $i \in \{1, 2, ..., N\}, j \in \{1, 2, ..., n_i\}$. Then*

(i) Each of these operators has an extension to a homomorphism $f_{i,j} : \mathcal{C}_{k_{i-1}}(X_{i-1}) \to \mathcal{C}_{k_i}(X_i)$

(ii) Let $\alpha_{i,j} \in \mathbb{R}$. If

$$\left(\sum_{j_N=1}^{n_N} \alpha_{N,j_N} \cdot f_{N,j_N}\right) \circ \left(\sum_{j_{N-1}=1}^{n_{N-1}} \alpha_{N-1,j_{N-1}} \cdot f_{N-1,j_{N-1}}\right) \circ \dots \circ \left(\sum_{j_1=1}^{n_1} \alpha_{1,j_1} \cdot f_{1,j_1}\right) = 0$$

in singular homology theory, then the analogous product in Milnor-Thurston homology theory vanishes also.

We will outline the construction of induced homomorphisms. For full a proof of the theorem see [4].

For $f_{i,j} : C^0(\Delta_{k_{i-1}}, X_{i-1}) \to C^0(\Delta_{k_i}, X_i)$ the induced homomorphism (also denoted by $f_{i,j}$) is simply the function which maps measure μ to its *image measure* (i.e. the measure $B \mapsto \mu(f_{i,j}^{-1}(B))$).

We can check that when D is determination set for μ, then $f_{i,j}(D)$ is determination set for $f_{i,j}(\mu)$. Consequently if μ is quasicompactly determined, so is $f_{i,j}(\mu)$. What is more $\|f_{i,j}(\mu)\| \le \|\mu\|$ thus $f_{i,j}$ can be seen as maps $\mathcal{C}_{k_{i-1}}(X_{i-1}) \to \mathcal{C}_{k_i}(X_i)$. We can prove that $f_{i,j}$ are compatible with addition and multiplication, so they are in fact homomorphisms.

□

The above theorem will be used as a tool for extending statements of singular theory onto Milnor-Thurston theory. The simple examples of its application could be the definition of the boundary operator, and the proof of homotopy invariance. In both cases the following lemma will be useful.

Lemma 21 *(Some properties of compact-open topology)*

(i) Let $f : X \to Y$ be continuous. Then $C^0(\Delta_k, X) \to C^0(\Delta_k, Y), \sigma \mapsto f \circ \sigma$ is continuous.

(ii) Let $\tau \in C^0(\Delta_l, \Delta_k)$. Then $C^0(\Delta_k, X) \to C^0(\Delta_l, X), \sigma \mapsto \sigma \circ \tau$ is continuous.

(iii) The map $C^0(\Delta_k, X) \to C^0(\Delta_k \times I, X \times I), \sigma \mapsto \sigma \times \mathrm{id}_I$ is continuous.

Proof. Statements (i) and (ii) are basic properties of compact-open topology. One can find a simple proof in [11, Chapter XII.2].

Let us prove (iii). Denote our map $\sigma \mapsto \sigma \times \mathrm{id}_I$ by g. From [11, Chapter XII.5.2] we know that the family

$$\{(A, U \times V) \mid A \subset \Delta_k \times I \text{ is compact },\quad U \text{ is open subset of } X, V \text{ is open subset of } I\}$$

is a subbasis for $C^0(\Delta_k \times I, X \times I)$, where

$$(A, W) := \{\tau \in C^0(\Delta_k \times I, X \times I) \mid \tau(A) \subset W\}.$$

Hence it is sufficient to prove that $g^{-1}((A, U \times V))$ is open.

Let π_1 denote the projection of $\Delta_k \times I$ to the first coordinate and let π_2 denote the projection to the second coordinate. The set $\pi_1(A)$ is compact, as a consequence

$$F = \{\tau \in C^0(\Delta_k, X) \mid \tau(\pi_1(A)) \subset U\}$$

is open in $C^0(\Delta_k, X)$.

We will prove, that $g^{-1}((A, U\times V)) = F$. Let $\tau \in g^{-1}((A, U\times V))$ and let $x \in \pi_1(A)$. Then there exists $y \in I$ such that $(x, y) \in A$. Finally $g(\tau)(x, y) \in U \times V$, hence $\tau(x) \in U$. Consequently $\tau \in F$. Hence $g^{-1}((A, U \times V)) \subset F$.

Conversely, let $\tau \in F$ and let $(x, y) \in A$. By $g(\tau)(x, y) = (\tau(x), y)$ we have $\tau(x) \in U$. Additionally $\pi_2(A) \in V$ unless $g^{-1}((A, U \times V))$ is empty. Thus $y \in V$. Finally $\tau \in g^{-1}((A, U \times V))$.

This way we proved, that $g^{-1}((A, U \times V)) = F$, which is open.

□

There is the canonical map $\delta_i : \Delta_{k-1} \to \Delta_k$ which embeds the standard simplex Δ_{k-1} as the ith face of Δ_k. According to Lemma 21 the function $\partial_i : \sigma \mapsto \sigma \circ \delta_i$ is continuous. Thus by the Theorem 20 the map ∂_i extends to the homomorphism $\mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$.

The boundary operator is defined in the following way:

$$\partial := \sum_{i=0}^{n} (-1)^i \partial_i.$$

Naturally $\partial^2 = 0$ in singular the homology theory, hence by Theorem 20 $\partial^2 = 0$ in Milnor-Thurston theory. Therefore $\mathcal{C}_*(X)$ is a chain-complex and we can define homology modules following the standard scheme described below.

Let $W \subset X$. Lemma 21 together with Theorem 20 are useful to see that inclusion map $i : W \to X$ induces a monomorphism $i : \mathcal{C}_*(W) \to \mathcal{C}_*(X)$. As was mentioned before, we can treat $\mathcal{C}_k(W)$ as submodule of $\mathcal{C}_k(X)$ this monomorphism i is therefore compatible with ∂. Hence it induces boundary operator ∂ on relative chain modules $\mathcal{C}_k(X, W)$.

4.3 The Milnor-Thurston homology modules

Now we can define homology modules using the standard scheme.

Definition 22 *(homology modules). Given the pair (X, W), where X is a topological space and W is its subspace we define the cycle module $\mathcal{Z}_k(X, W)$, the boundary module $\mathcal{B}_k(X, W)$ and the homology module $\mathcal{H}_k(X, W)$*

$$\begin{aligned}
\mathcal{Z}_k(X, W) &= \mathrm{Ker}\{\partial : \mathcal{C}_k(X, W) \to \mathcal{C}_{k-1}(X, W)\} \\
\mathcal{B}_k(X, W) &= \mathrm{Im}\{\partial : \mathcal{C}_{k+1}(X, W) \to \mathcal{C}_k(X, W)\} \\
\mathcal{H}_k(X, W) &= \mathcal{Z}_k(X, W)/\mathcal{B}_k(X, W)
\end{aligned}$$

For $W = \varnothing$ we have the absolute case.

This homology modules have the usual properties:

Theorem 23 *Let $f : (X, W) \to (Y, V)$ be a continuous mapping between pairs of topological spaces.*

(i) The map f induces a chain map between complexes $\mathcal{C}_*(X, W) \to \mathcal{C}_*(Y, V)$.

(ii) On the level of homology induced homomorphisms $f_{*k} : \mathcal{H}_k(X, W) \to \mathcal{H}_k(Y, V)$ are defined.

(iii) These homomorphisms satisfy $(f \circ g)_{*k} = f_{*k} \circ g_{*k}$ whenever f and g are continuous, composible mappings between pairs of topological spaces. Additionally $\mathrm{id}_{*n} = \mathrm{id}$.

Proof. *(i)* Let us start with the absolute case. We know, that f induces a continuous map on k-dimensional singular simplices $f_k : \sigma \mapsto f \circ \sigma$. We see that

$$(\partial_i \circ f_k)(\sigma) = f \circ \sigma \circ \delta_i = (f_{k-1} \circ \partial_i)(\sigma).$$

Hence f_k commutes with δ_i, so it commutes with ∂. Consequently the induced homomorphisms commute with ∂, hence f induces a chain mapping.

For the relative case we need to observe, that f_k commutes also with the inclusion operator.

(ii) It is a standard conclusion in homological algebra, that chain mappings induce homomorphisms on homology modules [6, p. 118].

(iii) It is easy to prove that $f_\bullet \circ g_\bullet = (f \circ g)_\bullet$. Then using standard methods of homological algebra we prove this property on the level of homology.

□

4.4 The coincidence of Milnor-Thurston homology theory with singular homology theory

It has been shown [4, 5] that Milnor-Thurson theory coincides with singular theory for simplicial complexes. One way to prove it is to check that Milnor-Thurston homology modules satisfy Eilenberg-Steenrod axioms. The proof is due to Zastrow, who uses the approach of Spanier's book [7].

4.4.1 The canonical map

There is the canonical chain map $C_*(X) \to \mathcal{C}_*(X)$, namely

$$\begin{aligned} \eta_k : C_k(X) &\to \mathcal{C}_k(X), \\ \sum_i \alpha_i \sigma_i &\mapsto \sum_i \alpha_i \mu_{\sigma_i} \end{aligned}$$

where $\alpha_i \in \mathbb{R}$, σ_i are singular simplices and μ_σ denotes an atomic measure.

We see, that the boundary operator in Milnor-Thurston theory is defined essentially by the same formula as in the singular theory. Hence η_k naturally commute with ∂, therefore η_k is a chain map. In the next sections it shall be proved, that this map on the level of homology is in fact an isomorphism.

4.4.2 Exactness Axiom

Theorem 24 *For Milnor-Thurston homology theory, there exists the long exact homology sequence.*

$$\dots \to \mathcal{H}_k(W) \to \mathcal{H}_k(X) \to \mathcal{H}_k(X,W) \to \mathcal{H}_{k-1}(W) \to \dots$$

for any topological space X and an arbitrary subspace W.

Proof. The Relative chain module is constructed as a quotient module. Hence there exists the short exact sequence

$$0 \to \mathcal{C}_*(A) \to \mathcal{C}_*(X) \to \mathcal{C}_*(X,A) \to 0.$$

Thus by the standard arguments of homological algebra (so called Zig-Zag Lemma) [6] there exists the long exact homology sequence.

□

4.4.3 Homotopy Invariance

Theorem 25 *Let $f \simeq g : (X,W) \to (Y,V)$ be continuous maps. Then $f_{*k} = g_{*k}$ for all $k \in \mathbb{N}_0$.*

Proof. *(1) Absolute case.* For every continuous $f, g : X \to Y$ such that $f \simeq g$, we have to construct a measure $\nu \in \mathcal{C}_{k+1}(Y)$ with $\partial\nu = f_\bullet(\mu) - g_\bullet(\mu)$ for a given $\mu \in \mathcal{Z}_k(X)$.

The standard approach to this problem is to create a chain homotopy, namely a map $D_k : \mathcal{C}_k(X) \to \mathcal{C}_{k+1}(Y)$ satisfying: $\partial_{k+1} \circ D_k + D_{k-1} \circ \partial_k = f_\bullet - g_\bullet$. We shall see that the construction of this chain-homotopy extends from singular theory to Milnor-Thurston theory.

The maps D_k in singular homology theory are constructed in the following way

$$D_k : \sigma \mapsto (H_\bullet \circ (\sigma \times \mathrm{id}_I)_\bullet)(c_{k+1})$$

where $\sigma \in C^0(\Delta_k, X)$.

The chain $c_k \in C_{k+1}(\Delta_k \times I)$ is any chain whose boundary contains singular simplices $\Delta_k \hookrightarrow \Delta_k \times \{0\}$, $\Delta_k \hookrightarrow \Delta_k \times \{1\}$ with opposite signs and the triangulation of $\partial\Delta_k \times I$ is consistent with the triangulation of c_{k-1}. One way to construct it is to use an appropriate triangulation of the prisma $\Delta_k \times I$.

Let $c_{k+1} = \sum_i \alpha_i \cdot \tau^i_{k+1}$ for singular simplices $\tau^i_{k+1} \in C^0(\Delta_{k+1}, \Delta_k \times I)$. Then using homomorphism property we can write D_k in the following way

$$D_k : \sigma \mapsto \sum_i \alpha_i \cdot (H_\bullet \circ (\sigma \times \mathrm{id}_I)_\bullet)(\tau^i_{k+1})$$

The map $C^0(\Delta_k, X) \to C_{k+1}(Y), \sigma \mapsto (H_\bullet \circ (\sigma \times \mathrm{id}_I)_\bullet)(\tau^i_{k+1})$ can be seen as the extension of the map $C^0(\Delta_k, X) \to C^0(\Delta_{k+1}, Y), \sigma \mapsto H \circ (\sigma \times \mathrm{id}_I) \circ \tau^i_{k+1}$ which is a composition of continuous maps by Lemma 21. Hence by Theorem 20 D_k can be extended to Milnor-Thurston chain modules. Consequently by standard arguments of homological algebra [6] we get $f_* = g_*$.

(2) Relative case. The construction of D_k is the same as in the absolute case. However in order to prove that D_k induces a chain homotopy on relative chains we have to show that $D_k(\mathcal{C}_k(W)) \subset \mathcal{C}_{k+1}(V)$. Suppose that $\mu \in \mathcal{C}_k(W)$, then it has a quasicompact determination set $D \subset C^0(\Delta_k, W)$. Because our homotopy respects the subspace structure, we see that for any $\sigma \in C^0(\Delta_k, W)$ its image due to the mapping $\sigma \mapsto H \circ (\sigma \times \mathrm{id}_I) \circ \tau^i_{k+1}$ lies in $C^0(\Delta_{k+1}, V)$. Thus the image of D under this mapping lies in $C^0(\Delta_{k+1}, V)$ and is a quasicompact determination set for the image measure.

Finally we see that D_k induces a chain homotopy on relative chains and by standard arguments of homological algebra we get $f_* = g_*$ on relative homology modules.

□

Corollary 26 *If $f : X \to Y$ is a homotopy equivalence, then f_* is isomorphism.*

Proof. It is an easy consequence of Theorem 25.

□

4.4.4 Initial Axiom

Theorem 27 *Let $*$ denote the space containing a single point. Then $\mathcal{H}_k(*) = 0$ for all $k \in \mathbb{N}$ and $\mathcal{H}_0(*) = \mathbb{R}$.*

Proof. We see that there is only one simplex in each dimension (namely the constant map). So every measure is of the form $\alpha \cdot \mu_\sigma$ for $\alpha \in \mathbb{R}$. Hence there exist isomorphisms between $\mathcal{C}_k(*)$ and $C_k(*)$. Therefore homology modules in Milnor-Thurston theory are trivial, just as in singular theory.

□

4.4.5 Excision Axiom

The excision theorem for Milnor-Thurston theory has a different form than for singular homology. Currently it is not known if the classical version of this axiom is true for measure homology. Nevertheless for normal spaces this theorem is equivalent with standard excision theorem.

Theorem 28 *Let $U \subset W \subset X$ and assume there exists V with*

$$\bar{U} \subset \overset{\circ}{V} \subset \bar{V} \subset W$$

For a proof see [4, Theorem 4.1].

□

4.4.6 The coincidence with singular homology theory

The preceding properties allow us to prove the coincidence of Milnor-Thurston homology theory with singular homology theory for certain topological spaces.

Theorem 29 *Let X be a topological space which can be triangulated as a simplicial complex and let Y be its subspace which can be realised as a subcomplex to some triangulation of X. Then there exist natural isomorphisms $H_k(X) \to \mathcal{H}_k(X)$, $H_k(X,Y) \to \mathcal{H}_k(X,Y)$.*

Proof. *(1) Finite case.* The proof is by the induction with respect to the number of simplices of the complex K. For the first step notice, that for the complex with only one simplex we have two pairs of topological spaces to be considered. Namely $(*,*)$ and $(*,\varnothing)$. In the first case all relative chain modules are zero for $n > 0$ in both theories, so homology modules obviously coincide. Similarly in the second case homology modules are zero for $n > 0$ by the Initial Axiom.

Now suppose that the theorem is true for complexes with at most k simplices. We will be considering the following exact sequences of homology modules

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_{n+1}(K,L) & \longrightarrow & H_n(L) & \longrightarrow & H_n(K) & \longrightarrow \\
 & \downarrow \eta_* & & \downarrow \eta_* & & \downarrow \eta_* & \\
\cdots \longrightarrow & \mathcal{H}_{n+1}(K,L) & \longrightarrow & \mathcal{H}_n(L) & \longrightarrow & \mathcal{H}_n(K) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & H_n(K,L) & \longrightarrow & H_{n-1}(L) & \longrightarrow & H_{n-1}(K) & \longrightarrow \cdots \\
 & \downarrow \eta_* & & \downarrow \eta_* & & \downarrow \eta_* & \\
\longrightarrow & \mathcal{H}_n(K,L) & \longrightarrow & \mathcal{H}_{n-1}(L) & \longrightarrow & \mathcal{H}_{n-1}(K) & \longrightarrow \cdots
\end{array} \quad (4)$$

Let K be a complex with $k+1$ simplices and let us start with $L = \Delta$ being a simplex of K with the largest possible dimension. Without loss of generality we can assume that $K \neq L$, otherwise the case is trivial, since every simplex is homotopy equivalent to a single point.

Let P be the barycentre of Δ. By the Excision Axiom and homotopy invariance

$$\mathcal{H}_n(K, \Delta) \cong \mathcal{H}_n(K \setminus P, \Delta \setminus P) \cong \mathcal{H}_k(K \setminus \mathring{\Delta}, \partial\Delta).$$

Hence the situation has been reduced to the case with one simplex less.

By the assumption we see that the vertical arrows corresponding to relative homology modules in diagram (4) are isomorphisms. In addition, L has less than $k+1$ simplices so arrows corresponding to homology modules of L are isomorphisms. By the Five Lemma [7] arrows corresponding to the homology modules of K are isomorphisms.

Now consider an arbitrary subcomplex L of the complex K. Then in diagram (4) the arrows corresponding to K and L are isomorphisms, so again by the Five Lemma relative modules are isomorphic. This completes the induction argument.

(2) Infinite case. Suppose $\eta_{*n} : H_n(K) \to \mathcal{H}_n(K)$ is not an isomorphism. It means there exists a "violating measure" which is either n-dimensional and has no preimage with respect to η_{*n} (in this case η_{*n} is not an epimorphism), or which is $(n+1)$-dimensional and its boundary is an image of a non-nullhomologic cycle in $Z_n(K)$ (η_{*n} is not a monomorphism).

Let D denote a determination set of the "violating measure". The image of this measure is quasicompact as an image of the quasicompact set $D \times \Delta_n$ with respect to a continuous mapping $F : D \times \Delta_n \to K, \sigma \times x \mapsto \sigma(x)$. Hence the image is contained in some finite subcomplex L. Thus the violating measure can be considered as a chain in $\mathcal{C}_n(L)$.

From considering the diagram

$$\begin{array}{ccc} H_n(L) & \longrightarrow & H_n(K) \\ \big\downarrow{\scriptstyle \eta_*} & & \big\downarrow{\scriptstyle \eta_*} \\ \mathcal{H}_n(L) & \longrightarrow & \mathcal{H}_n(K) \end{array}$$

we see that independently of whether η_* is not an epimorphism or it is not a monomorphism, we get a contradiction. Hence η_* must be an isomorphism.

□

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Appendix A Comments on Thurston's proof of Gromov theorem

In this appendix we present formal proofs of some facts used by Thurston in his proof of the Gromov Theorem [2, Theorem 6.2].

Appendix A.1 On the Gromov norm for homology theory with C^1 simplices

Thurston proves the inequality (1) using singular homology theory, and then he arguments that the inequality holds for measure homology also by the uniformity of straightening process. We will prove this fact using recent results.

We consider four homology theories: singular homology with C^1 simplices, measure homology with C^1 simplices, singular homology with continuous simplices and measure homology with continuous simplices. The homology modules are denoted: H^d, $\mathcal{H}^d$, H and $\mathcal{H}$ respectively. Zastrow [4, Theorem 3.4] proved that in case of a differentiable manifold M these four are isomorphic

$$H_k^d(M) \cong \mathcal{H}_k^d(M) \cong H_k(M) \cong \mathcal{H}_k(M),$$

and isomorphisms are induced by the canonical inclusions.

The norms on these homology modules are denoted: $\|\cdot\|_s^d$, $\|\cdot\|_m^d$, $\|\cdot\|_s$, $\|\cdot\|_m$, where the index s stands for *singular*, m stands for *measure* and d stands for *differentiable*. They are defined as an infimum to the norms of respective cycles (see definitions 10 and 11). Because of canonical inclusions there is an inequality between these norms

$$\|\cdot\|_m \leq \|\cdot\|_m^d \leq \|\cdot\|_s^d.$$

Löh [3] proved that the measure homology and the singular homology are isometrically isomorphic. Hence

$$\|\cdot\|_s = \|\cdot\|_m,$$

therefore

$$\|\cdot\|_s \leq \|\cdot\|_m^d \leq \|\cdot\|_s^d.$$

Let α be any singular chain of continuous symplices. Then straight(α) is a C^1-chain with

$$\|\text{straight}(\alpha)\| \leq \|\alpha\|,$$

which belongs to the same homology class as α (see Appendix B). Therefore

$$\| \cdot \|_s^d \leq \| \cdot \|_s$$

and finally

$$\| \cdot \|_m^d = \| \cdot \|_s^d.$$

Appendix A.2 On the volume covered by a chain

The fundamental class of an n-dimensional differentiable manifold M is a generator of $H_n(M, \mathbb{Z})$. Therefore we have two choices for $[M]$ depending on the orientation.

In the case of homology modules over $\mathbb{R}$, the fundamental class is defined by the canonical inclusion of $\mathbb{Z}$ to $\mathbb{R}$. Every n-dimensional homology class is a real multiplicity of the fundamental class.

The fundamental class can be seen as the set of cycles, which cover the whole manifold. We will define volume covered by a chain, and we will prove that it depends only on the homology class. It is a simple tool which allows us to determine homology class of a given n-cycle for a differentiable n-dimensional manifold.

Let $\mathcal{C}_n^d(M)$ denote compactly supported measures on $C^1(\Delta_n, M)$, where M is an n-dimensional hyperbolic manifold with volume form Ω.

Definition 30 *(volume covered by a chain). Let $\mu \in \mathcal{C}_n^d(M)$, then we call*

$$v(\mu) = \int_{C^1(\Delta_n, M)} \left(\int_{\Delta_n} \sigma^* \Omega \right) d\mu(\sigma)$$

the volume covered by the chain μ.

This definition can be naturally extended to continuous simplices, however in the proof of Gromov theorem we need only C^1 version.

Every hyperbolic manifold has a differential structure, therefore it can be triangulated [10, Theorem 10.6]. Consequently the fundamental class $[M]$ is homology class of the cycle α, where

$$\alpha = \sum_{i=1}^{m} \sigma_i,$$

for some singular simplices σ_i, with $v(\sigma_i(\Delta_n) \cap \sigma_j(\Delta_n)) = 0$ for each i, j and $\bigcup_{i=1}^m \sigma_i(\Delta_n) = M$. We see that (without loss of generality we can assume that the orientation of simplices agrees with orientation of our manifold)

$$v(\alpha) = \sum_{i=1}^{m} \int_{\Delta_n} \sigma_i^* \Omega = \sum_{i=1}^{m} v(\sigma_i(\Delta_n)) = v(M).$$

We shall prove that the volume covered by a cycle depends only on its homology class. Let $\partial = \sum_{i=0}^{n}(-1)^i\partial_i$ be a boundary operator.

Lemma 31 *Let σ be $(n+1)$-dimensional singular simplex. Then $v(\partial\sigma) = 0$.*

Proof. There is a theory of integration on chains of differential forms. A detailed description can be found in [9, Chapter 5]. Here we present main results.

Let ω be a k-form and let σ be a singular k-simplex. The integral over σ is defined as

$$\int_\sigma \omega := \int_{\Delta_k} \sigma^*\omega.$$

Next, let $c = \sum_i \alpha_i\sigma_i$ be some k-chain. We define the integral over c in the following way:

$$\int_c \omega := \sum_i \alpha_i \int_{\sigma_i} \omega.$$

Theorem 32 *(Stokes). Let ω be a k-form and let c be some $(k+1)$-chain. Then*

$$\int_{\partial c} \omega = \int_c d\omega,$$

where d denotes exterior derivative.

We can see that

$$v(\partial\sigma) = \sum_{i=0}^{n}(-1)^i \int_{\Delta_n} (\partial_i\sigma)^*\Omega = \sum_{i=0}^{n}(-1)^i \int_{\partial_i\sigma} \Omega = \int_{\partial\sigma} \Omega.$$

Hence by the Stokes theorem

$$v(\partial\sigma) = \int_\sigma d\Omega.$$

The volume form Ω in local coordinates $(x_1, x_2, ..., x_n)$ can be expressed as

$$\Omega = f(x_1, x_2, ...x_n)dx_1 \wedge dx_2 \wedge ... \wedge dx_n$$

where f is a function defined on some open neighbourhood in $\mathbb{R}^n$. Consequently

$$d\Omega = \sum_{\alpha=1}^{n} D_\alpha f(x_1, x_2, ...x_n)dx_\alpha \wedge dx_1 \wedge dx_2 \wedge ... \wedge dx_n = 0$$

Because $dx_i \wedge dx_i = 0$ for any i. Therefore $v(\partial\sigma) = 0$.

□

Theorem 33 *Let $\mu \in \mathcal{C}_n^d(M)$ be a cycle. Then $v(\mu)$ depends only on the homology class of μ.*

Proof. It is sufficient to prove that for any $\nu \in \mathcal{C}_{n+1}^d(M)$ we have $v(\partial\nu) = 0$. We see that

$$\begin{aligned} v(\partial\nu) &= \int_{C^1(\Delta_n,M)} \left(\int_{\Delta_n} \tau^*\Omega \right) d\partial\nu(\tau) \\ &= \sum_{i=0}^{n+1} (-1)^i \int_{C^1(\Delta_n,M)} \left(\int_{\Delta_n} \tau^*\Omega \right) d\partial_i\nu(\tau) \end{aligned}$$

When μ is a measure on a space X, f denotes a measurable function $X \to Y$ and $f\mu$ is an image measure, then $\int_Y g d(f\mu) = \int_X g \circ f d\mu$. Therefore

$$\begin{aligned} v(\partial\nu) &= \sum_{i=0}^{n+1} (-1)^i \int_{C^1(\Delta_{n+1},M)} \left(\int_{\Delta_n} (\partial_i\sigma)^*\Omega \right) d\nu(\sigma) \\ &= \int_{C^1(\Delta_{n+1},M)} \left(\sum_{i=0}^{n+1} (-1)^i \int_{\Delta_n} (\partial_i\sigma)^*\Omega \right) d\nu(\sigma) \\ &= \int_{C^1(\Delta_{n+1},M)} v(\partial\sigma) d\nu(\sigma). \end{aligned}$$

Hence by the Lemma 31

$$v(\partial\nu) = 0.$$

□

Appendix B The theory of hyperbolic manifolds

In this appendix we present useful properties of hyperbolic manifolds. The theory presented here is based on the introductory textbook of the subject [8].

The Minkowski n-space (or Lorentzian n-space) is $\mathbb{R}^n$ equipped with a metric tensor

$$(x, y)_n = -x_0y_0 + x_1y_1 + \dots + x_{n-1}y_{n-1}.$$

Naturally it has a structure of pseudo-Riemannian manifold.

A function which preserves the above form is called a *Lorentz transformation.* Hence a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation when

$$(f(x), f(y))_n = (x, y)_n.$$

One can prove that every Lorentz transformation is linear [8, Theorem 3.1.3].

The vector $x \in \mathbb{R}^n$ is called *positive time-like* vector if $(x,x)_n < 0$ and $x_0 > 0$. The Lorentz transformation f is called positive if it transforms positive time-like vectors to positive time-like vectors.

We construct the hyperbolic space $\mathbb{H}^n$ as a subspace of Minkowski $(n+1)$-space

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid (x,x)_{n+1} = -1, x_0 > 0\}.$$

For $x, y \in \mathbb{H}^n$ we define $\eta(x,y) := -\text{arcosh}((x,y)_{n+1})$. It has been proved that η satisfies axioms of a metric (see [8, p. 61]). Hence $\mathbb{H}^n$ is naturally a metric space.

We see that every positive Lorentz transformation of Minkowski $(n+1)$-space induces an isometry on $\mathbb{H}^n$. However the converse is also true [8, Theorem 3.2.3]:

Theorem 34 *Every isometry of $\mathbb{H}^n$ extends to a unique positive Lorentz transformation of $\mathbb{R}^{n+1}$.*

Let $\text{Isom}(\mathbb{H}^n)$ denote the group of isometries of the hyperbolic n-space. Let M be an n-manifold. A hyperbolic atlas Φ is the collection of functions

$$\Phi = \{\phi_i : U_i \to \mathbb{H}^n\}$$

called charts, satisfying the following conditions:

(i) Each U_i is a connected open subset of M,

(ii) Each chart map ϕ_i maps U_i into open subset of $\mathbb{H}^n$ homeomorphically,

(iii) The coordinate neighbourhoods U_i cover M,

(iv) If U_i and U_j overlap, then the function

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

agrees in a neighbourhood of each point of its domain with an element of $\text{Isom}(\mathbb{H}^n)$.

The notion of hyperbolic atlas is equivalent to $(\mathbb{H}^n, \text{Isom}(\mathbb{H}^n))$-atlas in Ratcliffe's book [8], where the more general theory is presented.

For every hyperbolic atlas Φ, there exists the unique maximal hyperbolic atlas [8, Theorem 8.3.1]. Hence we can make the following definition

Definition 35 *(hyperbolic manifold). An ordered pair (M, Φ), where M is n-manifold and Φ is a maximal hyperbolic atlas, is called a hyperbolic manifold.*

Every hyperbolic manifold is also a metric space (see [8, Theorem 8.3.4]). Additionally its manifold topology is determined by the metric structure, and every chart is a local isometry (see [8, Theorems 8.3.5 and 8.3.6]).

Functions that preserve hyperbolic structures are called hyperbolic maps (Ratcliffe uses the name $(\mathbb{H}^n, \mathrm{Isom}(\mathbb{H}^n))$-maps).

Definition 36 *Let M and N be hyperbolic manifolds of dimension n. We say that $f : M \to N$ is a hyperbolic map, if f is continuous and for each chart $\phi : U \to \mathbb{H}^n$ on M and each chart $\psi : V \to \mathbb{H}^n$ on N such that U and $f^{-1}(V)$ overlap, the function*

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \phi(f(U) \cap V)$$

agrees in a neighbourhood of each point of its domain with an element of $\mathrm{Isom}(\mathbb{H}^n)$.

Bijective hyperbolic map is called hyperbolic equivalence (Ratcliffe uses the name $(\mathbb{H}^n, \mathrm{Isom}(\mathbb{H}^n))$-equivalence [8]). It is obviously an isometry of hyperbolic manifolds. We say that hyperbolic manifolds M and N are equivalent, when there exists hyperbolic equivalence between them.

It has been proved that every complete connected hyperbolic manifold M is equivalent to $\mathbb{H}^n/\Gamma$ where Γ is some subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ (see [8, Theorem 8.5.9]). In addition the fundamental group $\pi_1(M)$ is isomorphic to Γ.

Additionally we have the very useful theorem (see [8, Theorem 8.1.3]):

Theorem 37 *Let Γ be a group of isometries of $\mathbb{H}^n$ that acts freely and discontinuously on $\mathbb{H}^n$. Then the quotient map*

$$p : \mathbb{H}^n \to \mathbb{H}^n/\Gamma$$

is a local isometry and a covering projection. Furthermore Γ is a group of covering transformations.

By the above theorem every complete connected hyperbolic manifold has $\mathbb{H}^n$ as its universal cover.

Appendix B.1 Barycentric coordinates and straight simplices

Let $\{x_i\}_{i=1}^k$ be points in the hyperbolic space $\mathbb{H}^n$. We can define convex combination of points with non-negative coefficients $\{q_i\}_{i=1}^k$, where $\sum_{i=1}^k q_i = 1$.

Let $y = \sum_{i=1}^{k} q_i x_i$, where Σ is addition in $\mathbb{R}^{n+1}$. We see that y is time-like vector

$$(y, y)_{n+1} = \sum_{i,j=1}^{k} q_i q_j (x_i, x_j)_{n+1} = -\sum_{i,j=1}^{k} q_i q_j \cosh \eta(x_i, x_j) < 0.$$

Thus we can define the convex combination of $\{x_i\}_{i=1}^{k} \subset \mathbb{H}^n$ with coefficients q_i in the following way

$$q_1 x_1 * q_2 x_2 * ... * q_n x_n := -y/(y, y)_{n+1}.$$

If f is an isometry, then it extends to the unique linear transformation in Minkowski space. Consequently we have the following

Lemma 38 *Let f be an isometry of $\mathbb{H}^n$, then*

$$f(q_1 x_1 * q_2 x_2 * ... * q_k x_k) = q_1 f(x_1) * q_2 f(x_2) * ... * q_k f(x_k).$$

Proof. We know that f extends to the positive Lorentz transformation, which will be also denoted by f. We have

$$f(q_1 x_1 * q_2 x_2 * ... * q_k x_k) = f(-y/(y, y)_{n+1}),$$

where

$$y = \sum_{i=1}^{k} q_i x_i.$$

Since Lorentz transformations are linear, we have

$$f(-y/(y, y)_{n+1}) = -\frac{1}{(y, y)_{n+1}} \sum_{i=1}^{k} q_i f(x_i).$$

The definition of Lorentz transformation yields $(y, y)_{n+1} = (f(y), f(y))_{n+1}$. As a consequence

$$-\frac{1}{(y, y)_{n+1}} \sum_{i=1}^{k} q_i f(x_i) = q_1 f(x_1) * q_2 f(x_2) * ... * q_k f(x_k),$$

which ends proof of the lemma.

□

Let $(q_0, q_1, ..., q_k)$ be coordinates in standard simplex Δ_k. Given points $\{x_i\}_{i=1}^k \subset \mathbb{H}^n$ we can define a function $\tau : \Delta_k \to \mathbb{H}^n$ by

$$\tau(q_0, q_1, ..., q_k) = q_0 x_0 * q_1 x_1 * ... * q_n x_n.$$

We shall call the function τ *straight simplex* with vertices $\{x_i\}_{i=1}^k$.

Given a hyperbolic manifold M and a singular simplex $\sigma : \Delta_k \to M$, we can define straightening process. Let $\tilde{\sigma}$ be any lifting of σ to $\mathbb{H}^n$ (the lifting exists because Δ_k is path connected, locally path connected and simply connected). Let straight$(\tilde{\sigma})$ be a straight simplex with the same vertices as $\tilde{\sigma}$. Then we define straight$_M(\sigma)$ as the projection of straight$(\tilde{\sigma})$ back to M.

This construction is independent of the choice of a lifting. Let $\hat{\sigma}$ be another lifting of σ. Then there exists covering transformation $f : \mathbb{H}^n \to \mathbb{H}^n$ such that $\hat{\sigma} = f \circ \tilde{\sigma}$. Each of this transformations is an isometry (see Theorem 37), thus by Lemma 38 we have straight$(\hat{\sigma}) = f \circ$ straight$(\tilde{\sigma})$, therefore straight$_M(\sigma)$ is independent of the lifting.

Lemma 39 *Let* $\tau \in C^0(\Delta_k, \mathbb{H}^n)$, *then*

$$\partial_i \text{straight}(\tau) = \text{straight}(\partial_i \tau).$$

Proof. Let $(r_0, r_1, ..., r_k) \in \Delta_k$, then by the definition of straight simplex

$$\text{straight}(\tau)(r_0, r_1, ..., r_k) = r_0 \tau(e_0) * r_1 \tau(e_1) * ... * r_k \tau(e_k), \tag{5}$$

where e_i denotes ith vector of the canonical base of $\mathbb{R}^{n+1}$. Let $(q_0, q_1, ..., q_{k-1}) \in \Delta_{k-1}$, then

$$(\partial_i \tau)(e_j) = \tau(\delta_i(e_j)).$$

As a consequence

$$\text{straight}(\partial_i \tau)(q_0, q_1, ..., q_{k-1}) = \\ q_0 \tau(e_0) * q_1 \tau(e_1) * ... * q_{i-1} \tau(e_{i-1}) * q_i \tau(e_{i+1}) * ... * q_{k-1} \tau(e_k).$$

When calculating ∂_istraight, we need to substitute q_j for r_j if $j < i$, 0 for r_i and q_j for r_{j+1} if $j \geq i$ in formula (5). This way we obtain

$$\partial_i \text{straight}(\tau)(q_0, q_1, ..., q_{k-1}) = \\ q_0 \tau(e_0) * q_1 \tau(e_1) * ... * q_{i-1} \tau(e_{i-1}) * q_i \tau(e_{i+1}) * ... * q_{k-1} \tau(e_k),$$

which ends proof of the lemma.

□

Consequently, we see that straight can be extended to a chain homomorphism. In addition, since ∂_i is just a composition of singular simplex with δ_i, we have

$$\partial_i(t\tilde{\sigma} * (1-t)\text{straight}(\tilde{\sigma})) = t\partial_i\tilde{\sigma} * (1-t)\text{straight}(\partial_i\tilde{\sigma}).$$

Thus $t\tilde{\sigma} * (1-t)\text{straight}(\tilde{\sigma})$ is homotopy between $\text{straight}(\tilde{\sigma})$ and $\tilde{\sigma}$ which respects chain complex structure. Its projection to M gives homotopy between $\text{straight}_M(\sigma)$ and σ that has the same property. Finally, we see that the homotopy allows us to construct a chain homotopy between the identity and straight_M.

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